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On Stability Estimates for the Inviscid Boussinesq Equations

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Abstract
We consider the (in)stability problem of the inviscid 2D Boussinesq equations near a combination of a shear flow $v = (y, 0)$ and a stratified temperature $\theta = \alpha y$ with $\alpha > \frac{1}{4}$. We show that for any $\epsilon > 0$ there exist non-trivial explicit solutions, which are initially perturbations of size ϵ, and grow to size 1 on a time scale ϵ^{-2}. Moreover, the (simplified) linearized problem around these non-trivial states exhibits improved upper bounds on the possible size of norm inflation for frequencies larger and smaller than ϵ^{-4}.

Keywords Boussinesq equations · Inviscid · Resonances · Stability

Mathematics Subject Classification 35Q35 · 35Q79 · 76B03 · 35B40

1 Introduction and Main Results

In this article we consider the stability of the incompressible, inviscid Boussinesq equations in a two-dimensional periodic channel

$$\begin{aligned} \partial_t v + v \cdot \nabla v + \nabla p &= \theta e_2, \\ \partial_t \theta + v \cdot \nabla \theta &= 0, \\ \operatorname{div}(v) &= 0, \\ (t, x, y) &\in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}, \end{aligned} \tag{1}$$

near the stationary solution

$$v = (y, 0), \ \theta = \alpha y, \tag{2}$$

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where $\alpha > \frac{1}{4}$ is a constant.

The Boussinesq equations are a common model of the evolution of a heat conducting fluid in terms of its velocity v and temperature θ and may additionally incorporate viscosity or thermal dissipation. In particular, questions of well-posedness and asymptotic behavior in regimes with partial dissipation (Tao et al., 2020; Masmoudi et al., 2022; Deng et al., 2021; Jiahong et al., 2019; Doering et al., 2018; Li and Titi, 2016; Cao and Jiahong, 2013) or the inviscid problem (Elgindi and Widmayer, 2015; Widmayer, 2018) have been an area with strong research activity in recent years.

The term

$$\theta e_2$$

models buoyancy and causes hotter fluid to rise above colder fluid, where $-e_2$ is the direction of gravity. It is well known that, in the case without shear ($v = 0$), this buoyancy might lead to the so-called Rayleigh-Bénard instability, if hotter fluid is below colder fluid, $\alpha < 0$.

In contrast, if $\alpha > 0$ is sufficiently large then the Miles–Howard criterion (Howard, 1961) rules out spectral instability, which is the setting considered in this article. As we state in Lemma 1.2 and recall in Sect. 2 the linearized equations around the stationary solution (2) are stable in arbitrary Sobolev regularity globally in time. However, for the nonlinear equations we construct explicit solutions growing as $\epsilon(1 + t^2)^{1/4}$ as t increases, which hence are only small on a time scale $t < \epsilon^{-2}$. Moreover, even when restricting to this time scale, higher, Gevrey regularity is required in order to establish stability (Bedrossian et al., 2021; Tao and Jiahong, 2019). In the corresponding viscous problem instead Sobolev regularity is required, however with a smallness condition depending on the size of the viscosity (Zhai and Zhao, 2023).

The aims of the present article are twofold:

- For related equations such as the Euler equations (Deng and Masmoudi, 2018; Deng and Zillinger, 2021), Vlasov–Poisson equations (Mouhot and Villani, 2011; Bedrossian, 2020) or partially viscous Boussinesq equations (Zillinger, 2021) it is known that the norm inflation of the nonlinear dynamics is tied to the interaction of non-trivial low frequency solutions, which we call traveling waves, and their interaction with high frequency perturbations. We thus construct these traveling waves for the present problem and discuss for which choices of perturbations and parameters one might expect the largest possible norm inflation.
- For these linearized equations we identify multiple frequency regimes depending on the initial size $\epsilon > 0$ of the waves and the time interval under consideration. For frequencies $|\xi| < \epsilon^{-4}$ we establish an upper bound for perturbations concentrated at frequency ξ by $\exp((\epsilon\xi)^{2/3})$. In particular, if $|\xi| \leq \epsilon^{-\alpha}$ with $1 < \alpha \leq 4$ this factor is bounded by $\exp(\xi^{2/3(1-1/\alpha)})$. This bound hence matches the control by $\exp(\sqrt{\xi})$, that is Gevrey 2 regularity, as in the nonlinear problem (Bedrossian et al., 2021) for $\xi = \epsilon^{-4}$, but exhibits improved bounds if ξ is smaller. As a complementary result, if $\xi > C\epsilon^{-4}$ with a sufficiently large r constant $C > 1$, we instead obtain an upper bound which is uniform in ξ and ϵ on the time scale under consideration.

We remark that for technical reasons we consider a simplified model, which fixes the underlying shear flow. As we discuss in Sect. 4 this simplification can be removed in time intervals where the main norm inflation takes place and for large times. For small times we provide a rough bound for the non-simplified model, but expect that it can be improved to a uniform bound with substantial additional technical effort.

Before stating our main results, we recall that the linearized problem around the stationary solution (2) is stable, when working in coordinates moving with the shear and choosing suitable unknowns. The following lemma is adapted from Bedrossian et al. (2021); Tao et al. (2020).

Lemma 1.1 *Let $\alpha > \frac{1}{4}$. Then the linearized Boussinesq equations around the stationary solution (2) are stable in the sense that for any initial data ω, θ with $\int \omega dx = \int \theta dx = 0$ the energy*

$$\alpha \|((\partial_x^{-2}\Delta)^{-1/4}\omega)(t, x - ty, y)\|_{L^2}^2 + \|((\partial_x^{-2}\Delta)^{1/4}\theta)(t, x - ty, y)\|_{L^2}^2$$

is bounded above and below for all times, uniformly in terms of its initial value, with a constant depending only on α.

As we discuss in Sect. 2.1 the choice of unknowns moving with the underlying shear flow

$$\begin{aligned} Z(t, x, y) &:= \sqrt{\alpha}\left((\partial_x^{-2}\Delta)^{-1/4}\omega\right)(t, x - ty, y), \\ Q(t, x, y) &:= \left((\partial_x^{-2}\Delta)^{1/4}\partial_x\theta\right)(t, x - ty, y) \end{aligned} \tag{3}$$

is natural. These unknowns have previously been used in Bedrossian et al. (2021) and we use the same notation. We remark that in the (partially) viscous setting other choices of unknowns are natural (Adhikari et al., 2022; Adhikari et al., 2010; Lai et al., 2021; Tao et al., 2020; Tao and Jiahong, 2019; Doering et al., 2018; Cao and Jiahong, 2013; Zhai and Zhao, 2023).

We further observe that the linearized problem around the stationary solution (2) in terms of (Z, Q) reads

$$\begin{aligned} \partial_t \begin{pmatrix} Z \\ Q \end{pmatrix} &= \begin{pmatrix} \frac{1}{2}\frac{\partial_x(\partial_y - t\partial_x)}{\partial_x^2 + (\partial_y - t\partial_x)^2} & \sqrt{\alpha}\partial_x(\partial_x^2 + (\partial_y - t\partial_x)^2)^{-1/2} \\ -\sqrt{\alpha}\partial_x(\partial_x^2 + (\partial_y - t\partial_x)^2)^{-1/2} & -\frac{1}{2}\frac{\partial_x(\partial_y - t\partial_x)}{\partial_x^2 + (\partial_y - t\partial_x)^2} \end{pmatrix} \begin{pmatrix} Z \\ Q \end{pmatrix}, \\ &=: A \begin{pmatrix} Z \\ Q \end{pmatrix} \end{aligned}$$

Since the operator on the right-hand side is a (time-dependent) constant coefficient Fourier multiplier the evolution of (Z, Q) decouples in Fourier space with respect to both x and y. Therefore all stability estimates hold frequency-wise and hence extend to arbitrary Sobolev, Besov or Gevrey spaces. However, this stability can be understood as an artifact of the fact that the stationary solution (2) is independent of x and that perturbations therefore decouple in frequency and cannot propagate along chains of resonances. For this reason, in order to capture instabilities of the nonlinear problem, instead of a stationary solution we consider nearby x-dependent explicit solutions.

Lemma 1.2 *(compare Proposition 2.1 in Zillinger (2021) and Bedrossian et al. (2021)) Let $\alpha \geq 0$ be given, then there exist non-trivial functions $f(t)$ and $g(t)$ such that*

$$\begin{aligned} \omega(t,x,y) &= -1 + f(t)\cos(x-ty), \\ \theta(t,x,y) &= \alpha y + g(t)\sin(x-ty), \\ v(t,x,y) &= (y,0) + \frac{1}{1+t^2}\nabla^{\perp}\cos(x-ty), \end{aligned} \tag{4}$$

are a solution of the nonlinear inviscid Boussinesq equations for all times. We call these solutions traveling waves. *Moreover, if $\alpha > \frac{1}{4}$ it holds that*

$$E(t) := \frac{|\alpha|}{\sqrt{1+t^2}}|f(t)|^2 + \sqrt{1+t^2}|g(t)|^2$$

satisfies

$$cE(0) \leq E(t) \leq CE(0)$$

for some constants $0 < c < C < \infty$ depending on α.

We remark that in terms of the unknowns (3) these traveling waves read

$$\begin{aligned} Z(t,x,y) &= \frac{f(t)}{(1+t^2)^{1/4}}\cos(x), \\ Q(t,x,y) &= g(t)(1+t^2)^{1/4}\sin(x). \end{aligned}$$

They are global in time, low-frequency solutions and remain uniformly bounded in any suitable Sobolev or Gevrey space for all times.

The functions $f(t)$ and $g(t)$ can be computed explicitly in terms of hypergeometric functions and the above results can hence be obtained by explicit computation (as was done in Zillinger (2021)). Furthermore, as shown in Lemma 1.1 the stability can also be obtained as a special case of an energy estimate similar to the one of Bedrossian et al. (2021).

In the following we will consider a simplified version of the linearization of the Boussinesq equations in terms of (Z, Q) around these traveling waves and establish upper bounds on the possible norm inflation. The reduction of the non-simplified linearized Boussinesq equations is discussed in Sect. 4.

We note that for the traveling waves of Lemma 1.2 for large times the vorticity grows as

$$\|\omega(t)\|_{L^\infty} \approx f(0)\sqrt{t}.$$

Therefore if the traveling wave is initially of size $\epsilon > 0$ it will remain a small perturbation of the stationary state (2) only on time scales

$$t < \delta^2 \epsilon^{-2},$$

where $0 < \delta < 0.1$ is a constant. Hence, similarly as in Bedrossian et al. (2021) we restrict to studying stability and norm inflation on that time interval.

Theorem 1.3 *(Stability and upper bounds on norm inflation) Let $0 < \delta < 0.1$ and $0 < \epsilon < 0.1$ be given and consider the simplified linearized Boussinesq equations around the traveling waves*

$$\begin{aligned} Z &= \frac{f(t)}{(1+t^2)^{1/4}} \cos(x), \\ Q &= g(t)(1+t^2)^{1/4} \sin(x), \end{aligned}$$

with $f(0) = g(0) = \epsilon$. That is, consider the linear problem

$$\begin{aligned} \partial_t \begin{pmatrix} Z \\ Q \end{pmatrix} + A \begin{pmatrix} Z \\ Q \end{pmatrix} &= - \begin{pmatrix} |\partial_x|^{1/2} \Delta_t^{-1/4} (\nabla^\perp |\partial_x|^{-1/2} \Delta_t^{-3/4} Z \cdot \nabla f(t) \cos(x)) \\ |\partial_x|^{-1/2} \Delta_t^{1/4} (\nabla^\perp |\partial_x|^{-1/2} \Delta_t^{-3/4} Z \cdot \nabla g(t) \cos(x)) \end{pmatrix}, \\ \Delta_t &= \partial_x^2 + (\partial_y - t\partial_x)^2. \end{aligned}$$

Then there exists $C > 0$ and $|\gamma| < \delta$ such that for any initial data (Z_0, Q_0) whose Fourier transform satisfies

$$\sum_k \int \exp\left(2C \min((\epsilon|\xi|^{1+\gamma})^{2/3-2\gamma}, \epsilon^{-2})\right) |\mathcal{F}(Z_0, Q_0)(k, \xi)|^2 d\xi \leq 1$$

the corresponding solution remains regular up to a loss in the constant C. That is, for all times $t > 0$ it holds that

$$\sum_k \int \exp\left(C \min((\epsilon|\xi|^{1+\gamma})^{2/3-2\gamma}, \epsilon^{-2})\right) (1 + \frac{\epsilon^{-2}}{|\xi|}) |\mathcal{F}(Z, Q)(t, k, \xi)|^2 d\xi \leq 1. \tag{5}$$

Here $k \in \mathbb{Z}$ denotes the frequency with respect to x and $\xi \in \mathbb{R}$ denotes the frequency with respect to y.

Moreover, there exists a constant $c = c(\alpha)$ such that if the Fourier transform of the initial data is supported in the region $|\xi| \geq c\epsilon^{-4}$ then the stability estimate improves to a uniform estimate

$$\|(Z, Q)(t)\|_{L^2} \leq 2\|(Z_0, Q_0)(t)\|_{L^2}. \tag{6}$$

Let us comment on these results:

- Since the traveling waves are independent of y, these equations decouple with respect to the Fourier frequency $\xi \in \mathbb{R}$ corresponding to y. We may hence interpret (5) as an *upper bound* on the possible *norm inflation* factor for frequency-localized initial data by

$$\exp\left(C \min((\epsilon|\xi|^{1+\gamma})^{2/3-2\gamma}), \epsilon^{-2}\right) (1 + \frac{\epsilon^{-2}}{|\xi|}).$$

  In particular, we emphasize that this multiplier strongly differs from the Euler case. As we discuss in Sect. 3.1 we expect that this bound is optimal in the sense that this norm inflation is attained (possibly with slightly smaller constant C) for all frequencies $\epsilon^{-1} \leq |\xi| \leq \epsilon^{-4}$. However, since estimates in certain time regimes are technically very involved (in particular for the non-simplified problem) in this article we only establish upper bounds.
- A corresponding nonlinear result has been established in Bedrossian et al. (2021) using different methods with an upper bound on the norm inflation by $\exp(C\xi^{\sigma})$ with $\sigma > \frac{1}{2}$. The present result recovers this bound with $\sigma = \frac{1}{2}$ for $\xi = \epsilon^{-4}$ in the linearized problem around traveling waves. As major novelties, in this article we prove that for the present model:
  - The upper bound on the norm inflation factor is different and, in particular, much smaller when $|\xi|$ is much smaller than ϵ^{-4}.
  - For large frequencies $|\xi| \geq c\epsilon^{-4}$ the norm inflation is bounded by a constant factor instead (see Proposition 3.7 for a more detailed statement).

  Compared to the estimates of Bedrossian et al. (2021) we further exploit that the underlying traveling wave is much smaller for small times and that the time cut-off imposes an upper bound on the frequencies of resonances.
- In our simplified equations we omit the term

$$\frac{1}{1+t^2}\begin{pmatrix} f(t)\Delta_t^{-1/4}(\cos(x)\partial_y \Delta_t^{1/4} Z) \\ f(t)\Delta_t^{1/4}(\cos(x)\partial_y \Delta_t^{-1/4} Q) \end{pmatrix}$$

  from the linearized Boussinesq equations. As we discuss in Sect. 4 in the main time regime $t > \xi^{2/3}\epsilon^{-1/3}$, where the resonance mechanism takes place, this simplification can be removed. For the regime of small times, for the non-simplified problem we instead obtain a rough growth bound by $\exp(c\sqrt{\xi})$. However, we expect that this bound can be improved to a uniform bound (as for the simplified model) with more technical effort.

The remainder of the article is structured as follows:

- In Sect. 2 we discuss the linearized problem around a ground state (2) as formulated in Lemma 1.1. In particular, we introduce the unknowns and system formulation used throughout the article.
- In Sect. 3.1 we discuss the underlying resonance mechanism for a toy model. In particular, this allows us to clearly present the norm inflation mechanism and compare it with the Euler equations or the partially viscous problem.

- Based on the insights derived from this model we show in Sects. 3.2.1 and 3.2.2 that norm inflation cannot happen outside a specific time interval depending on the size of ξ and ϵ.
- The main result of the paper is established in Sect. 3.2.3, where we establish bounds on the norm inflation achieved.
- Finally, in Sect. 4 we show that the previously derived norm inflation estimates also extend to the non-simplified model. In particular, the omitted terms are only non-negligible perturbations for small times, where they can be absorbed by a loss of Gevrey regularity. As we discuss, we do not expect this loss to be attained. However, since the main focus of this article lies in the resonance mechanism for large times, we do not pursue this further.

Notation

In this section we collect some notation used throughout the article for easier reference.

Our main object of interest are the (simplified) linearized Boussinesq equations around the traveling waves of Lemma 1.2 which we write in the form

$$\partial_t \begin{pmatrix} Z \\ Q \end{pmatrix} + A \begin{pmatrix} Z \\ Q \end{pmatrix} = R[f(t), g(t), Z, Q],$$

where

$$\begin{aligned} f(t) &\leq C\epsilon\sqrt{1+|t|}, \\ g(t) &\leq C\epsilon(1+|t|)^{-1/2} \end{aligned}$$

are the coefficients of the traveling waves of Lemma 1.2. These equations decouple after a Fourier transform in y. Hence we view these equations as equations for $(\mathcal{F}_y Z)(t,x,\xi)$, $(\mathcal{F}_y Q)(t,x,\xi)$ for any fixed frequency $\xi \in \mathbb{R}$ and with slight abuse of notation write $Z(t,x)$, $Q(t,x)$ again.

These equations may equivalently be expressed as coupled system for Fourier modes Z_k, Q_k as stated in Definition 3.3:

$$\begin{aligned} \begin{pmatrix} Z_k \\ Q_k \end{pmatrix}(t_2) = S_k(t_2,t_1) \begin{pmatrix} Z_k \\ Q_k \end{pmatrix}(t_1) \\ + \int_{t_1}^{t_2} S_k(t_2,t) \begin{pmatrix} c_k^+ Z_{k+1} + c_k^- Z_{k-1} \\ d_k^+ Z_{k+1} + d_k^- Z_{k-1} \end{pmatrix} dt, \end{aligned}$$

with the coefficient functions stated in (12) (for $k \pm 1 \neq 0$):

$$\begin{aligned} c_k^\pm &= \pm\frac{1}{2} f(t)\xi(1+(\xi/k-t)^2)^{-1/4}(1+(\xi/(k\pm 1)-t)^2)^{-3/4}, \\ d_k^\pm &= \pm\frac{1}{2} g(t)\xi\frac{k}{k\pm 1}(1+(\xi/k-t)^2)^{1/4}(1+(\xi/(k\pm 1)-t)^2)^{-3/4}. \end{aligned}$$

Here $c_k^{\pm}$ is used as shorthand notation for c_k^+ or c_k^-. Similarly, we use $c_{k\pm 1}^{\mp}$ to refer to c_{k+1}^- and c_{k-1}^+.

As noted after Lemma 1.1 for simplicity of notation the estimates of this article are stated for $L^2(dxdy)$ or $\ell^2(\mathbb{Z})$ (with respect to k). However, since the above system includes only nearest neighbor interaction all estimates extend to the case of weighted ℓ^2 spaces, provided the weight $\lambda(k)$ is such that $|\lambda(k)/\lambda(k \pm 1) - 1|$ is small enough. In particular, this allows for $\lambda(k) = 1 + c|k|^N$ for any $N \in \mathbb{N}$ and $\lambda(k) = \exp(c|k|^s)$ for any $0 < s < 1$ and hence to establish stability in Sobolev or Gevrey spaces.

Throughout this article in several estimates it suffices to control quantities only in terms of upper and lower bounds within a constant factor. Hence, in order to simplify notation, we sometimes approximate values. For instance, we write

$$\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)} \approx \frac{1}{k^2}$$

to denote that for any $k \in \mathbb{N}, k \neq 0$ the last two terms are comparable within a factor at most 10.

2 The Homogeneous Problem, Waves and Good Unknowns

As remarked following Lemma 1.1 the explicit solutions of the Boussinesq equations of the form

$$\begin{aligned} \omega(t, x+ty, y) &= -1 + f(t)\cos(x), \\ \theta(t, x+ty, y) &= \alpha y + g(t)\sin(x), \end{aligned}$$

may be found by inserting this ansatz into the Boussinesq equations, which reduce to an ODE for the coefficient functions f, g.

In the following we provide a different perspective on these solutions as low frequency waves. Thus consider the perturbations

$$\begin{aligned} W(t, x, y) &= \omega(t, x+ty, y) + 1, \\ F(t, x, y) &= \theta(t, x+ty, y) - \alpha y, \end{aligned}$$

in coordinates moving with the affine flow. Then the full nonlinear Boussinesq equations are given by

$$\begin{aligned} \partial_t W + \nabla^{\perp}\Phi \cdot \nabla W &= \partial_x F, \\ \partial_t F + \nabla^{\perp}\Phi \cdot \nabla F &= -\alpha \partial_x \Phi, \\ (\partial_x^2 + (\partial_y - t\partial_x)^2)\Phi &= W, \end{aligned} \tag{7}$$

where we used the cancellation of $\nabla^{\perp} \cdot \nabla$. In particular, the left-hand side has a very similar structure as the Boussinesq equations in vorticity formulation except that the equation satisfied by the stream function perturbation Φ now is time-dependent. With respect to these unknowns the traveling waves of Lemma 1.2 take the form

$$\begin{aligned} W &= f(t)\cos(x), \\ F &= g(t)\sin(x). \end{aligned}$$

They are explicit non-trivial solutions of the nonlinear problem, which are smooth, low frequency and initially are small perturbations of (0, 0).

In the following subsection we discuss the linearized equation around (0, 0) and introduce the associated (frequency-localized) solution operators. In particular, we show that the linearized problem around (0, 0) (which we call the homogeneous problem) is stable in arbitrary regularity. In contrast, as we sketch in Sect. 3.1 for a toy model the norm inflation of the corresponding non-linear problem is closely linked to the interaction of high and low frequencies by means of the nonlinearity

$$\nabla^{\perp}\Phi_{high}\cdot\nabla W_{low}.$$

In particular, this mechanism is not present in the linearized problem around (0, 0) but is present in the linearized problem around traveling waves. The main aim of the remainder of this article is to show that this linearized problem around such waves indeed captures this norm inflation mechanism and to identify the sharp regularity classes corresponding to this norm inflation.

2.1 Stability of the Homogeneous Problem

A natural first step towards understanding the nonlinear behavior of initially small solutions of (7) is to study the linearized problem

$$\begin{aligned} \partial_t W &= \partial_x F, \\ \partial_t F &= -\alpha\partial_x\Phi, \\ (\partial_x^2+(\partial_y-t\partial_x)^2)\Phi &= W. \end{aligned}$$

Given this form, we symmetrize the problem by introducing the good unknowns (3) (as in Bedrossian et al. (2021)):

$$\begin{aligned} Z(t,x,y) &= \sqrt{\alpha}((\partial_x^{-2}\Delta)^{-1/4}\omega)(t,x+ty,y), \\ Q(t,x,y) &= ((\partial_x^{-2}\Delta)^{+1/4}\partial_x\theta)(t,x+ty,y), \\ \Phi(t,x,y) &= (\Delta^{-1}\omega)(t,x+ty,y). \end{aligned}$$

We remark that the problem decouples after a Fourier transform in x and hence in our definition of Z, Q we may choose any power of ∂_x instead of $|\partial_x|^{1/2}$. This particular choice is made to simplify calculations for $\partial_x^{-2}\Delta$ and to exploit slightly improved cancellation properties in Proposition 4.4. For the x-average we omit the $|\partial_x|^{1/2}$ and define

$$\int Z(t,x,y)dx = \sqrt{\alpha}|\xi|^{-1/2}\int \omega dx,$$
$$\int Q(t,x,y)dx = |\xi|^{1/2}\int \theta dx.$$

The following proposition states the stability result of Lemma 1.1 in terms of these unknowns. We again emphasize that this linear stability result is known in the literature (Bedrossian et al., 2021; Zillinger, 2021; Doering et al., 2018; Tao et al., 2020; Bianchini et al., 2022). In the interest of a self-contained presentation and in order to highlight the effects of (missing) traveling waves, we include a statement in our notation and a full proof.

Proposition 2.1 *(see (Bianchini et al., 2022)) Let $\alpha > \frac{1}{4}$ and consider the linear system*

$$\partial_t \begin{pmatrix} Z \\ Q \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}L_t & \sqrt{\alpha}\partial_x \Delta_t^{-1/2} \\ -\sqrt{\alpha}\partial_x \Delta_t^{-1/2} & \frac{1}{2}L_t \end{pmatrix} \begin{pmatrix} Z \\ Q \end{pmatrix} = 0$$
$$\Delta_t := \partial_x^2 + (\partial_y - t\partial_x)^2$$
$$L_t := \partial_x(\partial_y - t\partial_x)\Delta_t^{-1}.$$

Then the energy

$$E(t) = \|Z\|_{L^2}^2 + \|Q\|_{L^2}^2 + \langle Z, \frac{1}{2\sqrt{\alpha}} L_t(\partial_x \Delta^{-1/2})^{-1} Q\rangle_{L^2}$$

is approximately constant in the sense that there exist constants $0 < c < C < \infty$, depending only on α, such that

$$cE(0) \leq E(t) \leq CE(0).$$

We remark that all operators involved are constant coefficient Fourier multipliers. The problem hence decouples in frequency and we may therefore replace the L^2 space in the definition of $E(t)$ by any Fourier-based Hilbert space such Sobolev spaces H^s, Besov spaces or Gevrey spaces and obtain the same result. We further remark that, as a decoupled ODE system in Fourier space, the solution operator could be computed explicitly. However, in view to later perturbed estimates, where upper bounds are sufficient, we instead employ an energy estimate approach as for instance used in Bedrossian et al. (2021).

Proof of Proposition 2.1 We observe that the operator on the right-hand side of the equation

$$\partial_t \begin{pmatrix} Z \\ Q \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}L_t & \sqrt{\alpha}\partial_x \Delta_t^{-1/2} \\ -\sqrt{\alpha}\partial_x \Delta_t^{-1/2} & \frac{1}{2}L_t \end{pmatrix} \begin{pmatrix} Z \\ Q \end{pmatrix}$$

has anti-symmetric off-diagonal entries. It thus follows that

$$\frac{d}{dt}(\|Z\|^2 + \|Q\|^2)/2 = -\langle L_t Z, \sqrt{\alpha} Z\rangle + \langle L_t Q, Q\rangle.$$

Since L_t is a bounded operator one may already obtain a rough upper bound by employing Gronwall's lemma (we remark that at this point we do not yet require $\alpha > \frac{1}{4}$). In order to improve this estimate we further use that also the diagonal entries are symmetric. Therefore we may compute that

$$\begin{aligned}&\frac{d}{dt}\langle Z, \sqrt{\alpha}\frac{1}{2}L_t(\partial_x \Delta_t^{-1/2})^{-1} Q\rangle_{L^2}\\&= \langle L_t Z, \sqrt{\alpha} Z\rangle - \langle L_t Q, Q\rangle\\&\quad + \langle Z, \partial_t(\frac{1}{2}L_t(\partial_x \Delta_t^{-1/2})^{-1})Q\rangle.\end{aligned}$$

Here the first two terms exactly cancel with the ones above and thus

$$\frac{d}{dt}E = \langle Z, \partial_t(\frac{1}{2}L_t(\partial_x \Delta^{-1/2})^{-1})Q\rangle.$$

Similarly as in Bedrossian et al. (2021); Zillinger (2021); Doering et al. (2018) we observe that the operator norm of

$$\frac{1}{2\sqrt{\alpha}}L_t(\partial_x \Delta_t^{-1/2})^{-1}$$

is strictly smaller than 2 if (and only if) $\alpha > \frac{1}{4}$. Therefore, in that case $E(t)$ is a positive definite bilinear form in (Z, Q) and it follows that

$$\frac{d}{dt}E \leq C \left\| \partial_t(\frac{1}{2}L_t(\partial_x \Delta_t^{-1/2})^{-1}) \right\| E.$$

The result hence follows by Gronwall's lemma and noting that the problem decouples in frequency. More precisely, instead of controlling the time integral of the operator norm of $\partial_t(\frac{1}{2}L_t(\partial_x \Delta^{-1/2})^{-1})$, it suffices to control the time integral of the Fourier symbol for each fixed frequency:

$$\frac{1}{2\sqrt{\alpha}}\int \left| \partial_t \frac{(\xi - kt)k}{k^2 + (\xi - kt)^2} \frac{(k^2 + (\xi - kt)^2)^{1/2}}{ik} \right| dt,$$

which is uniformly bounded. □

Having established stability of the linearized problem around $(Z, Q) = (0, 0)$ in the following we study the linearization around traveling waves.

3 Echo Chains and Gevrey Regularity

While the results of Sect. 2.1 establish linear stability of $(Z, Q) = (0, 0)$, nonlinear stability (see (Bedrossian et al., 2021)) and asymptotic behavior for large times are very challenging problems. As a first step towards understanding the (optimal) long time behavior of the nonlinear Boussinesq equation (7) near a shear and hydrostatic balance, in the following we consider a toy model highlighting the role of the nonlinearity and of traveling waves.

3.1 A Toy Model and Optimal Gevrey Classes

Resonance chains in phase-mixing problems often manifest as a low frequency part of the solution interacting with the high frequency part by means of the nonlinearity (see for instance (Deng and Masmoudi, 2018; Bedrossian et al., 2016; Deng and Zillinger, 2021; Zillinger, 2021)). Based on this heuristic in this section we introduce a toy model capturing this mechanism, which allows us to identify an expected optimal regularity class. The main aim of the remainder of the paper then is to show that the linearized equations around a traveling wave indeed exhibit this growth.

We recall that the nonlinear Boussinesq equations near a traveling wave read

$$
\begin{aligned}
&\partial_t \begin{pmatrix} Z \\ Q \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}L_t & \sqrt{\alpha}\partial_x \Delta_t^{-1/2} \\ -\sqrt{\alpha}\partial_x \Delta_t^{-1/2} & \frac{1}{2}L_t \end{pmatrix} \begin{pmatrix} Z \\ Q \end{pmatrix} \\
&\quad = \begin{pmatrix} f(t)|\partial_x|^{1/2}\Delta_t^{-1/4}(\sin(x)\partial_y|\partial_x|^{-1/2}\Delta_t^{-3/4}Z) \\ g(t)|\partial_x|^{-1/2}\Delta_t^{1/4}(\sin(x)\partial_y|\partial_x|^{-1/2}\Delta_t^{-3/4}Z) \end{pmatrix} \\
&\qquad + \begin{pmatrix} f(t)(1+t^2)^{-1}|\partial_x|^{1/2}\Delta_t^{-1/4}(\sin(x)\partial_y|\partial_x|^{-1/2}\Delta_t^{1/4}Z) \\ g(t)(1+t^2)^{-1}|\partial_x|^{-1/2}\Delta_t^{1/4}(\sin(x)\partial_y|\partial_x|^{1/2}\Delta_t^{-1/4}Q) \end{pmatrix} \\
&\qquad + \begin{pmatrix} |\partial_x|^{1/2}\Delta_t^{-1/4}(\nabla^{\perp}|\partial_x|^{-1/2}\Delta_t^{-3/4}Z \cdot \nabla|\partial_x|^{-1/2}\Delta_t^{1/4}Z) \\ |\partial_x|^{-1/2}\Delta_t^{1/4}(\nabla^{\perp}|\partial_x|^{-1/2}\Delta_t^{-3/4}Z \cdot \nabla|\partial_x|^{1/2}\Delta_t^{-1/4}Q) \end{pmatrix}.
\end{aligned}
$$

Here we consider Z and Q to be at high frequency and thus the right-hand side can be seen as paraproduct decomposition into low-high, high-low and high-high frequency products.

In order to derive our toy model in a first step we ignore all terms except the low-high term, which drives the resonance mechanism and arrive at

$$
\partial_t \begin{pmatrix} Z \\ Q \end{pmatrix} \approx \begin{pmatrix} f(t)|\partial_x|^{1/2}\Delta_t^{-1/4}(\sin(x)\partial_y|\partial_x|^{-1/2}\Delta_t^{-3/4}Z) \\ g(t)|\partial_x|^{-1/2}\Delta_t^{1/4}(\sin(x)\partial_y|\partial_x|^{-1/2}\Delta_t^{-3/4}Z) \end{pmatrix}.
$$

We observe that in this model the evolution for Z decouples and, since the coefficient functions do not depend on y the equations further decouples after a Fourier transform in y (which is also true for the linearized problem around a wave, but not for the nonlinear problem). Let thus $\xi > 1000$ be a given frequency in y and suppose that Z is localized at frequency k in x and ξ in y. Then the multiplier

$$\Delta_t^{-3/4} \rightsquigarrow (k^2)^{-3/4}(1+(t-\frac{\xi}{k})^2)^{-3/4}$$

is small unless $t \approx \frac{\xi}{k}$. As in Bedrossian et al. (2021) for the toy model we thus consider a two-dimensional system, which is supposed to approximate the evolution of $\mathcal{F}(Z)(k,\xi)$ and $\mathcal{F}(Z)(k-1,\xi)$ on a time interval, where $t \approx \frac{\xi}{k}$. More precisely, we replace powers of Δ_t by its Fourier symbol and for simplicity of the model approximate $t - \frac{\xi}{k+1} \approx \frac{\xi}{k} - \frac{\xi}{k+1} \approx \frac{\xi}{k^2}$ and $\frac{k}{k+1} \approx 1$. Then the toy model reads:

$$\begin{aligned}
\partial_t Z_R &= f(t)\frac{\xi}{k^2}\frac{1}{(1+(t-\frac{\xi}{k})^2)^{1/4}}((\frac{\xi}{k^2})^2)^{-3/4} Z_{NR},\\
\partial_t Z_{NR} &= f(t)\frac{\xi}{k^2}((\frac{\xi}{k^2})^2)^{-1/4}\frac{1}{(1+(t-\frac{\xi}{k})^2)^{3/4}} Z_R.
\end{aligned}$$

In Bedrossian et al. (2021) the authors construct a toy model in the same way, but further estimate $f(t) \leq 1$ from above. Recalling from Lemma 1.2 that

$$|f(t)| \leq C\epsilon\sqrt{1+t}$$

uniformly in time, this upper bound is achieved for t being comparable to ϵ^{-2}. However, for smaller times this upper bound is a (potentially large) overestimate, leading to a larger growth bound on the chain of resonances.

More precisely, we observe that by our choice of time interval $t \approx \frac{\xi}{k}$. This toy model thus suggests a growth by

$$\epsilon\sqrt{\frac{\xi}{k}}\sqrt{\frac{\xi}{k^2}} \leq \sqrt{\frac{\xi}{k^2}},$$

which is potentially much smaller:

Lemma 3.1 *Let $\xi \geq 100$ and $k \in \mathbb{N}$ be given and define*

$$\begin{aligned}
t_0 &= 2\xi,\\
t_k &= \frac{1}{2}(\frac{\xi}{k+1}+\frac{\xi}{k}).
\end{aligned}$$

Then on the time interval $I_k = (t_k, t_{k-1})$ we consider the simplified toy model

$$\begin{aligned}
\partial_t Z_R &= 0,\\
\partial_t Z_{NR} &= \epsilon(1+t^2)^{1/4}\frac{\xi}{k^2}((\frac{\xi}{k^2})^2)^{-1/4}\frac{1}{(1+(t-\frac{\xi}{k})^2)^{3/4}} Z_R.
\end{aligned}$$

Then if $Z_{NR}(t_k) = 0$ there exists a constant $1 < C < 10$ such that

$$Z_{NR}(t_{k-1}) = Z_R(t_k)\int \epsilon(1+t^2)^{1/4}\frac{\xi}{k^2}((\frac{\xi}{k^2})^2)^{-1/4}\frac{1}{(1+(t-\frac{\xi}{k})^2)^{3/4}}dt$$

$$\approx C\epsilon\sqrt{\frac{\xi}{k}}\sqrt{\frac{\xi}{k^2}}Z_R(t_k),$$

where we use $\approx$ *to denote upper and lower bounds within a factor* 10.

- We remark that here we neglected the evolution of Z_R, which in turn effects the evolution of Z_{NR}. As we discuss in Sect. 3.2.3 taking this coupling into account results in a slightly modified growth bound by

$$C\epsilon\sqrt{\frac{\xi}{k}}\left(\frac{\xi}{k^2}\right)^{\gamma}$$

  with $|\gamma - \frac{1}{2}| < \delta$ instead.
- By the restriction on the time scale it holds that $\epsilon\sqrt{\frac{\xi}{k}} \approx \epsilon\sqrt{t} \leq 1$. Thus the amount of growth may be estimated from above by $\sqrt{\frac{\xi}{k^2}}$, uniformly in ξ. However, this is an over estimate for most values of k and ξ.
- As we discuss following the proof of this lemma, the dependence on ϵ and k here strongly differs from the one of the Euler equations.

Proof of Lemma 3.1 The integral formula is immediate. We further observe that

$$\epsilon(1+t^2)^{1/4}\frac{\xi}{k^2}\left((\frac{\xi}{k^2})^2\right)^{-1/4} \approx \epsilon\left(\frac{\xi}{k}\right)^{1/2}\left(\frac{\xi}{k^2}\right)^{1/2} = \epsilon\frac{\xi}{k^{3/2}},$$

and that

$$\int_{\mathbb{R}} \frac{1}{(1+(t-\frac{\xi}{k})^2)^{3/4}}dt = \frac{\sqrt{\pi}\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \approx 5.2 \neq 0.$$

The result hence follows by observing that if $\frac{\xi}{k^2}$ is sufficiently large the integral over I_k is comparable to the integral over all of $\mathbb{R}$. □

Iterating this heuristic growth bound we may conjecture a total growth of

$$\sup_{k_0}\prod_{k=1}^{k_0}\frac{\epsilon\xi}{k^{3/2}} \rightsquigarrow \frac{1}{(\epsilon\xi)^{2/3}}\exp((\epsilon\xi)^{2/3}) \tag{8}$$

by choosing $k_0 \approx (\epsilon\xi)^{2/3}$ and using Stirling's approximation. Thus at first sight this toy model suggests stability for Gevrey $3/2$ regular initial data, uniformly in $0 < \epsilon < 1$ and for all times. However, since our evolution is restricted to the time interval

$$(0, \delta\epsilon^{-2})$$

this is an overestimate for large values of ξ. Indeed, for a resonance to happen the time $t \approx \frac{\xi}{k}$ needs to have passed. Thus if ξ is very large then in the above product we may only consider those k for which

$$\frac{\xi}{k} \leq \epsilon^{-2} \Leftrightarrow k \geq \xi\epsilon^2,$$

while by the above consideration our cascade should start at k_0 to maximize the product and we thus arrive at an estimate of the possible norm inflation by

$$\prod_{\xi\epsilon^2 \leq k \leq (\epsilon\xi)^{2/3}} \frac{\epsilon\xi}{k^{3/2}}. \tag{9}$$

Lemma 3.2 *For* $0 < \epsilon < 0.1$ *and* $0 < \xi < \epsilon^{-4}$ *consider the function*

$$G(\xi, \epsilon) := \prod_{\xi\epsilon^2 \leq k \leq (\epsilon\xi)^{2/3}} \frac{\epsilon\xi}{k^{3/2}}.$$

Then it holds that

$$G(\xi, \epsilon) \leq C \exp(3/2(\epsilon\xi)^{2/3}).$$

Moreover, this bound is attained in the sense that for any $1 < \sigma < 4$ *we may consider* $\xi = \epsilon^{-\sigma}$ *and there exists* $\epsilon_\sigma > 0$ *such that for* $0 < \epsilon < \epsilon_\sigma$

$$G(\epsilon^{-\sigma}, \epsilon) \geq C \exp(0.1(\epsilon\epsilon^{-\sigma})^{2/3})$$

We note that

$$\begin{aligned} &\xi\epsilon^2 \leq (\epsilon\xi)^{2/3} \\ &\Leftrightarrow \xi \leq \epsilon^{-4} \end{aligned}$$

and that the product is empty if ξ is larger than this. In particular, for $\xi \geq \epsilon^{-4}$ we may expect to obtain a uniform bound instead of norm inflation (this is shown in Proposition 3.6).

Proof of Lemma 3.2 For simplicity of notation let $k_0 = \lfloor (\epsilon\xi)^{2/3} \rfloor$. Then for the upper bound we may replace the starting point of the product by 1 to obtain

$$\begin{aligned} G(\xi, \epsilon) &\leq \prod_{1 \leq k \leq k_0} \frac{\epsilon\xi}{k^{3/2}} \\ &= \left((\epsilon\xi)^{2/3k_0}/k_0!\right)^{3/2}. \end{aligned}$$

For $k_0 \leq 100$ we may control this quantity by a constant uniformly in ϵ and ξ, since $(\epsilon\xi)^{2/3} \leq k_0 + 1$. It hence suffices to discuss the case when k_0 is large, where by Stirling's approximation formula it holds that

$$k_0! \sim \sqrt{2\pi k_0} k_0^{k_0} e^{-k_0}.$$

We may therefore further estimate

$$G(\xi, \epsilon) \leq e^{-3/2k_0}(2\pi k_0)^{-3} \left((\epsilon\xi)^{2/3}/k_0\right)^{k_0},$$

which yields the desired upper bound by noting that the last factor is controlled by $((k_0+1)/k_0)^{k_0}$ and hence uniformly bounded.

For the lower bound we argue similarly, but now have to take into account that the product starts at $k_1 := \lfloor \epsilon^2 \xi \rfloor$ and hence

$$G(\xi, \epsilon) = \left((\epsilon\xi)^{2/3k_0}/k_0!\right)^{3/2} \left(k_1!(\epsilon\xi)^{-2/3k_1}\right)^{3/2}.$$

We thus need to show that the second factor is not too small and hence cannot cancel the growth. Here we again may restrict to the case when k_1 is large and use Stirling's approximation to compute

$$\begin{aligned} k_1!(\epsilon\xi)^{-2/3k_1} &\sim (k_1(\epsilon\xi)^{-2/3})^{k_1} e^{-k_1}\sqrt{2\pi k_1} \\ &\approx (\epsilon^2\xi(\epsilon\xi)^{-2/3})^{k_1} e^{-k_1}\sqrt{2\pi k_1} \\ &= (\xi\epsilon^4)^{k_1/3} e^{-k_1}\sqrt{2\pi k_1} \\ &= \exp(-k_1(1+\log(\xi\epsilon^4)))\sqrt{2\pi k_1}. \end{aligned}$$

It thus suffices to estimate

$$\exp\left(3/2k_0 - 3/2k_1(1+\log(\xi\epsilon^4))\right).$$

Here we observe that for $\xi = \epsilon^{-\sigma}$ it holds that

$$\begin{aligned} k_0 &\approx \epsilon^{(1-\sigma)\frac{2}{3}}, \\ k_1 &\approx \epsilon^{2-\sigma}, \\ \log(\xi\epsilon^4) &\approx (4-\sigma)\log(\epsilon). \end{aligned}$$

Since $\sigma > 1$ the power of ϵ in the formula for k_1 is negative and for $\sigma < 4$ it holds that

$$(1-\sigma)\frac{2}{3} < 2 - \sigma.$$

Hence for $0 < \epsilon < \epsilon_\sigma$ sufficiently small $3/2k_0$ dominates. □

Thus on this time interval the total growth is limited in terms of ϵ (by a frequency cut-off) and letting ϵ tend to zero the optimal Gevrey regularity class is expected to be given by 2.

Conjucture *Let* $0 < \epsilon \leq 0.1$ *and consider the nonlinear Boussinesq equations perturbed around traveling waves of size* ϵ. *Then on the time interval* $(0, \epsilon^{-2})$ *the optimal space for stability is given by the Fourier weight*

$$\begin{cases} \exp((\epsilon\xi)^{2/3}) & \text{if } \epsilon^{-1} \leq \xi < 100\epsilon^{-4}, \\ 1 & \text{else}. \end{cases}$$

We remark that stability in Gevrey 2, that is a bound by $\exp(C\sqrt{\xi})$, has been established in Bedrossian et al. (2021), which coincides with the above weight for $\xi = \epsilon^{-4}$. The conjecture suggests an improvement to these estimates when ξ is much smaller or much larger than ϵ^{-4} and that for $\xi < \epsilon^{-4}$ the exponent of growth

$$(\epsilon\xi)^{2/3} < \xi^{1/2}$$

is attained. As first step towards proving this conjecture in this article we show that this statement is true for the (simplified) linearized Boussinesq equations around traveling waves. We remark that the above heuristic also suggests growth bounds for $t > \epsilon^{-2}$ for data in higher regularity classes (e.g. global in time for Gevrey $\frac{3}{2}$). However, at that point the toy model simplification

$$\partial_t Z_R \approx 0$$

ceases to be justified and the toy model has to be replaced. The question of stability on larger time scales than $(0, \epsilon^{-2})$ for more regular data thus remains an interesting problem for future research.

For comparison we also note that for the Euler equations and a wave initially of size ϵ the growth of the vorticity ω (instead of Z) is bounded by

$$\exp(\sqrt{\epsilon\xi})$$

and thus stability in Gevrey 2 regularity holds when considering arbitrarily large times (Deng and Masmoudi, 2018; Deng and Zillinger, 2021; Bedrossian and Masmoudi, 2015). When also restricting to the time interval $t < \epsilon^{-2}$ we require that ξ is such that

$$\frac{\xi}{\sqrt{\epsilon\xi}} \leq \epsilon^{-2}$$
$$\rightsquigarrow \xi \leq \epsilon^{-3}$$

and thus

$$\exp(\sqrt{\epsilon\xi}) \leq \exp(\min(\xi^{1/3}, \epsilon^{-1})).$$

From this heuristic model we can thus already see that the hydrostatic balance for $\alpha > \frac{1}{4}$ yields a strong change of the stability and norm inflation behavior of the Boussinesq equations compared to the Euler equations: The growth of the underlying traveling wave results in larger norm inflation.

3.2 The Inhomogeneous Problem and Upper Bounds

Building on the heuristic of the previous model in the following we consider the simplified linearized Boussinesq equations around a traveling wave

$$\partial_t \begin{pmatrix} Z \\ Q \end{pmatrix} + A \begin{pmatrix} Z \\ Q \end{pmatrix} = \begin{pmatrix} f(t)|\partial_x|^{1/2}\Delta_t^{-1/4}(\partial_y|\partial_x|^{-1/2}\Delta_t^{-3/4} Z\cos(x)) \\ g(t)|\partial_x|^{-1/2}\Delta_t^{1/4}(\partial_y|\partial_x|^{-1/2}\Delta_t^{-3/4} Z\cos(x)) \end{pmatrix}, \tag{10}$$

where we omitted the low frequency velocity contribution

$$\frac{1}{1+t^2} \begin{pmatrix} f(t)|\partial_x|^{1/2}\Delta_t^{-1/4}(\cos(x)\partial_y|\partial_x|^{-1/2}\Delta_t^{1/4} Z) \\ f(t)|\partial_x|^{-1/2}\Delta_t^{1/4}(\cos(x)\partial_y|\partial_x|^{-1/2}\Delta_t^{-1/4} Q) \end{pmatrix}.$$

As we discuss in Sect. 4 this term does not qualitatively change the dynamics at large times, but is technically challenging to control for small times.

As suggested by the notation we consider this problem as a (possibly large) perturbation of the inhomogeneous problem of Sect. 2. In particular, if we denote by

$$S(t_2, t_1)$$

the solution operator of the homogeneous problem, then we may equivalently express the above differential equation as the integral equation

$$\begin{aligned} \begin{pmatrix} Z \\ Q \end{pmatrix}(t_2) &= S(t_2, t_1) \begin{pmatrix} Z \\ Q \end{pmatrix}(t_1) \\ &\quad + \int_{t_1}^{t_2} S(t_2, t) \begin{pmatrix} f(t)|\partial_x|^{1/2}\Delta_t^{-1/4}(\partial_y|\partial_x|^{-1/2}\Delta_t^{-3/4} Z\cos(x)) \\ g(t)|\partial_x|^{1/2}\Delta_t^{1/4}(\partial_y|\partial_x|^{-1/2}\Delta_t^{-3/4} Z\cos(x)) \end{pmatrix} dt \end{aligned}$$

We next recall that the solution operator $S(\cdot, \cdot)$ is given by a Fourier multiplier and decouples in frequency. Hence, taking a Fourier transform in both x and y we arrive at a system with nearest neighbor interaction.

Definition 3.3 (Inhomogeneous system) The simplified linearized Boussinesq equations around a traveling waves for a perturbation frequency localized at $\xi \in \mathbb{R}$ read

$$\begin{aligned}\begin{pmatrix} Z_k \\ Q_k \end{pmatrix}(t_2) = S_k(t_2, t_1) \begin{pmatrix} Z_k \\ Q_k \end{pmatrix}(t_1) \\ + \int_{t_1}^{t_2} S_k(t_2, t) \begin{pmatrix} c_k^+ Z_{k+1} + c_k^- Z_{k-1} \\ d_k^+ Z_{k+1} + d_k^- Z_{k-1} \end{pmatrix} dt\end{aligned} \tag{11}$$

where we introduced the coefficient functions

$$\begin{aligned} c_k^{\pm} &= \pm \frac{1}{2} f(t) \frac{\xi}{(k \pm 1)^2} (1 + (\xi/k - t)^2)^{-1/4} (1 + (\xi/(k \pm 1) - t)^2)^{-3/4}, \\ d_k^{\pm} &= \pm \frac{1}{2} g(t) \frac{\xi}{(k \pm 1)^2} \frac{k}{k \pm 1} (1 + (\xi/k - t)^2)^{1/4} (1 + (\xi/(k \pm 1) - t)^2)^{-3/4}. \end{aligned} \tag{12}$$

and denote by Z_k, Q_k the Fourier modes at frequency $k \in \mathbb{Z}$ in x and frequency $\xi \in \mathbb{R}$ in y. As the system decouples in ξ we treat it as a fixed parameter and suppress it in our notation.

The toy model of Sect. 3.1 here omitted all terms except the main resonance mechanism due to c_{k-1}^+. Our main aim in the following is to show that this model indeed provides an accurate heuristic and that all other contributions can be controlled.

- The main time regime of interest is given by time intervals I_k, where $t \approx \frac{\xi}{k}$, for which c_{k-1}^+ is comparatively large. This regime is studied in Sect. 3.2.3. Here the coupling of modes leads to a modified growth behavior as compared to the toy model of Sect. 3.1.
- In Sects. 3.2.1 and 3.2.2 we show that in the remaining time intervals resonances are too small to have a large effect on the dynamics and the evolution is at most algebraically unstable.

3.2.1 The Long Time Regime

In this section we consider the regime of "large" times, where

$$2\xi < t < \delta\epsilon^{-2}.$$

As suggested by the heuristic model of Sect. 3.1 for such large times there are no resonances and hence the evolution is at most algebraically unstable.

Proposition 3.4 *let* $0 < \xi < 2\delta\epsilon^{-2}$ *be given and consider the time interval*

$$I = (2\xi, \delta\epsilon^{-2}).$$

Then on I *the solution to the system (11) grows at most algebraically in any Sobolev or suitable Gevrey space in the sense that the Fourier projections away from and onto the modes* $k = -1, 1$ *satisfy*

$$\|1_{|k|\neq 1}(Z, Q)(t)\| \leq C_\alpha \exp(10)\sqrt{\frac{t}{\xi}}\|(Z, Q)(2\xi)\|,$$

$$\|1_{|k|=1}(Z,Q)(t)\| \leq C_{\alpha,\gamma}\left(\frac{t}{\xi}\right)^{\gamma}\exp(10)\sqrt{\frac{t}{\xi}}\|(Z,Q)(2\xi)\|,$$

for any $t \in I$ *and any* $1/2 < \gamma < 1$.

This proposition implies a bound on the norm inflation on this time interval by $t^{3/2} \leq \epsilon^{-3}$, which is much smaller than the exponential growth bound expected on earlier time intervals. We further remark that this time interval is empty if $\xi > \epsilon^{-2}$ and that this proposition is hence only concerned with "small" frequencies.

Proof We recall from Proposition 2.1 of Sect. 2.1 that the solution operators

$$S_k(\cdot,\cdot) : \mathbb{C}^2 \to \mathbb{C}^2$$

are bounded by a constant C_α uniformly in k.

It thus suffices to control the corrections of (11)

$$\int_{2\xi}^{t_2} S_k(t_2,t)\begin{pmatrix} c_k^+ Z_{k+1} + c_k^- Z_{k-1} \\ d_k^+ Z_{k+1} + d_k^- Z_{k-1}\end{pmatrix} dt$$

in a suitable way to invoke Gronwall's lemma, where we will distinguish between the case where $|k| \geq 2$ and the cases $k = -1, 0, 1$.

For simplicity of presentation in the following we establish estimates in the unweighted space ℓ^2. The case of weighted spaces with a weight λ_k can be reduced to this case by considering modified coefficient functions of the form $\frac{\lambda_{k\pm1}}{\lambda_k}c_k^\pm$. More precisely, for instance for Sobolev spaces we may choose $\lambda_k = 1 + c|k|^s$, where c is a small constant and hence deduce that $\frac{\lambda_{k\pm1}}{\lambda_k}$ is bounded above and below by constants close to 1. Hence all estimates below extend to this case with possibly a small loss of constants.

In the following we estimate the coefficient functions. We observe that for $t > 2\xi$ all frequencies $k \neq 0$ are non-resonant in the sense that

$$|\xi - kt| \geq \frac{1}{2}t \geq |\xi|.$$

In particular, recalling the definition of the coefficient functions (12) as long as none of $k, k-1, k+1$ are zero, we may bound

$$|c_k^\pm| + |d_k^\pm| \leq Cf(t)\frac{\xi}{t^2},$$

for some universal constant C, where we with slight abuse of notation estimated $g(t) \leq f(t)/t$. We further recall that by our choice of time interval

$$f(t) \leq \sqrt{\delta} \ll 1$$

is small and note that

$$\int_{2\xi}^{\infty} \frac{\xi}{t^2} = \frac{1}{2}.$$

Hence, for these coefficient functions we obtain uniform L^1 estimate in time (more precisely, the supremum in k is still in L^1).

It hence only remains to discuss the cases $k \in \{-1, 0, 1\}$.

Here we observe that while

$$c_0^{\pm} \leq C \frac{f(t)\xi^{1/2}}{t^{3/2}}$$

is uniformly integrable,

$$c_{\pm 1}^{\mp} \leq C \frac{f(t)}{\xi^{1/2} t^{1/2}}$$

is not.

Therefore, we cannot hope for better growth estimates than for the simple ODE system

$$\partial_t \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & \frac{\xi}{t^2} \\ \frac{1}{\sqrt{\xi}\sqrt{t}} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

for $t > \xi$. Note that after rescaling we may without loss of generality set $\xi = 1$. We may then introduce $1/2 < \gamma < 1$ and consider

$$\partial_t \begin{pmatrix} a \\ (\frac{t}{\xi})^{-\gamma} b \end{pmatrix} = \begin{pmatrix} 0 & \frac{\xi^{1-\gamma}}{t^{2-\gamma}} \\ \frac{1}{\xi^{1/2-\gamma} t^{1/2+\gamma}} & -\frac{\gamma}{t} \end{pmatrix} \begin{pmatrix} a \\ (\frac{t}{\xi})^{-\gamma} b \end{pmatrix}.$$

We observe that the anti-diagonal entries are integrable in time by our choice of γ, while the bottom right-entry is negative. Hence, by Gronwall's lemma

$$|a|^2 + |(\frac{t}{\xi})^{-\gamma} b|^2$$

remains uniformly bounded, which implies that $|a|$ remains bounded while $|b|$ might grow algebraically.

The claimed estimate then follows with $b = |(Z_1, Z_{-1})|$ and $a = \|(Z_k)_{k \notin \{-1,1\}}\|$. □

3.2.2 The Small Time or High Frequency Regime

By the results of the preceding Sect. 3.2.1 any possible norm inflation has to happen for times

$$0 < t < 2\xi.$$

Thus similarly to the setting of the Euler or Vlasov–Poisson equations we partition this time interval into regions in which t is comparable to $\frac{\xi}{k}$ for some $k \in \mathbb{N}$.

Definition 3.5 Let $\xi > 0$ be given. Then for any $k \in \mathbb{N}$ we define

$$\begin{aligned} t_k &= \frac{1}{2}(\frac{\xi}{k+1} + \frac{\xi}{k}), \\ t_0 &= 2\xi, \end{aligned}$$

and the associated time intervals

$$I_k = (t_k, t_{k-1}).$$

We further define

$$k_0 = \lfloor (\epsilon|\xi|)^{2/3} \rfloor.$$

We recall from the toy model of Sect. 3.1 and from the structure of the coefficient functions $c_k^{\pm}, d_k^{\pm}$ stated in (12) that on a given time interval I_k the main resonance mechanism is expected to be determined by

$$\int_{I_k} c_{k\pm 1}^{\mp} \approx \frac{\epsilon\xi}{k^{3/2}} \approx \epsilon t^{3/2} \xi^{-1/2}.$$

In particular, this value is bigger than 1 for $k \leq k_0$ and smaller than 1 if $k \geq k_0 + 1$. We further remark that, if ξ is much bigger than ϵ^{-4} or if t is small, then we expect resonances to only result in small perturbation of the dynamics, as we prove in the following propositions. The main resonance mechanism in the remaining time interval is then studied in Sect. 3.2.3. As a first result we note that if $\xi > \epsilon^{-4}$ is very large all permissible choices of k (that is, with $t_k < \epsilon^{-2}$) are non-resonant and stability estimates can be obtained by a simple ode-type estimate.

Proposition 3.6 *(High frequency I) Let $\xi > \epsilon^{-4}$, then there exists as constant C depending only on α such that for any choice of initial data it holds that*

$$\|(Z, Q)(t)\| \leq \exp(C \min(t, (\epsilon\xi)^{2/3}, \epsilon^{-2}))\|(Z, Q)(0)\|.$$

Proof (Proof of Proposition 3.6) We recall that by Definition 3.3 we may equivalently consider a system of integral equations

$$\begin{aligned}\begin{pmatrix} Z_k \\ Q_k \end{pmatrix}(t) &= S_k(t,0)\begin{pmatrix} Z_k \\ Q_k \end{pmatrix}(0) \\ &\quad + \int_0^t S_k(t,\tau)\begin{pmatrix} c_k^+ Z_{k+1} + c_k^- Z_{k-1} \\ d_k^+ Z_{k+1} + d_k^- Z_{k-1} \end{pmatrix}(\tau)d\tau.\end{aligned}$$

In particular, using the bounds on S_k of Sect. 2.1 it follows that the solution satisfies the integral inequality

$$\|(Z,Q)(t)\| \leq C\|(Z,Q)(0)\| + \int_0^t C\|(Z,Q)(\tau)\| \sup_l(|c_l^\pm(\tau)| + |d_l^\pm(\tau)|)d\tau,$$

where C is a constant which may depend on α. The claimed bound hence follows by an application of Gronwall's inequality provided

$$\sup_{l,\tau\in(0,\epsilon^{-2})} (|c_l^\pm(\tau)| + |d_l^\pm(\tau)|) \leq 100. \tag{13}$$

Indeed, we observe that for $t \in I_k$ it holds that

$$|c_l^\pm| \leq |f(t)| \begin{cases} \frac{\xi}{k^2} & \text{if } l \geq k+1, \\ \sqrt{\frac{\xi}{k^2}} & \text{if } l \in \{k-1, k+1\}, \\ (\frac{\xi}{k^2})^{-1/2} & \text{if } l = k, \\ \frac{\xi}{k^2} + \xi^{-1} & \text{if } l \leq k-2. \end{cases}$$

Here we estimated

$$|\frac{\xi}{k} - \frac{\xi}{l}| \geq \frac{\xi}{k^2}$$

for $l \neq k$ and observed that since $\xi \geq \epsilon^{-4} > 2\epsilon^{-2}$ for $k \in \{-1, 0, 1\}$ we estimate $|\xi - kt| \geq \frac{1}{2}\xi$. It thus only remains to observe that

$$\frac{\xi}{k^2} = \left(\frac{\xi}{k}\right)^2 \xi^{-1} \approx t^2\xi^{-1} < 1$$

is uniformly bounded by assumption on ξ and that $f(t) < 1$ by our choice of time interval. The estimates on $d_l^\pm$ follow analogously by noting that $g(t)\sqrt{1+t}$ is uniformly bounded by our choice of time interval. Thus the estimate (13) holds, which concludes the proof. □

We remark that this bound is very rough and not expected to be sharp for most choices of ξ. Indeed as suggested by the model of Sect. 3.1 if ξ is much larger than ϵ^{-4} we obtain no norm inflation at all.

Proposition 3.7 *Let C_α denote the operator norm of semi-group of the homogeneous problem, that is*

$$C_\alpha = \sup_{l\in\mathbb{Z},t,s\in\mathbb{R}} |S_l(t,s)|.$$

Then for all $\xi > 2C_\alpha\epsilon^{-4}$ and all $t \in (0, \epsilon^{-2})$ it holds that

$$\|(Z,Q)(t)\| \leq CC_\alpha\|(Z,Q)\|.$$

The evolution is uniformly bounded.

Proof of Proposition 3.7 We claim that for this choice of ξ it holds that

$$\sup_l \int_{(0,\delta\epsilon^{-2})} |c_l^{\pm}|dt \leq \frac{1}{4}C_\alpha^{-1}. \tag{14}$$

Recalling the integral equation

$$\begin{pmatrix} Z_l \\ Q_l \end{pmatrix}(t) = S_l(t,0)\begin{pmatrix} Z_l \\ Q_l \end{pmatrix}(0) + \int_0^t S(t,\tau)(c_l^{\pm}, d_l^{\pm})\begin{pmatrix} Z_{l\pm1} \\ Q_{l\pm1} \end{pmatrix},$$

we thus deduce that

$$|(Z_l, Q_l)|(t) \leq C_\alpha|(Z_l, Q_l)|(0) + \frac{1}{2}\sup_{(0,t)}|(Z_{l\pm1}, Q_{l\pm1})|.$$

In particular, we may consider the supremum in $t \leq \tau$ on both sides and consider (suitably weighted) ℓ^2 norms to obtain that

$$\|(Z,Q)\|_{\ell^2,t} := \|\sup_{\tau\leq t}|(Z_l,Q_l)|(\tau)\|_{\ell^2}$$

satisfies

$$\|(Z,Q)\|_{\ell^2,t} \leq C_\alpha\|(Z,Q)(0)\|_{\ell^2} + \frac{1}{2}\|(Z,Q)\|_{\ell^2,t}.$$

Since the factor $\frac{1}{2}$ on the right-hand side is smaller than 1 we may subtract it from both sides and obtain that

$$\|(Z,Q)\|_{\ell^2,t} \leq 2C_\alpha\|(Z,Q)(0)\|_{\ell^2}.$$

Finally we observe that

$$\sup_{\tau\leq t}\|(Z_l,Q_l)(\tau)\|_{\ell^2} \leq \|(Z,Q)\|_{\ell^2,t}$$

and that for a time independent function (such as $(Z, Q)(0)$) equality holds.

It thus only remains to establish the estimates (14). Indeed we observe that if $I_{l\pm1} \subset (0, \delta\epsilon^{-2})$ then

$$\int_{I_{l\pm1}} |c_l^{\pm}|dt \leq f(t_l)\frac{\xi}{l^{3/2}} \leq \epsilon(t_l)^{3/2}\xi^{-1/2} \leq \epsilon^{-2}\xi^{-2} \leq \frac{1}{4C_\alpha}$$

On the remaining interval and for all other l we may estimate $f(t) \leq \delta$ and observe that

$$\int_{(0,\delta\epsilon^{-2})\setminus I_l} \xi \frac{1}{(l^2+(\xi-lt)^2)^{1/4}} \frac{1}{((l\pm1)^2+(\xi-(l\pm1)t)^2)^{3/4}} \leq 10.$$

and that for $l \neq 0$

$$\begin{aligned}
&\int_{I_l} \xi \frac{1}{(l^2+(\xi-lt)^2)^{1/4}} \frac{1}{((l\pm1)^2+(\xi-(l\pm1)t)^2)^{3/4}} \\
&\leq |\frac{\xi}{l^2}|^{-1/2} \int_{I_l} \frac{1}{(1+(\frac{\xi}{l}-t)^2)^{1/4}} \\
&\leq 2|\frac{\xi}{l^2}|^{-1/2}|\frac{\xi}{l^2}|^{+1/2} \leq 2.
\end{aligned}$$

The case $l = 0$ is estimated analogously. □

The same method of proof can also be applied for general ξ when restricting to suitably small times.

Proposition 3.8 *(The small time regime) Let $\xi < \epsilon^{-4}$ and define $T < \epsilon^{-2}$ such that*

$$\epsilon T^{3/2}\xi^{-1/2} = \frac{1}{4C_\alpha}.$$

Further suppose that $\delta < \frac{1}{C_\alpha}$ with C_α as in Proposition 3.7.

Then for all $0 \leq t \leq \min(T, \delta\epsilon^{-2})$ it holds that

$$\|(Z, Q)(t)\|_{\ell^2} \leq C\|(Z, Q)\|_{\ell^2}.$$

Proof of Proposition 3.8 We claim that for this choice of T it holds that

$$\begin{aligned}
\int_0^T |c_l^{\pm}| &\leq \frac{1}{4C_\alpha}, \\
\int_0^T |d_l^{\pm}| &\leq \frac{1}{4C_\alpha}.
\end{aligned}$$

The result then follows by the same argument as in the proof of Proposition 3.7.

Indeed, we observe that for k such that $t_k \leq T$ (that is for all k larger than k_1 with $t_{k_1} \approx T$) it holds that

$$\int_{I_{l\pm1}} |c_l^\pm| \leq \epsilon \frac{\xi}{k^{3/2}} \leq \epsilon t_k^{3/2} \xi^{-1/2} \leq \frac{1}{4C_\alpha}.$$

If instead k is such that $t_k < T$ (or if we integrate over $(0, T) \setminus I_{l\pm1}$) then integral is not (yet) resonant and hence

$$\int_0^T |c_l^\pm| \leq \delta \leq \frac{1}{4C_\alpha}.$$

□

For times larger than T resonances are possibly very large and thus the preceding argument does not work anymore, since estimates of the form

$$\|(Z, Q)\|_{\ell^2,t} \leq C + 2\|(Z, Q)\|_{\ell^2,t},$$

do not control the norm. In the following Sect. 3.2.3 we thus instead establish growth bounds on each interval I_k which mimic the growth of the toy model of Sect. 3.1 with slight changes to the exponent.

3.2.3 Main Echo Chains

In this section we consider the main norm inflation mechanism of the (simplified) linearized Boussinesq equations as compared to the toy model of Sect. 3.1. Here, similarly to the Euler setting (Deng and Zillinger, 2021), it turns out for large frequencies the back-coupling between resonant and non-resonant modes results in correction of the growth bounds, which has to be taken into account. As we discuss in Sect. 4 the following results remain valid for the non-simplified linearized Boussinesq equations as well.

As a preliminary step we consider a more accurate toy model and establish more accurate bounds. Similar bounds have previously been established in (Bedrossian et al. 2021, Proposition 4.1). We additionally highlight the time-dependence of $f(t)$.

Lemma 3.9 *Let* $\frac{\xi}{k^2} \geq 100$ *be given, let* $0 \leq f(t) < \delta$ *and consider the differential inequalities*

$$|\partial_t Z_{NR}| \leq f(t)\sqrt{\frac{\xi}{k^2}} \frac{1}{(1+t^2)^{3/4}} |Z_R|,$$
$$|\partial_t Z_R| \leq f(t) \left(\frac{\xi}{k^2}\right)^{-1/2} \frac{1}{(1+t^2)^{1/4}} |Z_{NR}|,$$

on the interval $(-\frac{\xi}{k^2}, \frac{\xi}{k^2})$*. Then there exists a constant* $0 < \gamma < 2\delta$ *such that*

$$|Z_{NR}(+\frac{\xi}{k^2}) - Z_{NR}(-\frac{\xi}{k^2})| \leq \|f\|_{L^\infty} \left(\frac{\xi}{k^2}\right)^{1/2+\gamma} \left(|Z_R(-\frac{\xi}{k^2})| + \|f\|_{L^\infty}|Z_{NR}(-\frac{\xi}{k^2})|\right),$$
$$|Z_R(+\frac{\xi}{k^2}) - Z_R(-\frac{\xi}{k^2})| \leq \|f\|_{L^\infty} \left(\frac{\xi}{k^2}\right)^{1/2+\gamma} \left(|Z_{NR}(-\frac{\xi}{k^2})| + \|f\|_{L^\infty}|Z_R(-\frac{\xi}{k^2})|\right).$$

That is, we obtain an upper bound on the norm inflation by $\|f\|_{L^\infty}(\frac{\xi}{k^2})^{1/2+\gamma}$.

We remark that if $f(t)$ is replaced by a constant and if we consider a differential equation instead of an inequality, then the ODE system

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = f \begin{pmatrix} 0 & (\frac{\xi}{k^2})^{1/2}\frac{1}{(1+t^2)^{3/4}} \\ (\frac{\xi}{k^2})^{-1/2}\frac{1}{(1+t^2)^{1/4}} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

can be solved explicitly by noting that u solves

$$(1+t^2)^{3/4}\partial_t(1+t^2)^{1/4}\partial_t u = (1+t^2)\partial_t^2 u + \frac{1}{2}t\partial_t u = f^2 u.$$

This is the defining equation of Legendre functions and the above estimates hence follow from the known asymptotics of these functions.

The main aim of this lemma is thus to provide a more robust energy-based proof, which also extends to differential inequalities with time-dependent coefficients.

Proof of Lemma 3.9 We consider the following energy:

$$\mathcal{E}(t) := \begin{cases} |\frac{(1+t^2)^{1/4}}{\sqrt{\xi/k^2}} Z_{NR}|^2 + |Z_R|^2 & \text{if } t < 0, \\ |\frac{1}{\sqrt{\xi/k^2}} Z_{NR}|^2 + |(1+t^2)^{-1/4} Z_R|^2 & \text{if } t > 0. \end{cases}$$

We note that $\mathcal{E}(t)$ is continuous and that both $(1+t^2)^{1/2}$ and $(1+t^2)^{-1/2}$ are decreasing on the respective time intervals. Hence by direct computation

$$\partial_t \mathcal{E}(t) \leq f(t)(1+t^2)^{-1/2}\mathcal{E}(t).$$

Integrating this inequality on $(-\frac{\xi}{k^2}, \frac{\xi}{k^2})$ we obtain that

$$\mathcal{E}(T) \leq \exp\left(\int_{-\frac{\xi}{k^2}}^{T} f(t)(1+t^2)^{-1/2}\right)\mathcal{E}(-\frac{\xi}{k^2}).$$

for all $T \in [-\frac{\xi}{k^2}, \frac{\xi}{k^2}]$. In particular, it holds that

$$\mathcal{E}(t) \leq \left(\frac{\xi}{k^2}\right)^{\gamma} \mathcal{E}\left(-\frac{\xi}{k^2}\right).$$

and by construction

$$\mathcal{E}(-\frac{\xi}{k^2}) \approx |Z_{NR}(-\frac{\xi}{k^2})|^2 + |Z_R(-\frac{\xi}{k^2})|^2.$$

We have thus established upper bounds for general initial data.

We next consider the differences compared to the initial data. By the fundamental theorem of calculus for any $-\frac{\xi}{k^2} \leq \tau \leq \frac{\xi}{k^2}$ it holds that

$$\begin{aligned}
|Z_{NR}(\tau) - Z_{NR}(-\frac{\xi}{k^2})| &\leq \int_{-\frac{\xi}{k^2}}^{\tau} f(t)(1+t^2)^{-3/4}|Z_R| \\
&\leq \int_{-\frac{\xi}{k^2}}^{\tau} f(t)((1+t^2)^{-1/2}1_{t<0} + (1+t^2)^{-3/4}1_{t>0})\sqrt{E}(t)dt \\
&\leq \|f(t)\|_{L^\infty}(1+(\frac{\xi}{k^2})^2)^\gamma \sqrt{E}\left(-\frac{\xi}{k^2}\right) \\
&\leq \|f(t)\|_{L^\infty}(1+(\frac{\xi}{k^2})^2)^\gamma (|Z_R(-\frac{\xi}{k^2})| + |Z_{NR}(-\frac{\xi}{k^2})|).
\end{aligned}$$

This is almost the desired bound except that we are still missing one factor of $\|f\|_{L^\infty}$. It is however already sufficient to estimate Z_R as

$$\begin{aligned}
|Z_R(\tau) - Z_R(-\frac{\xi}{k^2})| &\leq \|f(t)\|_{L^\infty} \int_{-\frac{\xi}{2}}^{\tau} (1+t^2)^{-1/4}(|Z_{NR}(t) - Z_{NR}(-\frac{\xi}{k^2})| + |Z_{NR}(\frac{\xi}{k^2})|)dt \\
&\leq \|f(t)\|_{L^\infty} \left(\frac{\xi}{k^2}\right)^{1/2} (|Z_{NR}(t) - Z_{NR}(-\frac{\xi}{k^2})| + |Z_{NR}(\frac{\xi}{k^2})|).
\end{aligned}$$

Finally we may return to the bound for Z_{NR} and split

$$|Z_{NR}(\tau) - Z_{NR}(-\frac{\xi}{k^2})| \leq \int_{-\frac{\xi}{k^2}}^{\tau} f(t)(1+t^2)^{-3/4}(|Z_R(t) - Z_R(-\frac{\xi}{k^2})| + |Z_R(\frac{\xi}{k^2})|$$

and insert the just derived bound. □

Having established this improved model we next show that also the (simplified) linearized Boussinesq equations exhibit this modified growth (as compared to the toy model of Sect. 3.1; the non-simplified equations are studied in Proposition 4.2). Here in addition to the above growth bounds we have to take into account the evolution by the homogeneous semigroup (see Sect. 2.1). We further recall that for the linearized Boussinesq system

$$|f(t)| \approx \epsilon\sqrt{\frac{\xi}{k}}.$$

Proposition 3.10 *Let* $0 < \xi < \epsilon^{-4}$, $\alpha > \frac{1}{4}$ *and* $0 < \epsilon < \delta$ *and consider the linearized Boussinesq equations (11) on the time interval*

$$I_k = (t_k, t_{k-1})$$

with $1 \leq k \leq k_0$ *and* t_k, t_{k-1} *as in Definition 3.5. Then there exists* $C = C(\alpha)$ *and* $0 < \gamma < \delta$ *such that for all choices of data at time* t_k *and all* $t \in \overline{I_k}$ *it holds that*

$$\|(Z, Q)(t)\|_{\ell^2} \leq C\epsilon(\frac{\xi}{k})^{1/2}(\frac{\xi}{k^2})^{\gamma} \,\|(Z, Q)(t_k)\|_{\ell^2} .$$

Corollary 3.11 *Under the same assumptions as in Proposition 3.10 for any* $l \leq k_0$ *it holds that*

$$\|(Z, Q)(t_l)\|_{\ell^2} \leq \left\|(Z, Q)(t_{k_0})\right\|_{\ell^2} \prod_{l \leq k \leq k_0} C\epsilon \left(\frac{\xi}{k}\right)^{1/2} \left(\frac{\xi}{k^2}\right)^{\gamma}$$

Thus the total possible norm inflation on $(t_{k_0}, \delta\epsilon^{-2})$ *is bounded by the exponential factor stated in Theorem 1.3.*

Proof of Corollary 3.11 The result follows by repeated application of the estimate of Proposition 3.10. In particular, choosing t_l maximal we obtain the products discussed in Sect. 3.1 with an additional correction in the exponent. □

Proof of Proposition 3.10 Based on the structure of the homogeneous problem as studied in Sect. 2.1 we consider

$$E_l(t) = |Z_l(t)|^2 + |Q_l(t)|^2 + \frac{1}{2\sqrt{\alpha}} \frac{(\xi - lt)}{\sqrt{l^2 + (\xi - lt)^2}} \Re Z_l \overline{Q_l}.$$

Then by the estimates of Proposition 2.1 it holds that

$$\begin{aligned}
\partial_t E_l(t) \leq\, & C \partial_t \left(\frac{1}{2\sqrt{\alpha}} \frac{(\xi - lt)}{\sqrt{l^2 + (\xi - lt)^2}} \right) E_l(t) \\
& + 2\overline{Z_l}(c_l^+ Z_{l+1} + c_l^- Z_{l-1}) \\
& + 2\overline{Q_l}(d_l^+ Q_{l+1} + d_l^- Q_{l-1}) \\
& + \frac{1}{2\sqrt{\alpha}} \frac{(\xi - lt)}{\sqrt{l^2 + (\xi - lt)^2}} \overline{Z_l}(d_l^+ Q_{l+1} + d_l^- Q_{l-1}) \\
& + \frac{1}{2\sqrt{\alpha}} \frac{(\xi - lt)}{\sqrt{k^2 + (\xi - lt)^2}} \overline{Q_l}(c_l^+ Z_{l+1} + c_l^- Z_{l-1}).
\end{aligned}$$

In order to remove the first term we introduce

$$\tilde{E}_l(t) = E_l(t) \exp \left(\int \left| \partial_t \left(\frac{1}{2\sqrt{\alpha}} \frac{(\xi - lt)}{\sqrt{l^2 + (\xi - lt)^2}} \right) \right| dt \right)$$

and observe that

$$\partial_t \tilde{E}_l \leq C\sqrt{\tilde{E}_l}(|c_l^+| + |d_l^+|)\sqrt{\tilde{E}_{l+1}} + C\sqrt{\tilde{E}_l}(|c_l^-| + |d_l^-|)\sqrt{\tilde{E}_{l-1}}.$$

Recalling that

$$|c_l^{\pm}| \leq |f(t)| \begin{cases} (\xi/k)^{1/2}(1 + (t - \frac{\xi}{k})^2)^{-3/4} & \text{if } c_l^{\pm} = c_{k\pm 1}^{\mp}, \\ (\xi/k)^{-1/2}(1 + (t - \frac{\xi}{k})^2)^{-1/4} & \text{else.} \end{cases}$$

we are thus in the framework of Lemma 3.9. More precisely, we may define

$$Z_{NR} = \sqrt{\tilde{E}_{k+1}^2 + \tilde{E}_{k-1}^2},$$
$$Z_R = \sqrt{\sum_{l \notin \{k-1, k+1\}} \tilde{E}_l^2}.$$

Then by the above estimates these functions satisfy the assumptions of Lemma 3.9 with

$$0 \leq |f(t)| \leq \epsilon\sqrt{\xi/k} \leq \delta.$$

In particular, it follows that

$$\|(Z, Q)\|_{\ell^2} \approx |Z_R|^2 + |Z_{NR}|^2$$

grows at most by a factor

$$1 + \|f(t)\|_{L^\infty} \left(\frac{\xi}{k^2}\right)^{1/2+\gamma}.$$

Since we are in the regime where the latter factor is bounded below, we may omit the 1 at the cost of a constant factor, which proves the result. □

4 On the Model Reduction

In this section we discuss the non-simplified linearized Boussinesq equations

$$\partial_t \begin{pmatrix} Z \\ Q \end{pmatrix} + A \begin{pmatrix} Z \\ Q \end{pmatrix} = \begin{pmatrix} f(t)\Delta_t^{-1/4}(\partial_y \Delta_t^{-3/4} Z \cos(x)) \\ g(t)\Delta_t^{1/4}(\partial_y \Delta_t^{-3/4} Z \cos(x)) \end{pmatrix} + \frac{1}{1+t^2} \begin{pmatrix} f(t)\Delta_t^{-1/4}(\cos(x)\partial_y \Delta_t^{1/4} Z) \\ f(t)\Delta_t^{1/4}(\cos(x)\partial_y \Delta_t^{-1/4} Q) \end{pmatrix}, \tag{15}$$

which we may also express in an integral system as in Definition (11) by introducing the coefficients

$$\begin{aligned} g_k^{\pm} &= \pm\frac{f(t)\xi}{2(1+t^2)}(k^2+(\xi-kt)^2)^{1/4}((k\pm 1)^2+(\xi-(k\pm 1)t)^2)^{-1/4}, \\ h_k^{\pm} &= \pm\frac{f(t)\xi}{2(1+t^2)}(k^2+(\xi-kt)^2)^{-1/4}((k\pm 1)^2+(\xi-(k\pm 1)t)^2)^{1/4}. \end{aligned} \tag{16}$$

We observe that by our choice of time interval

$$\frac{f(t)}{2(1+t^2)} \leq \min(\epsilon\frac{\sqrt{t}}{1+t^2}, \delta\frac{1}{1+t^2})$$

is small and integrable and that

$$\frac{f(t)}{1+t^2}\cos(x)\partial_y$$

is a transport operator which corresponds to a change of variables

$$\begin{aligned} (x, y) &\mapsto (x, y - F(t)\sin(x)), \\ F(t) &= \int_0^t \frac{f(t)}{1+t^2}d\tau \leq 2\epsilon, \end{aligned} \tag{17}$$

which is an analytic change of variables and a small perturbation of the identity (for ϵ small).

In view of this smallness and in order to simplify the analysis of the model and the presentation of the resonance mechanism, throughout this article we have considered the simplified linearized Boussinesq equations which omitted these terms.

In the following we show that this simplification indeed does not change the results of the long-time regime of Sect. 3.2.1 and the echo chains of Sect. 3.2.3. For the small time regime of Sect. 3.2.2 we obtain (much) rougher bounds for the full model. We expect that with (considerable) technical effort it should be possible to improve these bounds after incorporating an additional change of variables (see the discussion following Proposition 4.2).

We begin by discussing the "large time" regime of Sect. 3.2.1:

$$2\xi < t < \delta\epsilon^{-2}.$$

Proposition 4.1 *Let $\epsilon, \delta, f(t), \gamma(t)$ be as in Proposition 3.4. Then the solution of the linearized Boussinesq equations (15) exhibits at most algebraic growth on the time interval $(2\xi, \delta\epsilon^{-2})$. More precisely, for all $t \in (2\xi, \delta\epsilon^{-2})$ the projections onto and away from the Fourier modes $k = -1, 1$ satisfy:*

$$\|1_{|k|\neq 1}(Z, Q)(t)\| \leq C_\alpha \exp(10)\sqrt{\frac{t}{\xi}}\|(Z, Q)(2\xi)\|,$$

$$\|1_{|k|=1}(Z, Q)(t)\| \leq C_{\alpha,\gamma}(\frac{t}{\xi})^{\gamma} \exp(10)\sqrt{\frac{t}{\xi}}\|(Z, Q)(2\xi)\|,$$

where $1/2 < \gamma < 1$ *is a constant.*

Proof of Proposition 4.1 We observe that for all $k \notin -1, 0, 1$ for $t > 2\xi$ the fractions

$$(k^2 + (\xi - kt)^2)^{1/4}((k \pm 1)^2 + (\xi - (k \pm 1)t)^2)^{-1/4}$$

are uniformly bounded. Hence, for these values of k we may bound

$$|g_k^{\pm}| + |h_k^{\pm}| \leq \frac{\delta\xi}{1 + t^2}.$$

We further observe that

$$\int_{2\xi}^{\infty} \frac{\delta\xi}{1 + t^2} dt \leq 2\delta$$

is integrable. The result hence follows by the same proof as for Proposition 3.4 by noting that in the remaining cases

$$|g_k^{\pm}| + |h_k^{\pm}| \leq \frac{\delta\xi}{1 + t^2}(t/\xi)^{1/2}.$$

□

We next turn to the resonant regime of Sect. 3.2.3 which consists of the time intervals $I_k = (t_k, t_{k-1})$ for which

$$\epsilon\sqrt{\xi/k}\sqrt{\xi/k^2}$$

is large.

Proposition 4.2 *(Bound on norm inflation) Under the assumptions of Proposition 3.10 also for the linearized Boussinesq equations the possible norm inflation is controlled in the sense that for all* $t \in \overline{I_k}$ *it holds that*

$$\|(Z, Q)(t)\|_{\ell^2} \leq C\epsilon(\frac{\xi}{k})^{1/2}(\frac{\xi}{k^2})^{\gamma} \|(Z, Q)(t_k)\|_{\ell^2},$$

where $C = C(\alpha)$ *and* $0 < \gamma < \delta$ *are constants*

Proof of Proposition 4.2 We consider the same energies and unknowns as in the proof of Proposition 3.10, where in the computation of $\partial_t \tilde{E}_l$ we obtain additional terms controlled by

$$(|h_l^{\pm}| + |g_l^{\pm}|)\tilde{E}_{l\pm1}.$$

We now observe that for $l \notin \{k-1, k, k+1\}$ it holds that

$$(|h_l^{\pm}| + |g_l^{\pm}|) \leq \frac{f(t)}{1+t^2}\xi$$

and for $t \in I_k$ we may further bound

$$\begin{aligned}\frac{f(t)}{1+t^2}\xi &\leq c\delta \frac{1}{1+(\xi/k)^2}\xi \leq c\delta(\xi/k^2)^{-1} \\ &\leq c\delta(\xi/k^2)^{-1/2}(1+(t-\frac{\xi}{k})^2)^{-1/4}.\end{aligned}$$

For $l \in \{k-1, k, k+1\}$ we argue similarly and control

$$\begin{aligned}&\frac{f(t)}{1+(\xi/k)^2}\xi(1+(\xi/k-t)^2)^{-1/4}(1+(\xi/(k+1)-t)^2)^{1/4} \\ &\leq f(t)(\xi/k^2)^{-1}(1+(\xi/k-t)^2)^{-1/4}(\xi/k^2)^{1/2} \\ &\leq f(t)(\xi/k^2)^{-1/2}(1+(\xi/k-t)^2)^{-1/4}\end{aligned}$$

Thus for all l we may control

$$(|h_l^{\pm}| + |g_l^{\pm}|)\tilde{E}_{l\pm 1} \leq cf(t)(\xi/k^2)^{-1/2}(1+(\xi/k-t)^2)^{-1/4}\tilde{E}_{l\pm 1}$$

in the same way as a non-resonant contribution $c_l^{\pm} \neq c_{k\pm 1}^{\mp}$. The result hence follows by the same argument as in Proposition 3.10. □

It hence only remains to discuss the "small time" regime of Sect. 3.2.2. Here we observe that

$$\int \frac{f(t)}{1+t^2}\xi \leq \int \epsilon \frac{\sqrt{t}}{1+t^2}\xi \leq c\epsilon\xi$$

in general is *not* small enough to employ the contraction argument of Proposition 3.8 unless ξ is smaller than ϵ^{-1}. Indeed also for the change of variables (17) we cannot expect good bounds in high Sobolev or Gevrey norms for frequencies larger than ϵ^{-1}. In order to obtain better bounds we thus have to take these changes into account.

One option here is to consider the unknowns (Z, Q) in the coordinates (17). However, here $F(t)\sin(x)$ introduces further nearest neighbor coupling in x, which makes terms such as $\Delta_t^{-1/4}$ very technically challenging to study. A second option, which sidesteps this issue, is to restrict to studying stability estimates in $\ell^2(\mathbb{Z})$ (that is, with respect to Fourier modes in x for a fixed frequency in y), following the argument of Bedrossian et al. (2021). Since $f(t)/(1+t^2)\cos(x)\partial_y$ is an anti-symmetric operator in L^2, this space allows us to exploit cancellation and hence to estimate

$$h_l^{\pm} - \frac{f(t)}{2(1+t^2)}\xi$$

instead.

As a first preliminary result we consider an adaptation of the "intermediate time" estimate of (Bedrossian et al. 2021, Section 6.3.2), which allows for loss of regularity in Gevrey σ with $\sigma < 2$ in the time interval where $t > \sqrt{\xi}$.

Lemma 4.3 *Let ξ and T be as in Proposition 3.8 and suppose that $\sqrt{\xi} < T$. Then on the time interval $(\sqrt{\xi}, T)$ the maximal possible norm inflation is bounded by*

$$\exp(\delta c_\sigma \xi^\sigma)$$

for any $\sigma > \frac{1}{2}$.

Proof Arguing similarly as in the proof of Proposition 3.8 we consider the energy

$$\exp(\lambda(t)\xi^\sigma)\|(Z, Q)\|_{L^2}^2$$

with $\lambda(t)$ decreasing in time and bounded below, still to be determined. Computing the time derivative it then suffices to show that $\lambda(t)$ can be chosen such that

$$\dot{\lambda}(t)\xi^\sigma + |g_l^\pm| + |h_l^\pm|.$$

Indeed, we claim that

$$|g_l^\pm| + |h_l^\pm| \leq f(t)$$

and observe that since $t < T$ and $t > \sqrt{\xi}$ it holds that

$$\begin{aligned} f(t)\sqrt{\xi}k &\leq 1, \\ \Leftrightarrow f(t) &\leq \xi^{1/2}/t \leq \xi^\sigma t^{-1+2(\sigma-1/2)}. \end{aligned}$$

The result then follows by noting that $t^{-1+2(\sigma-1/2)}$ is integrable and hence

$$\lambda(\tau) := \lambda(\sqrt{\xi}) + \int_{\sqrt{\xi}}^{\tau} t^{-1+2(\sigma-1/2)} dt$$

yields the desired result.

It remains to prove the claim. For this purpose we observe that away from resonant frequencies, that is for $l \notin \{k-1, k, k+1\}$ it holds that

$$(l^2 + (\xi - lt)^2)^{1/4}((l \pm 1)^2 + (\xi - (l \pm 1)t)^2)^{-1/4} \leq 2$$

is bounded and hence

$$|g_l^\pm| + |h_l^\pm| \leq \frac{f(t)}{1+t^2}\xi \leq f(t),$$

where we used $t > \sqrt{\xi}$ in the last step.

For the resonant frequency we estimate

$$(l^2 + (\xi - lt)^2)^{1/4}((l \pm 1)^2 + (\xi - (l \pm 1)t)^2)^{-1/4} \leq 2(1 + \frac{\xi}{k^2}).$$

Then since $t > \sqrt{\xi}$ it holds that.

$$\begin{aligned}\frac{1}{1+t^2}\xi(1 + \frac{\xi}{k^2}) &\leq \frac{1}{1+t^2}(\xi + t^2)\\ &\leq \frac{1}{1+t^2}2t^2 \leq 2.\end{aligned}$$

This concludes the proof of the claim. □

We next need to consider the “small time regime” where $t < \min(\sqrt{\xi}, T)$, where we adapt the argument of Section 6.3.1 in Bedrossian et al. (2021) (in their notation we estimate a term similar to $\mathcal{T}_N^{p,1}$). On that time interval the bound by $\sqrt{\xi}/t$ is not sufficient. We thus need to exploit the L^2 cancellation, which involves

$$(1 + (\frac{\xi}{l} - t)^2)^{1/4}(1 + (\frac{\xi}{l \pm 1} - t)^2)^{-1/4} - 1.$$

Lemma 4.4 *Let ξ, T and $\sigma > \frac{1}{2}$ be as in Proposition 3.8. Then for all $0 < t < \min(\sqrt{\xi}, T)$ it holds that*

$$\begin{aligned}&\langle Z, |\frac{1}{1+t^2}(f(t)|\partial_x|^{1/2}\Delta_t^{-1/4}(\cos(x)\partial_y|\partial_x|^{-1/2}\Delta_t^{1/4}Z\rangle_{\ell^2}\\ &+ \langle Q, f(t)|\partial_x|^{-1/2}\Delta_t^{+1/4}(\cos(x)\partial_y|\partial_x|^{1/2}\Delta_t^{-1/4}Q\rangle_{L^2}\end{aligned}$$

is bounded by

$$\exp(c_\sigma \xi^\sigma)\|(Z, Q)(0)\|_{\ell^2}^2$$

We remark that for this estimate we only establish stability in the unweighted ℓ^2 space, since the proof exploit that the shear $f(t)\cos(x)\partial_y$ is anti-symmetric on L^2.

Proof We observe that for $t \to \infty$ or $l \to \infty$

$$|\partial_x|^{1/4}\Delta_t^{1/4} \to 1.$$

We may hence exploit the fact that the operator $f(t)\cos(x)\partial_y$ is anti-symmetric and thus have to estimate

$$f(t)/(1 + t^2)\xi((1 + (\xi/l - t)^2)^{1/4}(1 + (\xi/(l \pm 1) - t)^2)^{-1/4} - 1).$$

We claim that this term can be estimated from above:

$$f(t)/(1+t^2)\xi((1+(\xi/l-t)^2)^{1/4}(1+(\xi/(l\pm 1)-t)^2)^{-1/4}-1) \le f(t) \quad (18)$$

This is sufficient to conclude since $f(t) \le \delta$ by assumption and for $1+t \le \sqrt{\xi}$ we may insert a factor

$$1 = (1+t)^{2\sigma}(1+t)^{-2\sigma} \le \xi^{\sigma}(1+t)^{-2\sigma}$$

and $(1+t)^{-2\sigma}$ is integrable since $\sigma > 1/2$.

It thus remains to prove the claim (18). We may rewrite the last factor in the term to be estimated as

$$\begin{aligned} &(1+(\xi/l-t)^2)^{1/4}(1+(\xi/(l\pm 1)-t)^2)^{-1/4}-1 \\ &= (1+(\xi/(l\pm 1)-t)^2)^{-1/4}((1+(\xi/l-t)^2)^{1/4}-(1+(\xi/(l\pm 1)-t)^2)^{1/4}). \end{aligned}$$

We first discuss the case when l and $l \pm 1$ do not equal k. In this case by the intermediate value theorem there exists

$$\frac{\xi}{l} < \eta < \frac{\xi}{l\pm 1}$$

such that

$$(1+(\xi/l-t)^2)^{1/4}-(1+(\xi/(l\pm 1)-t)^2)^{1/4} \le (1+(\eta-t)^2)^{-3/4}\frac{\xi}{l(l\pm 1)}.$$

Since both l and $l \pm 1$ are non-resonant it follows that $|\eta - t| \ge \frac{\xi}{l(l\pm 1)}$ and $|\frac{\xi}{l} - t| \ge \frac{\xi}{l(l\pm 1)}$. Summarizing for this case we obtain that

$$\xi((1+(\xi/l-t)^2)^{1/4}(1+(\xi/(l\pm 1)-t)^2)^{-1/4}-1) \le C$$

and thus obtain a bound by $f(t)/(1+t^2) \le f(t)$.

For the remaining resonant cases the potentially largest one is given by $l = k$. In that case we may estimate

$$\begin{aligned} &\xi((1+(\xi/l-t)^2)^{1/4}(1+(\xi/(l\pm 1)-t)^2)^{-1/4}-1) \\ &\qquad \le (\frac{\xi}{k})^2(1+(\frac{\xi}{k}-t)^2)^{-1/4}. \end{aligned}$$

We thus obtain a bound of the total contribution by

$$f(t)(1+(\frac{\xi}{k}-t)^2)^{-1/4} \le f(t),$$

which concludes the proof. □

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Declarations

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