A Family of Semitoric Systems with Four Focus–Focus Singularities and Two Double Pinched Tori

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Abstract

We construct a one-parameter family $F_t = (J, H_t)_{0 \leq t \leq 1}$ of integrable systems on a compact 4-dimensional symplectic manifold (M, ω) that changes smoothly from a toric system F_0 with eight elliptic–elliptic singular points via toric type systems to a semitoric system F_t for $t^- < t < t^+$. These semitoric systems F_t have precisely four elliptic–elliptic and four focus–focus singular points. Moreover, at $t = \frac{1}{2}$, the system has precisely two focus–focus fibres each of which contains exactly two focus–focus points, giving these fibres the shape of double pinched tori. We exemplarily parametrise one of these fibres explicitly.

Keywords Integrable systems · Semitoric systems · Focus-focus singularities · Pinched torus · Hamiltonian–Hopf bifurcation · Hamiltonian S^1 –action

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1 Introduction

Integrable systems lie at the intersection of many areas in mathematics and physics like, for example, dynamical systems, ODEs, PDEs, symplectic geometry, Lie theory, classical mechanics, mathematical physics, etc. Integrable systems display a ‘certain amount of order’ due to being ‘not chaotic’ and having their flow lines stay in the fibres

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of the momentum map. Their global behaviour, nevertheless, can be very intricate. When it comes to their singularities, nondegenerate singularities can be classified by means of a local normal form that splits a singularity into hyperbolic, elliptic, regular, and focus components where in fact the latter always comes as a pair, usually thus referred to as focus–focus block.

In this paper, we will construct a special one-parameter family of integrable systems of two degrees of freedom on a 4-dimensional manifold which displays an interesting bifurcation behaviour and where all mentioned types of components will appear except for the hyperbolic ones. Since a completely integrable system gives rise to a Lagrangian fibration this explicit family may also become of interest for the study of low dimensional singular Lagrangian fibrations and, for instance, the (non)displaceability of its fibers under Hamiltonian isotopies, i.e. questions of symplectic rigidity.

Let us now be more precise. Given a 4-dimensional, connected, symplectic manifold (M, ω) , a *semitoric system* is a completely integrable system $F := (J, H) : M \rightarrow \mathbb{R}^2$ where F only has nondegenerate singularities that have no hyperbolic components and where $J : M \rightarrow \mathbb{R}$ is proper and induces an effective Hamiltonian $\mathbb{S}^1$ -action. For instance, coupled spin oscillators and coupled angular momenta are semitoric.

Contrary to toric systems on 4-dimensional manifolds that give rise to an $\mathbb{S}^1 \times \mathbb{S}^1$ -action and only admit elliptic–elliptic and elliptic–regular singular points, semitoric systems give rise to an $\mathbb{S}^1 \times \mathbb{R}$ -action and admit in addition focus–focus singularities.

Whereas toric systems are classified (up to equivariant symplectomorphism) by precisely one invariant, namely by the image of their momentum map (cf. Delzant 1988), the classification of semitoric systems is more involved: apart from one invariant based on a ‘straightened’ image of the momentum map, Pelayo and Vũ Ngọc (2009, 2011) showed that there are four other invariants necessary for a symplectic classification, namely the total number of focus–focus points, the position of their images in the straightened image of the momentum map, a Birkhoff normal form type invariant (called ‘Taylor series invariant’) that describes the impact a focus–focus point has on the neighbourhood of its fibre, and, last but not least, the so-called twisting index invariant that describes the dynamical interaction between the different focus–focus fibres. Originally this classification applied only to semitoric systems with maximally one focus–focus point per fibre. Recently, Palmer et al. (2019) generalised the invariants to allow for more than one focus–focus point per fibre. For an overview of the research results on semitoric systems during the past decade, we refer the reader to the recent surveys by Alonso and Hohloch (2019) and Pelayo (2021) and the references therein.

During the last decade, semitoric systems gained more and more attention and popularity. This is due to the fact that, although their classification is more involved than the one of toric systems, it is still ‘doable and constructive’: Pelayo and Vũ Ngọc (2011) showed, how to construct for given data that are admissible as invariants, a semitoric system having these data as invariants. They constructed the systems by gluing together coordinate patches containing the necessary singular points etc. But patching together plus smoothing along the seams of the patches is not very suitable for *explicit observations* how a semitoric system and its invariants change under variation of parameters.

For this, it is much more practical to have systems that are defined globally by one explicit formula on an explicitly given symplectic manifold. Recently, there has been progress in this direction: Alonso et al. (2019, 2020) completed the computation of the classifying invariants for coupled spin oscillators with varying radius and coupled angular momenta with two varying radii and a coupling parameter. Hohloch and Palmer (2018) generalised the coupled angular momenta to an explicit semitoric family with two coupling parameters that has two distinct focus–focus points for certain intervals of the parameters. Le Floch and Palmer (2019) pushed this further to explicit families of semitoric systems on Hirzebruch surfaces. By using invariant functions to perturb a given toric system, they came up with a method that seemed to work for more general situations than Hirzebruch surfaces.

This technique plus the question how difficult it is to construct *explicit* semitoric systems with *more than two* focus–focus points motivated the present paper: We construct a one-parameter family of integrable systems F_t that is at time $t = 0$ a toric system, then, as soon as $t > 0$, it becomes a system of toric type until some $0 < t^- < \frac{1}{2}$. At $t = t^-$, four singular points undergo a Hamiltonian–Hopf bifurcation, changing from elliptic–elliptic to focus–focus. As soon as $t > t^-$, the system is semitoric with four focus–focus points until some t^+ with $\frac{1}{2} < t^+ < 1$ where the focus–focus points undergo again a Hamiltonian–Hopf bifurcation that renders them elliptic–elliptic. Moreover, at time $t = \frac{1}{2}$, the system has precisely two focus–focus fibres each of which contains exactly two focus–focus points, effectively creating thus two singular fibres that are double pinched tori. For $t^+ < t \leq 1$, the system is again of toric type.

The precise construction of F_t goes as follows: First, starting with the octagon Δ in Fig. 1, we follow the steps of Delzant’s (1988) construction as described in detail in Cannas da Silva (2008) to obtain a 4-dimensional, compact, connected, symplectic manifold $(M, \omega) := (M_\Delta, \omega_\Delta)$ by using symplectic reduction by a Hamiltonian $\mathbb{T}^6$ -action of the 10-dimensional preimage of a certain map from $\mathbb{C}^8$ to $\mathbb{R}^6$ (the details are given in Sect. 3). Points on (M, ω) are usually written as equivalence classes of the form $[z] = [z_1, \dots, z_8]$ with $z_k = x_k + iy_k \in \mathbb{C}$ for $1 \leq k \leq 8$. The momentum map of the toric system on (M, ω) is in fact surprisingly simple:

Theorem 1.1 *Let Δ be the octagon displayed in Fig. 1. Then, $F = (J, H) : (M, \omega) \rightarrow \mathbb{R}^2$ given by*

$$J([z_1, \dots, z_8]) = \frac{1}{2}|z_1|^2, \quad H([z_1, \dots, z_8]) = \frac{1}{2}|z_3|^2$$

is a momentum map of an effective Hamiltonian 2-torus action satisfying $F(M) = \Delta$. Thus, in particular, F has precisely eight elliptic–elliptic singular points.

This theorem is restated as Theorem 3.6 and proven throughout Sect. 3. In addition, in Proposition 3.7, we compute the precise coordinates of the eight elliptic–elliptic fixed points of $F = (J, H)$.

Now, we look for a suitable function to perturb the integral H of $F = (J, H)$ with. To this aim, consider the function $Z : \mathbb{C}^8 \rightarrow \mathbb{C}$ given by $Z(z_1, \dots, z_8) := \overline{z_2 z_3 z_4} z_6 z_7 z_8$. We will see in Lemma 4.3 that it descends to a function $Z : (M, \omega) \rightarrow \mathbb{C}$

Fig. 1 The octagon $\Delta \subset \mathbb{R}^2$, obtained by chopping off the corners of the square $[0, 3] \times [0, 3]$, is a Delzant polytope

and so do its real part $\Re(Z) =: X$ and imaginary part $\Im(Z) =: Y$. Both functions are in addition invariant under J so that they also pass to the reduced spaces $M^{red,j}$ for $j \in J(M)$. Now, we vary H via linear combination $(1 - 2t)H + t\gamma X$ for $\gamma > 0$ sufficiently small and obtain

Theorem 1.2 *Let $(M, \omega, F = (J, H))$ be the toric system from Theorem 1.1 and let $0 < \gamma < \frac{1}{48}$ and set $F_t := (J, H_t) := (J, (1 - 2t)H + t\gamma X) : (M, \omega) \rightarrow \mathbb{R}^2$. Then, $(M, \omega, F_t)_{0 \leq t \leq 1}$ is toric for $t = 0$, of toric type for $0 < t < t^-$, semitoric for $t^- < t < t^+$, and again of toric type for $t^+ < t \leq 1$ where*

$$0 < t^- := \frac{1}{2(1 + 24\gamma)} < \frac{1}{2} < t^+ := \frac{1}{2(1 - 24\gamma)} < 1.$$

For all $t \in [0, 1]$, the system F_t has precisely eight fixed points of which four are always elliptic–elliptic. The other four pass at $t = t^-$ from elliptic–elliptic via a Hamiltonian–Hopf bifurcation to focus–focus. At $t = t^+$, these four focus–focus points turn again back into elliptic–elliptic points via a Hamiltonian–Hopf bifurcation. For the special situation occurring at $t = \frac{1}{2}$, see Proposition 1.3.

The momentum map image $F_t(M)$ is plotted for various values of t in Fig. 2. Theorem 1.2 is restated as Theorem 4.7 in Sect. 4. Moreover, Proposition 4.8 computes the explicit coordinates of the eight fixed points of F_t for time $0 \leq t \leq 1$, thus extending Proposition 3.7 from the toric situation at $t = 0$ to $t \in [0, 1]$.

Let us remark that in the terminology of Kane et al. (2018), the systems $(F_t)_{t^- < t < t^+}$ are minimal of type (6) with $k = -2$ and $c = 4$ and $d = 4$, see (Kane et al. 2018, Theorem 2.4 and the table in Theorem 4.15).

The proof of Theorem 1.2 resp. Theorem 4.7 is done in several steps spread over the following sections:

- In Sect. 5, we show the existence and determine the positions of the fixed points.
- In Sect. 6, we show that the fixed points are nondegenerate and we determine their type.

Fig. 2 Image of the momentum map $F_t(M)$ of the family (M, ω, F_t) from Theorem 1.2 plotted with *Mathematica* for fifteen time steps between $t = 0$ and $t = 1$ with $\gamma = \frac{1}{60}$

- In Sect. 7, we determine the rank one points and show that they are nondegenerate and of elliptic-regular type.
- In Sect. 8, we summarise all steps and prove Theorem 4.7 and hereby also Theorem 1.2.

At $t = \frac{1}{2}$, the fibres over $(1, 0)$ and $(2, 0)$ contain precisely two focus-focus points each. These fibres can be thought of as double pinched tori as sketched in Fig. 3 and we exemplarily parametrise the one over $(1, 0)$ explicitly.

Fig. 3 The fibre $F_{\frac{1}{2}}^{-1}(1, 0)$ seen as double pinched torus. The parametrisation with r and ϑ in Proposition 1.3 attains A for $r = 0$ and B for $r = \sqrt{6}$. The sign in front of r tells the two ‘bulges’ of the double pinched torus apart. ϑ describes the rotational coordinate around each bulge. The plot is done with *Mathematica*

Proposition 1.3 *At $t = \frac{1}{2}$, the system $F_{\frac{1}{2}}$ has precisely two focus–focus fibres, each of which contains precisely two focus–focus points so that each of these two fibres has the shape of a double pinched torus (see Fig. 3). Exemplarily, $F_{\frac{1}{2}}^{-1}(1, 0)$ can be parametrised as*

$$\left\{ \left[\sqrt{2}, \sqrt{6-r^2}, \sqrt{6-r^2}, \sqrt{8-r^2}, 2, \sqrt{2+r^2}, \pm r i e^{-i\vartheta}, r e^{i\vartheta} \right] \in M \mid \vartheta \in [0, 2\pi[, r \in [0, \sqrt{6}] \right\}.$$

This statement is reformulated as Proposition 4.9 and proven in Sect. 8.

The successful construction of the 1-parameter family F_t suggests that Le Floch’s and Palmer’s (2019) method interpolation with invariant functions may work in more generality. In particular, it should make the construction of semitoric systems possible with any wished number of focus–focus points and with a *globally defined, explicitly given* momentum map. In practice, the verification of such examples by hand may be lengthy and awkward since the number of equations defining the symplectic manifold grows.

Control of the Taylor series invariant or twisting index is unfortunately not (yet) possible with these techniques. Nevertheless, constructing focus–focus fibres containing two focus–focus points seemed easy enough. But since these focus–focus points originated from elliptic–elliptic points underlying vertices of the momentum polytope, one needs to employ additional methods if one wants to obtain fibres containing more than two focus–focus points.

Figures

The figures are either done with *Mathematica* or with *Xfig*.

2 Fundamental Definitions, Facts, and Conventions

2.1 Symplectic Conventions

Within this paper, we usually consider $\mathbb{R}^{2n}$ equipped with the coordinates $(x_1, y_1, x_2, y_2, \dots, x_n, y_n)$. Sometimes we identify $(x_k, y_k) = x_k + iy_k = z_k$ for $1 \leq k \leq n$ and consider $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ with coordinates $(z_1, \dots, z_n)$. The standard symplectic form ω_{st} on $\mathbb{R}^{2n}$ is thus represented by a $(2n \times 2n)$ -matrix having submatrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ along the diagonal and zeros otherwise.

Given a symplectic manifold (M, ω) and a smooth function $f : M \rightarrow \mathbb{R}$, we define the Hamiltonian vector field $\mathcal{X}^f$ of f via $\omega(\mathcal{X}^f, \cdot) = df$. When we consider $\mathcal{X}^f$ at a point $p \in M$, we usually write $\mathcal{X}^f(p)$ and, in case of the symplectic form, ω_p .

The Poisson bracket of two functions $f, g : M \rightarrow \mathbb{R}$ induced by ω is given by

$$\{f, g\} := -\omega(\mathcal{X}^f, \mathcal{X}^g) = -df(\mathcal{X}^g) = dg(\mathcal{X}^f).$$

2.2 Completely Integrable Systems

A $2n$ -dimensional completely integrable system is a triple (M, ω, f) where (M, ω) is a $2n$ -dimensional symplectic manifold and $f = (f_1, \dots, f_n) \in C^\infty(M, \mathbb{R}^n)$ is a map that satisfies

- $\{f_j, f_k\} = 0$ for all $1 \leq j, k \leq n$.
- $\mathcal{X}^{f_1}(p), \dots, \mathcal{X}^{f_n}(p)$ are linearly independent for almost all $p \in M$.

The function f is called the *momentum map* of the integrable system (M, ω, f) .

Let (M, ω, f) be a $2n$ -dimensional completely integrable system. A point $p \in M$ is called *regular* for f if $\mathcal{X}^{f_1}(p), \dots, \mathcal{X}^{f_n}(p)$ are linearly independent. Otherwise p is said to be *singular*. The *rank* of the singular point p is the rank of $(\mathcal{X}^{f_1}(p), \dots, \mathcal{X}^{f_n}(p))$ or, equivalently, the rank of the Jacobian $df(p)$. A singular point of rank zero is usually called a *fixed point*. A value $r \in \mathbb{R}^n$ is called *regular* if the whole fibre or level set $f^{-1}(r)$ only contains regular points. Otherwise $r \in \mathbb{R}^n$ is called *singular*. A fibre is said to be *regular* if it contains only regular points and otherwise *singular*.

2.3 Nondegeneracy and Local Normal Form in $2n$ Dimensions

A singular point $p \in M$ of rank zero of a completely integrable system $(M, \omega, f = (f_1, \dots, f_n))$ is *nondegenerate* if the Hessians $d^2 f_1(p), \dots, d^2 f_n(p)$ span a Cartan subalgebra of the Lie algebra of quadratic forms on $T_p M$. We refer to Bolsinov and Fomenko (2004) for the precise definition of nondegenerate points of higher rank. For the case $\dim M = 4$, we recall it below.

Williamson (1936) classified Cartan subalgebras of $\mathfrak{sp}(n, \mathbb{R})$. This in turn yields a classification of the possible subalgebras generated by the Hessians in $T_p M$ seen as $\mathfrak{sp}(n, \mathbb{R})$. Vey (1978), Eliasson (1984, 1990), Miranda and Zung (2004), Vũ Ngọc and Wacheux (2013), Chaperon (2013), and others extended Williamson's pointwise classification to the following local classification and even more general versions.

Theorem 2.1 (Local normal form) *Let $p \in M$ be a nondegenerate singular point of a $2n$ -dimensional completely integrable system $(M, \omega, f = (f_1, \dots, f_n))$. Then there exist local symplectic coordinates $(x, y) := (x_1, \dots, x_n, y_1, \dots, y_n)$ around p such that there exists $q = (q_1, \dots, q_n) : M \rightarrow \mathbb{R}^n$ where each q_j is given by one of*

1. *elliptic:* $q_j(x, y) = \frac{1}{2}(x_j^2 + y_j^2)$,
2. *hyperbolic:* $q_j(x, y) = x_j y_j$,
3. *focus-focus:* $\begin{cases} q_j(x, y) = x_j y_{j+1} - x_{j+1} y_j, \\ q_{j+1}(x, y) = x_j y_j + x_{j+1} y_{j+1}, \end{cases}$
4. *non-singular:* $q_j(x, y) = y_j$,

such that $\{f_j, q_k\} = 0$ for all $1 \leq j, k \leq n$.

Equivalently, the different types of nondegenerate singular points can be distinguished by the eigenvalues in the associated Cartan subalgebra:

Proposition 2.2 (Vũ Ngọc 2006, Chapter 3) *Let C be a regular element in the Cartan subalgebra generated by the Hessians of the components of the momentum map (i.e. C has $2n$ distinct eigenvalues) at a fixed point. Then there appear three distinct types of groups of eigenvalues of C :*

1. *elliptic block:* a pair of imaginary eigenvalues $\pm i\beta$,
2. *hyperbolic block:* a pair of real eigenvalues $\pm\alpha$,
3. *focus-focus block:* a quadruple of complex eigenvalues $\pm\alpha \pm i\beta$,

where $\alpha, \beta \in \mathbb{R}^{\neq 0}$.

2.4 Nondegeneracy and Local Normal Form in 4 Dimensions

In this paper, we mainly work on 4-dimensional symplectic manifolds, i.e. there are only singular points of rank zero or rank one possible. In this situation, nondegeneracy of rank zero points (= fixed points) can be verified as follows.

Lemma 2.3 (Bolsinov and Fomenko 2004) *Let $(M, \omega, f = (f_1, f_2))$ be a 4-dimensional completely integrable system having a fixed point $p \in M$. Let ω_p be the matrix of the symplectic form with respect to a basis of $T_p M$ and let $d^2 f_1(p)$ and $d^2 f_2(p)$ be the matrices of the Hessians of f_1 and f_2 with respect to the same basis. Then, the fixed point p is nondegenerate if and only if $d^2 f_1(p)$ and $d^2 f_2(p)$ are linearly independent and there exists a linear combination of $\omega_p^{-1} d^2 f_1(p)$ and $\omega_p^{-1} d^2 f_2(p)$ which has four distinct eigenvalues.*

Denote by $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ the distinct eigenvalues of a nondegenerate fixed point p as considered in Lemma 2.3. Then, Theorem 2.1 and Proposition 2.2 imply that the type of p is one of the following:

- *elliptic–elliptic* if $\lambda_1, \lambda_2 = \pm i\alpha$ and $\lambda_3, \lambda_4 = \pm i\beta$,
- *elliptic–hyperbolic* if $\lambda_1, \lambda_2 = \pm i\alpha$ and $\lambda_3, \lambda_4 = \pm\beta$,
- *hyperbolic–hyperbolic* if $\lambda_1, \lambda_2 = \pm\alpha$ and $\lambda_3, \lambda_4 = \pm\beta$,
- *focus–focus* if $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta, \lambda_3 = -\alpha + i\beta$ and $\lambda_4 = -\alpha - i\beta$,

where $\alpha, \beta \in \mathbb{R}^{\neq 0}$ and $\alpha \neq \beta$ for the elliptic–elliptic and hyperbolic–hyperbolic cases.

Nondegeneracy of rank one points can be characterised as follows on 4-dimensional symplectic manifolds (M, ω) , see Bolsinov & Fomenko (Bolsinov and Fomenko 2004, Section 1.8.2). Let p be a singular point of rank one of a 4-dimensional completely integrable system $(M, \omega, f = (f_1, f_2))$. Then, there are $\mu, \lambda \in \mathbb{R}$ such that $\mu df_1(p) + \lambda df_2(p) = 0$ and $L_p := \text{Span}\{\mathcal{X}^{f_1}(p), \mathcal{X}^{f_2}(p)\} \subset T_p M$ is the tangent line in p of the orbit generated by the $\mathbb{R}^2$ -action. Denote by $L_p^\perp$ the symplectic orthogonal complement to L_p in $T_p M$. Note that $L_p \subset L_p^\perp$. The Poisson commutativity $\{f_1, f_2\} = 0$ implies that they are invariant under the $\mathbb{R}^2$ -action. Therefore, $\mu d^2 f_1(p) + \lambda d^2 f_2(p)$ descends to the quotient $L_p^\perp / L_p$. This allows to define

Definition 2.4 (Bolsinov and Fomenko 2004) A rank one critical point p of a 4-dimensional completely integrable system $(M, \omega, f = (f_1, f_2))$ is *nondegenerate* if $\mu d^2 f_1(p) + \lambda d^2 f_2(p)$ is invertible on $L_p^\perp / L_p$.

The possible types of nondegenerate rank one points on 4-dimensional manifolds are

- *elliptic–regular* if the eigenvalues of $\omega_p^{-1}(\mu d^2 f_1(p) + \lambda d^2 f_2(p))$ on $L_p^\perp / L_p$ are of the form $\pm i\alpha$, $\alpha \in \mathbb{R}^{\neq 0}$,
- *hyperbolic–regular* if the eigenvalues of $\omega_p^{-1}(\mu d^2 f_1(p) + \lambda d^2 f_2(p))$ on $L_p^\perp / L_p$ are of the form $\pm\alpha$, $\alpha \in \mathbb{R}^{\neq 0}$.

If one of the integrals has a periodic flow, the search for rank one points can be done by means of reduced spaces as described in Lemma 2.7 later on.

2.5 Symplectic Reduction

Let us recall the following important notations and results:

Theorem 2.5 (Marsden and Weinstein 1974) Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ inducing a Hamiltonian action on a symplectic manifold (M, ω) . Let $f : M \rightarrow \mathfrak{g}^*$ be the momentum map of this action. Let $r \in \mathfrak{g}^*$ be a regular value of f that is fixed by the coadjoint action and denote by $\kappa : f^{-1}(r) \hookrightarrow M$ the inclusion. Assume that G acts freely and properly on $f^{-1}(r)$. Then,

- $M^{\text{red}, r} := f^{-1}(r)/G$ is a manifold of dimension $\dim M - 2 \dim G$.
- The quotient map $\tau : f^{-1}(r) \rightarrow f^{-1}(r)/G = M^{\text{red}, r}$ is a principal G -bundle.
- Then, there is a unique symplectic structure $\omega^{\text{red}, r}$ on $M^{\text{red}, r}$ such that $\tau^* \omega^{\text{red}, r} = \kappa^* \omega$.

The symplectic manifold $(M^{\text{red}, r}, \omega^{\text{red}, r})$ is called the *symplectic reduction of M at level r* by the action generated by G , also known as *symplectic quotient* or *Marsden–Weinstein quotient*.

We will use Theorem 2.5 (Marsden-Weinstein) in particular in the situation of a 4-dimensional integrable systems where one of the integrals induces an $\mathbb{S}^1$ -action:

Corollary 2.6 *Let $(M, \omega, F = (J, H))$ be a 4-dimensional completely integrable system and let J have a periodic flow. Let $j \in \mathbb{R}$ be a regular value of J and assume that the $\mathbb{S}^1$ -action induced by J is free and proper on $J^{-1}(j)$. Then, the symplectic reduction in M at level j by the $\mathbb{S}^1$ -action is given by the Marsden-Weinstein quotient*

$$M^{\text{red},j} := J^{-1}(j)/\mathbb{S}^1$$

and the Poisson commutativity of J and H assures that H descends to a smooth function $H^{\text{red},j} : M^{\text{red},j} \rightarrow \mathbb{R}$, called reduced Hamiltonian of H .

The following statement can be found in Le Floch and Palmer (2019, Lemma 2.6) who in turn refer to Toth and Zelditch (2003, Definition 3) and Hohloch and Palmer (2018, Corollary 2.5). It means that the reduced Hamiltonian function is particularly useful to study singular points of rank one of the original system.

Lemma 2.7 *Let $(M, \omega, F = (J, H))$ be a completely integrable system on a 4-dimensional symplectic manifold and let J have a periodic flow. Let $j \in \mathbb{R}$ be a regular value of J and assume that the $\mathbb{S}^1$ -action induced by J is free and proper on $J^{-1}(j)$. Let $p \in J^{-1}(j)$ and set $[p] := \tau(p) \in M^{\text{red},j}$. Then*

1. p is a singular point of rank one of $F = (J, H)$.
 $\Leftrightarrow [p]$ is a singular point of $H^{\text{red},j}$, i.e. $dH^{\text{red},j}([p]) = 0$.
2. p is a nondegenerate singular point of rank one of F .
 $\Leftrightarrow [p]$ is a nondegenerate singular point of $H^{\text{red},j}$.
 $\Leftrightarrow dH^{\text{red},j}([p]) = 0$ and $d^2H^{\text{red},j}$ is invertible at $[p]$ in $M^{\text{red},j}$.
3. p is an elliptic-regular (respectively hyperbolic-regular) point of F .
 $\Leftrightarrow [p]$ is an elliptic (respectively hyperbolic) singular point of $H^{\text{red},j}$.

So far, we considered symplectic reduction of regular values. If the value is singular one encounters stratified spaces and needs to work with singular reduction, see Sjamaar and Lerman (1991), or, for an overview, Alonso (2019, pp. 26–28), or do it by hand for simple examples as in Lemma 4.1.

2.6 Toric Systems and Delzant's Construction

Let (M, ω, f) be a $2n$ -dimensional completely integrable system. If f is in fact the momentum map of a Hamiltonian n -torus action, we call (M, ω, f) a *toric manifold*. According to the convexity theorem by Atiyah (1982), Guillemin and Sternberg (1982), $f(M)$ is a convex polytope spanned by the images of the fixed points. It is often referred to as *momentum polytope*. A convex polytope in $\mathbb{R}^n$ is *Delzant* if (i) its edges have rational slope, (ii) at each vertex, precisely n edges meet, (iii) at each vertex, the n tangent directions considered as vectors in $\mathbb{Z}^n$ span $\mathbb{Z}^n$.

Theorem 2.8 (Delzant 1988) *A $2n$ -dimensional toric manifold having an effective Hamiltonian torus action is classified up to equivariant isomorphism by its momentum*

polytope which is in fact Delzant. Conversely, for each Delzant polytope, there exists, up to equivariant symplectomorphism, a toric manifold having the Delzant polytope as momentum polytope.

Crucial for the present paper is the fact that Delzant's proof is constructive, i.e. given a Delzant polytope Δ , there is an explicit way how to construct a toric manifold (M, ω, f) with $f(M) = \Delta$. We briefly sketch the construction as outlined by Cannas da Silva (2008) since we will make ample use of it in Sect. 3:

- (1) Given a Delzant polytope $\Delta \subset \mathbb{R}^n$ with k edges, write it as intersection of k halfspaces (whose boundaries lies on the edges of Δ) by means of the primitive, integral, outer normal vectors.
- (2) Define a map $\vartheta : \mathbb{R}^k \rightarrow \mathbb{R}^n$ by sending the standard basis of $\mathbb{R}^k$ to the k outer normal vectors of the edges. This map is also welldefined as map $\vartheta : \mathbb{Z}^k \rightarrow \mathbb{Z}^n$ and descends to $\vartheta : \mathbb{R}^k / \mathbb{Z}^k = \mathbb{T}^k \rightarrow \mathbb{R}^n / \mathbb{Z}^n = \mathbb{T}^n$.
- (3) The kernel $N := \ker(\vartheta)$ of $\vartheta : \mathbb{T}^k \rightarrow \mathbb{T}^n$ is isomorphic to an $(k - n)$ -torus. Consider the inclusion $N \hookrightarrow \mathbb{T}^k$ and extend it to a map $\ell : \mathbb{R}^{k-n} \rightarrow \mathbb{R}^n$. Denote its dual map by $\ell^* : \mathbb{R}^n \rightarrow \mathbb{R}^{k-n}$.
- (4) Let $c \in \mathbb{Z}^k$ be the vector whose entries are given by the minimal distance to the origin of the boundaries of the halfspaces defining Δ . Consider the standard k -torus action on $\mathbb{C}^k$ with momentum map

$$L := L_c : \mathbb{C}^k \rightarrow \mathbb{R}^k, \quad L_c(z_1, \dots, z_k) := -\frac{1}{2}(|z_1|^2, \dots, |z_k|^2) + c$$

and define the map

$$\tilde{L} := \ell^* \circ L : \mathbb{C}^k \rightarrow \mathbb{R}^{k-n}.$$

- (5) Note that $0 \in \mathbb{R}^{k-n}$ is a regular value of $\tilde{L}$ and that N acts freely and properly on $\tilde{L}^{-1}(0)$. According to Theorem 2.5 (Marsden-Weinstein), $M^{red,0} := \tilde{L}^{-1}(0)/N$ is a symplectic manifold with symplectic form $\omega^{red,0}$ which satisfies $\tau^* \omega^{red,0} = \kappa^* \omega_{st}$ where $\tau : \tilde{L}^{-1}(0) \rightarrow \tilde{L}^{-1}(0)/N$ is the quotient map and $\kappa : \tilde{L}^{-1}(0) \hookrightarrow (\mathbb{C}^k, \omega_{st})$ the inclusion and ω_{st} the standard symplectic form on $\mathbb{C}^k$.
- (6) Find a right inverse $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^k$ of the map $\vartheta : \mathbb{R}^k \rightarrow \mathbb{R}^n$ and consider the concatenation

$$\tilde{F} := \sigma^* \circ L \circ \kappa : \tilde{L}^{-1}(0) \rightarrow \mathbb{R}^n.$$

Since $\tilde{F}$ is invariant under the action of N it descends to a map $F : M^{red,0} \rightarrow \mathbb{R}^n$. Then $F : (M^{red,0}, \omega^{red,0}) \rightarrow \mathbb{R}^n$ is the momentum map of a Hamiltonian $\mathbb{T}^n$ -action satisfying $F(M^{red,0}) = \Delta$.

2.7 Systems of Toric Type

The following notion appeared first in Vũ Ngọc (2007, Definition 2.1) for systems that are ‘toric up to a diffeomorphism of the momentum polytope’.

Definition 2.9 A completely integrable system (M, ω, F) is of *toric type* if $F : M \rightarrow \mathbb{R}^2$ is proper and if there exists an effective, Hamiltonian $\mathbb{T}^2$ -action on M whose momentum map is of the form $f \circ F$ where $f : F(M) \rightarrow f(F(M))$ is a diffeomorphism.

Properness of F is automatically satisfied if the underlying manifold M is compact. Systems of toric type are a special case of so-called *weakly toric systems* introduced in Hohloch et al. (2018).

2.8 Semitoric Systems and Semitoric Transition Families

Toric systems only admit elliptic–elliptic or elliptic–regular points. In order to admit more types of singular points while keeping the class of systems as accessible as possible Vũ Ngọc (2003, 2007) began to focus on the following class of systems:

Definition 2.10 A 4-dimensional completely integrable system $(M, \omega, F = (J, H))$ is *semitoric* if

- 1) $J : M \rightarrow \mathbb{R}$ is proper;
- 2) J is the momentum map of an effective Hamiltonian $\mathbb{S}^1$ -action (i.e. the flow of $\mathcal{X}^J$ is periodic with minimal period 2π in our convention);
- 3) all singular points of F are nondegenerate and have no hyperbolic components.

A semitoric system is called *simple* if each fibre of J contains at most one focus–focus point. Simple semitoric systems were studied and classified by Pelayo and Vũ Ngọc (2009; 2011) by means of five invariants.

The semitoric systems constructed later in Sect. 4 have always two focus–focus points in a fibre of J whenever the fibre contains focus–focus points, so they are not simple. However, had we started the construction with a Delzant octagon where the vertices of the horizontal edges do not lie on the same vertical line, then we would expect to get a simple semitoric system.

We will study the transition of elliptic–elliptic points to focus–focus points and back. Note that, at the moment of transition, the singular points have to be degenerate, i.e. the system is at this very moment *not* semitoric. The following definition is adapted from Le Floch and Palmer (2019) who study transitions in parameter depending semitoric families over Hirzebruch surfaces.

Definition 2.11 Let (M, ω) be a 4-dimensional symplectic manifold. Let $J : (M, \omega) \rightarrow \mathbb{R}$ be smooth with periodic Hamiltonian flow and let $H : [0, 1] \times M \rightarrow \mathbb{R}$ be smooth with $H_t := H(t, \cdot)$ for $t \in [0, 1]$ such that $F_t := (J, H_t) : (M, \omega) \rightarrow \mathbb{R}^2$ is a completely integrable system. We call $(M, \omega, F_t = (J, H_t))$ a *semitoric family with fixed $\mathbb{S}^1$ -action and $k \in \mathbb{N}_0$ degenerate times $t_1, \dots, t_k$* if $(M, \omega, F_t = (J, H_t))$ is semitoric for all $t \in [0, 1] \setminus \{t_1, \dots, t_n\}$.

Coupled angular momenta are an example for such a family. Hohloch and Palmer (2018) generalized them to a two-parameter family admitting two focus–focus points while leaving the $\mathbb{S}^1$ -action unchanged. Le Floch and Palmer (2019) studied semitoric families with fixed $\mathbb{S}^1$ -action on Hirzebruch surfaces.

Here, we are in particular interested in fixed points changing from elliptic–elliptic to focus–focus and back (so-called *Hamiltonian–Hopf bifurcations*). The following definition is inspired by Le Floch and Palmer (2019).

Definition 2.12 A *semitoric family with transition points* $p_1, \dots, p_n \in M$ and *transition times* $0 < t^- < t^+ < 1$ is a semitoric family with fixed $\mathbb{S}^1$ -action $(M, \omega, F_t)_{0 \leq t \leq 1}$ and degenerate times t^- and t^+ , such that

- 1) for $0 < t < t^-$ and $t^+ < t < 1$, the points $p_1, \dots, p_n$ are elliptic–elliptic;
- 2) for $t^- < t < t^+$, the points $p_1, \dots, p_n$ are focus–focus;
- 3) for $t = t^-$ and $t = t^+$, the points $p_1, \dots, p_n$ are degenerate and there are no degenerate singular points in $M \setminus \{p_1, \dots, p_n\}$;
- 4) if p_i is a maximum (resp. minimum) of $H_0|_{J^{-1}(J(p_i))}$ then p_i is a minimum (resp. maximum) of $H_1|_{J^{-1}(J(p_i))}$.

Briefly, we just speak of a *semitoric transition family* in such a situation.

The following statements describe what happens at the transition times.

Proposition 2.13 (Hohloch and Palmer (2018)) *Let $(M, \omega, F_t)_{0 \leq t \leq 1}$ be a semitoric transition family with fixed $\mathbb{S}^1$ -action. Let $p \in M$ and $t_0 \in]0, 1[$ be such that p is a fixed point for all t in a neighbourhood of t_0 where p is of focus–focus type for $t > t_0$ and of elliptic–elliptic type for $t < t_0$. Then, p is a degenerate fixed point for $t = t_0$.*

Note that singular points might change from rank zero to rank one points:

Proposition 2.14 (Le Floch and Palmer (2019)) *Let $(M, \omega, (J, H_t))$ be a semitoric family with fixed $\mathbb{S}^1$ -action and suppose that $p \in M$ is a rank zero singular point for some $t_0 \in [0, 1]$. Then, p is a singular point for all $t \in [0, 1]$ but not necessarily of rank zero. If p does not belong to a fixed surface of J , then p is a fixed point for all $t \in [0, 1]$.*

3 The Toric System Constructed from the Octagon in Fig. 1

Using Delzant's construction sketched in Sect. 2.6, we will construct in this section the toric system associated to the polytope Δ in Fig. 1.

3.1 The Symplectic Manifold Associated to the Octagon Δ

Consider the octagon Δ in Fig. 1. A brief look at the coordinates of the vertices

$$(0, 2) \quad (0, 1) \quad (1, 0) \quad (2, 0) \quad (3, 1) \quad (3, 2) \quad (2, 3) \quad (1, 3)$$

of Δ yields for the edges, listed counterclockwise, as spanning vectors

$$(-1, 0) \quad (-1, -1) \quad (0, -1) \quad (1, -1) \quad (1, 0) \quad (1, 1) \quad (0, 1) \quad (-1, 1)$$

which in turn yields that Δ is a Delzant polygon. Thus, we may use Delzant's construction (see Sect. 2.6) to obtain a symplectic manifold (M, ω) and momentum map $F : (M, \omega) \rightarrow \mathbb{R}^2$ with $F(M) = \Delta$. To this aim, write Δ as intersection of eight halfplanes:

$$\Delta := \left\{ (x, y) \in \mathbb{R}^2 \mid \begin{array}{ll} \langle (x, y), (-1, 0) \rangle \leq 0, & \langle (x, y), (1, 0) \rangle \leq 3 \\ \langle (x, y), (-1, -1) \rangle \leq -1, & \langle (x, y), (1, 1) \rangle \leq 5 \\ \langle (x, y), (0, -1) \rangle \leq 0, & \langle (x, y), (0, 1) \rangle \leq 3 \\ \langle (x, y), (1, -1) \rangle \leq 2, & \langle (x, y), (-1, 1) \rangle \leq 2 \end{array} \right\} \quad (1)$$

Note that all appearing outer-pointing normal vectors of the half planes are primitive. Denote by $e_1, \dots, e_8$ the standard basis of $\mathbb{R}^8$. Consider the linear map $\vartheta : \mathbb{R}^8 \rightarrow \mathbb{R}^2$ that assigns to each basis vector an outer normal vector as follows (starting at the left edge and going counter-clockwise):

$$\begin{array}{llll} e_1 \mapsto (-1, 0) & e_3 \mapsto (0, -1) & e_5 \mapsto (1, 0) & e_7 \mapsto (0, 1) \\ e_2 \mapsto (-1, -1) & e_4 \mapsto (1, -1) & e_6 \mapsto (1, 1) & e_8 \mapsto (-1, 1) \end{array} \quad (2)$$

The map $\vartheta : \mathbb{R}^8 \rightarrow \mathbb{R}^2$ is surjective and induces a surjective linear map $\vartheta : \mathbb{Z}^8 \rightarrow \mathbb{Z}^2$. Therefore, passing to the quotients $\mathbb{R}^8/\mathbb{Z}^8 =: \mathbb{T}^8$ and $\mathbb{R}^2/\mathbb{Z}^2 =: \mathbb{T}^2$ yields a well-defined quotient map

$$\vartheta : \mathbb{T}^8 \rightarrow \mathbb{T}^2$$

that is linear. Its kernel is given by

$$\begin{aligned} N := \ker(\vartheta) &= \{(x_1, \dots, x_8) \in \mathbb{T}^8 \mid x_1 \cdot (-1, 0) + \dots + x_8 \cdot (-1, 1) = (0, 0)\} \\ &= \left\{ (x_1, \dots, x_8) \in \mathbb{T}^8 \mid \begin{array}{l} x_5 = x_1 + x_2 - x_4 - x_6 + x_8 \\ x_7 = x_2 + x_3 + x_4 - x_6 - x_8 \end{array} \right\} \simeq \mathbb{T}^6 \end{aligned}$$

which yields an inclusion $N \hookrightarrow \mathbb{T}^8$. By renaming two variables, we obtain the (injective) linear map

$$\begin{aligned} \ell : \mathbb{R}^6 &\rightarrow \mathbb{R}^8, \\ (x_1, \dots, x_6) &\mapsto (x_1, x_2, x_3, x_4, x_1 + x_2 - x_4 - x_5 + x_6, x_5, x_2 + x_3 + x_4 - x_5 - x_6, x_6). \end{aligned}$$

To simplify notation, identify in the following the dual vector space $(\mathbb{R}^m)^*$ with $\mathbb{R}^m$ for all $m \in \mathbb{N}_0$. Recall that the dual map $f^* : \mathbb{R}^m \rightarrow \mathbb{R}^n$ of a linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is defined via the standard dual pairing

$$\langle y, f(x) \rangle = \langle f^*(y), x \rangle \quad \forall x \in \mathbb{R}^n, \forall y \in \mathbb{R}^m.$$

Thus, if $f(x) = Cx$ for an $(m \times n)$ -matrix C , then $f^*(y) = C^T y$. Therefore, we find

$$\ell^* : \mathbb{R}^8 \rightarrow \mathbb{R}^6, \quad (y_1, \dots, y_8) \mapsto C^T \begin{pmatrix} y_1 \\ \vdots \\ y_8 \end{pmatrix} \quad \text{with } C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

i.e. explicitly

$$\ell^*(y) = (y_1 + y_5, y_2 + y_5 + y_7, y_3 + y_7, y_4 - y_5 + y_7, -y_5 + y_6 - y_7, y_5 - y_7 + y_8).$$

Note that, since ℓ is injective, ℓ^* is surjective.

The goal is to consider N as an action on $\mathbb{C}^8$ and then to pass to the quotient. As a first step, we note

Remark 3.1 Consider the symplectic manifold $(\mathbb{C}^8, \omega_{st} := -\frac{i}{2} \sum_{k=1}^8 dz_k \wedge d\bar{z}_k)$ with the standard $\mathbb{T}^8$ -action

$$\mathbb{T}^8 \times \mathbb{C}^8 \rightarrow \mathbb{C}^8, \quad (t, z) := (t_1, \dots, t_8, z_1, \dots, z_8) \mapsto (e^{it_1} z_1, \dots, e^{it_8} z_8).$$

This action is Hamiltonian and admits the family of momentum maps

$$L_c : \mathbb{C}^8 \rightarrow \mathbb{R}^8, \quad L_c(z) = -\frac{1}{2}(|z_1|^2, \dots, |z_8|^2) + c$$

where $c \in \mathbb{R}^8$.

Now choose $c = (0, -1, 0, 2, 3, 5, 3, 2)$, which are the constants used to define the eight half planes in (1), and set $L := L_{(0, -1, 0, 2, 3, 5, 3, 2)} : \mathbb{C}^8 \rightarrow \mathbb{R}^8$.

Recall $\mathbb{T}^6 \simeq N \subset \mathbb{T}^8$ and use the inclusion $N \hookrightarrow \mathbb{T}^8$ as motivation to define the action

$$\begin{aligned} &(\mathbb{T}^6 \simeq N) \times \mathbb{C}^8 \rightarrow \mathbb{C}^8, \\ &(t, z) \mapsto (e^{it_1} z_1, e^{it_2} z_2, e^{it_3} z_3, e^{it_4} z_4, e^{i(t_1+t_2-t_4-t_5+t_6)} z_5, e^{it_5} z_6, e^{i(t_2+t_3+t_4-t_5-t_6)} z_7, e^{it_6} z_8). \end{aligned} \quad (3)$$

The corresponding momentum map is given by

$$\tilde{L} := \ell^* \circ L : \mathbb{C}^8 \xrightarrow{L} \mathbb{R}^8 \xrightarrow{\ell^*} \mathbb{R}^6$$

which is in coordinates given by

$$\begin{aligned} (z_1, \dots, z_8) &\xrightarrow{\ell} -\frac{1}{2}(|z_1|^2, \dots, |z_8|^2) + (0, -1, 0, 2, 3, 5, 3, 2) \\ &\xrightarrow{\ell^*} -\frac{1}{2}(|z_1|^2 + |z_5|^2, |z_2|^2 + |z_5|^2 + |z_7|^2, |z_3|^2 + |z_7|^2, |z_4|^2 - |z_5|^2 + |z_7|^2, \\ &\quad -|z_5|^2 + |z_6|^2 - |z_7|^2, |z_5|^2 - |z_7|^2 + |z_8|^2) + (3, 5, 3, 2, -1, 2) \end{aligned}$$

Note that for the action of N on $(\mathbb{C}^8, \omega_0)$ with momentum map $\tilde{L} : \mathbb{C}^8 \rightarrow \mathbb{R}^6$, the value $0 \in \mathbb{R}^6$ is regular since ℓ^* is surjective. The regular level set $\tilde{L}^{-1}(0)$ is described by six equations:

$$\tilde{L}^{-1}(0) = \left\{ (z_1, \dots, z_8) \in \mathbb{C}^8 \mid \begin{array}{ll} (A) & |z_1|^2 + |z_5|^2 = 6 \\ (B) & |z_2|^2 + |z_5|^2 + |z_7|^2 = 10 \\ (C) & |z_3|^2 + |z_7|^2 = 6 \\ (D) & |z_4|^2 - |z_5|^2 + |z_7|^2 = 4 \\ (E) & |z_5|^2 - |z_6|^2 + |z_7|^2 = 2 \\ (F) & |z_5|^2 - |z_7|^2 + |z_8|^2 = 4 \end{array} \right\} \quad (4)$$

The six equations (A) – (F) in (4) are referred to as the **manifold equations** of M . Consider the inclusion $\kappa : \tilde{L}^{-1}(0) \hookrightarrow \mathbb{C}^8$ and note that N acts freely and properly on $\tilde{L}^{-1}(0)$ (for a proof, see Cannas da Silva (2008)). Therefore, we may apply Theorem 2.5 (Marsden-Weinstein) to this situation and conclude that

$$M := M^{red,0} = \tilde{L}^{-1}(0)/N$$

is a 4-dimensional symplectic manifold, denoting the quotient map by $\tau : \tilde{L}^{-1}(0) \rightarrow M$. On the quotient, we have the symplectic form $\omega := \omega^{red,0}$ which satisfies

$$\tau^* \omega = \kappa^* \omega_{st} \quad (5)$$

3.2 Explicit Charts of the Manifold M

For the constructions later on, we need explicit charts for M and some additional notation.

Notation 3.2 *The notation mod 8 means that integers are counted modulo 8 and that they are considered as element of the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Note that this set starts with 1 instead of 0.*

To define suitable charts for the manifold M , we will make use of the following technical observation.

Lemma 3.3 *Let $[z] = [z_1, \dots, z_8] \in M = \tilde{L}^{-1}(0)/N$ and $k \in \{1, \dots, 8\}$ and $z_k = 0$. Then, there are precisely two distinct cases possible:*

1. $z_j \neq 0$ for $j \in \{k+2, \dots, k+7\} \bmod 8$,
2. $z_j \neq 0$ for $j \in \{k+1, \dots, k+6\} \bmod 8$.

This means that maximally two coordinates of z may vanish and, if so, these must be consecutive ones.

Proof We show the assertion for $k = 1$. The other cases follow similarly. Let $z_1 = 0$ and consider the manifold equations (A) – (F) from (4). Equation (A) leads to $|z_5|^2 = 6 \neq 0$ implying $z_5 \neq 0$. That z_3, z_4, z_6, z_7 must be nonzero is proven by contradiction:

$$\begin{aligned} \text{If } z_3 = 0 &\stackrel{(C)}{\Rightarrow} |z_7|^2 = 6 \stackrel{(B)}{\Rightarrow} 0 \leq |z_2|^2 = -2 \quad \text{!} \\ \text{If } z_4 = 0 &\stackrel{(D)}{\Rightarrow} |z_7|^2 = 10 \stackrel{(C)}{\Rightarrow} 0 \leq |z_3|^2 = -4 \quad \text{!} \\ \text{If } z_6 = 0 &\stackrel{(E)}{\Rightarrow} 0 \leq |z_7|^2 = -4 \quad \text{!} \\ \text{If } z_7 = 0 &\stackrel{(F)}{\Rightarrow} 0 \leq |z_8|^2 = -2 \quad \text{!} \end{aligned}$$

Thus, if $z_1 = 0$, only z_2 or z_8 may vanish. But z_2 and z_8 cannot vanish at the same time:

$$\begin{aligned} \text{If } z_2 = 0 &\stackrel{(A)}{\Rightarrow} |z_5|^2 = 6 \stackrel{(B)}{\Rightarrow} |z_7|^2 = 4 \stackrel{(F)}{\Rightarrow} |z_8|^2 = 2 \Rightarrow z_8 \neq 0. \\ \text{If } z_8 = 0 &\stackrel{(A)}{\Rightarrow} |z_5|^2 = 6 \stackrel{(F)}{\Rightarrow} |z_7|^2 = 2 \stackrel{(B)}{\Rightarrow} |z_2|^2 = 2 \Rightarrow z_2 \neq 0. \end{aligned}$$

Moreover, since the action of N on $\tilde{L}^{-1}(0)$ preserves the norm $|z_j|$ for each entry $1 \leq j \leq 8$, the action does not affect the property of being zero. $\square$

This motivates us to define, for $1 \leq \nu \leq 8$, the open sets

$$U_\nu := \{ [z_1, \dots, z_8] \in M \mid z_k \neq 0 \text{ for all } k \in \{\nu+2, \dots, \nu+7\} \bmod 8 \}$$

In other words, U_ν is the only subset of M where z_ν and $z_{\nu+1}$ may vanish. This is independent of the chosen representative for the point $[z_1, \dots, z_8] \in M$. In particular, we have an open covering $M = \bigcup_{\nu=1}^8 U_\nu$.

Now, we will construct explicit charts $\psi_\nu : U_\nu \rightarrow \mathbb{R}^4$ for all $1 \leq \nu \leq 8$. Given $[z] \in U_\nu$, there are at least six variables nonzero among $z_1, \dots, z_8$. Now use the $N \simeq \mathbb{T}^6$ -action to pick strictly positive real numbers as representatives for them. Writing $z_k = x_k + iy_k$ for $1 \leq k \leq 8$, this means $y_k = 0$ and $x_k > 0$ for these six representatives. Therefore, for $\nu = 1$, we represent U_1 by points of the form

$$(z_1, \dots, z_8) = (x_1, y_1, x_2, y_2, x_3, 0, x_4, 0, x_5, 0, x_6, 0, x_7, 0, x_8, 0).$$

Using the manifold equations from (4), the variables $x_3, \dots, x_8$ can be written in U_1 as functions of x_1, y_1, x_2, y_2 as follows: abbreviating $|z_1|^2 = x_1^2 + y_1^2$ and $|z_2|^2 = x_2^2 + y_2^2$, we obtain

$$\begin{aligned} x_3 &= \sqrt{2 - |z_1|^2 + |z_2|^2}, & x_5 &= \sqrt{6 - |z_1|^2}, & x_7 &= \sqrt{4 + |z_1|^2 - |z_2|^2}, \\ x_4 &= \sqrt{6 - 2|z_1|^2 + |z_2|^2}, & x_6 &= \sqrt{8 - |z_2|^2}, & x_8 &= \sqrt{2 + 2|z_1|^2 - |z_2|^2}. \end{aligned} \quad (6)$$

Therefore,

$$\psi_1 : U_1 \rightarrow \mathbb{R}^4, \quad [x_1, y_1, x_2, y_2, x_3, 0, x_4, 0, x_5, 0, x_6, 0, x_7, 0, x_8, 0] \mapsto (x_1, y_1, x_2, y_2)$$

is a chart for U_1 with inverse

$$\psi_1^{-1} : \mathbb{R}^4 \rightarrow U_1, \quad (x_1, y_1, x_2, y_2) \mapsto [x_1, y_1, x_2, y_2, x_3, 0, x_4, 0, x_5, 0, x_6, 0, x_7, 0, x_8, 0]$$

where $x_3, \dots, x_8$ in ψ_1^{-1} are determined by (6). Applying analogous reasoning to the remaining indices, we obtain, for all $1 \leq \nu \leq 8$, charts of the form

$$\begin{aligned} \psi_\nu : U_\nu \rightarrow V_\nu := \psi_\nu(U_\nu) \subseteq \mathbb{R}^4, \\ [x_1, 0, x_2, 0, \dots, x_\nu, y_\nu, x_{\nu+1}, y_{\nu+1}, \dots, x_8, 0] \mapsto (x_\nu, y_\nu, x_{\nu+1}, y_{\nu+1}). \end{aligned}$$

3.3 The Symplectic Form ω on M in Local Coordinates

In this subsection, we will compute the symplectic form ω on M constructed in Sect. 3.1 in the local charts (U_ν, ψ_ν) defined in Sect. 3.2.

Let $z \in \tilde{L}^{-1}(0)$. Then, there is (at least one) index $1 \leq \nu \leq 8$ such that $\tau(z) = [z] \in U_\nu$. Over U_ν , the quotient map τ has the right inverse

$$\tau_\nu^{-1} : U_\nu \rightarrow \tilde{L}^{-1}(0), \quad [z] \mapsto (x_1, 0, x_2, 0, \dots, x_\nu, y_\nu, x_{\nu+1}, y_{\nu+1}, \dots, x_8, 0)$$

defined analogously to the chart ψ_ν, ψ_ν^{-1} in Sect. 3.2, i.e. we have $\tau \circ \tau_\nu^{-1} = \text{Id}_{U_\nu}$ and thus in turn $\psi_\nu \circ \tau \circ \tau_\nu^{-1} \circ \psi_\nu^{-1} = \text{Id}_{V_\nu}$ where $\psi_\nu(U_\nu) =: V_\nu$. Denote by $\omega_\nu := (\psi_\nu^{-1})^* \omega$ the symplectic form in local coordinates on $V_\nu \subseteq \mathbb{R}^4$. Then, the relation $\tau^* \omega = \kappa^* \omega_{st}$ in (5) becomes

$$\kappa^* \omega_{st} = \tau^* \omega = \tau^* \psi_\nu^* \omega_\nu = (\psi_\nu \circ \tau)^* \omega_\nu$$

and, pulling $\kappa^* \omega_{st}$ back with $\tau_\nu^{-1} \circ \psi_\nu^{-1}$ to V_ν , we obtain

$$(\kappa \circ \tau_\nu^{-1} \circ \psi_\nu^{-1})^* \omega_{st} = (\tau_\nu^{-1} \circ \psi_\nu^{-1})^* \kappa^* \omega_{st} = (\tau_\nu^{-1} \circ \psi_\nu^{-1})^* (\psi_\nu \circ \tau)^* \omega_\nu = \omega_\nu$$

Consider the symplectic form ω_{st} on $\mathbb{C}^8 \simeq \mathbb{R}^{16}$ as represented by the (16×16) -matrix with submatrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on the diagonal and everywhere else zero. For $\nu = 1$, we now compute the pullback ω_1 of the matrix of ω_{st} explicitly. The cases $\nu = 2, \dots, 8$ are similar. We find

$$\begin{aligned} \kappa \circ \tau_1^{-1} \circ \psi_1^{-1} : V_1 \subseteq \mathbb{R}^4 \rightarrow \mathbb{C}^8 \simeq \mathbb{R}^{16}, \\ (x_1, y_1, x_2, y_2) \mapsto (x_1, y_1, x_2, y_2, x_3, 0, x_4, 0, x_5, 0, x_6, 0, x_7, 0, x_8, 0) \end{aligned}$$

and calculate the Jacobian

$$d(\kappa \circ \tau_1^{-1} \circ \psi_1^{-1}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-x_1}{\sqrt{2-x_1^2+x_2^2-y_1^2+y_2^2}} & \frac{-y_1}{\sqrt{2-x_1^2+x_2^2-y_1^2+y_2^2}} & \frac{x_2}{\sqrt{2-x_1^2+x_2^2-y_1^2+y_2^2}} & \frac{y_2}{\sqrt{2-x_1^2+x_2^2-y_1^2+y_2^2}} \\ 0 & 0 & 0 & 0 \\ \frac{-2x_1}{\sqrt{6+x_2^2-2(x_1^2+y_1^2)+y_2^2}} & \frac{-2y_1}{\sqrt{6+x_2^2-2(x_1^2+y_1^2)+y_2^2}} & \frac{x_2}{\sqrt{6+x_2^2-2(x_1^2+y_1^2)+y_2^2}} & \frac{y_2}{\sqrt{6+x_2^2-2(x_1^2+y_1^2)+y_2^2}} \\ 0 & 0 & 0 & 0 \\ \frac{-x_1}{\sqrt{6-x_1^2-y_1^2}} & \frac{-y_1}{\sqrt{6-x_1^2-y_1^2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-x_2}{\sqrt{8-x_2^2-y_2^2}} & \frac{-y_2}{\sqrt{8-x_2^2-y_2^2}} \\ 0 & 0 & 0 & 0 \\ \frac{x_1}{\sqrt{4+x_1^2-x_2^2+y_1^2-y_2^2}} & \frac{y_1}{\sqrt{4+x_1^2-x_2^2+y_1^2-y_2^2}} & \frac{-x_2}{\sqrt{4+x_1^2-x_2^2+y_1^2-y_2^2}} & \frac{-y_2}{\sqrt{4+x_1^2-x_2^2+y_1^2-y_2^2}} \\ 0 & 0 & 0 & 0 \\ \frac{2x_1}{\sqrt{2-x_2^2+2(x_1^2+y_1^2)-y_2^2}} & \frac{2y_1}{\sqrt{2-x_2^2+2(x_1^2+y_1^2)-y_2^2}} & \frac{-x_2}{\sqrt{2-x_2^2+2(x_1^2+y_1^2)-y_2^2}} & \frac{-y_2}{\sqrt{2-x_2^2+2(x_1^2+y_1^2)-y_2^2}} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which, considering ω_{st} and ω_1 as matrices, yields

$$\omega_1 = (d(\kappa \circ \tau_1^{-1} \circ \psi_1^{-1}))^T \omega_{st} d(\kappa \circ \tau_1^{-1} \circ \psi_1^{-1}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \omega_{st} \quad (7)$$

Analogous calculations show $\omega_v = \omega_{st}$ on $V_v \subseteq \mathbb{R}^4$ for all $1 \leq v \leq 8$.

3.4 Momentum Map of the Toric System on (M, ω)

Now let us resume Delzant's construction (see Sect. 2.6 for an overview of the steps) to obtain a momentum map F on M satisfying $F(M) = \Delta$. First, we have to find a map $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}^8$ that is a right inverse of the map $\vartheta : \mathbb{R}^8 \rightarrow \mathbb{R}^2$ defined in (2).

Lemma 3.4 *The map $\sigma := \frac{1}{6}\vartheta^* : \mathbb{R}^2 \rightarrow \mathbb{R}^8$ is a right inverse of ϑ .*

Proof Written in matrix form, $\vartheta : \mathbb{R}^8 \rightarrow \mathbb{R}^2$ is given by

$$\vartheta(x) = \begin{pmatrix} -1 & -1 & 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & 0 & 1 & 1 & 1 \end{pmatrix} x =: Cx.$$

Therefore the dual map is given by $\vartheta^* : \mathbb{R}^2 \rightarrow \mathbb{R}^8$, $\vartheta^*(y) = C^T y$. Since $CC^T = 6 \text{Id}_{\mathbb{R}^2}$, we find $\text{Id}_{\mathbb{R}^2} = \frac{1}{6} \vartheta \circ \vartheta^* = \vartheta \circ (\frac{1}{6} \vartheta^*)$ such that $\sigma := \frac{1}{6} \vartheta^*$ is a right inverse of ϑ . $\square$

The next step in Delzant's construction is to compute the concatenation

$$\tilde{L}^{-1}(0) \xrightarrow{\kappa} \mathbb{C}^8 \xrightarrow{L} \mathbb{R}^8 \xrightarrow{\sigma^*} \mathbb{R}^2$$

where we remark that $\sigma^* = \frac{1}{6} \vartheta$.

Lemma 3.5 *The map $\tilde{F} := \frac{1}{6} \vartheta \circ L \circ \kappa : \tilde{L}^{-1}(0) \rightarrow \mathbb{R}^2$ is explicitly given by $\tilde{F}(z_1, \dots, z_8) = \frac{1}{2}(|z_1|^2, |z_3|^2)$ and is invariant under the action of N . Thus, $\tilde{F}$ descends to a map $F : M \rightarrow \mathbb{R}^2$.*

Proof Let $z = (z_1, \dots, z_8) \in \tilde{L}^{-1}(0)$ and compute

$$L(\kappa(z_1, \dots, z_8)) = L(z_1, \dots, z_8) = -\frac{1}{2}(|z_1|^2, \dots, |z_8|^2) + (0, -1, 0, 2, 3, 5, 3, 2)$$

and

$$\begin{aligned} \frac{1}{6} \vartheta(L(z)) &= \frac{1}{6} \vartheta \left(-\frac{1}{2} (|z_1|^2, \dots, |z_8|^2) + (0, -1, 0, 2, 3, 5, 3, 2) \right) \\ &= \frac{1}{12} (|z_1|^2 + |z_2|^2 - |z_4|^2 - |z_5|^2 - |z_6|^2 + |z_8|^2, \\ &\quad |z_2|^2 + |z_3|^2 + |z_4|^2 - |z_6|^2 - |z_7|^2 - |z_8|^2) \\ &\quad + \frac{1}{6} (9, 9) \end{aligned}$$

and now we use twice the manifold equations from (4) to obtain

$$\begin{aligned} &= \frac{1}{12} \left(6 - |z_5|^2 + 10 - |z_5|^2 - |z_7|^2 - 4 - |z_5|^2 + |z_7|^2 - |z_5|^2 + 2 - |z_5|^2 \right. \\ &\quad \left. - |z_7|^2 + 4 - |z_5|^2 + |z_7|^2 + 18, \right. \\ &\quad \left. 10 - |z_5|^2 - |z_7|^2 + 6 - |z_7|^2 + 4 + |z_5|^2 - |z_7|^2 + 2 - |z_5|^2 \right. \\ &\quad \left. - |z_7|^2 - |z_7|^2 - 4 + |z_5|^2 - |z_7|^2 + 18 \right) \\ &= \frac{1}{12} \left(-6 |z_5|^2 + 36, -6 |z_7|^2 + 36 \right) \\ &= \frac{1}{12} \left(-6 (6 - |z_1|^2) + 36, -6 (6 - |z_3|^2) + 36 \right) \\ &= \frac{1}{2} (|z_1|^2, |z_3|^2). \end{aligned}$$

Since the action of N given in (3) does not change the absolute value of z_1 and z_3 , the map $F(z_1, \dots, z_8) = \frac{1}{2} (|z_1|^2, |z_3|^2)$ is invariant under the action of N and passes thus to the quotient $M = \tilde{L}^{-1}(0)/N$. $\square$

According to Delzant's construction, the map $F : (M, \omega) \rightarrow \mathbb{R}^2$ is the desired toric momentum map having the original polytope as image of the momentum map:

Theorem 3.6 *Let Δ be the octagon displayed in Fig. 1. Then, $F = (J, H) : (M, \omega) \rightarrow \mathbb{R}^2$ with*

$$J([z_1, \dots, z_8]) = \frac{1}{2}|z_1|^2, \quad H([z_1, \dots, z_8]) = \frac{1}{2}|z_3|^2$$

is a momentum map of an effective Hamiltonian 2-torus action and satisfies $F(M) = \Delta$. The $\mathbb{S}^1$ -action of J is given by

$$\mathbb{S}^1 \times M \rightarrow M, \quad (t, [z_1, \dots, z_8]) \mapsto [e^{-it} z_1, z_2, \dots, z_8]$$

and the $\mathbb{S}^1$ -action of H by

$$\mathbb{S}^1 \times M \rightarrow M, \quad (t, [z_1, \dots, z_8]) \mapsto [z_1, z_2, e^{-it} z_3, z_4, \dots, z_8].$$

Proof Constructed via Delzant's construction, F is the momentum map of an effective Hamiltonian 2-torus action. Note that $\{J, H\} = 0$ is actually additionally proven, explicitly in local coordinates, in the proof of Proposition 4.6 when showing Poisson commutativity of the integrals of the (future) semitoric system.

Let us now verify $F(M) = \Delta$. Recall the manifold equations (A) – (F) from (4) and note that equations (A) and (C) imply

$$0 \leq J \leq 3 \quad \text{and} \quad 0 \leq H \leq 3 \quad \text{and} \quad |z_5|^2 = 6 - 2J$$

which we will use in the following. We conclude

$$(B) \Rightarrow |z_2|^2 + |z_7|^2 = 4 + 2J \Rightarrow |z_7|^2 \leq 4 + 2J \stackrel{(C)}{\Rightarrow} |z_3|^2 \geq 2 - 2J \Rightarrow H \geq 1 - J,$$

$$(F) \Rightarrow |z_7|^2 = |z_8|^2 + 2 - 2J \geq 2 - 2J \stackrel{(C)}{\Rightarrow} |z_3|^2 \leq 4 + 2J \Rightarrow H \leq 2 + J,$$

$$(D) \Rightarrow |z_7|^2 = -|z_4|^2 + 10 - 2J \leq 10 - 2J \stackrel{(C)}{\Rightarrow} |z_3|^2 \geq -4 + 2J \Rightarrow H \geq -2 + J,$$

$$(E) \Rightarrow |z_7|^2 = |z_6|^2 - 4 + 2J \geq -4 + 2J \stackrel{(C)}{\Rightarrow} |z_3|^2 \leq 10 - 2J \Rightarrow H \leq 5 - J.$$

The equations $0 \leq J \leq 3$ and $0 \leq H \leq 3$ describe the square $[0, 3] \times [0, 3]$ and the equations

$$H \geq 1 - J, \quad H \leq 2 + J, \quad H \geq -2 + J, \quad H \leq 5 - J$$

describe the edges after chopping off corners of the square $[0, 3] \times [0, 3]$ in Fig. 1. Thus, indeed, $(J, H)(M) = F(M) = \Delta$. $\square$

The Atiyah–Guillemin–Sternberg convexity theorem tell us that the *images* of the elliptic–elliptic fixed points of $F = (J, H) : (M, \omega) \rightarrow \mathbb{R}^2$ are precisely the vertices of the octagon Δ , but it does not say *where* in the 4-dimensional manifold (M, ω) the *preimages* of these vertices lie.

Fig. 4 The momentum polytope $F(M) = \Delta$ with the images of the fixed points

Proposition 3.7 $F = (J, H)$ has precisely the following eight fixed points:

$P^{min} := [0, 0, \sqrt{2}, \sqrt{6}, \sqrt{6}, 2\sqrt{2}, 2, \sqrt{2}]$	is the fixed point with $z_1 = z_2 = 0$,
$B := [\sqrt{2}, 0, 0, \sqrt{2}, 2, 2\sqrt{2}, \sqrt{6}, \sqrt{6}]$	is the fixed point with $z_2 = z_3 = 0$,
$D := [2, \sqrt{2}, 0, 0, \sqrt{2}, \sqrt{6}, \sqrt{6}, 2\sqrt{2}]$	is the fixed point with $z_3 = z_4 = 0$,
$Q^{min} := [\sqrt{6}, \sqrt{6}, \sqrt{2}, 0, 0, \sqrt{2}, 2, 2\sqrt{2}]$	is the fixed point with $z_4 = z_5 = 0$,
$Q^{max} := [\sqrt{6}, 2\sqrt{2}, 2, \sqrt{2}, 0, 0, \sqrt{2}, \sqrt{6}]$	is the fixed point with $z_5 = z_6 = 0$,
$C := [2, 2\sqrt{2}, \sqrt{6}, \sqrt{6}, \sqrt{2}, 0, 0, \sqrt{2}]$	is the fixed point with $z_6 = z_7 = 0$,
$A := [\sqrt{2}, \sqrt{6}, \sqrt{6}, 2\sqrt{2}, 2, \sqrt{2}, 0, 0]$	is the fixed point with $z_7 = z_8 = 0$,
$P^{max} := [0, \sqrt{2}, 2, 2\sqrt{2}, \sqrt{6}, \sqrt{6}, \sqrt{2}, 0]$	is the fixed point with $z_8 = z_1 = 0$.

They are mapped under F to the vertices of the octagon as displayed in Fig. 4.

Proof We first look for the fixed points of J and H separately since fixed points of $F = (J, H)$ are precisely the points that are fixed points for both J and H .

Fixed points of J : The point $[z] = [z_1, \dots, z_8] \in M$ will be fixed by the action of J if the following holds true:

Case 1: $z_1 = 0$ in the first coordinate yields a fixed point of the J -action.

Case 2: $z_5 = 0$ also gives rise to a fixed point, since we can take $(t_1, \dots, t_6) = (-t, 0, \dots, 0)$ in the N -action, which only gives another representation for the same point in M . This means we can actually see the J -action as a special case of the N -action.

Case 3: Seeing the J -action as an N -action with $(t_1, \dots, t_6) = (-t, 0, \dots, 0)$ means that t rotates both z_1 and z_5 . There are several possibilities to compensate the rotation of z_5 . Let us for example set the parameter $t_2 = t$ (so that z_5 is kept invariant). Then t_2 also affects z_2 and z_7 , so a fixed point should satisfy $z_2 = 0$ and either $z_7 = 0$ or one of the z_3, z_4, z_6, z_8 is zero, where we choose a third parameter (for example $t_3 = -t$) in the N -action. We know from Lemma 3.3 that only two subsequent entries can be zero, so the only remaining possibility is $z_2 = 0$ and $z_3 = 0$.

Case 4: The same idea works when we compensate the rotation of z_5 by choosing another parameter in the N -action: picking $t_4 = -t$ leads to $z_4 = 0$ and $z_3 = 0$.

Moreover, setting $t_5 = -t$ gives rise to $z_6 = 0$ and $z_7 = 0$. Last, considering $t_6 = t$ implies $z_8 = 0$ and $z_7 = 0$.

Fixed points of H : Similar to the previous case, a point $[z] = [z_1, \dots, z_8]$ will be fixed by the H -action in the following cases:

Case 1: $z_3 = 0$.

Case 2: $z_7 = 0$.

Case 3: $z_2 = 0$ and $z_1 = 0$; $z_4 = 0$ and $z_5 = 0$; $z_6 = 0$ and $z_5 = 0$; $z_8 = 0$ and $z_1 = 0$.

Conclusion: When considering these conditions together, we see that the fixed points are precisely those points with two subsequent entries equal to zero. If we consider such a fixed point with $z_v = 0$ and $z_{v+1} = 0$, then this point lies in $U_v \subseteq M$ and can be parametrised by $(x_v, y_v, x_{v+1}, y_{v+1}) = (0, 0, 0, 0)$. All other entries can be written as positive real numbers and are found by means of the manifold equations. $\square$

4 A Semitoric Family with Four Focus–Focus Points and Two Double Pinched Tori

Within this section, let $(M, \omega, F = (J, H))$ be the toric manifold constructed in Sect. 3 that satisfies $F(M) = \Delta$ where Δ is the octagon from Fig. 1.

The aim of this section is to replace the second integral H by a parameter depending family H_t such that $(M, \omega, F_t = (J, H_t))$ is a semitoric family with fixed $\mathbb{S}^1$ -action having four transition points $A, B, C, D \in M$ which map under $F_0 = (J, H_0)$ to the points $(1, 3)$, $(1, 0)$, $(2, 3)$, $(2, 0)$ as sketched in Fig. 4.

4.1 Geometric Interpretation of $M^{\text{red},j}$

Following Karshon (1999), we call a value j of J *extremal* if j is a global minimum or maximum of J , and *interior* otherwise. A fixed point of J in the fibre $J^{-1}(j)$ is *extremal* if j is extremal. Analogously we define *interior* fixed points. The following statement is a compilation of Lemma 2.12 in Le Floch and Palmer (2019) and Proposition 3.4 in Hohloch et al. (2015).

Lemma 4.1 *Let $(M, \omega, F = (J, H))$ be a semitoric system.*

1. *Let $j \in J(M)$ be an interior value. If j is regular for J , then $M^{\text{red},j}$ is diffeomorphic to a 2-sphere. If j is singular for J then $M^{\text{red},j}$ is homeomorphic to a 2-sphere (but not diffeomorphic).*
2. *Let $j \in J(M)$ be an extremal value of J . If j corresponds to a vertical edge of the momentum polytope of $F = (J, H)$ then $M^{\text{red},j}$ is diffeomorphic to a 2-sphere. Otherwise $M^{\text{red},j}$ is a point.*

Figure 1 shows that $j \in \{0, 3\}$ are the extremal values of J and $j \in \{1, 2\}$ singular interior values. All $j \in [0, 3] \setminus \{0, 1, 2, 3\}$ are regular interior values. We conclude.

Corollary 4.2 *Let $(M, \omega, F = (J, H))$ be the toric system from Theorem 3.6. Then,*

$$M^{\text{red},j} \stackrel{\text{diffeo}}{\simeq} \mathbb{S}^2 \text{ for } j \in [0, 3] \setminus \{1, 2\} \text{ and } M^{\text{red},j} \stackrel{\text{homeo}}{\simeq} \mathbb{S}^2 \text{ for } j \in \{1, 2\}.$$

The situation is shown in Fig. 5. Let us explain the geometric intuition of the shape of the reduced spaces and positions of the singular points of $H^{\text{red},j}$. Later in Sect. 4.2, we will give a parametrisation of $M^{\text{red},j}$.

Recall that, according to Lemma 2.7, nondegenerate elliptic–regular points of $F = (J, H)$ on the 4-dimensional manifold (M, ω) that are regular for J correspond to nondegenerate elliptic points of $H^{\text{red},j}$ on the 2-dimensional space $M^{\text{red},j}$. Hence there are, for $j \in]0, 3[\setminus \{1, 2\}$, precisely two elliptic points of $H^{\text{red},j}$ on $M^{\text{red},j}$. They are located at the ‘north and south pole’ of $M^{\text{red},j} \stackrel{\text{diffeo}}{\simeq} \mathbb{S}^2$ since, as a maximum and minimum of $H^{\text{red},j}$, the north and south pole are elliptic points of $H^{\text{red},j}$. But there are not more than two elliptic points possible for $H^{\text{red},j}$ since there are only two elliptic–regular points in $F^{-1}(j)$.

Now consider the interior singular values $j \in \{1, 2\}$. The proof of Lemma 2.12 in Le Floch and Palmer (2019) (see also the overview in Alonso (2019, pp. 26–28) on singular reduction) shows that an elliptic–elliptic or focus–focus point on M causes $M^{\text{red},j}$ to have a singularity (a ‘non-smooth peak’) such that $M^{\text{red},j}$ is only homeomorphic to a 2-sphere but not diffeomorphic. Since there are two elliptic–elliptic points in the level set of $j \in \{1, 2\}$ of J , the reduced space $M^{\text{red},j}$ has two ‘non-smooth peaks’ as shown in Fig. 5.

Now consider the extremal values $j \in \{0, 3\}$ where, according to Lemma 4.1, the reduced space is diffeomorphic to a 2-sphere. Thus there are two (elliptic) points $[p_{\max}], [p_{\min}] \in M^{\text{red},j}$ where $H^{\text{red},j}$ attains its maximum and minimum, i.e. $dH^{\text{red},j}$ vanishes in $[p_{\max}]$ and $[p_{\min}]$. Since j is extremal, we have $dJ(p) = 0$ for all $p \in J^{-1}(j)$. Altogether we conclude that $[p_{\max}]$ and $[p_{\min}]$ are mapped to elliptic–elliptic points of $F = (J, H)$ which are precisely the upper and lower vertex of the vertical edges over $j = 0$ and $j = 3$ in the octagon Δ . The other points in $M^{\text{red},j}$ correspond to elliptic–regular rank one points of M and are mapped to the vertical segments of the momentum polytope between the vertices.

4.2 Parametrisation of $M^{\text{red},j}$

After these geometric considerations, we explain now how to obtain the coordinates used for displaying the reduced spaces $M^{\text{red},j}$ in Fig. 5. The idea is to ‘foliate’ the 2-dimensional space $M^{\text{red},j}$ by the level sets of $H^{\text{red},j}$ and consider $H^{\text{red},j}$ as a height function of $M^{\text{red},j}$. Then the possible values $h_j \in H^{\text{red},j}(M^{\text{red},j})$ lead to one coordinate. The other coordinate comes from parametrising the (usually 1-dimensional) level sets of $H^{\text{red},j}$. Therefore, we need a map that is well-defined on the reduced spaces $M^{\text{red},j}$ and whose level sets are transverse to the ones of $H^{\text{red},j}$.

Lemma 4.3 *Denote by $\Re$ and $\Im$ the real and imaginary parts of a complex valued function and consider $Z : \mathbb{C}^8 \rightarrow \mathbb{C}$ given by $Z(z_1, \dots, z_8) := \overline{z_2} \overline{z_3} \overline{z_4} z_6 z_7 z_8$. Then*

Fig. 5 $M^{red,j}$ plotted for $j \in \{0, 0.5, 1, 1.5, 2, 2.5, 3\}$ with *Mathematica*. For $j = 1$ and $j = 2$, the reduced space $M^{red,j}$ contains two singular points, i.e. it is homeomorphic to a 2-sphere with two ‘peaks’, otherwise $M^{red,j}$ is diffeomorphic to a 2-sphere

the functions

$$X : \mathbb{C}^8 \rightarrow \mathbb{R}, \quad X := \Re(Z) \quad \text{and} \quad Y : \mathbb{C}^8 \rightarrow \mathbb{R}, \quad Y := \Im(Z)$$

are N -invariant and thus descend as well-defined functions to M . Moreover, X and Y are also J -invariant and therefore descend to the reduced spaces $M^{red,j}$ for $j \in J(M)$.

Proof Recall the action of N from (3). Let $(t_1, \dots, t_6) \in \mathbb{T}^6 \simeq N$ and $(z_1, \dots, z_8) \in \mathbb{C}^8$ and calculate

$$\begin{aligned} & Z(e^{it_1} z_1, e^{it_2} z_2, e^{it_3} z_3, e^{it_4} z_4, e^{i(t_1+t_2-t_4-t_5+t_6)} z_5, e^{it_5} z_6, e^{i(t_2+t_3+t_4-t_5-t_6)} z_7, e^{it_6} z_8) \\ &= e^{i(-t_2-t_3-t_4+t_5+t_2+t_3+t_4-t_5-t_6+t_6)} (\overline{z_2} \overline{z_3} \overline{z_4} z_6 z_7 z_8) = \overline{z_2} \overline{z_3} \overline{z_4} z_6 z_7 z_8 = Z(z_1, \dots, z_8) \end{aligned}$$

i.e. Z is N -invariant. Now recall the $\mathbb{S}^1$ -action induced by J from Theorem 3.6, let $t \in \mathbb{S}^1$ and $(z_1, \dots, z_8) \in \mathbb{C}^8$, and compute

$$Z(e^{-it} z_1, z_2, \dots, z_8) = \overline{z_2} \overline{z_3} \overline{z_4} z_6 z_7 z_8 = Z(z_1, \dots, z_8)$$

which shows the J -invariance of Z . Hence, also $X = \Re(X)$ and $Y = \Im(Z)$ are N - and J -invariant. $\square$

By means of the implicit function theorem, we would like to use $Z = (X, Y)$ to obtain coordinates on $M^{\text{red},j}$ of the form $(u, H^{\text{red},j}(u))$ or $(u^{-1}(H^{\text{red},j}), H^{\text{red},j})$. Thus, we need to relate $Z = (X, Y)$ to H and J .

Lemma 4.4 *Let $F = (J, H) : (M, \omega) \rightarrow \mathbb{R}^2$ be the toric system from Theorem 3.6 and X and Y the functions defined in Lemma 4.3. Then,*

$$X^2 + Y^2 = 2^6 H(H + J - 1)(H - J + 2)(-H - J + 5)(3 - H)(2 - H + J).$$

Proof Use the definition of J and H to find $|z_1|^2 = 2J$ and $|z_3|^2 = 2H$ and use the manifold equations to get furthermore

$$\begin{aligned} |z_2|^2 &= -2 + 2H + 2J, & |z_5|^2 &= 6 - 2J, & |z_7|^2 &= 6 - 2H, \\ |z_4|^2 &= 4 + 2H - 2J, & |z_6|^2 &= 10 - 2H - 2J, & |z_8|^2 &= 4 - 2H + 2J. \end{aligned}$$

We now calculate

$$\begin{aligned} X^2 + Y^2 &= |Z|^2 = |\overline{z_2} \overline{z_3} \overline{z_4} z_6 z_7 z_8|^2 = |z_2|^2 |z_3|^2 |z_4|^2 |z_6|^2 |z_7|^2 |z_8|^2 \\ &= 2^6 H(H + J - 1)(H - J + 2)(-H - J + 5)(3 - H)(2 - H + J). \end{aligned}$$

$\square$

Thus $Z = (X, Y)$ is related to J and H via a rotation invariant formula. Therefore, we may set $Y = 0$, solve for X and obtain $M^{\text{red},j}$ parametrised as a surface of revolution in $\mathbb{R}^3$ with coordinates $(X, Y, H^{\text{red},j})$.

Corollary 4.5 *On the section $Y = 0$, consider X from Lemma 4.3 by means of Lemma 4.4 as function*

$$X(J, H) = 8\sqrt{H(H + J - 1)(H - J + 2)(-H - J + 5)(3 - H)(2 - H + J)}.$$

Then, for fixed $j \in J(M)$, the reduced space $M^{\text{red},j}$ can be parametrised by

$$\begin{aligned} \mathbb{R}/2\pi\mathbb{Z} \times H^{\text{red},j}(M^{\text{red},j}) &\subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3, \\ (t, h_j) &\mapsto (X(j, h_j) \cos(t), X(j, h_j) \sin(t), h_j) \end{aligned}$$

where $X(j, h_j) = 8\sqrt{h_j(h_j + j - 1)(h_j - j + 2)(-h_j - j + 5)(3 - h_j)(2 - h_j + j)}$.

The function $h_j \mapsto \pm X(j, h_j)$ is plotted for various values of j in Fig. 6. Rotating the image of $h_j \mapsto X(j, h_j)$ around the horizontal axis gives the space $M^{\text{red},j}$ that is plotted in Fig. 5 in the (rotated) coordinate system $(X, Y, H^{\text{red},j})$.

Fig. 6 The Section $Y = 0$ of the reduced space $M^{red,j}$ plotted with *Mathematica* for seven steps between $j = 0$ and $j = 3$, parametrised via $h_j \mapsto \pm X(j, h_j)$

4.3 The Family (J, H_t) of Integrable Systems

We know from Theorem 3.6 that $(M, \omega, F = (J, H))$ is toric. The idea is now to obtain a family H_t by interpolation between H and X , i.e. we consider a new ‘height function’ that changes from H to X when varying the parameter.

Proposition 4.6 *Let $t \in \mathbb{R}$ and $\gamma \in \mathbb{R}^{\neq 0}$ and define on the symplectic manifold (M, ω) constructed in Sect. 3.1 the map*

$$H_t : (M, \omega) \rightarrow \mathbb{R}, \quad H_t := (1 - 2t)H + t\gamma X.$$

Then, $F_t := (J, H_t) : (M, \omega) \rightarrow \mathbb{R}^2$ is a completely integrable system.

Proof There are two ways to show that J and H_t Poisson commute w.r.t. the Poisson bracket $\{\cdot, \cdot\}$ induced by ω . Let us first argue abstractly:

Since $F = (J, H) : (M, \omega) \rightarrow \mathbb{R}^2$ is a toric momentum map according to Theorem 3.6, we have in particular $\{J, H\} = 0$. Moreover, X can be seen as function $X(J, H)$ depending solely on the Poisson commuting functions J and H and some constants. Thus, $\{X(J, H), J\} = 0 = \{X(J, H), H\}$ such that linearity of the Poisson bracket implies

$$0 = (1 - 2t)\{J, H\} + t\gamma\{J, X\} = \{J, (1 - 2t)H + t\gamma X\} = \{J, H_t\}.$$

Later on in this paper, we will need (parts of) the explicit calculations in local coordinates of $\{J, H_t\} = 0$, so we add here the proof in local coordinates. We only consider

the chart (U_1, ψ_1) since the cases $2 \leq \nu \leq 8$ go analogously. We find

$$(J \circ \psi_1^{-1})(x_1, y_1, x_2, y_2) = \frac{1}{2} (x_1^2 + y_1^2) \quad \text{and} \quad d(J \circ \psi_1^{-1}) = (x_1, y_2, 0, 0)$$

and, using $x_3 = \sqrt{2 - |z_1|^2 + |z_2|^2}$ from (6) and $y_3 = 0$, we get

$$\begin{aligned} (H \circ \psi_1^{-1})(x_1, y_1, x_2, y_2) &= \frac{1}{2} (x_3^2 + y_3^2) = \frac{1}{2} (2 - x_1^2 - y_1^2 + x_2^2 + y_2^2), \\ d(H \circ \psi_1^{-1})(x_1, y_1, x_2, y_2) &= (-x_1, -y_1, x_2, y_2). \end{aligned} \quad (8)$$

Recall that the symplectic form ω_1 on $\psi_1(U_1) = V_1$ is the standard symplectic form ω_{st} on $\mathbb{R}^4$. We calculate

$$\begin{aligned} \mathcal{X}^{J \circ \psi_1^{-1}}(x_1, y_1, x_2, y_2) &= (y_1, -x_1, 0, 0)^T, \\ \mathcal{X}^{H \circ \psi_1^{-1}}(x_1, y_1, x_2, y_2) &= (-y_1, x_1, y_2, -x_2)^T, \\ \{J \circ \psi_1^{-1}, H \circ \psi_1^{-1}\} &= -\omega_1(\mathcal{X}^{J \circ \psi_1^{-1}}, \mathcal{X}^{H \circ \psi_1^{-1}}) \\ &= -d(J \circ \psi_1^{-1})\mathcal{X}^{H \circ \psi_1^{-1}} = (x_1, y_1, 0, 0) \begin{pmatrix} -y_1 \\ x_1 \\ y_2 \\ -x_1 \end{pmatrix} = 0. \end{aligned}$$

Thus, J and H Poisson commute. Moreover, using the definition of ψ_1^{-1} , we find

$$\begin{aligned} (Z \circ \psi_1^{-1})(x_1, y_1, x_2, y_2) &= Z(x_1, y_1, x_2, y_2, x_3, 0, x_4, 0, x_5, 0, x_6, 0, x_7, 0, x_8, 0) \\ &= (x_2 - iy_2)x_3x_4x_6x_7x_8 = x_2x_3x_4x_6x_7x_8 + ix_3x_4x_6x_7x_8y_2 \end{aligned}$$

and by using the relations defining $x_3, \dots, x_8$ we get

$$\begin{aligned} (X \circ \psi_1^{-1})(x_1, y_1, x_2, y_2) &= x_2x_3x_4x_6x_7x_8 \\ &= x_2 \sqrt{(2 - (x_1^2 + y_1^2) + (x_2^2 + y_2^2)) (6 - 2(x_1^2 + y_1^2) + (x_2^2 + y_2^2)) (8 - (x_2^2 + y_2^2))} \\ &\quad \sqrt{(4 + (x_1^2 + y_1^2) - (x_2^2 + y_2^2)) (2 + 2(x_1^2 + y_1^2) - (x_2^2 + y_2^2))} \end{aligned} \quad (9)$$

which can be seen, in the first two variables, as a function depending on $\xi_1 = x_1^2$ and $\eta_1 = y_1^2$ and that is symmetric in ξ_1 and η_1 . Thus, the partial derivatives of $X \circ \psi_1^{-1}$ in (x_1, y_1, x_2, y_2) w.r.t. x_1 and y_1 coincide except for the factor x_1 resp. y_1 , i.e. we obtain

$$d(X \circ \psi_1^{-1}) = (x_1 f, y_1 f, g, h)$$

Fig. 7 The reduced space $M^{red,j}$ for the singular value $j = 1$. At $t = 0$, we have $H_0^{red,1} = H^{red,1}$ and the singular levels are just the two singular elliptic points. At $t = \frac{1}{2}$, we have $H_1^{red,1} = (\frac{\gamma}{2}X)^{red,1}$ and the singular level is the blue curve connecting the two singular focus–focus points. This plot is done with *Mathematica*

for suitable functions $f, g, h : V_1 \rightarrow \mathbb{R}$ with coordinates (x_1, y_1, x_2, y_2) . This yields at the point (x_1, y_1, x_2, y_2)

$$\mathcal{X}^{X \circ \psi_1^{-1}} = (y_1 f, -x_1 f, h, -g)^T$$

and thus

$$\{J \circ \psi_1^{-1}, X \circ \psi_1^{-1}\} = -d(J \circ \psi_1^{-1}) \left(\mathcal{X}^{X \circ \psi_1^{-1}} \right) = (x_1, y_1, 0, 0) \begin{pmatrix} y_1 f \\ -x_1 f \\ h \\ -g \end{pmatrix} = 0.$$

Since $H_t = (1 - 2t)H + t\gamma X$, linearity of the Poisson bracket yields $\{J \circ \psi_1^{-1}, H_t \circ \psi_1^{-1}\} = 0$ for all t and all γ .

Now, we show that $\mathcal{X}^J$ and $\mathcal{X}^{H_t}$ are almost everywhere linearly independent. For $t = 0$, we have $H_0 = H$ and thus the claim follows from $F = (J, H)$ being an integrable system. For $t \neq 0$, consider $\mathcal{X}^{H_t \circ \psi_1^{-1}} = (1 - 2t)\mathcal{X}^{H \circ \psi_1^{-1}} + t\gamma \mathcal{X}^{X \circ \psi_1^{-1}}$ and note that $\mathcal{X}^{J \circ \psi_1^{-1}}$ and $\mathcal{X}^{X \circ \psi_1^{-1}}$ are linearly dependent if and only if $\partial_{x_2}(X \circ \psi_1^{-1}) = 0 = \partial_{y_2}(X \circ \psi_1^{-1})$. This is only possible if $y_2 = 0$ and x_2 satisfies an equation, so these (x_1, y_1, x_2, y_2) live in a two-dimensional subset, and hence, the linear independence holds almost everywhere. $\square$

4.4 Intuition on Focus–Focus Points of the Family $F_t = (J, H_t)$

On the reduced space, the family H_t has the following geometric meaning: For $t = 0$, we have $H_0^{red,j} = H^{red,j}$ which is our ‘vertical’ height coordinate. For $t \in]0, \frac{1}{2}[$, the ‘height’ $H_t^{red,j}$ is measured ‘tilted more and more towards the horizontal’ X -coordinate which is reached (up to scaling by $\frac{\gamma}{2}$) at $t = \frac{1}{2}$ with $H_{\frac{1}{2}}^{red,j} = (\frac{\gamma}{2}X)^{red,j}$. Then, the singular points of the manifold are interior points of this function and will be singular points of focus–focus type. This idea is visualised in Fig. 7 for the level $J = 1 = j$.

4.5 Intuition on the Rank 1 Points of the Family $F_t = (J, H_t)$

At the interior levels $j \in]0, 3[\setminus \{1, 2\}$, the reduced space $M^{red,j}$ is diffeomorphic to a 2-sphere. Thus, no matter how far $H_t^{j,red}$ is ‘tilted’ from H towards X , the maximum and minimum of $H_t^{red,j}$ are each reached at a unique point on $M^{red,j}$.

By Lemma 2.7, these points correspond to the rank one points of (M, ω, F_t) which will turn out to be nondegenerate of elliptic–regular type and map to the top and bottom edge points in the momentum polytope $F_t(M)$. As $H_t^{red,j}$ ‘tilts’ from $H^{red,j} = H$ to X as t increases, also the rank one points on M may change.

For the levels $j \in \{0, 3\}$, the minimum resp. maximum of $H_t^{red,j}$ is also attained at a unique point on $M^{red,j}$. Both correspond to rank zero points of elliptic–elliptic type on M .

When t changes from 0 to 1, other rank one points will become the rank zero points, but, according to Proposition 2.14, none of these points ever becomes regular.

For the levels $j \in \{1, 2\}$, the reduced manifold has two singular points, which are the maximum and minimum of $H_t^{red,j}$ as long as $t < t^-$ and $t^+ < t$ for certain ‘degenerate times’ $0 < t^- < \frac{1}{2} < t^+ < 1$. When t is in between the degenerate times, two other points on the reduced space will become the extremal values of $H_t^{red,j}$. These points correspond to nondegenerate elliptic–regular rank one points on M which are mapped to the upper and lower boundary of the octagon. The fixed points are then focus–focus points and mapped to the interior.

4.6 Main Results

We showed in Theorem 3.6 that $(M, \omega, F = (J, H))$ is a toric system with $F(M) = \Delta$. The family $F_t = (J, H_t)$ was shown to be an integrable system in Proposition 4.6. But (M, ω, F_t) has even nicer properties:

Theorem 4.7 *Let $(M, \omega, F = (J, H))$ be the toric system from Theorem 3.6 and let $(M, \omega, F_t = (J, (1 - 2t)H + t\gamma X))$ be the family of integrable systems from Proposition 4.6 and let $0 < \gamma < \frac{1}{48}$. Then, $(M, \omega, F_t)_{0 \leq t \leq 1}$ is a semitoric transition family with fixed $\mathbb{S}^1$ -action having four transition points and two transition times*

$$0 < t^- := \frac{1}{2(1 + 24\gamma)} < \frac{1}{2} < t^+ := \frac{1}{2(1 - 24\gamma)} < 1.$$

The proof is spread over the following sections and summarised in Sect. 8. The image $F_t(M)$ of the momentum map is plotted for $\gamma = \frac{1}{60}$ in Fig. 2: The images of four elliptic–elliptic fixed points ‘pass into the interior’ of the momentum polytope at $t = t^-$, becoming focus–focus points. At $t = \frac{1}{2}$, the focus–focus points form two pairs where each pair is mapped to the same value in the momentum polytope, i.e. the points of a pair lie in the same fibre. At $t = t^+$, the images of the focus–focus points become again boundary points of the momentum polytope, i.e. they switch back to being elliptic–elliptic. We compute now the coordinates in M of the eight fixed points of $F_t = (J, H_t)$.

Proposition 4.8 *For all $t \in [0, 1]$, the system $F_t = (J, H_t)$ has precisely eight fixed points. They are given by four points not depending on t , namely*

$$\begin{aligned} A &= [\sqrt{2}, \sqrt{6}, \sqrt{6}, 2\sqrt{2}, 2, \sqrt{2}, 0, 0], & C &= [2, 2\sqrt{2}, \sqrt{6}, \sqrt{6}, \sqrt{2}, 0, 0, \sqrt{2}], \\ B &= [\sqrt{2}, 0, 0, \sqrt{2}, 2, 2\sqrt{2}, \sqrt{6}, \sqrt{6}], & D &= [2, \sqrt{2}, 0, 0, \sqrt{2}, \sqrt{6}, \sqrt{6}, 2\sqrt{2}] \end{aligned}$$

as in Proposition 3.7, and four points that change with t as follows:

$$\begin{aligned} P_t^{min} &= \left[0, u_-(t), \sqrt{2+u_-^2(t)}, \sqrt{6+u_-^2(t)}, \sqrt{6}, \sqrt{8-u_-^2(t)}, \sqrt{4-u_-^2(t)}, \sqrt{2-u_-^2(t)} \right], \\ P_t^{max} &= \left[0, u_+(t), \sqrt{2+u_+^2(t)}, \sqrt{6+u_+^2(t)}, \sqrt{6}, \sqrt{8-u_+^2(t)}, \sqrt{4-u_+^2(t)}, \sqrt{2-u_+^2(t)} \right], \\ Q_t^{min} &= \left[\sqrt{6}, \sqrt{8-v_+^2(t)}, \sqrt{4-v_+^2(t)}, \sqrt{2-v_+^2(t)}, 0, v_+(t), \sqrt{2+v_+^2(t)}, \sqrt{6+v_+^2(t)} \right], \\ Q_t^{max} &= \left[\sqrt{6}, \sqrt{8-v_-^2(t)}, \sqrt{4-v_-^2(t)}, \sqrt{2-v_-^2(t)}, 0, v_-(t), \sqrt{2+v_-^2(t)}, \sqrt{6+v_-^2(t)} \right], \end{aligned}$$

where $u_{\pm} : [0, 1] \rightarrow [-\sqrt{2}, \sqrt{2}]$ are the unique smooth solutions of the equation

$$(1-2t)u\sqrt{f(u^2)} + \gamma t f(u^2) + \gamma t u^2 f'(u^2) = 0 \quad (10)$$

where

$$f(\zeta) := (2+\zeta)(6+\zeta)(8-\zeta)(4-\zeta)(2-\zeta).$$

$u_{\pm}$ has initial values $u_-(0) = 0$ and $u_+(0) = \sqrt{2}$. Moreover, $v_{\pm} : [0, 1] \rightarrow [-\sqrt{2}, \sqrt{2}]$ are the unique smooth solutions of

$$-(1-2t)v\sqrt{f(v^2)} + \gamma t f(v^2) + \gamma t v^2 f'(v^2) = 0. \quad (11)$$

$v_{\pm}$ has initial values $v_-(0) = -\sqrt{2}$ and $v_+(0) = 0$. For $t = 0$, we recover $P_0^{min} = P^{min}$, $P_0^{max} = P^{max}$, $Q_0^{min} = Q^{min}$, $Q_0^{max} = Q^{max}$ from Proposition 3.7.

This will be proven in Sect. 5. Let us now shed some light on the situation at $t = \frac{1}{2}$.

Proposition 4.9 *At $t = \frac{1}{2}$, the focus-focus points A and B lie both in $F_{\frac{1}{2}}^{-1}(1, 0)$ and the focus-focus points C and D lie both in $F_{\frac{1}{2}}^{-1}(2, 0)$ and both fibres have the form of a double pinched torus as displayed in Fig. 3. Exemplarily, we compute the fibre $F_{\frac{1}{2}}^{-1}(1, 0)$ as*

$$\left\{ \left[\sqrt{2}, \sqrt{6-|z_8|^2}, \sqrt{6-|z_8|^2}, \sqrt{8-|z_8|^2}, 2, \sqrt{2+|z_8|^2}, \pm i\overline{z_8}, z_8 \right] \in M \mid |z_8| \in [0, \sqrt{6}] \right\}.$$

Using polar coordinates, this yields for $F_{\frac{1}{2}}^{-1}(1, 0)$ the parametrisation

$$\left\{ \left[\sqrt{2}, \sqrt{6-r^2}, \sqrt{6-r^2}, \sqrt{8-r^2}, 2, \sqrt{2+r^2}, \pm r i e^{-i\vartheta}, r e^{i\vartheta} \right] \in M \mid \vartheta \in [0, 2\pi[, r \in [0, \sqrt{6}] \right\}$$

This statement is given in Sect. 8.

5 The Whereabouts of the Fixed Points of $F_t = (J, H_t)$

In this section, we will determine the explicit coordinates of the eight fixed points of $F_t = (J, H_t)$ and hereby prove Proposition 4.8.

We only have to consider the case $t > 0$ since $t = 0$ is already treated in Proposition 3.7. For $t > 0$, the family $F_t = (J, H_t)$ will turn out to be of toric type or semitoric — apart from the two transition times where the four transition points pass through a degeneracy.

Proof of Proposition 4.8 $p = [p_1, \dots, p_8] \in M$ can only be a fixed point of $F_t = (J, H_t)$, if it is a fixed point of J . Thus, according to Proposition 3.7, p lies in $\{A, B, C, D\}$ or it satisfies $p_1 = 0$ or $p_5 = 0$.

Case $p \in \{A, B, C, D\}$: If p is one of these fixed points, p is also a fixed point of H according to Proposition 3.7. Write $z_k = x_k + i y_k$ and recall $X(z_1, \dots, z_8) = \Re(\overline{z_2} \overline{z_3} \overline{z_4} z_6 z_7 z_8)$. Let $\hat{z}_k$ stand for the omission of z_k resp. $\overline{z}_k$ and define $\text{sign}(\hat{z}_k) = 1$ if $\hat{z}_k = z_k$ and $\text{sign}(\hat{z}_k) = -1$ if $\hat{z}_k = \overline{z}_k$. We calculate

$$\begin{aligned} \partial_{x_k} X(z_1, \dots, z_8) &= \partial_{x_k} (x_k \Re(\overline{z_2} \cdots \hat{z}_k \cdots z_8) - \text{sign}(\hat{z}_k) y_k \Im(\overline{z_2} \cdots \hat{z}_k \cdots z_8)) \\ &= \Re(\overline{z_2} \cdots \hat{z}_k \cdots z_8) \end{aligned}$$

and

$$\partial_{y_k} X(z_1, \dots, z_8) = -\text{sign}(\hat{z}_k) \Im(\overline{z_2} \cdots \hat{z}_k \cdots z_8).$$

Note that each of the points A, B, C, D has two vanishing coordinate entries, thus we find for $p \in \{A, B, C, D\}$

$$dX(p) = (\partial_{x_1} X(p), \partial_{y_1} X(p), \partial_{x_2} X(p), \dots, \partial_{y_8} X(p)) = (0, 0, 0, \dots, 0).$$

Thus, p is also a fixed point of X . Since $H_t = (1-2t)H + t\gamma X$ is a linear combination, p is also a fixed point of H_t .

Case: p has $p_1 = 0$: First we show that any fixed point p with $p_1 = 0$ has to lie in the set $U_1 \subseteq M$. We argue by contradiction: $p_1 = 0$ would also allow $p \in U_8$, i.e. $p_8 = 0$. By definition of U_8 , we have $z_2, \dots, z_7 \neq 0$ and we may choose a representative with nonvanishing real part, thus $\partial_{x_8} X(p) = \text{Re}(\overline{p_2} \overline{p_3} \overline{p_4} p_6 p_7) \neq 0$ and thus the derivative of X does not vanish in p . Picking in particular $p \in U_8$ to be the fixed point $P^{\max} = [0, \sqrt{2}, 2, 2\sqrt{2}, \sqrt{6}, \sqrt{6}, \sqrt{2}, 0]$ of the toric system

$F_0 = (J, H)$, we find

$$dH_t(p) = (1 - 2t)dH(p) + t\gamma dX(p) = 0 + t\gamma dX(p) \neq 0$$

for $t \neq 0$. This shows that any fixed point p of F_t with $p_1 = 0$ has to lie in U_1 .

Let us now work in the coordinate chart (U_1, ψ_1) . By assumption, we have $0 = z_1 = x_1 + iy_1$ and we now determine the coordinates $z_2 = x_2 + iy_2$ of any possible fixed point $p = \psi_1(0, 0, x_2, y_2) \in U_1$. We already calculated $H \circ \psi_1^{-1}$ in (8) and, plugging in $(0, 0, x_2, y_2)$, yields

$$(H \circ \psi_1^{-1})(0, 0, x_2, y_2) = \frac{1}{2} (2 + x_2^2 + y_2^2) = \frac{1}{2} (2 + |z_2|^2).$$

We also calculated $X \circ \psi_1^{-1}$ in (9) implying

$$\begin{aligned} (X \circ \psi_1^{-1})(0, 0, x_2, y_2) &= x_2 \sqrt{(2 + (x_2^2 + y_2^2))(6 + (x_2^2 + y_2^2))(8 - (x_2^2 + y_2^2))(4 - (x_2^2 + y_2^2))(2 - (x_2^2 + y_2^2))} \\ &= x_2 \sqrt{(2 + |z_2|^2)(6 + |z_2|^2)(8 - |z_2|^2)(4 - |z_2|^2)(2 - |z_2|^2)}. \end{aligned}$$

Now set

$$\begin{aligned} \mathbf{g} : \mathbb{R}^2 &\rightarrow \mathbb{R}, & \mathbf{g}(x_2, y_2) &:= x_2^2 + y_2^2, \\ \mathbf{f} : \mathbb{R} &\rightarrow \mathbb{R}, & \mathbf{f}(\zeta) &:= (2 + \zeta)(6 + \zeta)(8 - \zeta)(4 - \zeta)(2 - \zeta). \end{aligned}$$

Then we get for $H_t = (1 - 2t)H + t\gamma X$ in local coordinates

$$\begin{aligned} (H_t \circ \psi_1^{-1})(0, 0, x_2, y_2) &= \frac{1 - 2t}{2} (2 + |z_2|^2) + \gamma t x_2 \sqrt{(2 + |z_2|^2)(6 + |z_2|^2)(8 - |z_2|^2)(4 - |z_2|^2)(2 - |z_2|^2)} \\ &= \frac{1 - 2t}{2} (2 + \mathbf{g}(x_2, y_2)) + \gamma t x_2 \sqrt{\mathbf{f}(\mathbf{g}(x_2, y_2))}. \end{aligned}$$

Now we look for critical points, i.e. (x_2, y_2) such that $(d(H_t \circ \psi_1^{-1}))(0, 0, x_2, y_2) = 0$. We compute

$$\partial_{x_2}(H_t \circ \psi_1^{-1}) = 0 \Leftrightarrow (1 - 2t)x_2 + \gamma t \sqrt{\mathbf{f}(\mathbf{g}(x_2, y_2))} + \gamma t x_2^2 \frac{\mathbf{f}'(\mathbf{g}(x_2, y_2))}{\sqrt{\mathbf{f}(\mathbf{g}(x_2, y_2))}} = 0 \quad (12)$$

and

$$\partial_{y_2}(H_t \circ \psi_1^{-1}) = 0 \Leftrightarrow (1 - 2t)y_2 + \gamma t x_2 y_2 \frac{\mathbf{f}'(\mathbf{g}(x_2, y_2))}{\sqrt{\mathbf{f}(\mathbf{g}(x_2, y_2))}} = 0. \quad (13)$$

First we show that (12) and (13) force $y_2 = 0$ if $t > 0$: We argue by contradiction and assume that $y_2 \neq 0$. Then dividing the right hand side of (13) by y_2 yields

$$\gamma t x_2 \frac{f'(g(x_2, y_2))}{\sqrt{f(g(x_2, y_2))}} = 2t - 1.$$

We substitute this in the right hand side of (12) and get

$$0 = (1 - 2t)x_2 + \gamma t \sqrt{f(g(x_2, y_2))} + x_2(2t - 1) = \gamma t \sqrt{f(g(x_2, y_2))}.$$

But this cannot be true since $\sqrt{f(g(x_2, y_2))} = \frac{1}{x_2}(X \circ \psi_1^{-1})(0, 0, x_2, y_2) \stackrel{(9)}{=} x_3 x_4 x_6 x_7 x_8 \neq 0$ in U_1 and $\gamma \neq 0$ and $t > 0$.

Thus, we conclude $y_2 = 0$ which implies $|z_2|^2 = x_2^2 = g(x_2, 0)$ and $p = \psi_1^{-1}(0, 0, x_2, 0)$. Thus, we still need to determine the possible values of x_2 to determine p completely. This we approach by inserting $y_2 = 0$ on the right-hand side of (12) and by multiplying it with $\sqrt{f(g(x_2, 0))} = \sqrt{f(x_2^2)}$ which yields

$$(1 - 2t) x_2 \sqrt{f(x_2^2)} + \gamma t f(x_2^2) + \gamma t x_2^2 f'(x_2^2) = 0. \quad (14)$$

Note that equation (14) is equation (10) in Proposition 4.8. Using the manifold equations (4), we get a restriction for $|x_2|$: Since $z_1 = 0$, we get $|z_5|^2 = 6$ and furthermore $|z_2|^2 + |z_7|^2 = 4$ and $|z_7|^2 = 2 + |z_6|^2 \geq 2$ and thus

$$|x_2|^2 = |z_2|^2 \leq 2.$$

Renaming x_2 as u to drop the index and simplify notation, we are thus looking for (smooth) solutions $u : [0, 1] \rightarrow [-\sqrt{2}, \sqrt{2}]$ solving $\mathcal{F}(t, u(t)) = 0$ where

$$\mathcal{F} : [0, 1] \times [-\sqrt{2}, \sqrt{2}] \rightarrow \mathbb{R}, \quad \mathcal{F}(t, u) := (1 - 2t)u\sqrt{f(u^2)} + \gamma t f(u^2) + \gamma t u^2 f'(u^2).$$

A look at the formulas of f and $\mathcal{F}$ shows immediately that $\mathcal{F}(0, \pm\sqrt{2}) = 0 = \mathcal{F}(0, 0)$. This are the only zeros for $t = 0$ as the plot of the graph of $\mathcal{F}$ in Fig. 8 shows. Moreover, the zero $(0, -\sqrt{2})$ is isolated within $[0, 1] \times [-\sqrt{2}, \sqrt{2}]$ whereas $(0, 0)$ and $(0, \sqrt{2})$ are the limit for $t \rightarrow 0$ of two unique smooth curves $t \mapsto (t, u_-(t))$ and $t \mapsto (t, u_+(t))$ that satisfy $\mathcal{F}(t, u_{\pm}(t)) = 0$ and $u_-(t) \in]-\sqrt{2}, 0]$ and $u_+(t) \in [0, \sqrt{2}]$ for $t \in [0, 1]$ as displayed in Fig. 9.

Letting $\gamma > 0$ tend to zero leads to a slimmer and slimmer ‘waist’ of the graph of $\mathcal{F}$ and a steeper and steeper slope of $u_{\pm}$ as displayed in Figs. 10 and 11.

Substituting the variable x_2 in (14) with $u_{\pm}$, we obtain the fixed points

$$P_t^{min} := \psi_1^{-1}(0, 0, u_-(t), 0) \quad \text{and} \quad P_t^{max} := \psi_1^{-1}(0, 0, u_+(t), 0).$$

Fig. 8 The graph (orange) of $(t, u) \mapsto \mathcal{F}(t, u)$ intersected with a horizontal plane through zero (blue) seen from two different angles. Both plots are done with *Mathematica* for $\gamma = \frac{1}{50}$

Fig. 9 Left: Plot of the level sets of $\mathcal{F}(t, u)$ with the two solutions (blue) satisfying $\mathcal{F}(t, u_{\pm}(t)) = 0$. Right: The positive solution u_+ and the negative solution u_- of $\mathcal{F}(t, u) = 0$. Both plots are done with *Mathematica* for $\gamma = \frac{1}{50}$

Fig. 10 Left: The graph (orange) of $(t, u) \mapsto \mathcal{F}(t, u)$ intersected with a horizontal plane through zero (blue). Right: The positive solution u_+ and the negative solution u_- of $\mathcal{F}(t, u) = 0$. Both plots are done with *Mathematica* for $\gamma = \frac{1}{1000}$

Fig. 11 Left: The graph (orange) of $(t, u) \mapsto \mathcal{F}(t, u)$ intersected with a horizontal plane through zero (blue) seen from two different angles. Right: The positive solution u_+ and the negative solution u_- of $\mathcal{F}(t, u) = 0$. Both plots are done with *Mathematica* for $\gamma = \frac{1}{10000}$

Using the definition of ψ_1^{-1} , i.e. the formulas in (6), we obtain the coordinates of $P_t^{\min}$ and $P_t^{\max}$ as given in Proposition 4.8. Letting $t \rightarrow 0$, we recover $P^{\max} = P_0^{\max}$ and $P^{\min} = P_0^{\min}$ of the toric system, as computed in Proposition 3.7.

Case: p has $p_5 = 0$: We proceed analogously to the case $p_1 = 0$, i.e. first, we show p to live in U_5 and then we work in the chart

$$\psi_5^{-1}(x_5, y_5, x_6, y_6) = [x_1, 0, x_2, 0, x_3, 0, x_4, 0, x_5, y_5, x_6, y_6, x_7, 0, x_8, 0]$$

where $x_1, x_2, x_3, x_4, x_7, x_8$ are determined by the manifold equations from (4) as

$$\begin{aligned} x_1 &= \sqrt{6 - |z_5|^2}, & x_3 &= \sqrt{4 + |z_5|^2 - |z_6|^2}, & x_7 &= \sqrt{2 - |z_5|^2 + |z_6|^2}, \\ x_2 &= \sqrt{8 - |z_6|^2}, & x_4 &= \sqrt{2 + 2|z_5|^2 - |z_6|^2}, & x_8 &= \sqrt{6 - 2|z_5|^2 + |z_6|^2} \end{aligned}$$

where $|z_5|^2 = x_5^2 + y_5^2$ and $|z_6|^2 = x_6^2 + y_6^2$. We find $p = \psi_5^{-1}(0, 0, x_6, y_6)$ and

$$H_t(p) = \frac{1-2t}{2}(4 - |z_6|^2) + \gamma t x_6 \sqrt{(8 - |z_6|^2)(4 - |z_6|^2)(2 - |z_6|^2)(2 + |z_6|^2)(6 + |z_6|^2)}.$$

The system of equations describing $d(H_t \circ \psi_5^{-1})(0, 0, x_6, y_6) = 0$ leads to $y_6 = 0$ and forces x_6 to satisfy for $t > 0$

$$-(1 - 2t)x_6 \sqrt{f(x_6^2)} + \gamma t f(x_6^2) + \gamma t x_6^2 f'(x_6^2) = 0. \quad (15)$$

$z_5 = 0$ and the manifold equations (4) imply $|z_6|^2 = |z_7|^2 - 2$ and $|z_7|^2 = 4 - |z_4|^2 \leq 4$ which leads to $|z_6|^2 = x_6^2 \leq 2$.

Renaming x_6 as v , we are thus looking for smooth solutions $v : [0, 1] \rightarrow [-\sqrt{2}, \sqrt{2}]$ solving $\tilde{\mathcal{F}}(t, v(t)) = 0$ where

Fig. 12 Left: The graph (orange) of $(t, v) \mapsto \tilde{\mathcal{F}}(t, v)$ intersected with a horizontal plane through zero (blue). Right: The positive solution v_+ and the negative solution v_- of $\tilde{\mathcal{F}}(t, v) = 0$. Both plots are done with *Mathematica* for $\gamma = \frac{1}{50}$

$$\tilde{\mathcal{F}} : [0, 1] \times [-\sqrt{2}, \sqrt{2}] \rightarrow \mathbb{R}, \quad \tilde{\mathcal{F}}(t, v) := -(1 - 2t)v\sqrt{f(v^2)} + \gamma t f(v^2) + \gamma t v^2 f'(v^2).$$

The situation is plotted in Fig. 12: similar as above, we obtain two smooth solutions $v_- : [0, 1] \rightarrow [-\sqrt{2}, 0[$ and $v_+ : [0, 1] \rightarrow [0, \sqrt{2}[$ satisfying $\tilde{\mathcal{F}}(t, v_{\pm}(t)) = 0$ and $v_-(0) = -\sqrt{2}$ and $v_+(0) = 0$. This in turn gives us the fixed points

$$Q_t^{\min} := \psi_5^{-1}(0, 0, v_-(t), 0) \quad \text{and} \quad Q_t^{\max} := \psi_5^{-1}(0, 0, v_+(t), 0).$$

Using the definition of ψ_5^{-1} , we obtain the coordinates of the $Q_t^{\min}$ and $Q_t^{\max}$ given in Proposition 4.8. Moreover, for $t \rightarrow 0$, we recover $Q^{\max} = Q_0^{\max}$ and $Q^{\min} = Q_0^{\min}$ of the toric system, see Proposition 3.7. $\square$

6 The Types of the Fixed Points of $F_t = (J, H_t)$

In this section, we will show that $P_t^{\min}, P_t^{\max}, Q_t^{\min}, Q_t^{\max}$ are elliptic–elliptic singular points for all $t \in [0, 1]$ and that A, B, C, D are turning from elliptic–elliptic to focus–focus at time t^- and back to elliptic–elliptic at t^+ passing through a degeneracy at $t^{\pm}$.

Proposition 6.1 *Let $0 < \gamma < \frac{1}{48}$ and set*

$$t^- := \frac{1}{2(1 + 24\gamma)} \quad \text{and} \quad t^+ := \frac{1}{2(1 - 24\gamma)}.$$

Then the rank zero points A, B, C, D are of elliptic–elliptic type for $0 \leq t < t^-$ as well as for $t^+ < t \leq 1$ and of focus–focus type for $t^- < t < t^+$.

Moreover, for $t = t^{\pm}$, the points A, B, C, D are degenerate (see Proposition 2.13).

Proof *Step 1: Type of the point A:* Since the point A has coordinates $A_7 = 0 = A_8$, we work with the chart (U_7, ψ_7) where

$$\psi_7^{-1} : \mathbb{R}^4 \rightarrow U_7, \quad \psi_7^{-1}(x_7, y_7, x_8, y_8) = [x_1, 0, x_2, 0, \dots, x_7, y_7, x_8, y_8]$$

with $|z_7|^2 = x_7^2 + y_7^2$ and $|z_8|^2 = x_8^2 + y_8^2$ and

$$\begin{aligned} x_1 &= \sqrt{2 - |z_7|^2 + |z_8|^2}, & x_3 &= \sqrt{6 - |z_7|^2}, & x_5 &= \sqrt{4 + |z_7|^2 - |z_8|^2}, \\ x_2 &= \sqrt{6 - 2|z_7|^2 + |z_8|^2}, & x_4 &= \sqrt{8 - |z_8|^2}, & x_6 &= \sqrt{2 + 2|z_7|^2 - |z_8|^2}. \end{aligned}$$

We have $A = \psi_7^{-1}(0, 0, 0, 0)$ in this chart, and we find

$$\begin{aligned} (H_t \circ \psi_7^{-1})(x_7, y_7, x_8, y_8) &= \frac{(1-2t)}{2}(6 - |z_7|^2) \\ &\quad + t\gamma(x_7x_8 - y_7y_8)\sqrt{(6-2|z_7|^2+|z_8|^2)(6-|z_7|^2)(8-|z_8|^2)(2+2|z_7|^2-|z_8|^2)} \\ &=: \mathcal{K}(x_7, y_7, x_8, y_8) + \mathcal{L}(x_7, y_7, x_8, y_8)\sqrt{\mathcal{M}(x_7, y_7, x_8, y_8)}. \end{aligned} \quad (16)$$

To determine the Hessian $d^2(H_t \circ \psi_7^{-1})(0, 0, 0, 0)$, we calculate

$$d^2\mathcal{K}(x_7, y_7, x_8, y_8) = \begin{pmatrix} -(1-2t) & 0 & 0 & 0 \\ 0 & -(1-2t) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and, for $\sigma, \tau \in \{x_7, y_7, x_8, y_8\}$, we get

$$\begin{aligned} \partial_\sigma (\mathcal{L}\sqrt{\mathcal{M}}) &= (\partial_\sigma \mathcal{L})\sqrt{\mathcal{M}} + \mathcal{L} \frac{1}{2\sqrt{\mathcal{M}}} \partial_\sigma \mathcal{M}, \\ \partial_{\sigma\tau}^2 (\mathcal{L}\sqrt{\mathcal{M}}) &= (\partial_{\sigma\tau}^2 \mathcal{L})\sqrt{\mathcal{M}} + (\partial_\sigma \mathcal{L}) \frac{1}{2\sqrt{\mathcal{M}}} \partial_\tau \mathcal{M} + (\partial_\tau \mathcal{L}) \frac{1}{2\sqrt{\mathcal{M}}} \partial_\sigma \mathcal{M} \\ &\quad + \mathcal{L} \partial_\tau \left(\frac{1}{2\sqrt{\mathcal{M}}} \right) \partial_\sigma \mathcal{M} + \mathcal{L} \frac{1}{2\sqrt{\mathcal{M}}} \partial_{\sigma\tau}^2 \mathcal{M}. \end{aligned} \quad (17)$$

We calculate

$$\begin{aligned} d\mathcal{L}(x_7, y_7, x_8, y_8) &= (t\gamma x_8, -t\gamma y_8, t\gamma x_7, -t\gamma y_7), \\ d^2\mathcal{L}(x_7, y_7, x_8, y_8) &= \begin{pmatrix} 0 & 0 & t\gamma & 0 \\ 0 & 0 & 0 & -t\gamma \\ t\gamma & 0 & 0 & 0 \\ 0 & -t\gamma & 0 & 0 \end{pmatrix}. \end{aligned}$$

Now note that $\partial_\sigma \mathcal{L}$ and $\partial_\sigma \mathcal{M}$ and $\partial_{\sigma\tau}^2 \mathcal{M}$ contain at least one of the variables x_7, y_7, x_8, y_8 as a factor and that $\mathcal{L}(0, 0, 0, 0) = 0$ and $\sqrt{\mathcal{M}(0, 0, 0, 0)} = 24$. This

leads to

$$d^2 \left(\mathcal{L}\sqrt{\mathcal{M}} \right) (0, 0, 0, 0) = \begin{pmatrix} 0 & 0 & 24t\gamma & 0 \\ 0 & 0 & 0 & -24t\gamma \\ 24t\gamma & 0 & 0 & 0 \\ 0 & -24t\gamma & 0 & 0 \end{pmatrix}$$

and therefore we get at $\psi_7(A) = (0, 0, 0, 0)$

$$d^2(H_t \circ \psi_7^{-1})(0, 0, 0, 0) = \begin{pmatrix} 2t-1 & 0 & 24t\gamma & 0 \\ 0 & 2t-1 & 0 & -24t\gamma \\ 24t\gamma & 0 & 0 & 0 \\ 0 & -24t\gamma & 0 & 0 \end{pmatrix}.$$

According to (7), in the chart (U_7, ψ_7) , the symplectic form is denoted by ω_7 and shown to be the standard symplectic form in (7). We get

$$(\omega_7)_{(0,0,0,0)}^{-1} d^2(H_t \circ \psi_7^{-1})(0, 0, 0, 0) = \begin{pmatrix} 0 & 2t-1 & 0 & -24t\gamma \\ 1-2t & 0 & -24t\gamma & 0 \\ 0 & -24t\gamma & 0 & 0 \\ -24t\gamma & 0 & 0 & 0 \end{pmatrix}.$$

First, consider the case $t \neq \frac{1}{2}$: Set $c := 24\gamma$ and compute the characteristic polynomial χ of $\omega_7^{-1} d^2(H_t \circ \psi_7^{-1})$ in $(0, 0, 0, 0)$ as

$$\chi(\xi) = \xi^4 + (1 - 4t + 2(2 - c^2)t^2)\xi^2 + c^4t^4.$$

Now look at the polynomial $\tilde{\chi}$ defined by requiring $\tilde{\chi}(\xi) = \chi(\xi^2)$, i.e.

$$\tilde{\chi}(\xi) = \xi^2 + (1 - 4t + 2(2 - c^2)t^2)\xi + c^4t^4.$$

$\tilde{\chi}$ has discriminant δ given by

$$\delta := -(2t-1)^2(4(c^2-1)t^2+4t-1) = -4(c^2-1)(2t-1)^2 \left(t - \frac{1}{2(1-c)}\right) \left(t - \frac{1}{2(1+c)}\right).$$

Setting

$$t^- := \frac{1}{2(1+c)} = \frac{1}{2(1+24\gamma)} \quad \text{and} \quad t^+ := \frac{1}{2(1-c)} = \frac{1}{2(1-24\gamma)}$$

yields

$$\delta = 4(1-c^2)(2t-1)^2(t-t^+)(t-t^-).$$

The assumption $0 < \gamma < \frac{1}{48}$ implies that $0 < t^- < \frac{1}{2} < t^+ < 1$ and in particular that $c < \frac{1}{2}$ and hence $1 - c^2 > 0$. Moreover, since $t \neq \frac{1}{2}$, the factor $(2t - 1)^2$ is strictly positive. Thus, the sign of the discriminant δ depends on the position of t relative to t^- and t^+ , leaving us with two cases:

Consider the case $t^- < t < t^+$: Here we have $\delta < 0$, so that $\tilde{\chi}$ has two complex conjugate roots $a + ib$ and $a - ib$ with nonzero imaginary part b . Hence, the four zeros of χ are of the form $\pm\alpha \pm i\beta$, i.e. A is nondegenerate and of focus–focus type according to the list after Lemma 2.3.

Consider the case $0 < t < t^-$ or $t^+ < t \leq 1$: Here we have $\delta > 0$ so that $\tilde{\chi}$ has two real roots

$$\lambda_{\pm} := \frac{-(1 - 4t + 2(2 - c^2)t^2) \pm \sqrt{\delta}}{2}.$$

We have

$$\tilde{b} := 2(2 - c^2)t^2 - 4t + 1 = 2(2 - c^2) \left(t - \frac{1}{2 - c\sqrt{2}} \right) \left(t - \frac{1}{2 + c\sqrt{2}} \right)$$

which is positive since

$$t^- < \frac{1}{2 + c\sqrt{2}} < \frac{1}{2 - c\sqrt{2}} < t^+$$

and $t < t^-$ or $t^+ < t$. Hence, we find $\lambda_- < 0$. Moreover, $\tilde{b} > \sqrt{\delta}$ since $b^2 - \Delta = 4c^4t^4 > 0$. This implies $\lambda_+ < 0$ as well. Therefore, the zeros of χ are of the form $\pm i\alpha, \pm i\beta$ and A is nondegenerate of elliptic–elliptic type according to the list after Lemma 2.3.

Second, consider the case $t = \frac{1}{2}$: Here we find $\delta = 0$, so the eigenvalues are not distinct and we cannot conclude the type of A in the same way as above. We now start with

$$(\omega_7^{-1})_{(0,0,0,0)} d^2(H_t \circ \psi_7^{-1})(0, 0, 0, 0) = \begin{pmatrix} 0 & 0 & 0 & -12\gamma \\ 0 & 0 & -12\gamma & 0 \\ 0 & -12\gamma & 0 & 0 \\ -12\gamma & 0 & 0 & 0 \end{pmatrix}$$

and consider in addition $(J \circ \psi_7^{-1})(x_7, y_7x_8, y_8) = \frac{1}{2}(2 - x_7^2 - y_7^2 + x_8^2 + y_8^2)$. We find

$$d^2(J \circ \psi_7^{-1})(0, 0, 0, 0) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and thus

$$(\omega_7^{-1})_{(0,0,0,0)} d^2(J \circ \psi_7^{-1})(0, 0, 0, 0) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The linear combination

$$\begin{aligned} & (\omega_7)_{(0,0,0,0)}^{-1} d^2(J \circ \psi_7^{-1})(0, 0, 0, 0) - \frac{1}{12\gamma} (\omega_7)_{(0,0,0,0)}^{-1} d^2(H_t \circ \psi_7^{-1})(0, 0, 0, 0) \\ &= \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \end{aligned}$$

has four distinct eigenvalues $\pm 1 \pm i$, so $A = \psi_7^{-1}(0, 0, 0, 0)$ is nondegenerate of focus–focus type.

Proof for the point B: Note that $B = [\sqrt{2}, 0, 0, \sqrt{2}, 2, 2\sqrt{2}, \sqrt{6}, \sqrt{6}]$ lies in the chart (U_2, ψ_2) given by

$$\psi_2^{-1} : \mathbb{R}^4 \rightarrow U_2, \quad \psi_2^{-1}(x_2, y_2, x_3, y_3) = [x_1, 0, x_2, y_2, x_3, y_3, x_4, 0, \dots, x_8, 0]$$

with $|z_2|^2 = x_2^2 + y_2^2$ and $|z_3|^2 = x_3^2 + y_3^2$ and

$$\begin{aligned} x_1 &= \sqrt{2 + |z_2|^2 - |z_3|^2}, & x_5 &= \sqrt{4 - |z_2|^2 + |z_3|^2}, & x_7 &= \sqrt{6 - |z_3|^2}, \\ x_4 &= \sqrt{2 - |z_2|^2 + 2|z_3|^2}, & x_6 &= \sqrt{8 - |z_2|^2}, & x_8 &= \sqrt{6 + |z_2|^2 - 2|z_3|^2}. \end{aligned}$$

and is given by $B = \psi_2^{-1}(0, 0, 0, 0)$. Proceeding as above for the fixed point A , we find

$$(\omega_2^{-1})_{(0,0,0,0)} d^2(H_t \circ \psi_2^{-1})(0, 0, 0, 0) = \begin{pmatrix} 0 & 0 & 0 & -24\gamma t \\ 0 & 0 & -24\gamma t & 0 \\ 0 & -24\gamma t & 0 & 1 - 2t \\ -24\gamma t & 0 & 2t - 1 & 0 \end{pmatrix}.$$

This matrix has the same characteristic polynomial as the corresponding matrix in the calculations for the point A above. Hence, all calculations and implications are identical to the ones made for the point A .

Proof for the points C and D: Analogous to the ones for A and B . $\square$

We now study the type of the remaining four fixed points.

Proposition 6.2 *The points $P_t^{\min}$, $P_t^{\max}$, $Q_t^{\min}$ and $Q_t^{\max}$ are elliptic–elliptic for all $t \in [0, 1]$.*

Proof We showed in Proposition 4.8 that we recover via $P_0^{min} = P^{min}$, $P_0^{max} = P^{max}$, $Q_0^{min} = Q^{min}$, $Q_0^{max} = Q^{max}$ the fixed points of the toric system $(M, \omega, F = (J, H))$ which are all elliptic–elliptic. Thus, it is sufficient to prove Proposition 6.2 for $t > 0$.

First, consider P_t^{min} and P_t^{max} : As in the proof of Proposition 4.8, we work in the chart (U_1, ψ_1) . Here we have $\psi_1(P_t^{min}) = (0, 0, u_-(t), 0)$ and $\psi_1(P_t^{max}) = (0, 0, u_+(t), 0)$ where $t \mapsto u_{\pm}(t) \in [-\sqrt{2}, \sqrt{2}]$ are the two real solutions of (10). In the chart (U_1, ψ_1) with identification $x_1^2 + y_1^2 = |z_1|^2$ and $x_2^2 + y_2^2 = |z_2|^2$, we get

$$(H_t \circ \psi_1^{-1})(x_1, y_1, x_2, y_2) = \frac{1-2t}{2}(2 - |z_1|^2 + |z_2|^2) \\ + \gamma t x_2 \sqrt{(2 - |z_1|^2 + |z_2|^2)(6 - 2|z_1|^2 + |z_2|^2)(8 - |z_2|^2)(4 + |z_1|^2 - |z_2|^2)(2 + 2|z_1|^2 - |z_2|^2)}$$

and set analogously to (16)

$$=: \mathcal{K}(x_1, y_1, x_2, y_2) + \mathcal{L}(x_1, y_1, x_2, y_2) \sqrt{\mathcal{M}(x_1, y_1, x_2, y_2)}.$$

We need to calculate the Hessian of $H_t \circ \psi_1^{-1}$ in the point $(0, 0, u, 0)$ for $u := u_{\pm}(t)$. All following calculations only care about the variables x_1, y_1, x_2, y_2 of the chart while holding true for all $t \in [0, 1]$. Thus, for sake of readability, we drop the dependence on the parameter t in some of the expressions. We find

$$d^2 \mathcal{K}(x_1, y_1, x_2, y_2) = \begin{pmatrix} 2t-1 & 0 & 0 & 0 \\ 0 & 2t-1 & 0 & 0 \\ 0 & 0 & 1-2t & 0 \\ 0 & 0 & 0 & 1-2t \end{pmatrix}$$

and

$$\mathcal{L}(x_1, y_1, x_2, y_2) = \gamma t x_2, \quad d\mathcal{L}(x_1, y_1, x_2, y_2) = (0, 0, \gamma t, 0), \quad d^2 \mathcal{L} = \mathbf{0}.$$

The formula for $\partial_{\sigma\tau}^2(\mathcal{L}\sqrt{\mathcal{M}})$ is the same as in (17). For $\sigma, \tau \in \{x_1, y_1, x_2, y_2\}$, note that $\partial_{\sigma\tau}^2 \mathcal{L} = 0$ and that $\partial_{\sigma} \mathcal{L} = 0$ for $\sigma \neq x_2$ and $\partial_{x_2} \mathcal{L} = \gamma t$. We have

$$\mathcal{M}(u) := \mathcal{M}(0, 0, u, 0) = (2 + u^2)(6 + u^2)(8 - u^2)(4 - u^2)(2 - u^2) \\ = -u^{10} + 6u^8 + 44u^6 - 216u^4 - 160u^2 + 768$$

and the first partial derivatives of $\mathcal{M}$ are of the form

$$\partial_{\sigma} \mathcal{M}(x_1, y_1, x_2, y_2) = \sigma \operatorname{Rest}(x_1^2 + y_1^2, x_2^2 + y_2^2)$$

for $\sigma \in \{x_1, y_1, x_2, y_2\}$ where $\operatorname{Rest}(\cdot, \cdot)$ consists of expressions depending on $x_1^2 + y_1^2$ and $x_2^2 + y_2^2$. Thus, we get $\partial_{\sigma} \mathcal{M}(0, 0, u, 0) = 0$ for $\sigma \in \{x_1, y_1, y_2\}$. We compute for $\sigma = x_2$

$$\partial_{x_2} \mathcal{M}(0, 0, u, 0) = -10u^9 + 48u^7 + 264u^5 - 864u^3 - 320u = \frac{d}{du} \mathcal{M}(u) =: \mathcal{M}'(u).$$

The mixed second partial derivatives are of the form

$$\partial_{\sigma\tau}^2 \mathcal{M}(x_1, y_1, x_2, y_2) = \sigma \tau \operatorname{Rest}(x_1^2 + y_1^2, x_2^2 + y_2^2)$$

for all $\sigma, \tau \in \{x_1, y_1, x_2, y_2\}$ with $\sigma \neq \tau$ where $\operatorname{Rest}(\cdot, \cdot)$ consists of expressions depending on $x_1^2 + y_1^2$ and $x_2^2 + y_2^2$. Therefore we get $\partial_{\sigma\tau} \mathcal{M}(0, 0, u, 0) = 0$ for all $\sigma, \tau \in \{x_1, y_1, x_2, y_2\}$ with $\sigma \neq \tau$. Thus, a look at (17) shows that

$$\partial_{\sigma\tau}^2 \left(\mathcal{L}\sqrt{\mathcal{M}} \right) (0, 0, u, 0) = 0 \quad \forall \sigma, \tau \in \{x_1, y_1, x_2, y_2\} \text{ with } \sigma \neq \tau.$$

To get the remaining second derivatives, we first compute

$$\begin{aligned} \partial_{x_1 x_1}^2 \mathcal{M}(0, 0, u, 0) &= 12u^8 - 84u^6 - 256u^4 + 1200u^2 + 640 \\ &= \partial_{y_1 y_1}^2 \mathcal{M}(0, 0, u, 0), \\ \partial_{x_2 x_2}^2 \mathcal{M}(0, 0, u, 0) &= -90u^8 + 336u^6 + 1320u^4 - 2592u^2 - 320, \\ \partial_{y_2 y_2}^2 \mathcal{M}(0, 0, u, 0) &= -10u^8 + 48u^6 + 264u^4 - 864u^2 - 320 \end{aligned}$$

and thus

$$\begin{aligned} \partial_{x_1 x_1}^2 \left(\mathcal{L}\sqrt{\mathcal{M}} \right) (0, 0, u, 0) &= \frac{\gamma t u (6u^8 - 42u^6 - 128u^4 + 600u^2 + 320)}{\sqrt{\mathcal{M}(u)}} \\ &= \partial_{y_1 y_1}^2 \left(\mathcal{L}\sqrt{\mathcal{M}} \right) (0, 0, u, 0), \\ \partial_{y_2 y_2}^2 \left(\mathcal{L}\sqrt{\mathcal{M}} \right) (0, 0, u, 0) &= \frac{\gamma t u (-5u^8 + 24u^6 + 132u^4 - 432u^2 - 160)}{\sqrt{\mathcal{M}(u)}}. \end{aligned}$$

Setting

$$\begin{aligned} g(u) &:= 15u^{18} - 153u^{16} - 652u^{14} + 9216u^{12} - 3984u^{10} - 126288u^8 + 159552u^6 \\ &\quad + 510336u^4 - 803840u^2 - 184320 \end{aligned} \quad (18)$$

we obtain

$$\partial_{x_2 x_2}^2 \left(\mathcal{L}\sqrt{\mathcal{M}} \right) (0, 0, u, 0) = \frac{2 \gamma t u g(u)}{\sqrt{\mathcal{M}^3(u)}}.$$

Setting

$$\begin{aligned} a(u) &:= 2t - 1 + \frac{\gamma t u (6u^8 - 42u^6 - 128u^4 + 600u^2 + 320)}{\sqrt{\mathcal{M}(u)}}, \\ b(u) &:= (1 - 2t) + \frac{2 \gamma t u g(u)}{\sqrt{\mathcal{M}^3(u)}}, \end{aligned}$$

$$c(u) := (1 - 2t) + \frac{\gamma t u (-5u^8 + 24u^6 + 132u^4 - 432u^2 - 160)}{\sqrt{\mathcal{M}(u)}},$$

we obtain

$$d^2(H_t \circ \psi_1^{-1})(0, 0, u, 0) = \begin{pmatrix} a(u) & 0 & 0 & 0 \\ 0 & a(u) & 0 & 0 \\ 0 & 0 & b(u) & 0 \\ 0 & 0 & 0 & c(u) \end{pmatrix}.$$

We calculate the type of the singular points by finding the eigenvalues of the matrix

$$(\omega_1^{-1})|_{(0,0,u,0)} d^2(H_t \circ \psi_1^{-1})(0, 0, u, 0) = \begin{pmatrix} 0 & a(u) & 0 & 0 \\ -a(u) & 0 & 0 & 0 \\ 0 & 0 & 0 & c(u) \\ 0 & 0 & -b(u) & 0 \end{pmatrix}.$$

The characteristic polynomial is given by

$$\chi(\xi) = (\xi^2 + a^2(u))(b(u)c(u) + \xi^2).$$

Hence the eigenvalues are of the form

$$\lambda^\pm = \pm ia(u) \quad \text{and} \quad \mu^\pm = \pm i\sqrt{b(u)c(u)}.$$

Now we show that $\lambda^\pm$ and $\mu^\pm$ are purely imaginary. Let us first investigate $\lambda^\pm$. Since $u \in]-\sqrt{2}, \sqrt{2}[$ for $t > 0$, we have $\mathcal{M}(u) > 0$ so that $\sqrt{\mathcal{M}(u)}$ and $a(u)$ are real numbers. Since $u = u_\pm(t)$ is also real valued, $\lambda^\pm = \lambda_t^\pm$ is purely imaginary for all $t \in]0, 1[$. Now, we investigate $\mu^\pm$. First, recall the function f from (10) and note that $f(u^2) = \mathcal{M}(u)$ and $2uf'(u^2) = \mathcal{M}'(u)$. Thus, we can rewrite

$$c(u) = 1 - 2t + \frac{\gamma t u \mathcal{M}'(u)}{2u\sqrt{\mathcal{M}(u)}} \stackrel{(10)}{=} -\frac{\gamma t \mathcal{M}(u)}{u\sqrt{\mathcal{M}(u)}} = -\frac{\gamma t \sqrt{\mathcal{M}(u)}}{u}.$$

The numerator $\gamma t \sqrt{\mathcal{M}(u)}$ is a strict positive number for $t > 0$, and hence, we conclude that $c(u)$ and u have opposite signs. Moreover,

$$\begin{aligned} b(u) &= 1 - 2t + \frac{2t\gamma u g(u)}{\sqrt{\mathcal{M}^3(u)}} = 1 - 2t + \frac{\gamma t \mathcal{M}'(u)}{2\sqrt{\mathcal{M}(u)}} - \frac{\gamma t \mathcal{M}'(u)}{2\sqrt{\mathcal{M}(u)}} + \frac{2t\gamma u g(u)}{\sqrt{\mathcal{M}^3(u)}} \\ &\stackrel{(10)}{=} -\frac{\gamma t \sqrt{\mathcal{M}}}{u} - \frac{\gamma t \mathcal{M}'(u)}{2\sqrt{\mathcal{M}}} + \frac{2\gamma t u g(u)}{\sqrt{\mathcal{M}^3(u)}} \\ &= \frac{\gamma t}{u\sqrt{\mathcal{M}^3(u)}} \left(-\mathcal{M}^2(u) - \frac{u\mathcal{M}(u)\mathcal{M}'(u)}{2} + 2u^2 g(u) \right) \\ &=: \frac{\gamma t \mathcal{P}(u)}{u\sqrt{\mathcal{M}^3(u)}}. \end{aligned}$$

Fig. 13 Left: $\mathcal{P}$ plotted from $-\sqrt{2}$ to $\sqrt{2}$, displaying the maxima at $u = 0, \pm\sqrt{2}$ and the minima at $u = \pm 1$. Right: $\mathcal{P}$ plotted over an interval strictly larger than $[-\sqrt{2}, \sqrt{2}]$. Already slightly outside $[-\sqrt{2}, \sqrt{2}]$, the function $\mathcal{P}$ descends rapidly, becoming ‘even negativer’. Both plots are done with *Mathematica*

Explicitly, we have

$$\begin{aligned}\mathcal{P}(u) &= -589824 - 995328u^4 + 678912u^6 + 193728u^8 - 173952u^{10} - 6624u^{12} + 14112u^{14} \\ &\quad - 1044u^{16} - 240u^{18} + 24u^{20}, \\ \mathcal{P}'(u) &= 96(-2+u)(-1+u)u^3(1+u)(2+u)(-8+u^2)(-2+u^2)(2+u^2)(6+u^2) \\ &\quad (-54-10u^2+5u^4).\end{aligned}$$

$\mathcal{P}$ is strictly negative over $[-\sqrt{2}, \sqrt{2}]$ with maximum $\mathcal{P}(0) = \mathcal{P}(\pm\sqrt{2}) = -589824 < 0$ and minimum $\mathcal{P}(\pm 1) = -880236 < 0$, see Fig. 13.

Hence, $b(u)$ and u have opposite signs. Altogether we conclude $b(u)c(u) > 0$ and thus $\mu^\pm = \mu_t^\pm$ is purely imaginary for all $t \in [0, 1]$.

Finally, a look at the list after Proposition 4.8 shows that $P_t^{\min}$ and $P_t^{\max}$ are nondegenerate points of elliptic–elliptic type for all $t \in [0, 1]$.

Second, consider $Q_t^{\min}$ and $Q_t^{\max}$: Here we work in the chart (U_5, ψ_5) . Keeping $x_5^2 + y_5^2 = |z_5|^2$ and $x_6^2 + y_6^2 = |z_6|^2$ in mind, we calculate

$$\begin{aligned}(H_t \circ \psi_5^{-1})(x_5, y_5, x_6, y_6) &= \frac{1-2t}{2}(4 + |z_5|^2 - |z_6|^2) \\ &\quad + \gamma t x_6 \sqrt{(2 - |z_5|^2 + |z_6|^2)(6 - 2|z_5|^2 + |z_6|^2)(8 - |z_6|^2)(4 + |z_5|^2 - |z_6|^2)(2 + 2|z_5|^2 - |z_6|^2)}\end{aligned}$$

and the Hessian of $(H_t \circ \psi_5^{-1})$ in the point $(0, 0, u, 0) \in \psi_5(U_5)$ with $u := u_\pm(t)$ is given by

$$d^2(H_t \circ \psi_5^{-1})(0, 0, u, 0) = \begin{pmatrix} \tilde{a}(u) & 0 & 0 & 0 \\ 0 & \tilde{a}(u) & 0 & 0 \\ 0 & 0 & \tilde{b}(u) & 0 \\ 0 & 0 & 0 & \tilde{c}(u) \end{pmatrix}$$

where

$$\tilde{a}(u) := 1 - 2t + \frac{2\gamma t u(160 + 300u^2 - 64u^4 - 21u^6 + 3u^8)}{\sqrt{768 - 160u^2 - 216u^4 + 44u^6 + 6u^8 - u^{10}}}$$

$$\tilde{b}(u) := -1 + 2t + \frac{2\gamma tu g(u)}{(768 - 160u^2 - 216u^4 + 44u^6 + 6u^8 - u^{10})^{3/2}}$$

with $g(u)$ the same function as in (18) and

$$\tilde{c}(u) := -1 + 2t + \frac{\gamma tu(-160 - 432u^2 + 132u^4 + 24u^6 - 5u^8)}{\sqrt{768 - 160u^2 - 216u^4 + 44u^6 + 6u^8 - u^{10}}}.$$

Note that $\tilde{a}(u)$ coincides with $a(u)$ except for the minus sign at the term $(1 - 2t)$ right at the beginning. The same holds true for $\tilde{b}(u)$ and $b(u)$ and $\tilde{c}(u)$ and $c(u)$. Thus, we obtain almost the same matrix up to these terms. Now use (11) instead of (10), where this opposite sign also appears. Similar calculations as above show that Q_t^{min} and Q_t^{max} are also nondegenerate singular points of elliptic–elliptic type. $\square$

7 The Singular Points of Rank One and Their Types

For $t = 0$, the system $F_0 = (J, H_0) = (J, H)$ is toric (see Theorem 3.6) so that its rank one points are elliptic–regular and mapped to the edges of the momentum polytope $F_0(M) = \Delta$. In what follows, we will study the case $t > 0$: we will first investigate the case $dJ(p) \neq 0$ and then the case $dJ(p) = 0$.

7.1 Case $dJ(p) \neq 0$

Geometrically this means that, at such a rank one point p , either $dH_t(p) = 0$ or $dH_t(p)$ and $dJ(p)$ are linearly dependent.

Lemma 7.1 *Let $t \in]0, 1]$ and let $p = [p_1, p_2, \dots, p_8]$ be a rank one point of $F_t = (J, H_t)$ with $dJ(p) \neq 0$. Then, $p_2, p_3, p_4, p_6, p_7, p_8$ are all nonzero.*

Proof Let p be a rank one point of $F_t = (J, H_t)$. Then, there exists $(\nu, \mu) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ with $\nu dH_t(p) + \mu dJ(p) = 0$. We conclude that $\nu \neq 0$ since $\nu = 0$ would force also $\mu = 0$ because of $dJ(p) \neq 0$, but $(\mu, \nu) = (0, 0)$ is excluded. Thus, we may divide by ν and rewrite the condition to be a rank one point as $dH_t(p) - \lambda dJ(p) = 0$ for some $\lambda \in \mathbb{R}$, i.e. we look for the extrema of the function $H_t - \lambda J : M \rightarrow \mathbb{R}$. Now summarise the six manifold equations of M in (4) in a function $K = (K_1, \dots, K_6) : \mathbb{C}^8 \rightarrow \mathbb{R}^6$. Thus, we may see p as critical point of $H_t - \lambda J$ in $\mathbb{C}^8$ having to satisfy $K(p) = 0$.

This is an optimisation problem with constraints which can be solved by using Lagrange multipliers. Thus a rank one singular point is a point $p = [p_1, \dots, p_8] \in \mathbb{C}^8$

for which there are $\lambda, \sigma_1, \dots, \sigma_6 \in \mathbb{R}$ such that p satisfies

$$\begin{cases} \nabla H_t(p) = \lambda \nabla J(p) + \sigma_1 \nabla K_1(p) + \dots + \sigma_6 \nabla K_6(p) \\ |p_1|^2 + |p_5|^2 = 6 \\ |p_2|^2 + |p_5|^2 + |p_7|^2 = 10 \\ |p_3|^2 + |p_7|^2 = 6 \\ |p_4|^2 - |p_5|^2 + |p_7|^2 = 4 \\ |p_5|^2 - |p_6|^2 + |p_7|^2 = 2 \\ |p_5|^2 - |p_7|^2 + |p_8|^2 = 4 \end{cases} \quad (19)$$

Write the gradient in $\mathbb{C}^8$ as $(\partial_{z_1}, \partial_{\bar{z}_1}, \dots, \partial_{z_8}, \partial_{\bar{z}_8})$ and rewrite H_t and J as

$$H_t(z_1, \dots, z_8) = \frac{1-2t}{2} z_3 \bar{z}_3 + t \gamma \Re(\bar{z}_2 \bar{z}_3 \bar{z}_4 z_6 z_7 z_8) \quad \text{and} \quad J(z_1, \dots, z_8) = \frac{1}{2} z_1 \bar{z}_1$$

Then, the first equation of (19) yields w.r.t. derivatives $\partial_{z_1}, \dots, \partial_{z_8}$

$$\begin{aligned} 0 &= \frac{\lambda}{2} \bar{p}_1 + \sigma_1 \bar{p}_1 \\ t \gamma p_3 \bar{p}_4 \bar{p}_6 \bar{p}_7 \bar{p}_8 &= \sigma_2 \bar{p}_2 \\ \frac{1-2t}{2} \bar{p}_3 + t \gamma p_2 \bar{p}_4 \bar{p}_6 \bar{p}_7 \bar{p}_8 &= \sigma_3 \bar{p}_3 \\ t \gamma p_2 \bar{p}_3 \bar{p}_6 \bar{p}_7 \bar{p}_8 &= \sigma_4 \bar{p}_4 \\ 0 &= (\sigma_1 + \sigma_2 - \sigma_4 + \sigma_5 + \sigma_6) \bar{p}_5 \\ t \gamma \bar{p}_2 \bar{p}_3 \bar{p}_4 \bar{p}_7 \bar{p}_8 &= -\sigma_5 \bar{p}_6 \\ t \gamma \bar{p}_2 \bar{p}_3 \bar{p}_4 p_6 p_8 &= (\sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 - \sigma_6) \bar{p}_7 \\ t \gamma \bar{p}_2 \bar{p}_3 \bar{p}_4 p_6 p_7 &= \sigma_6 \bar{p}_8 \end{aligned}$$

and we get the same equations with all p_i conjugate for the derivatives $\partial_{\bar{z}_1}, \dots, \partial_{\bar{z}_8}$.

Suppose now that $p_2 = 0$. Since $t \neq 0$ and $\gamma > 0$, it follows from the second equation that at least one of p_3, p_4, p_6, p_7, p_8 must be zero. A similar conclusion is true if we assume one of p_3, p_4, p_6, p_7, p_8 to vanish. Lemma 3.3 implies that never more than two subsequent coordinates p_k for $k \in \{2, 3, 4, 6, 7, 8\}$ are zero. But in this case, we get one of the fixed points A, B, C, D which are rank zero points instead of rank one points with $dJ(p) = 0$. Thus, a singular point p of rank one with $dJ(p) \neq 0$ has entries $p_2, p_3, p_4, p_6, p_7, p_8 \neq 0$. $\square$

Recall from the proof of Proposition 3.7 that the set of points with $dJ = 0$ consists of the fixed points A, B, C, D together with all points z satisfying $z_1 = 0$ or $z_5 = 0$. Thus, if $dJ(z) \neq 0$ for a point $z \in M$, we have $z_1, z_5 \neq 0$. Together with Lemma 7.1, this means that the coordinate entries of a singular point of rank one never vanish.

W.l.o.g. let us work in the chart (U_1, ψ_1) . Given $z \in U_1$ with $dJ(z) \neq 0$ (and thus no entries equal to zero), write it as $z = \psi_1^{-1}(x_1, y_1, x_2, y_2) =$

Fig. 14 Domain of definition of ρ depending on the value of j plotted with *Mathematica* between the maximally admissible bounds $0 \leq j \leq 3$ and $0 \leq \rho \leq \sqrt{8}$ for the constraints given in (20)

$[x_1, y_1, x_2, y_2, x_3, 0, \dots, x_8, 0]$ and rotate $z_1 = x_1 + iy_1$ with the $\mathbb{S}^1$ -action of J until $y_1 = 0$. Abbreviate $J(z) =: j$ and note that we have in this situation $x_1 = \sqrt{2j}$. Now consider the image of z under the quotient map to the reduced space $M^{\text{red},j}$. Note that the possible choices of $z_2 = x_2 + iy_2$ in the chart $z = \psi_1^{-1}(x_1, y_1, x_2, y_2)$ now describe $M^{\text{red},j}$ completely. Since $z_2 \neq 0$, we may write $z_2 = \rho e^{i\vartheta}$ in polar coordinates. We have $\vartheta \in [0, 2\pi[$, but the actual values of $\rho > 0$ are implicitly determined by the manifold equations (4) which read in the new coordinates

$$\left\{ \begin{array}{ll} x_1 = \sqrt{2j}, & x_5 = \sqrt{6 - 2j}, \\ x_2 = \rho \cos(\vartheta), & x_6 = \sqrt{8 - \rho^2}, \\ y_2 = \rho \sin(\vartheta), & x_7 = \sqrt{4 + 2j - \rho^2}, \\ x_3 = \sqrt{2 - 2j + \rho^2}, & x_8 = \sqrt{2 + 4j - \rho^2}, \\ x_4 = \sqrt{6 - 4j + \rho^2}, & y_1 = y_3 = y_4 = y_5 = y_6 = y_7 = y_8 = 0. \end{array} \right. \quad (20)$$

Thus, we have always in particular $0 < \rho < \sqrt{8}$ and, by definition of J , also $0 \leq j \leq 3$. The possible values for ρ depend on the value of j and are plotted in Fig. 14. The function $H_t^{\text{red},j}$ on $M^{\text{red},j}$ becomes in these coordinates

$$H_t^{\text{red},j}(\rho, \vartheta) = \frac{1 - 2t}{2}(2 - 2j + \rho^2) + \gamma t \rho \cos(\vartheta) \sqrt{g(\rho, j)}$$

where

$$g(\rho, j) := (2 - 2j + \rho^2)(6 - 4j + \rho^2)(8 - \rho^2)(4 + 2j - \rho^2)(2 + 4j - \rho^2) = (x_3 x_4 x_6 x_7 x_8)^2.$$

This change of coordinates is of use for

Proposition 7.2 *For $t \in]0, 1]$, the rank one points of $F_t = (J, H_t)$ on $M \setminus \{J^{-1}(0) \cup J^{-1}(3)\}$ are nondegenerate of elliptic-regular type.*

Proof By assumption, dJ does not vanish in the rank one points under consideration. According to Lemma 2.7, the point (ρ, ϑ) is a rank one singular point of F_t if $dH_t^{red,j}(\rho, \vartheta) = 0$, i.e. the following equations hold true:

$$\partial_{\vartheta} H_t^{red,j} = -\gamma t \rho \sin(\vartheta) \sqrt{g(\rho, j)} = 0, \quad (21)$$

$$\partial_{\rho} H_t^{red,j} = (1 - 2t)\rho + \gamma t \cos(\vartheta) \left(\sqrt{g(\rho, j)} + \frac{\rho \partial_{\rho} g(\rho, j)}{2\sqrt{g(\rho, j)}} \right) = 0. \quad (22)$$

First consider (21). We have $\gamma, t, \rho > 0$ and, furthermore, by Lemma 7.1, $\sqrt{g(\rho, j)} = x_3 x_4 x_6 x_7 x_8 \neq 0$. Thus the first equation holds true for $\sin \vartheta = 0$, i.e. $\vartheta \in \{0, \pi\} \subset [0, 2\pi[$. Now abbreviate

$$h(\rho, j) := \sqrt{g(\rho, j)} + \frac{\rho \partial_{\rho} g(\rho, j)}{2\sqrt{g(\rho, j)}}$$

and calculate the entries of the Hessian $d^2 H_t^{j,red}$ of $H_t^{j,red}$ as

$$\begin{aligned} \partial_{\rho\vartheta}^2 H_t^{red,j} &= -\gamma t \sin(\vartheta) h(\rho, j), \\ \partial_{\vartheta\vartheta}^2 H_t^{red,j} &= -\gamma t \rho \cos(\vartheta) \sqrt{g(\rho, j)}, \\ \partial_{\rho\rho}^2 H_t^{red,j} &= 1 - 2t + \gamma t \cos(\vartheta) \partial_{\rho} h(\rho, j). \end{aligned}$$

For $\vartheta \in \{0, \pi\}$, we find $\partial_{\rho\vartheta}^2 H_t^{red,j} = 0$ and $\partial_{\vartheta\vartheta}^2 H_t^{red,j} < 0$ for $\vartheta = 0$ and $\partial_{\vartheta\vartheta}^2 H_t^{red,j} > 0$ for $\vartheta = \pi$. To determine the sign of $\partial_{\rho\rho}^2 H_t^{red,j}$, we rewrite equation (22) as

$$1 - 2t = \frac{-\gamma t \cos(\vartheta) (2g(\rho, j) + \rho \partial_{\rho} g(\rho, j))}{2\rho \sqrt{g(\rho, j)}}$$

and calculate

$$\partial_{\rho} h(\rho, j) = \frac{2g(\rho, j) (\rho \partial_{\rho\rho}^2 g(\rho, j) + 2\partial_{\rho} g(\rho, j)) - \rho \partial_{\rho} g(\rho, j)^2}{4g(\rho, j)^{\frac{3}{2}}}.$$

Fig. 15 Left: Plot of $f(\rho, j)$ over the region of admissible (ρ, j) from Fig. 14. The graph is approaching the horizontal plane of height zero from below. Center and Right: The horizontal plane of height zero plotted in blue and $f(\rho, j)$ plotted in orange for the range between -200000 and 0 . The ‘orange peaks’ rise at the boundary of the admissible region. On the boundary itself, the function may vanish (e.g. $f(0, 1) = 0$). All three plots are realised with *Mathematica*

Combining those results yields

$$\partial_{\rho\rho}^2 H_t^{red,j} = \frac{\gamma t \cos(\vartheta) f(\rho, j)}{4\rho g(\rho, j)^{\frac{3}{2}}}$$

with

$$f(\rho, j) := 2\rho^2 g(\rho, j) \partial_{\rho\rho}^2 g(\rho, j) + 2\rho g(\rho, j) \partial_{\rho} g(\rho, j) - \rho^2 (\partial_{\rho} g(\rho, j))^2 - 4g(\rho, j)^2.$$

As Fig. 15 shows, we have $f(\rho, j) < 0$ for all admissible (j, ρ) in the interior of the domain plotted in Fig. 14.

Hence, we conclude $\partial_{\rho\rho}^2 H_t^{red,j} < 0$ for $\vartheta = 0$ and $\partial_{\rho\rho}^2 H_t^{red,j} > 0$ for $\vartheta = \pi$.

Putting all results together, we see that the Hessian $d^2 H_t^{j,red}(\rho, \vartheta)$ has strictly positive determinant in both cases. Therefore by Lemma 2.7, the point (ρ, ϑ) is a nondegenerate rank one point of F_t of elliptic–regular type. $\square$

7.2 Case $dJ(p) = 0$

Geometrically, this means that at such a rank one point p , we have $dH_t(p) \neq 0$, i.e. we are dealing with points that are fixed under the flow of J , but are non-fixed by the flow of H_t . Recall from the proof of Proposition 3.7 that the set of points with $dJ = 0$ consists of the fixed points A, B, C, D together with all points z satisfying $z_1 = 0$ or $z_5 = 0$.

Proposition 7.3 *For $t \in]0, 1]$, the rank one points of $F_t = (J, H_t)$ in $J^{-1}(0) \cup J^{-1}(3)$ are nondegenerate of elliptic–regular type.*

Proof Consider first $J^{-1}(0)$: Here a rank one point p satisfies $0 = p_1$ and hence $p \in U_1$ or $p \in U_8$. Let us first assume $p \in U_1$ and write $p = \psi_1^{-1}(0, 0, x_2, y_2)$. According to the discussion around Definition 2.4, we need to consider the Hessian of J to determine nondegeneracy and type of rank one points. In the chart (U_1, ψ_1)

evaluated in $(0, 0, x_2, y_2)$, we have

$$d^2(J \circ \psi_1^{-1})(0, 0, x_2, y_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (\omega_1^{-1})_{(0,0,x_2,y_2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and thus

$$(\omega_1^{-1})_{(0,0,x_2,y_2)} d^2(J \circ \psi_1^{-1})(0, 0, x_2, y_2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now we determine the space $L_p \subseteq T_p M$ given by the tangent line of the orbit through p . Since $dJ(p) = 0$, we have $L_p = \text{Span}\{\mathcal{X}^{H_t}(p)\}$. Similar to in the proof of Proposition 4.6 we obtain $d(H_t \circ \psi_1^{-1})(0, 0, x_2, y_2) = (0, 0, f_1, f_2)$ for some functions f_1 and f_2 depending on x_2, y_2 . We calculate the Hamiltonian vector field

$$(\mathcal{X}^{H_t \circ \psi_1^{-1}})^T = -d(H_t \circ \psi_1^{-1})\omega_1^{-1} = (0, 0, f_2, -f_1).$$

Therefore, we get

$$L_p = \text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ f_2 \\ -f_1 \end{pmatrix} \right\} \quad \text{and} \quad L_p^\perp = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ f_2 \\ -f_1 \end{pmatrix} \right\}$$

so that

$$L_p^\perp / L_p \simeq \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Therefore $\omega_1^{-1} d^2(J \circ \psi_1^{-1})$ descends to $L_p^\perp / L_p$ as $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which has purely imaginary eigenvalues $\pm i$. Thus, by Definition 2.4 and the discussion afterwards, the rank one points are nondegenerate of elliptic-regular type.

The only rank one points in $J^{-1}(0)$ that lie in U_8 but not in U_1 are points z with $z_1 = 0 = z_8$. When we insert this into the manifold equations (4), we get immediately $|z_1|^2 = 0 = |z_8|^2$ and then find successively $|z_5|^2 = 6$, $|z_7|^2 = 2$, $|z_3|^2 = 4$, $|z_4|^2 = 8$, $|z_6|^2 = 6$ and eventually $|z_2|^2 = 2$. This determines a unique point $p := [0, \sqrt{2}, 2, 2\sqrt{2}, \sqrt{6}, \sqrt{6}, \sqrt{2}, 0] \in U_8$. Similar calculations as above now

in the chart (U_8, ψ_8) for $p = \psi_8^{-1}(0, 0, 0, 0)$ lead to

$$(\omega_8^{-1})_{(0,0,0,0)} d^2(J \circ \psi_8^{-1})(0, 0, 0, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and $\mathcal{X}^{H_t \circ \psi_8^{-1}}(0, 0, 0, 0) = (f_2, -f_1, 0, 0)^T$ for some functions f_1 and f_2 . Then, $(\omega_8^{-1})_{(0,0,0,0)} d^2(J \circ \psi_8^{-1})(0, 0, 0, 0)$ descends to the associated quotient $L_p^\perp/L_p$ as $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus, p also is of elliptic-regular type.

Now consider $J^{-1}(3)$: Here a rank one point p satisfies $0 = p_5$. We work in the chart (U_5, ψ_5) where $p = \psi_5^{-1}(0, 0, x_6, y_6)$. Using the relation $|z_1|^2 + |z_5|^2 = 6$ we get

$$(J \circ \psi_5^{-1})(x_5, y_5, x_6, y_6) = \frac{6 - x_5^2 - y_5^2}{2} \quad \text{and} \quad d^2(J \circ \psi_5^{-1})(0, 0, x_6, y_6) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and hence

$$(\omega_5^{-1})_{(0,0,x_6,y_6)} d^2(J \circ \psi_5^{-1})(0, 0, x_6, y_6) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Analogously to above, we find $d(H_t \circ \psi_5^{-1}) = (0, 0, f_1, f_2)$ where f_1, f_2 are some suitable functions. Moreover, $(\omega_5^{-1})_{(0,0,x_6,y_6)} d^2(J \circ \psi_5^{-1})(0, 0, x_6, y_6)$ descends to the associated $L_p^\perp/L_p$ as $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This matrix has eigenvalues $\pm i$, so the rank one points in $J^{-1}(3)$ in U_5 are also nondegenerate and of elliptic-regular type.

The chart (U_5, ψ_5) contains all possible rank one points in $J^{-1}(3)$ except those with $0 = p_4$ for which we have to use the chart (U_4, ψ_4) . Analogous calculations as above show that there is only one such point and that it is of elliptic-regular type. $\square$

8 The Proofs of Theorem 4.7 and Proposition 4.9

Proof of Theorem 4.7 In Theorem 3.6, we showed that (M, ω) is a 4-dimensional compact symplectic manifold and that $F = (J, H)$ is, up to equivariant symplectomorphism, the toric system satisfying $F(M) = \Delta$ where Δ is the octagon from Fig. 1. In Proposition 4.6, we extended $(M, \omega, F = (J, H))$ to a family $(M, \omega, F_t = (J, H_t))$ of integrable systems with $F_0 = F$. Since J does not depend on t , its induced $\mathbb{S}^1$ -action is unchanged under variation of t . Moreover, since M is compact, $J : M \rightarrow \mathbb{R}$ is proper.

It remains to show that apart from a finite number of transition times, the system $F_t = (J, H_t)$ is semitoric, i.e. singularities are nondegenerate and have no hyperbolic components and that, at the transition times, certain singular points change from elliptic–elliptic to focus–focus or vice versa.

In Sect. 5, we deduced the eight fixed points of $F_t = (J, H_t)$ for $t \in]0, 1]$ denoted by $A, B, C, D, P_t^{\min}, P_t^{\max}, Q_t^{\min}, Q_t^{\max}$ and their coordinates, hereby proving Proposition 4.8. In Proposition 6.1, we showed that there are two transition times $0 < t^- < \frac{1}{2} < t^+ < 1$ where A, B, C, D switch from being nondegenerate and elliptic–elliptic via a degeneracy at t^- to being nondegenerate and focus–focus and then via a degeneracy at t^+ to being nondegenerate and elliptic–elliptic. In Proposition 6.2, we showed that $P_t^{\min}, P_t^{\max}, Q_t^{\min}, Q_t^{\max}$ are nondegenerate and elliptic–elliptic for all $t \in]0, 1]$.

Proposition 7.2 proves that the rank one points of $F_t = (J, H_t)$ on $M \setminus (J^{-1}(0) \cup J^{-1}(3))$ are nondegenerate and of elliptic–regular type for all $t \in]0, 1]$. Proposition 7.3 shows that the rank one points in $J^{-1}(0)$ and $J^{-1}(3)$ are nondegenerate and of elliptic–regular type for all $t \in]0, 1]$. This finishes the proof of Theorem 4.7. $\square$

It remains to prove Proposition 4.9, namely that, at $t = \frac{1}{2}$, the focus–focus points A and B lie both in $F_{\frac{1}{2}}^{-1}(1, 0)$ and that the focus–focus points C and D lie both in $F_{\frac{1}{2}}^{-1}(2, 0)$ and that this gives the fibres the form of a double pinched torus.

Proof of Proposition 4.9 Zung (1996, 1997) showed that a focus–focus fibre containing exactly n focus–focus points consists of a chain of n spheres where each of the spheres intersects transversally two others and that the intersection points are given by the n focus–focus points. Such a fibre is said to have the form of an n -pinched torus. In the special case $n = 1$, the fibre is a sphere with one point of self-intersection and is said to be a single pinched torus.

The exact coordinates of the fixed points A, B, C, D are calculated in Proposition 4.8 and Proposition 6.1 proves that they are of focus–focus type for $t^- < t < t^+$ where $0 < t^- < \frac{1}{2} < t^+ < 1$. We find $F_{\frac{1}{2}} = (J, \frac{\gamma}{2}X)$ and calculate $F_{\frac{1}{2}}(A) = (1, 0) = F_{\frac{1}{2}}(B)$ and $F_{\frac{1}{2}}(C) = (2, 0) = F_{\frac{1}{2}}(D)$. Thus, according to Zung (1996, 1997), the fibres over $(1, 0)$ and $(2, 0)$ have the form of double pinched tori.

We now parametrise the fibre $F_{\frac{1}{2}}^{-1}(1, 0)$ as follows. Given any $z = [z_1, \dots, z_8] \in F_{\frac{1}{2}}^{-1}(1, 0) \subset M$, we have $J(z) = 1$ and thus $|z_1|^2 = 2$. Moreover, we find $0 = X(z) = \Re(\overline{z_2} \overline{z_3} \overline{z_4} z_6 z_7 z_8) = 0$. Since the last two coordinate entries of A are zero, A lies in the chart (U_7, ψ_7) . Thus, let us determine the points of $F_{\frac{1}{2}}^{-1}(1, 0)$ that lie in U_7 . Recall

$$\psi_7^{-1} : \mathbb{R}^4 \rightarrow U_7, \quad \psi_7^{-1}(x_7, y_7, x_8, y_8) = [x_1, 0, x_2, 0, \dots, x_6, 0, x_7, y_7, x_8, y_8]$$

where

$$\begin{aligned}x_1 &= \sqrt{2 - |z_7|^2 + |z_8|^2}, & x_3 &= \sqrt{6 - |z_7|^2}, & x_5 &= \sqrt{4 + |z_7|^2 - |z_8|^2}, \\x_2 &= \sqrt{6 - 2|z_7|^2 + |z_8|^2}, & x_4 &= \sqrt{8 - |z_8|^2}, & x_6 &= \sqrt{2 + 2|z_7|^2 - |z_8|^2}.\end{aligned}$$

Thus, we may consider $z_1, \dots, z_6$ as lying in $\mathbb{R}^+$. Consider $z_7 = x_7 + iy_7$ and $z_8 = x_8 + iy_8$. From $|z_1|^2 = 2$ we conclude $z_1 = \sqrt{2}$ and thus $|z_7|^2 = |z_8|^2$. The equation $\Re(\bar{z}_2 \bar{z}_3 \bar{z}_4 z_6 z_7 z_8) = 0$ becomes $z_2 z_3 z_4 z_6 (x_7 x_8 - y_7 y_8) = 0$ and since $z_1, \dots, z_6 \neq 0$ in U_7 , we deduce $x_7 x_8 = y_7 y_8$. Keeping the above deductions in mind, we calculate

$$x_7^2 |z_8|^2 = x_7^2 x_8^2 + x_7^2 y_8^2 = y_7^2 y_8^2 + x_7^2 y_8^2 = y_8^2 |z_7|^2 = y_8^2 |z_8|^2.$$

If $|z_8|^2 \neq 0$, this implies $x_7 = \pm y_8$ and thus $y_7 = \pm x_8$, where we need to choose the same sign to satisfy $x_7 x_8 = y_7 y_8$. Hence, we get two possible values for z_7 , namely

$$z_7 = y_8 + ix_8 = i \bar{z}_8 \quad \text{or} \quad z_7 = -y_8 - ix_8 = -i \bar{z}_8.$$

If $|z_8|^2 = 0$, then $z_8 = 0$ and by $|z_7|^2 = |z_8|^2$ also $z_7 = 0$. Then $z_7 = \pm i \bar{z}_8$ still holds, but does no longer give two different values. Moreover, the manifold equations (4) imply $|z_8|^2 = |z_7|^2 \leq 6$. Note that the inequality $|z_8|^2 < 6$ is strict for points in U_7 , since otherwise $z_2 = 0$ or $z_3 = 0$.

It remains to consider the points that do not lie in U_7 , i.e. at least one of the values $z_1, \dots, z_6$ is zero. We check the following cases:

- $z_1 = 0$ is not possible due to $J(z) = \frac{1}{2}|z_1|^2 = 1$.
- If $z_2 = 0$ we still have $z_1 \neq 0$, but z lies in U_2 and thus we have $z_1 = \sqrt{2}$. We use the manifold equations (4) to find the other coordinates of the point and get

$$z = [\sqrt{2}, 0, 0, \sqrt{2}, 2, 2\sqrt{2}, \sqrt{6}, \sqrt{6}]$$

which are the coordinates of the focus–focus point B .

- If $z_3 = 0$, then $|z_1|^2 = 2$ and the manifold equations imply $|z_5|^2 = 4$, $|z_7|^2 = 6$ and eventually $z_2 = 0$, so we recover the focus–focus point B again.
- If $z_4 = 0$ we calculate $|z_1|^2 = 2$, $|z_5|^2 = 4$, $|z_7|^2 = 8$ which leads to the contradiction $|z_3|^2 = -2$, so $z_4 \neq 0$ must be true.
- If $z_5 = 0$ then $|z_1|^2 = 6$ which contradicts $J(z) = \frac{1}{2}|z_1|^2 = 1$, so $z_5 \neq 0$ must be true.
- If $z_6 = 0$ then the manifold equations (4) imply $|z_1|^2 = 2$, $|z_5|^2 = 4$ which lead to the contradiction $|z_7|^2 = -2$, so $z_6 \neq 0$ must be true.

Therefore the only point in $F_{\frac{1}{2}}^{-1}(1, 0)$ which does not lie in U_7 is the focus–focus point B . This point can be included in the previous description of the fibre when we allow $|z_8| = \sqrt{6}$. Indeed, both the positive and negative choice of the sign of z_7 results in the point B . Altogether we conclude that the fibre $F_{\frac{1}{2}}^{-1}(1, 0)$ is of the form

$$\left\{ \left[\sqrt{2}, \sqrt{6 - |z_8|^2}, \sqrt{6 - |z_8|^2}, \sqrt{8 - |z_8|^2}, 2, \sqrt{2 + |z_8|^2}, \pm i \overline{z_8}, z_8 \right] \in M \mid |z_8| \in [0, \sqrt{6}] \right\}.$$

Concerning the reformulation in polar coordinates, we remark that $r = 0$ recovers the point A and $r = \sqrt{6}$ gives the point B . For all other values of r , the fibre splits up in two parts, one for the plus and one for the minus sign in the seventh coordinate. The rotation related to the $\mathbb{S}^1$ -action induced by J is described by the parameter ϑ , so indeed, we get a 2-torus pinched at the points A and B , as visualised in Fig. 3. $\square$

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