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Two-step Shapley-solidarity value for cooperative games with coalition structure

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Abstract

This paper proposes an alternative for the two-step Shapley value for cooperative games with coalition structure that has earlier been proposed by Kamijo. The value is based on the idea that within a union of players, worth should be distributed based on the solidarity principle. Specifically, we propose a two-step Shapley-solidarity value, in which the surplus of a union's Shapley value in the quotient game is distributed equally among the union's members, and players obtain the solidarity value of the respective subgame within their union. We provide an intuitive procedural characterization for this value and give three axiomatizations to pinpoint the differences to comparable values.

Keywords Cooperative games · Coalition structure · Solidarity · Procedural characterization · Axiomatization

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1 Introduction

A transferable utility cooperative game (TU game) formulates a situation in which a finite set of n players can cooperate with each other and generate joint revenues. The cooperative behavior among players is captured by the assumption that coalitions $S \subseteq \{1, \dots, n\}$ of players can form, and the worth of a coalition S represents the revenue that can be obtained by that coalition. Assuming that the grand coalition $N = \{1, \dots, n\}$ is being formed, one of the major questions in cooperative game theory is how the worth of the grand coalition should be distributed among the players. An answer to this question is a mapping of any given game to a payoff vector $(\psi_1, \dots, \psi_n)$, which is called a *value* of a cooperative game. Arguably the most well-known value in cooperative game theory is the Shapley value (Shapley 1953b), which offers each player her expected marginal contribution when assuming all possible $n!$ orders of the n players happen with the same probability.

It should be clear that the assumption of free cooperation between any subsets of players is sometimes not realistic. One concept to capture such situations is that players are partitioned into subgroups $C_1, \dots, C_m$, for example due to physical characteristics such as geographic location, or because players actively organize themselves into subgroups in order to improve their bargaining position, such as cartels and syndicates. Such pre-defined subgroups then mean that the cooperation among players can happen within a given subgroup C_k, while cooperation outside the subgroups happens on the level of the subgroups themselves. The partition is usually called a *coalition structure*, and the subgroups C_k are referred to as *unions*.

TU games with coalition structure were first considered by Aumann and Dreze (1974). They assume the grand coalition is divided into disjoint and independent unions, and there is no side payment between unions. In the *Aumann-Drèze* (AD) value (Aumann and Dreze 1974), every player receives the payoff allocated to her by the Shapley value in the subgame she is playing within her union. The possibility of cooperation between unions begins with Owen (1977), who interpreted the unions as "bargaining blocks". Owen assumes that the coalition of all players is being formed, and hence, the worth of the grand coalition is distributed. The Owen value (Owen 1977) is defined by taking two levels of interaction among players into account, first among unions and then within each union: First the unions get the Shapley value of the game played in the so-called quotient game, which is the game where the unions are the players. Then, to distribute the each union's Shapley payoff over its players, Owen defines an *induced internal game* in which he considers the worth of a coalition $S \subseteq C_k$ to be the union's Shapley value of the quotient game where the union C_k is replaced with S. The payoff is then again distributed using the Shapley value. Following the Owen procedure, several other values have been extended to TU games with coalition structure, including the Banzhaf value (Banzhaf 1965), the τ-value (Tijs 1981), the equal division surplus value (Driessen and Funaki 1991), etc. We refer to Owen (1982), Casas-Méndez et al. (2003) and Alonso-Meijide et al. (2020) for these.

In this paper, along the lines of the previously mentioned papers, we suggest a new value for cooperative games with coalition structure. This value is closely related to another value for cooperative games with coalition structure that has been suggested by Kamijo (2009), the so-called two-step Shapley value. That value exhibits a certain conceptual simplicity, e.g., when compared to Owen's value: In the first step, all players of a union equally share the *Shapley net surplus* of the union containing them, i.e., there is an equal distribution of the difference between the Shapley value obtained by this union in the quotient game, and the worth of it. That means the union is left with its worth, which is again distributed using the Shapley value. However, we believe that it lies in the nature of games with coalition structure that players within one union should exhibit a higher degree of solidarity, which is not captured by using the Shapley value for the game within unions. In the following, we elaborate a bit more on Kamijo's value, as well as other closely related values from the literature, in order to motivate our proposal.

Kamijo's two-step Shapley value actually establishes an "interpolation" between the approaches suggested by Owen and Aumann and Drèze. On the one hand, it affirms Owen's assumption that the grand coalition is being formed, and unions first play the quotient game to distribute the worth of the grand coalition among them. This is the same as in the first step of Owen's approach. On the other hand, it also retains the idea of separation between unions, because cooperation of the players within a union is simply modelled by the corresponding subgame restricted to the players of a union, as by Aumann and Drèze. In the second step, every player is just assigned the Shapley value of the respective subgame, and the sum of the two parts gives the two-step Shapley value (Kamijo 2009). With a change to the weighted Shapley value (Shapley 1953a) in the first step, this was generalized even further to the so-called collective value (Kamijo 2013).

Observe that *solidarity* among players *within* one union is embedded in Kamijo's approach only from the perspective of interaction among unions, as the union's Shapley surplus is equally divided among its members in the first step. However, solidarity is not really reflected by using the Shapley value for the games *within* the unions, because the Shapley value is known to be a purely performance-based value, which has been formalized elegantly by an axiomatization based on marginality as given by Young (1985). Here, marginality refers to the fact that a player's payoff *only* depends on her own marginal contribution.

Almost parallel to our work, Hu (2020) also suggested to address this issue, i.e., incorporating a higher degree of solidarity among the players within a union, by using the equal division value for the subgames per union, which then gives rise to the so-called weighted Shapley-egalitarian value (Hu 2020). The equal division value, however, is totally independent of players' performance. Another value that suggests itself in this context is the solidarity value by Sprumont (1990) and Nowak and Radzik (1994). Unlike equal division, the solidarity value takes individuals' differences into consideration, yet implements the solidarity principle as well: It employs the *average* marginal contribution instead of marginal contribution as in the Shapley value, and in this way, the value is equipped with the feature of solidarity by providing support to "weaker" players: a player who contributes less than the average marginal contribution is supported by stronger partners.

That said, it should be mentioned that the solidarity value has previously been adopted into games with coalition structure by Calvo and Gutiérrez (2013). They also use the Shapley value for the quotient game of all unions, but following Owen's procedure, consider the induced internal games to distribute the unions' Shapley payoffs among the players, and this is based on the solidarity value. The resulting value is the Shapley-solidarity value.

In this paper, inspired by the conceptual simplicity of Kamijo's two-step approach, and the idea to incorporate a realistic level of solidarity among the players of a given union, we suggest to marry Kamijo's two-step approach with the idea to use the solidarity value for the game played by players within any given union. Note that this is different from Owen's value as well as Calvo and Gutiérrez's Shapley-solidarity value, as we follow Kamijo's approach and only distribute the *net surplus* of the unions' Shapley payoffs in the first step. Hence, for the second step, there is no need to revert to Owen's induced internal game, and we simply use the solidarity value to distribute the union's worth among its players. Arguably, this is conceptually simpler.

In lack of a better name and to avoid confusion with the values proposed earlier, we call this new value the *two-step Shapley-solidarity* value. Our main contributions are an intuitive procedural interpretation for this new value, and to give three axiomatizations that highlight the differences between this value and specifically the two-step Shapley value. As to the technical contribution of the paper, in order to get our axiomatizations done, we use a property that we call coalitional A-null player property, and moreover, we have to revert to a new basis of the space of all games.

The rest of the paper is organized as follows. Section 2 contains some preliminaries, fixes the notation used, and gives a quick recap of the most relevant values that have been proposed earlier. After introducing the two-step Shapley-solidarity value in Sect. 3, we provide a procedural interpretation for it in Sect. 4. Finally, three axiomatic characterizations are presented in Sect. 5. Section 6 gives some final conclusions.

2 Definitions and notations

2.1 TU games and values

A cooperative game with transferable utility (or TU game) is a pair (N, v) consisting of a nonempty and finite set of players N and the characteristic function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. An element $i \in N$ and a subset S of N are called a player and a coalition, respectively. Especially, we call N the grand coalition. With some abuse of notation, we omit the braces for singletons. Thus, we write $S \cup i$ for $S \cup \{i\}$, $S \backslash i$ for $S \backslash \{i\}$, etc. For each $S \subseteq N$, $v(S)$ is the worth of coalition S. The cardinality of S is denoted by the corresponding lower case letter s or $|S|$.

Let $\mathcal{G}^N$ be the family of all TU games over N. For any two TU games $(N, v), (N, w) \in \mathcal{G}^N$, $\alpha \in \mathbb{R}$, the characteristic functions of TU games $(N, v + w)$ and $(N, \alpha v) \in \mathcal{G}^N$ are respectively given by $(v + w)(S) = v(S) + w(S)$ and $(\alpha v)(S) = \alpha v(S)$ for all $S \subseteq N$. For $T \in 2^N \backslash \emptyset$, the subgame of (N, v) when the player set is restricted to T is denoted by TU game $(T, v_{|T}) \in \mathcal{G}^T$, where $v_{|T}(S) = v(S)$ for all $S \subseteq T$. The TU game $(N, u_T) \in \mathcal{G}^N$, where $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise, is called a **unanimity game**. Any TU game can be uniquely represented by unanimity games,

$$v = \sum_{T \subseteq N, T \neq \emptyset} c_T u_T, \tag{1}$$

where $c_T = \sum_{S \subseteq T} (-1)^{t-s} v(S)$ is called the Harsanyi dividend (Harsanyi 1963).

A player $i \in N$ is a **null player** in $(N, v) \in \mathcal{G}^N$ if $v(S) = v(S \cup i)$ for all $S \subseteq N \backslash i$ and a **dummy player** if $v(S \cup i) = v(S) + v(i)$ for all $S \subseteq N \setminus i$. A player $i \in N$ is an **A-null player** in $(N, v) \in \mathcal{G}^N$ if the average marginal contribution of the singleton players in any coalition $S \ni i$ is zero, that is, $\frac{1}{s} \sum_{j \in S} (v(S) - v(S \backslash j)) = 0$ for all $S \subseteq N$ with $i \in S$. Replacing individual marginal contributions by average marginal contributions can be seen as a means of solidarity among the players in a coalition. An A-null player does not contribute anything to any coalition in this average sense. Two players $i, j \in N$ are **symmetric** in $(N, v) \in \mathcal{G}^N$ if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \backslash \{i, j\}$. A permutation on N is a mapping $\pi : N \to N$ that associates every player i with a position $\pi(i)$. Let Π_N denote the collection of all $n!$ permutations on N. Given $\pi \in \Pi_N$, the set of players that are in positions before player i, called the predecessors of i, is denoted by $P^\pi(N, i) = \{j \in N \mid \pi(j) \leq \pi(i)\}$.

A value on $\mathcal{G}^N$ is an operator that assigns a payoff vector $\varphi(N, v) = (\varphi_i(N, v))_{i \in N} \in \mathbb{R}^n$ to every TU game $(N, v) \in \mathcal{G}^N$. The Shapley value (Shapley 1953b) is probably the best known value. For any $(N, v) \in \mathcal{G}^N$, it is given by

$$Sh_i(N, v) = \sum_{S \subseteq N \backslash i} \frac{s!(n - s - 1)!}{n!} [v(S \cup \{i\}) - v(S)], \ i \in N,$$

while the solidarity value (Sprumont 1990) is given by

$$Sol_i(N, v) = \sum_{S \ni i} \frac{(s - 1)!(n - s)!}{n!} \left[\frac{1}{s} \sum_{j \in S} (v(S) - v(S \setminus j)) \right], \ i \in N.$$

In order to describe values of games by their characteristic properties, consider the following axioms of a value φ on $\mathcal{G}^N$:

- Efficiency (**E**): For all $(N, v) \in \mathcal{G}^N$, $\sum_{i \in N} \varphi_i(N, v) = v(N)$.
- Symmetry (**S**): For all $(N, v) \in \mathcal{G}^N$ and $\{i, j\} \subseteq N$, if i, j are symmetric, then $\varphi_i(N, v) = \varphi_j(N, v)$.
- Additivity (**A**): For all $(N, v), (N, w) \in \mathcal{G}^N$, $\varphi_i(N, v + w) = \varphi_i(N, v) + \varphi_i(N, w)$.
- Null player axiom (**NP**): For all $(N, v) \in \mathcal{G}^N$ and $i \in N$, if i is a null player in (N, v), then $\varphi_i(N, v) = 0$.
- A-Null player axiom (**ANP**): For all $(N, v) \in \mathcal{G}^N$ and $i \in N$, if i is an A-null player in (N, v), then $\varphi_i(N, v) = 0$.

While the Shapley value is characterized by **E**, **A**, **S** and **NP**, the characterization of the solidarity value can be obtained by replacing **NP** with **ANP**.

Theorem 1 (Shapley 1953b) *A value φ on $\mathcal{G}^N$ satisfies efficiency, additivity, symmetry and null player axiom if and only if $\varphi(N, v) = Sh(N, v)$ for each $(N, v) \in \mathcal{G}^N$.*

Theorem 2 (*Nowak and Radzik* 1994) *A value φ on $\mathcal{G}^N$ satisfies efficiency, additivity, symmetry and A-null player axiom if and only if $\varphi(N, v) = Sol(N, v)$ for each $(N, v) \in \mathcal{G}^N$.*

2.2 TU games with coalition structure and some coalitional values

Given a finite set of players N, a coalition structure over N is a partition of the player set N, i.e., $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ is a coalition structure if $\bigcup_{h \in M} C_h = N$, where $M = \{1, 2, \dots, m\}$, and $C_h \cap C_r = \emptyset$ when $h \neq r$. We call an element $C_h \in \mathcal{C}$ a union. There are two trivial coalition structures, namely $\mathcal{C}^N = \{N\}$ and $\mathcal{C}^n = \{\{i\} \mid i \in N\}$, where only the grand coalition forms in $\mathcal{C}^N$ and each union is a singleton in $\mathcal{C}^n$. Denote by $\mathcal{C}_N$ the set of all possible coalition structures over N. For any $S \subseteq N$, we denote the restriction of $\mathcal{C}$ on the player set S as $\mathcal{C}_{|S}$, i.e., $\mathcal{C}_{|S} = \{C_h \cap S \mid C_h \in \mathcal{C} \text{ and } C_h \cap S \neq \emptyset\}$.

For a given coalition structure $\mathcal{C}$, a permutation $\pi \in \Pi_N$ is consistent with $\mathcal{C}$ if $i \in C_h \in \mathcal{C}$ and $j \in C_h \in \mathcal{C}$ and $k \in N$, $\pi(i) < \pi(k) < \pi(j)$ implies that the player k also belongs to union C_h. The set of all the permutations on N that are consistent with $\mathcal{C}$ is denoted by $\Pi_{N,\mathcal{C}}$.

A cooperative game with a coalition structure is a triple $(N, v, \mathcal{C})$ where (N, v) is a TU game and $\mathcal{C}$ is a coalition structure over N. We denote by $\mathcal{CG}^N$ the collection of all TU games with coalition structure over player set N, and by $\mathcal{CG}$ the collection of all TU games with coalition structure. Given a non-empty coalition S, denote the restriction of $(N, v, \mathcal{C}) \in \mathcal{CG}^N$ to S as the TU game with coalition structure $(S, v_{|S}, \mathcal{C}_{|S})$. Given a TU game with a coalition structure $(N, v, \mathcal{C}) \in \mathcal{CG}^N$, the **quotient game** $(M, v^{\mathcal{C}})$ is defined as

$$v^{\mathcal{C}}(Q) = v(\cup_{h \in Q} C_h),\ Q \subseteq M.$$

We say that $C_h \in \mathcal{C}$ is a null coalition in $(N, v, \mathcal{C})$ if h is a null player in $(M, v^{\mathcal{C}})$, and $C_h, C_r \in \mathcal{C}$ are symmetric coalitions in $(N, v, \mathcal{C})$ if h and r are symmetric players in $(M, v^{\mathcal{C}})$.

A coalitional value is a function $\psi \colon \mathcal{CG}^N \to \mathbb{R}^n$ that assigns to each cooperative game with coalition structure $(N, v, \mathcal{C})$ a payoff vector.

The Shapley-solidarity value (SS value) employs different rules between guiding cooperation among the players within a union and interaction among unions. Firstly, at the union level, all unions play the corresponding quotient game and get their payoffs prescribed by the Shapley value, i.e., for each $(N, v, \mathcal{C})$ and $C_k \in \mathcal{C}$,

$$\sum_{i \in C_k} SS_i(N, v, \mathcal{C}) = Sh_k\left(M, v^{\mathcal{C}}\right).$$

For players within a union $C_k \in \mathcal{C}$, they play an **induced internal game** (C_k, v_{C_k}) in which the worth of a coalition is reassessed. For coalition $S \subseteq C_k$, assume that its complement $\bar{S}$, where $\bar{S} = C_k \setminus S$, leaves the game. The worth of S in the induced internal game is specified by the Shapley value of the quotient game when replacing the union C_k with coalition S. That is

$$v_{C_k}(S) = Sh_k\left(M, (v_{|N \setminus \bar{S}})^{\mathcal{C}_{|N \setminus \bar{S}}}\right).$$

Then, a player $i \in C_k$ gets payoff according to the solidarity value (Calvo and Gutiérrez 2013),

$$SS_i(N, v, \mathcal{C}) = Sol_i(C_k, v_{C_k}).$$

Unlike the two-step procedure above, Kamijo (2009) introduced the two-step Shapley value with a different, arguably simpler, two-step approach. To be more specific, one considers the *surplus* $Sh_k(M, v^{\mathcal{C}}) - v(C_k)$ in the first step, which is equally divided among the players in C_k. Then, there is no need to resort to a new worth to assess a coalition's power when considering the intra-union bargaining. For $C_k \in \mathcal{C}$, one can just consider the corresponding subgame $(C_k, v_{|C_k})$ and Kamijo proposes to use the Shapley value to distribute $v(C_k)$. For each $(N, v, \mathcal{C}) \in \mathcal{CG}^N$ and $i \in C_k$, the two-step Shapley value (Kamijo 2009) is therefore given by

$$TSh_i(N, v, \mathcal{C}) = Sh_i(C_k, v_{|C_k}) + \frac{Sh_k\left(M, v^{\mathcal{C}}\right) - v\left(C_k\right)}{|C_k|}.$$

3 The two-step Shapley-solidarity value

Similar to the two-step Shapley value and the collective value, the two-step Shapley-solidarity value proposed here also distributes the worth of the grand coalition in two steps. Firstly, players within one union act collectively to bargain with other unions, all unions play the quotient game and obtain a payoff prescribed by the Shapley value. The surplus of the difference between the obtained payoff and the worth of the union is distributed equally among union members. Then, players within one union negotiate the worth that they can guarantee on their own, namely the worth of the union they belong to, and they obtain the solidarity value for the subgame restricted on the corresponding union.

Definition 1 For each $(N, v, \mathcal{C}) \in \mathcal{CG}^N$ and $i \in C_k \in \mathcal{C}$, the two-step Shapley-solidarity value is given by

$$TSS_i(N, v, \mathcal{C}) = Sol_i(C_k, v_{|C_k}) + \frac{Sh_k(M, v^{\mathcal{C}}) - v(C_k)}{|C_k|}. \tag{2}$$

Clearly, the so defined two-step Shapley-solidarity value will degenerate to the Shapley value, respectively the solidarity value in the two extreme cases when the coalition structure is either all singleton players, or the grand coalition.

Remark 1 For $(N, v, \mathcal{C}) \in \mathcal{CG}^N$ and if $\mathcal{C} = \mathcal{C}^N$, $TSS(N, v, \mathcal{C}) = Sol(N, v)$, and for $(N, v, \mathcal{C}) \in \mathcal{CG}^N$ and $\mathcal{C} = \mathcal{C}^n$, $TSS(N, v, \mathcal{C}) = Sh(N, v)$.

In that sense, the level of solidarity increases with more players joining unions. The same property is shared by the Shapley-solidarity value of Calvo and Gutiérrez (2013). Except for the equal distribution of the surplus of a union which is not present there, the main difference lies in another intra-union game, i.e., Owen's induced internal game. In this game the unions' internal behavior is actually re-assessed from a "non-solidarity" perspective. Intuitively speaking, a coalition S contained in one union C_k takes into consideration that the remaining members $C_k \setminus S$ might defect. Hence, they re-evaluate their worth to be what they can earn in the quotient game while assuming the remaining members are breaking away from their union.

Compared with the two close relatives, the two-step Shapley value and the Shapley-solidarity value, the two-step Shapley-solidarity value embeds more of a solidarity principle in the intra-union game, as it avoids the possible divergence among union members for the evaluation of their internal cooperation, and as it uses the solidarity value instead of the Shapley value. In this sense, the outcome should reflect a larger level of solidarity within unions. This can also be illustrated with the following, simple example of a four player game.

Example 1 Consider player set $N = \{1, 2, 3, 4\}$ and TU game (N, v) where the characteristic function v is given by $v(\{3\}) = v(\{1, 2, 3\}) = 1$, $v(\{4\}) = v(\{3, 4\}) = v(\{1, 2, 4\}) = v(\{1, 3, 4\}) = v(\{2, 3, 4\}) = v(N) = L$ $(L \in \mathbb{R}_+)$, and $v(S) = 0$ otherwise. Now, players 1, 2 and 3 form the union C_I and player 4 remains alone, which gives rise to the coalition structure $\mathcal{C} = \{C_I = \{1, 2, 3\}, C_{II} = \{4\}\}$.

The above description gives rise to the TU game with coalition structure $(N, v, \mathcal{C})$. Note that players 1 and 2 can be considered the "weak" players for union C_I, because they cannot generate any worth on their own, no matter if they choose to act alone or cooperate. The corresponding quotient game $(M, v^{\mathcal{C}})$ is a two-person TU game where $M = \{I, II\}$. Hence, it is easy to get that

$$Sh_I(M, v^{\mathcal{C}}) = \frac{1}{2},\ Sh_{II}\left(M, v^{\mathcal{C}}\right) = L - \frac{1}{2}.$$

Following Owen's procedure, we get the induced internal game for union C_I, namely (C_I, v_{C_I}), where $v_{C_I}(\{1\}) = v_{C_I}(\{2\}) = -L/2$, $v_{C_I}(\{3\}) = 1/2$, $v_{C_I}(\{1, 2\}) = v_{C_I}(\{1, 3\}) = v_{C_I}(\{2, 3\}) = 0$ and finally, $v_{C_I}(\{1, 2, 3\}) = 1/2$. Obviously, players 1 and 2 are symmetric in (C_I, v_{C_I}). For Kamijo's two-step approach,

Table 1 Three payoff vectors for Example 1

Values	$TSh(N, v, \mathcal{C})$ Kamijo (2009)	$SS(N, v, \mathcal{C})$ Calvo and Gutiérrez (2013)	$TSS(N, v, \mathcal{C})$
Payoffs	$(0, 0, \frac{1}{2}, L - \frac{1}{2})$	$(\frac{1}{8} - \frac{L}{24}, \frac{1}{8} - \frac{L}{24}, \frac{1}{4} + \frac{L}{12}, L - \frac{1}{2})$	$(\frac{1}{12}, \frac{1}{12}, \frac{1}{3}, L - \frac{1}{2})$

players 1, 2 and 3 bargain with their union worth based on the restricted subgame $(C_I, v_{|C_I})$, and the symmetric relationship between player 1 and 2 holds in this subgame as well. Meanwhile, there is no need to consider the intra-bargaining for union C_{II} since it only contains player 4. Then, we can compute the three coalitional values for the TU game with the coalition structure in Example 1 as shown in Table 1.

Player 4 obtains the Shapley value in the quotient game as the final payoff since she forms a union alone, and there exists no difference in her payoff assigned by the three coalitional values. We focus on the payoffs of players in union C_I: The symmetry of players 1 and 2 accounts for their same payoff in all three coalitional values. Hence, the payoff difference between them and player 3 directly reflects the level of solidarity of union C_I. For this example, the difference is 1/4 within players of union C_I for the two-step Shapley-solidarity value, compared to 1/2 for the two-step Shapley value. We see the same effect also when compared to the Shapley-solidarity value, as long as $L > 1$. Moreover, it turns out that the Shapley-solidarity value has a payoff difference of $(L + 1)/8$ within the players of union C_I which grows linearly in L, even though the subgame within union C_I has worths 0 and 1 only.

4 Procedural characterization of the two-step Shapley-solidarity value

Along the lines of Shapley's procedural characterization of the Shapley value via average marginal contributions for all $n!$ permutations of players, we here provide a corresponding characterization of the two-step Shapley-solidarity value. First, with the restriction of coalition structure, it is assumed that the grand coalition forms in a consistent permutation, which indicates that the players within the same union enter the grand coalition consecutively. For each $\pi_c \in \Pi_{N,\mathcal{C}}$ and $i \in C_k$, we denote by $p^{\pi_c}(N, i)$ and $p^{\pi_c}(C_k, i)$ the predecessors of player i with respect to N and C_k respectively, i.e., $p^{\pi_c}(N, i) = \{j \in N \mid \pi_c(j) \leq \pi_c(i)\}$, $p^{\pi_c}(C_k, i) = \{j \in C_k \mid \pi_c(j) \leq \pi_c(i)\}$. The predecessors of a union $C_k \in \mathcal{C}$ is denoted by $p^{\pi_c}(N, C_k) = \{j \in N \mid \pi_c(j) < \min_{i \in C_k} \pi_c(i)\}$. In the following, we present a procedure in which the allocation scenario is envisaged to generate the two-step Shapley-solidarity value.

Given a TU game with coalition structure $(N, v, \mathcal{C}) \in \mathcal{CG}^N$, the procedure consists of the following steps:

Step 1 The players enter the grand coalition in a consistent permutation, and all consistent permutations have the same probability.

Step 2 Every entering player $i \in C_k \in \mathcal{C}$ joins in and forms the new coalition $p^{\pi_c}(N, i)$. The player brings the marginal contribution $M_i^{\pi_c}(N) := v(p^{\pi_c}(N, i)) - v(p^{\pi_c}(N, i) \setminus i)$. With a near-sighted union solidarity principle in mind, the player takes her marginal contribution with respect to the union she belongs to, namely $M_i^{\pi_c}(C_k) := v(p^{\pi_c}(C_k, i)) - v(p^{\pi_c}(C_k, i) \setminus i)$, and splits it equally among her union predecessors $p^{\pi_c}(C_k, i)$.

Step 3 The residual (negative or positive) brought by player i's joining, $M_i^{\pi_c}(N) - M_i^{\pi_c}(C_k)$, is equally shared by the union successors of player i, i.e., $s^{\pi_c}(C_k, i) = \{j \in C_k \mid \pi_c(j) > \pi_c(i)\}$.

Step 4 The last player of a union $i \in C_k \in \mathcal{C}$, so when $|p^{\pi_c}(C_k, i)| = |C_k|$, is then to be treated in a special way, and obtains a residual of $M_i^{\pi_c}(N) - M_i^{\pi_c}(C_k) - v(p^{\pi_c}(N, C_k))$, which is denoted by $\gamma_i^{\pi_c}$.

As shown in the procedure, each player focuses only on the corresponding union members under the restriction of the coalition structure. Either one player's marginal contribution or the residual is shared only by the players who are in the same union. This is exactly an embodiment of solidarity within a union. Besides, with the last union member joining in, this union is complete and the last player of the union thereby affords a payment to the union's predecessors to prevent their coalition's worth from being infringed. In view of the fact that the last union player has no union successors, a residual of $M_i^{\pi_c}(N) - M_i^{\pi_c}(C_k) - v(p^{\pi_c}(N, C_k))$ is shared "by herself".

In order to show that this procedural description coincides with the two-step Shapley-solidarity value, note that for each $(N, v, \mathcal{C}) \in \mathcal{CG}^N$ and each $\pi_c \in \Pi_{N,\mathcal{C}}$, Steps 1–4 determine a payoff $\psi_i^{\pi_c}(N, v, \mathcal{C})$ for each $i \in C_k \in \mathcal{C}$ as follows:

$$\psi_i^{\pi_c}(N, v, \mathcal{C}) = \begin{cases} M_i^{\pi_c}(C_k) + \beta_i^{\pi_c}, & \pi_c(i) = |p^{\pi_c}(N, C_k)| + 1; \\ \frac{M_i^{\pi_c}(C_k)}{|p^{\pi_c}(C_k, i)|} + \beta_i^{\pi_c} + \alpha_i^{\pi_c}, & 1 < \pi_c(i) - |p^{\pi_c}(N, C_k)| < |C_k|; \\ \frac{M_i^{\pi_c}(C_k)}{|p^{\pi_c}(C_k, i)|} + \alpha_i^{\pi_c} + \gamma_i^{\pi_c}, & \pi_c(i) = |p^{\pi_c}(N, C_k)| + |C_k|, \end{cases} \tag{3}$$

where

$$\alpha_i^{\pi_c} = \sum_{r=|p^{\pi_c}(N,C_k)|+1}^{\pi_c(i)-1} \frac{M_{\pi_c^{-1}(r)}^{\pi_c}(N) - M_{\pi_c^{-1}(r)}^{\pi_c}(C_k)}{|p^{\pi_c}(N, C_k)| + |C_k| - r},$$

and

$$\beta_i^{\pi_c} = \sum_{z=\pi_c(i)+1}^{|p^{\pi_c}(N,C_k)|+|C_k|} \frac{M_{\pi_c^{-1}(z)}^{\pi_c}(C_k)}{|p^{\pi_c}(C_k, \pi_c^{-1}(z))|}.$$

Then, for $i \in C_k \in \mathcal{C}$ the *procedural outcome* is given by

$$\psi_i(N, v, \mathcal{C}) := \frac{1}{|\Pi_{N,\mathcal{C}}|} \sum_{\pi_c \in \Pi_{N,\mathcal{C}}} \psi_i^{\pi_c}(N, v, \mathcal{C}). \tag{4}$$

Next, we will show the coincidence between the procedural outcome and the two-step Shapley-solidarity value.

Theorem 3 *For each TU game with coalition structure* $(N, v, \mathcal{C}) \in \mathcal{CG}^N$, *the procedural outcome given by Eq.*(4), $\psi(N, v, \mathcal{C})$ *coincides with the two-step Shapley-solidarity value* $TSS(N, v, \mathcal{C})$.

The proof is mainly technical, and can be found in Appendix A.

5 Axiomatizations

As we can see, the two-step Shapley-solidarity value and the two-step Shapley value exactly differ in what principle is agreed to be used in bargaining on each union's worth. Hence, we next propose axiomatizations to further indicate the precise similarities and differences between these two values.

5.1 Coalitional A-null player

To begin with, we proceed with recalling some axioms in coalition structure setting.

Axiom 1 Efficiency (**E**). For each $(N, v, \mathcal{C}) \in \mathcal{CG}^N$, $\sum_{i\in N} \psi_i(N, v, \mathcal{C}) = v(N)$.

Axiom 2 Additivity (**A**). For $(N, v, \mathcal{C}), (N, w, \mathcal{C}) \in \mathcal{CG}^N$, $\psi_i(N, v + w, \mathcal{C}) = \psi_i(N, v, \mathcal{C}) + \psi_i(N, w, \mathcal{C})$.

Axiom 3 Coalitional symmetry (**CS**). For each $(N, v, \mathcal{C}) \in \mathcal{CG}^N$, and $\{h, r\} \subseteq M$, if C_h and C_r are symmetric coalitions in $(N, v, \mathcal{C})$, then $\sum_{i\in C_h} \psi_i(N, v, \mathcal{C}) = \sum_{i\in C_r} \psi_i(N, v, \mathcal{C})$.

Axiom 4 Internal equity (**IE**). For each $(N, v, \mathcal{C}) \in \mathcal{CG}^N$, each $k \in M$ and $\{i, j\} \subseteq C_k$, if i and j are symmetric players in $(C_k, v_{|C_k})$, then $\psi_i(N, v, \mathcal{C}) = \psi_j(N, v, \mathcal{C})$.

Efficiency, additivity and coalition symmetry are standard and most of coalitional values meet these requirements. Coalitional symmetry requires to treat symmetric unions equally. Kamijo (2009) introduces the internal equity axiom to consider the symmetric situation from the player's level. It states that two players who are judged to be symmetric in the internal situation should be treated equally and thus receive equal payoff. The following coalitional null player axiom is proposed in Kamijo (2009), with which the two-step Shapley value is characterized.

Axiom 5 Coalitional null player (**CNP**). For each $(N, v, \mathcal{C}) \in \mathcal{CG}^N$, each $k \in M$ and $i \in C_k$, if i is a null player in (N, v), and k is a dummy player in $(M, v^{\mathcal{C}})$, then $\psi_i(N, v, \mathcal{C}) = 0$.

In the statement of the coalitional null player, a null player gets nothing if the union she belongs to is a dummy player in quotient game. Otherwise, it is possible that a null player receives nonzero payoff. Hence, it is not necessarily the case that a zero payoff is given to all null players, which means the null player axiom (Owen 1977) needs not hold. Actually, identifying which kind of players is supposed to get zero payoff or how to deal with the payoff of a null player is one of the key issues in axiomatizations with additivity. This pops up in quite a number of papers which apply variants of the null player axiom to characterize values or coalitional values. For example, the δ-reducing player proposed by van den Brink and Funaki (2015), the p-null player proposed by Béal et al. (2017) for a class of solidarity values, two types of null players proposed by Borkotokey et al. (2020) for a class of k-lateral Shapley values, the partial A-null player introduced by Hu and Li (2018) for the Shapley-solidarity value, and the α-indemnificatory null player introduced by Zou et al. (2020) for the α-egalitarian Owen value, just to name a few.

In line with these works, we introduce the coalitional A-null player axiom for cooperative games with coalition structure. It states that if a player is an A-null player in the subgame with a player set consisting of her union members, and the union she belongs to is a dummy player in the quotient game, then this player should obtain zero payoff.

Axiom 6 Coalitional A-null player (**CANP**). For each $(N, v, \mathcal{C}) \in \mathcal{CG}^N$, each $k \in M$ and $i \in C_k$, if i is an A-null player in $(C_k, v_{|C_k})$, and k is a dummy player in $(M, v^{\mathcal{C}})$, then $\psi_i(N, v, \mathcal{C}) = 0$.

With the aid of this axiom, we obtain an axiomatization of the two-step Shapley-solidarity value. Before we give the formal axiomatization, some definitions and lemmas are needed. For $\mathcal{C} \in \mathcal{C}_N$, we firstly define a family of TU games $\{(N, \tilde{u}_T)\}_{T \in 2^N \setminus \emptyset}$ with respect to $\mathcal{C}$ as follows.

Given $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$, for any $T \subseteq C_k$, $k \in M$,

$$\tilde{u}_T(S) = \begin{cases} \binom{s}{t}^{-1} \cdot \binom{|C_k|}{t}, & T \subseteq S \subseteq C_k; \\ 1, & T \subseteq S \nsubseteq C_k; \\ 0, & T \nsubseteq S, \end{cases}$$

and if $T \nsubseteq C_k$, $\forall k \in M$,

$$\tilde{u}_T(S) = \begin{cases} 1, & T \subseteq S; \\ 0, & T \nsubseteq S. \end{cases}$$

Note that $(N, \tilde{u}_T)$ is an ordinary unanimity game when $T \nsubseteq C_k$. Next, we show that the family of $\{(N, \tilde{u}_T)\}_{T \in 2^N \setminus \emptyset}$ forms a basis of $\mathcal{G}^N$.

Lemma 4 *For each $\mathcal{C} \in \mathcal{C}_N$, the family of TU games $\{(N, \tilde{u}_T)\}_{T \in 2^N \setminus \emptyset}$ is a basis of the linear space $\mathcal{G}^N$.*

Proof It is well-known that $\mathcal{G}^N$ is a $(2^n - 1)$-dimensional linear space. Similar to the spirit of the proof of Lemma 2.2 in Nowak and Radzik (1994), we just show that TU games $\{(N, \tilde{u}_T)\}_{T \in 2^N \setminus \emptyset}$ consist of a set of $2^n - 1$ independent vectors in $\mathcal{G}^N$. To that end, let $S_1, S_2, \cdots, S_{2^n-1}$ be a fixed sequence containing all non-empty set of N such that $n = |S_1| \geq |S_2| \geq \cdots \geq |S_{2^n-1}|$. Moreover, define a $(2^n - 1) \times (2^n - 1)$ matrix $A = [a_{i,j}]$ whose entries are given by

$$a_{i,j} = \tilde{u}_{S_i}(S_j),\ i,j = 1, 2, \ldots, 2^n - 1.$$

Notice that A is a triangular matrix and its diagonal entries equal $\binom{|C_k|}{t}$ if $T \subsetneq C_k$ $(k \in M)$ and 1 otherwise. Hence, we see that $\det(A) = \prod_{k=1}^{m} \prod_{\emptyset \neq T \subsetneq C_k} \binom{|C_k|}{t} \neq 0$. It follows that vectors $\{\tilde{u}_T\}_{T \in 2^N \setminus \emptyset}$ are independent, and thus, $\{(N, \tilde{u}_T)\}_{T \in 2^N \setminus \emptyset}$ forms a basis of $\mathcal{G}^N$. This holds for all coalition structures $\mathcal{C}$. □

Then, we have the following, main theorem.

Theorem 5 *A coalitional value ψ on $\mathcal{CG}^N$ satisfies efficiency, additivity, coalitional symmetry, internal equity, and coalitional A-null player if and only if $\psi(N, v, \mathcal{C})$ is the two-step Shapley-solidarity value.*

Proof *Existence*. Firstly, we show that the two-step Shapley-solidarity value satisfies the above five axioms. Efficiency and additivity are trivial due to the definition of the two-step Shapley-solidarity value. Coalition symmetry and internal equity can be easily verified since both Shapley value and solidarity value satisfy symmetry. It also turns out to be true that the two-step Shapley-solidarity value satisfies coalitional A-null player axiom, because the solidarity value satisfies the A-null player axiom and the Shapley value assigns a dummy player his stand-alone worth.

Uniqueness. Let ψ be a coalitional value over $\mathcal{CG}^N$ which satisfies the five axioms. Lemma 4 immediately implies that, given $\mathcal{C} \in \mathcal{C}_N$, for each $(N, v) \in \mathcal{G}^N$, there exists $\{\lambda_T \mid \lambda_T \in \mathbb{R}, T \in 2^N \setminus \emptyset\}$ such that $v = \sum_{T \in 2^N \setminus \emptyset} \lambda_T \tilde{u}_T$. According to additivity, it is now sufficient to prove that, for each TU game $(N, \lambda_T \tilde{u}_T, \mathcal{C})$, $\psi(N, \lambda_T \tilde{u}_T, \mathcal{C})$ is uniquely determined by efficiency, coalitional symmetry, internal equity and the coalitional A-null player axiom.

For $\mathcal{C} \in \mathcal{C}_N$ and each $T \in 2^N \setminus \emptyset$, denote $D = \{k \in M \mid C_k \cap T \neq \emptyset\}$. Note that the corresponding quotient game $(M, (\lambda_T \tilde{u}_T)^{\mathcal{C}})$ for $(N, \lambda_T \tilde{u}_T, \mathcal{C})$ is equivalent to the unanimity game $(M, \lambda_T u_D)$ since, for each $T \in 2^N \setminus \emptyset$, there is

$$(\lambda_T \tilde{u}_T)^{\mathcal{C}}(Q) = \lambda_T \tilde{u}_T(\cup_{k \in Q} C_k) = \begin{cases} \lambda_T, & T \subseteq \cup_{k \in Q} C_k; \\ 0, & \text{otherwise}, \end{cases}$$

for all $Q \subseteq M$. Hence, each $k \notin D$ is a null player in $(M, (\lambda_T \tilde{u}_T)^{\mathcal{C}})$. Moreover, for each $k \notin D$, the subgame $(C_k, (\lambda_T \tilde{u}_T)_{|C_k})$ is a null game, namely $(\lambda_T \tilde{u}_T)_{|C_k}(S) = 0$ for all $S \subseteq C_k$. By the coalitional A-null player axiom, we have $\psi_i(N, \lambda_T \tilde{u}_T, \mathcal{C}) = 0$ for each $i \in C_k$ $(k \notin D)$. For $k \in D$, there is $\sum_{i \in C_k} \psi_i(N, \lambda_T \tilde{u}_T, \mathcal{C}) = \frac{\lambda_T}{d}$, which derives from efficiency and coalitional symmetry.

Now, let us focus on the internal distribution of the payoff that one union obtains from their collective bargaining. For each $T \in 2^N \backslash \emptyset$, there is the corresponding D, and we consider the following two cases.

(i) $d = 1$. Let $D = \{k\}$, notice that each player $i \in C_k \backslash T$ is an A-null player in $(C_k, (\lambda_T \tilde{u}_T)_{|C_k})$ since, for each coalition $S \subseteq C_k$ satisfying $T \subseteq S$ and $i \in S$,

$$\begin{aligned}
(\lambda_T \tilde{u}_T)_{|C_k}(S) &= \lambda_T \binom{s}{t}^{-1} \cdot \binom{|C_k|}{t} \\
&= \lambda_T \frac{t!(s-t)!}{s!} \cdot \frac{|C_k|!}{t!(|C_k|-t)!} \\
&= \lambda_T \frac{1}{s} \cdot (s-t) \cdot \frac{t!(s-t-1)!}{(s-1)!} \frac{|C_k|!}{t!(|C_k|-t)!} \\
&= \frac{1}{s} \sum_{j \in S} (\lambda_T \tilde{u}_T)_{|C_k}(S \setminus j).
\end{aligned}$$

Besides, k is a dummy player in the quotient game $(M, (\lambda_T \tilde{u}_T)^{\mathcal{C}})$. By the coalitional A-null player axiom, we have $\psi_i(N, \lambda_T \tilde{u}_T, \mathcal{C}) = 0$ for each $i \in C_k \setminus T$. Furthermore, the symmetry of any two players $i, j \in T$ in subgame $(C_k, (\lambda_T \tilde{u}_T)_{|C_k})$ immediately implies $\psi_i(N, \lambda_T \tilde{u}_T, \mathcal{C}) = \frac{\lambda_T}{t}$.

(ii) $d \geq 2$. For each C_k $(k \in D)$ and $\{i, j\} \subseteq C_k$, we have $(\lambda_T \tilde{u}_T)_{|C_k}(S \cup i) = (\lambda_T \tilde{u}_T)_{|C_k}(S \cup j)$ for each $S \subseteq C_k \setminus \{i, j\}$. Thus, by internal equity, there is $\psi_i(N, \lambda_T \tilde{u}_T, \mathcal{C}) = \frac{\lambda_T}{d \cdot |C_k|}$ for each $i \in C_k$ $(k \in D)$.

Hence, it is clear that $\psi(N, \lambda_T \tilde{u}_T, \mathcal{C})$ is unique, which completes the proof. □

5.2 Quasi-balanced contributions for grand coalition

This section provides two other axiomatizations which are related to the principle of balanced contributions. The balanced contributions property was firstly proposed by Myerson (1980) to characterize the Shapley value. It requires that any two players must have the same impacts on mutual payoff when one of them departs from the game. Subsequently, Xu et al. (2016) introduce quasi-balanced contributions to verify that the solidarity value is the unique efficient value satisfying this property.

Axiom 7 Quasi-balanced contributions (**QBC**). For each $(N, v) \in \mathcal{G}^N$ and $\{i, j\} \subseteq N$,

$$\varphi_i(N, v) - \varphi_i(N \setminus j, v) + \frac{1}{n}v(N \setminus j) = \varphi_j(N, v) - \varphi_j(N \setminus i, v) + \frac{1}{n}v(N \setminus i).$$

Moreover, there are counterparts for the coalitional structure setting to describe the mutual influence of two unions and two players within the same union, namely coalitional balanced contributions and intracoalitional balanced contributions as introduced by Calvo et al. (1996).

Axiom 8 Intracoalitional balanced contributions (**IBC**). For each $(N, v, \mathcal{C}) \in \mathcal{CG}^N$, and $i, j \in C_h \in \mathcal{C}$ with $i \neq j$,

$$\psi_i(N, v, \mathcal{C}) - \psi_i(N \setminus j, v_{|N\setminus j}, \mathcal{C}_{|N\setminus j}) = \psi_j(N, v, \mathcal{C}) - \psi_j(N \setminus i, v_{|N\setminus i}, \mathcal{C}_{|N\setminus i}).$$

Axiom 9 Coalitional balanced contributions (**CBC**). For each $(N, v, \mathcal{C}) \in \mathcal{CG}^N$, and each $C_h, C_r \in C$ with $r \neq h$,

$$\sum_{i\in C_h} \psi_i(N, v, \mathcal{C}) - \sum_{i\in C_h} \psi_i(N \setminus C_r, v_{|N\setminus C_r}, \mathcal{C}_{|N\setminus C_r})$$
$$= \sum_{i\in C_r} \psi_i(N, v, \mathcal{C}) - \sum_{i\in C_r} \psi_i(N \setminus C_h, v_{|N\setminus C_h}, \mathcal{C}_{|N\setminus C_h}).$$

Intracoalitional balanced contributions means that given two players in the same union, the amounts that both players gain or lose when the other leaves the game should be equal. Correspondingly, from the perspective of unions, there is the coalitional balanced contributions property. These two axioms together with efficiency give rise to the Owen value (Calvo et al. 1996). Calvo and Gutiérrez (2010) also prove that the two-step Shapley value can be characterized with the **CBC** axiom, in which there are two other axioms being involved, called population solidarity within unions and coherence.

Axiom 10 Population solidarity within unions (**PSU**). For each $(N, v, \mathcal{C}) \in \mathcal{CG}^N$, each $C_h, C_r \in \mathcal{C}$ with $r \neq h$, and each $\{i, j, k\} \subseteq N$ with $\{i, j\} \subseteq C_h$ and $k \in C_r$,

$$\psi_i(N, v, \mathcal{C}) - \psi_i(N \setminus k, v_{|N\setminus k}, \mathcal{C}_{|N\setminus k}) = \psi_j(N, v, \mathcal{C}) - \psi_j(N \setminus k, v_{|N\setminus k}, \mathcal{C}_{|N\setminus k}).$$

Axiom 11 Coherence (**C**). For each $(N, v) \in \mathcal{G}^N$, $\psi(N, v, \mathcal{C}^N) = \psi(N, v, \mathcal{C}^n)$.

Population solidarity within unions states that players in the same union follow the solidarity principle in such a way that all members in the union experience the same gains or losses when the game changes due to addition or deletion of players outside the union. Coherence means that it is indistinguishable between games in which all players belong to one union and when all of them act as singletons. The following theorem is due to Calvo and Gutiérrez (2010).

Theorem 6 (Calvo and Gutiérrez 2010) *The two-step Shapley value is the only value that satisfies efficiency, coalitional balanced contributions, population solidarity within unions and coherence.*

Besides, they also introduce the axiom of null coalition which requires null coalitions should get nothing.

Axiom 12 Null coalition (**NC**). For each $(N, v, \mathcal{C}) \in \mathcal{CG}^N$, and $k \in M$, if k is a null coalition, then $\sum_{i \in C_k} \psi_i(N, v, \mathcal{C}) = 0$.

Then the following theorem holds.

Theorem 7 (Calvo and Gutiérrez 2010) *A coalitional value ψ satisfies efficiency, additivity, coalitional symmetry, null coalitional axiom, population solidarity within unions and coherence if and only if $\psi(N, v, \mathcal{C}) = TSh(N, v, \mathcal{C})$.*

Both Theorems 6 and 7 invoke the coherence axiom which is violated by the two-step Shapley-solidarity value. Next, we will show if we replace the coherence in the above two theorems with a coalitional version of the quasi-balanced contributions for TU games with coalition structures, called the quasi-balanced contributions for the grand coalition, we can get corresponding axiomatizations of the two-step Shapley-solidarity value. First, we formulate the mentioned axiom.

Axiom 13 Quasi-balanced contributions for the grand coalition (**QCGC**). For each $(N, v, \mathcal{C}) \in \mathcal{CG}^N$ with $|\mathcal{C}| = 1$, and $i, j \in C_k \in \mathcal{C}$,

$$\begin{aligned}\psi_i(N, v, \mathcal{C}) - \psi_i(N \setminus j, v_{|N \setminus j}, \mathcal{C}_{|N \setminus j}) + \frac{1}{n} v(N \setminus j) \\ = \psi_j(N, v, \mathcal{C}) - \psi_j(N \setminus i, v_{|N \setminus i}, \mathcal{C}_{|N \setminus i}) + \frac{1}{n} v(N \setminus i).\end{aligned}$$

Note that this axiom has exactly the same requirement as the condition for the solidarity value. As we know, the solidarity value can be characterized by quasi-balanced contributions and efficiency. Hence, quasi-balanced contributions for the grand coalition is the corresponding feature for TU games with coalition structure in which the coalition structure is just one union.

Theorem 8 *A coalitional value ψ satisfies efficiency, additivity, coalitional symmetry, null coalitional axiom, population solidarity within unions and quasi-balanced contributions for the grand coalition if and only if $\psi(N, v, \mathcal{C}) = TSS(N, v, \mathcal{C})$.*

Proof The proof follows the same spirit as the proof of Theorem 4 in Calvo and Gutiérrez (2010) (Theorem 7 above). For clarity, we here restate it in order to highlight the difference.

Existence. It is straightforward to verify that the two-step Shapley-solidarity value satisfies efficiency, additivity, coalitional symmetry, null coalition axiom and the population solidarity within unions. As for the quasi-balanced contributions for the grand coalition, if $\mathcal{C} = \{C_1\} = \{N\}$, then $M = \{1\}$ and $Sh_1(M, v^{\mathcal{C}}) = v(N)$, and there is $TSS_i(N, v, \mathcal{C}) = Sol_i(N, v)$ for each $i \in N$. Hence, the two-step Shapley-solidarity value satisfies the quasi-balanced contributions for the grand coalition because the solidarity value satisfies the quasi-balanced contributions.

Uniqueness. Let ψ be a coalitional value satisfying the above six axioms. Given $(N, v, \mathcal{C}) \in \mathcal{CG}^N$, define value ϕ on $\mathcal{G}^M$ by, for each $k \in M$, $\phi_k(M, v^{\mathcal{C}}) = \sum_{i \in C_k} \psi_i(N, v, \mathcal{C})$.

It turns out that the value ϕ is well-defined by efficiency, additivity, coalitional symmetry and null coalition axiom, and there is $\phi_k(M, v^{\mathcal{C}}) = Sh_k(M, v^{\mathcal{C}})$. Thus, when $\mathcal{C} = \mathcal{C}^n$, $\psi_i(N, v, \mathcal{C}^n) = \phi_i(M, v^{\mathcal{C}^n}) = Sh_i(N, v) = TSS_i(N, v, \mathcal{C}^n)$. On the other hand, when $\mathcal{C} = \mathcal{C}^N$, because ψ satisfies efficiency and quasi-balanced contributions for the grand coalition, then by Theorem 4.2 in Xu et al. (2016), we can obtain $\psi_i(N, v, \mathcal{C}^N) = Sol_i(N, v) = TSS_i(N, v, \mathcal{C}^N)$.

Now we focus on the cases when the coalition structure is not trivial. Assume that $|\mathcal{C}| \geq 2$, for each $\{h, r\} \subseteq M$, each $i \in C_h$ and $k \in C_r$, according to population solidarity within unions, there exists $\gamma_h \in \mathbb{R}$ such that $\psi_i(N, v, \mathcal{C}) - \psi_i(N \backslash k, v_{|N \backslash k}, \mathcal{C}_{|N \backslash k}) = \gamma_h$, and hence, for each $i \in C_h$,

$$\psi_i(N, v, \mathcal{C}) = \psi_i(N \setminus k, v_{|N \backslash k}, \mathcal{C}_{|N \backslash k}) + \frac{1}{|C_h|}[Sh_h(M, v^{\mathcal{C}}) - Sh_h(M, v^{\mathcal{C}_{|N \backslash k}})].$$

Using the population solidarity within unions repeatedly until only C_h is in the game, we obtain

$$\begin{aligned}\psi_i(N, v, \mathcal{C}) &= \psi_i(C_h, v_{|C_h}, \mathcal{C}_{|C_h}) + \frac{1}{|C_h|}[Sh_h(M, v^{\mathcal{C}}) - Sh_h(\{h\}, v^{\mathcal{C}_{|C_h}})] \\ &= Sol_i(C_h, v_{|C_h}) + \frac{1}{|C_h|}[Sh_h(M, v^{\mathcal{C}}) - v(C_h)] \\ &= TSS_i(N, v, \mathcal{C})\end{aligned}$$

for each $i \in C_h \in \mathcal{C}$. Hence, there is $\psi(N, v, \mathcal{C}) = TSS(N, v, \mathcal{C})$ for each $(N, v, \mathcal{C}) \in \mathcal{CG}^N$, which completes the proof. □

Theorem 9 *A coalitional value ψ satisfies efficiency, coalitional balanced contributions, population solidarity within unions and quasi-balanced contributions for the grand coalition if and only if $\psi(N, v, \mathcal{C}) = TSS(N, v, \mathcal{C})$.*

Proof *Existence*. It is left to show the two-step Shapley-solidarity value satisfies the coalitional balanced contribution. By definition, for each $(N, v, \mathcal{C}) \in \mathcal{CG}^N$ and each $C_h, C_r \in C$ with $r \neq h$, $\sum_{i \in C_h} TSS_i(N, v, \mathcal{C}) = Sh_h(M, v^{\mathcal{C}})$ and $\sum_{i \in C_r} TSS_i(N, v, \mathcal{C}) = Sh_r(M, v^{\mathcal{C}})$. Hence, the coalitional balanced contributions of the two-step Shapley-solidarity value immediately follows from the balanced contributions of the Shapley value (Myerson 1980).

Uniqueness. Let ψ be a coalitional value satisfying the above four axioms. We show $\psi(N, v, \mathcal{C}) = TSS(N, v, \mathcal{C})$ for all $(N, v, \mathcal{C}) \in \mathcal{CG}^N$ by induction on $|\mathcal{C}|$.

Let $|\mathcal{C}| = 1$. This means that the coalition structure is trivial and $\mathcal{C} = \mathcal{C}^N$. Given a TU game with coalition structure $(N, v, \mathcal{C}^N) \in \mathcal{CG}^N$, quasi-balanced contributions for the grand coalition together with efficiency implies $\psi_i(N, v, \mathcal{C}^N) = Sol_i(N, v)$ for all $i \in N$. Hence, we have $\psi(N, v, \mathcal{C}) = TSS(N, v, \mathcal{C})$ for $(N, v, \mathcal{C}) \in \mathcal{CG}^N$ with $|\mathcal{C}| = 1$.

Now, assume $\psi(N, v, \mathcal{C}) = TSS(N, v, \mathcal{C})$ holds for all TU games with coalition structure $(N, v, \mathcal{C}) \in \mathcal{CG}^N$ when $|\mathcal{C}| \leq m$, we prove $\psi(N, v, \mathcal{C}) = TSS(N, v, \mathcal{C})$ can also be established for $(N, v, \mathcal{C})$ with $|\mathcal{C}| = m + 1$.

Let $(N, v, \mathcal{C})$ be a TU game with coalition structure where $|\mathcal{C}| = m + 1$. Since both ψ and TSS satisfy **CBC**, we have

$$\begin{aligned}&\sum_{i\in C_h} \psi_i(N, v, \mathcal{C}) - \sum_{i\in C_r} \psi_i(N, v, \mathcal{C})\\ &\quad= \sum_{i\in C_h} \psi_i(N \setminus C_r, v_{|N\setminus C_r}, \mathcal{C}_{|N\setminus C_r}) - \sum_{i\in C_r} \psi_i(N \setminus C_h, v_{|N\setminus C_h}, \mathcal{C}_{|N\setminus C_h}),\end{aligned}$$

and

$$\begin{aligned}&\sum_{i\in C_h} TSS_i(N, v, \mathcal{C}) - \sum_{i\in C_r} TSS_i(N, v, \mathcal{C})\\ &\quad= \sum_{i\in C_h} TSS_i(N \setminus C_r, v_{|N\setminus C_r}, \mathcal{C}_{|N\setminus C_r}) - \sum_{i\in C_r} TSS_i(N \setminus C_h, v_{|N\setminus C_h}, \mathcal{C}_{|N\setminus C_h}).\end{aligned}$$

Moreover, according to the induction hypothesis, we have

$$\begin{aligned}&\sum_{i\in C_h} \psi_i(N \setminus C_r, v_{|N\setminus C_r}, \mathcal{C}_{|N\setminus C_r}) - \sum_{i\in C_r} \psi_i(N \setminus C_h, v_{|N\setminus C_h}, \mathcal{C}_{|N\setminus C_h})\\ &\quad= \sum_{i\in C_h} TSS_i(N \setminus C_r, v_{|N\setminus C_r}, \mathcal{C}_{|N\setminus C_r}) - \sum_{i\in C_r} TSS_i(N \setminus C_h, v_{|N\setminus C_h}, \mathcal{C}_{|N\setminus C_h}).\end{aligned}$$

The above three equations yield

$$\sum_{i\in C_h} \psi_i(N, v, \mathcal{C}) - \sum_{i\in C_h} TSS_i(N, v, \mathcal{C}) = \sum_{i\in C_r} \psi_i(N, v, \mathcal{C}) - \sum_{i\in C_r} TSS_i(N, v, \mathcal{C}),$$

for all $C_h, C_r \in \mathcal{C}$. Then, fixing h in the left part in the above equation and summing over $r \in M$ of the right, we have

$$\begin{aligned}&|M|\left(\sum_{i\in C_h} \psi_i(N, v, \mathcal{C}) - \sum_{i\in C_h} TSS_i(N, v, \mathcal{C})\right)\\ &\quad= \sum_{r\in M}\sum_{i\in C_r} \psi_i(N, v, \mathcal{C}) - \sum_{r\in M}\sum_{i\in C_r} TSS_i(N, v, \mathcal{C}).\end{aligned}$$

Combining with efficiency, we get

$$\sum_{i\in C_h} \psi_i(N, v, \mathcal{C}) = \sum_{i\in C_h} TSS_i(N, v, \mathcal{C}), \tag{5}$$

for all $h \in M$.

Then, it remains to show $\psi_i(N,v,\mathcal{C}) = TSS_i(N,v,\mathcal{C})$ for all $i \in C_h \in \mathcal{C}$. This can be obtained by induction on $|C_h|$. Given $C_h \in \mathcal{C}$ with $|C_h| = 1$, Eq.(5) yields $\psi_i(N,v,\mathcal{C}) = TSS_i(N,v,\mathcal{C})$ for $\{i\} = C_h$. We now assume $|C_h| \geq 2$. For each $C_h, C_r \in \mathcal{C}$, and $\{i,j\} \subseteq C_h$, by repeatedly using **PSU** on ψ and TSS until the players within union C_r are ruled out, we have

$$\psi_i(N,v,\mathcal{C}) - \psi_j(N,v,\mathcal{C}) = \psi_i(N \setminus C_r, v_{|N\setminus C_r}, \mathcal{C}_{|N\setminus C_r}) - \psi_j(N \setminus C_r, v_{|N\setminus C_r}, \mathcal{C}_{|N\setminus C_r}),$$

and

$$\begin{aligned} &TSS_i(N,v,\mathcal{C}) - TSS_j(N,v,\mathcal{C}) \\ &\quad = TSS_i(N \setminus C_r, v_{|N\setminus C_r}, \mathcal{C}_{|N\setminus C_r}) - TSS_j(N \setminus C_r, v_{|N\setminus C_r}, \mathcal{C}_{|N\setminus C_r}). \end{aligned}$$

Again, by induction hypothesis, we have

$$\begin{aligned} \psi_i(N \setminus C_r, v_{|N\setminus C_r}, \mathcal{C}_{|N\setminus C_r}) &= TSS_i(N \setminus C_r, v_{|N\setminus C_r}, \mathcal{C}_{|N\setminus C_r}), \\ \psi_j(N \setminus C_r, v_{|N\setminus C_r}, \mathcal{C}_{|N\setminus C_r}) &= TSS_j(N \setminus C_r, v_{|N\setminus C_r}, \mathcal{C}_{|N\setminus C_r}). \end{aligned}$$

Hence, there is

$$\psi_i(N,v,\mathcal{C}) - TSS_i(N,v,\mathcal{C}) = \psi_j(N,v,\mathcal{C}) - TSS_j(N,v,\mathcal{C}).$$

Then, fixing i and summing over $j \in C_h$, we obtain

$$|C_h|(\psi_i(N,v,\mathcal{C}) - TSS_i(N,v,\mathcal{C})) = \sum_{j\in C_h}(\psi_j(N,v,\mathcal{C}) - TSS_j(N,v,\mathcal{C})).$$

By Eq.(5), we conclude that $\psi_i(N,v,\mathcal{C}) = TSS_i(N,v,\mathcal{C})$ for all $i \in C_h \in \mathcal{C}$, which completes the proof. □

6 Conclusion

The two-step Shapley-solidarity value is in our opinion a conceptually simple value for cooperative games with coalition structure that captures the solidarity concept within unions. The given axiomatizations exactly pinpoint this and show similarities and also the subtle difference when compared to the two-step Shapley value as defined by Kamijo (2009). The given Example 1 also highlights the difference to the two closest relatives, but of course, other examples can be constructed to show opposite effects, too. It is an interesting question for further research to find subclasses of games to turn the "empirical" observations of Example 1 into a firm theorem. In this context, observe that for *anonymous* games where $v(S) = |S|$, and a specific class of *simple* games, namely when $v(S) = 1$ for all $|S| \geq 2$ and $v(S) = 0$ otherwise, all three values that we defined in Table 1 are identical. Moreover, for *additive* games where $v(S) = \sum_{i\in S} v(i)$, the two-step Shapley value is given by the stand-alone worth of the players, so $TSh_i(N,v) = v(i)$, and the Shapley-solidarity value and two-step Shapley-solidarity value are identical, but different from the former.

There is another coalitional value which is akin to our proposed value and the two-step Shapley value. It can be obtained by using the solidarity value for both, the induced internal game and the quotient game in Kamijo's two step approach. Formally, it is given by, for each $(N, v, \mathcal{C}) \in \mathcal{CG}^N$ and $i \in C_k \in \mathcal{C}$,

$$TS_i(N, v, \mathcal{C}) = Sol_i(C_k, v_{|C_k}) + \frac{Sol_k(M, v^{\mathcal{C}}) - v(C_k)}{|C_k|}.$$

Let us call it the two-step solidarity value.

By definition, this value supports a higher degree of solidarity among unions compared to the other two values, due to the fact that the revenue distribution among unions is by the solidarity value instead of the Shapley value. When applying the two-step solidarity value to the game in Example 1, one obtains the payoff vector $TS(N, v, \mathcal{C}) = (\frac{L}{12}, \frac{L}{12}, \frac{L}{12} + \frac{1}{4}, \frac{3L}{4} - \frac{1}{4})$. Although this value leads to the same difference within players of union C_I as the two-step Shapley-solidarity value does, it has a smaller difference of $\frac{L-1}{2}$ between the two unions, compared to the difference $L - 1$ in the two-step Shapley-solidarity value.

This difference can be confirmed from the axiomatic perspective as well. With a modification of the axiom of **CANP**, we obtain an axiom based on A-dummy players. Here, a player $i \in N$ is called an A-dummy player if $\frac{1}{s} \sum_{j \in S}(v(S) - v(S \setminus j)) = v(i)$ for all $S \subseteq N$ with $i \in S$.

Axiom 14 Coalitional A-dummy player (CADP). For each $(N, v, \mathcal{C}) \in \mathcal{CG}^N$, each $k \in M$ and $i \in C_k$, if i is an A-null player in $(C_k, v_{|C_k})$, and k is an A-dummy player in $(M, v^{\mathcal{C}})$, then $\psi_i(N, v, \mathcal{C}) = 0$ and $\sum_{i \in C_k} \psi_i(N, v, \mathcal{C}) = v(C_k)$.

CADP replaces the condition for dummy players of **CANP** with A-dummy players, and requires the total payoff for a union which is an A-dummy player in the quotient game being equal to the union's worth. Then, the two-step solidarity value can be axiomatized by **E**, **A**, **CS**, **IE** and **CADP**. The proof is similar to the proof of Theorem 5, except that we work with yet another basis for the set of TU games, namely $\{(N, \tilde{w}_T)\}_{T \subseteq 2^N \setminus \emptyset}$, with respect to coalition structure $\mathcal{C}$, where $\tilde{w}_T$ is defined by, for any $T \subseteq C_k$, $k \in M$,

$$\tilde{w}_T(S) = \begin{cases} \binom{s}{t}^{-1} \cdot \binom{|C_k|}{t}, & T \subseteq S \subseteq C_k; \\ \frac{|\mathcal{C}_{|S}|+1}{2}, & T \subseteq S \nsubseteq C_k; \\ 0, & T \nsubseteq S, \end{cases}$$

and

$$\tilde{w}_T(S) = \begin{cases} \binom{|\mathcal{C}_{|S}|}{|\mathcal{C}_{|T}|}^{-1}, & T \subseteq S; \\ 0, & T \nsubseteq S, \end{cases}$$

for all $T \nsubseteq C_k$ and all $k \in M$. Here, recall that $\mathcal{C}_{|S}$ is the restriction of $\mathcal{C}$ on the player set S for all $S \subseteq N$, namely $\mathcal{C}_{|S} = \{C_h \cap S \mid C_h \in \mathcal{C} \text{ and } C_h \cap S \neq \emptyset\}$.

Appendix A Proof of Theorem 3

We recall Theorem 3:

Theorem 3. *For each TU game with coalition structure $(N, v, \mathcal{C}) \in \mathcal{CG}^N$, the procedural outcome given by Eq.(4), $\psi(N, v, \mathcal{C})$ coincides with the two-step Shapley-solidarity value $TSS(N, v, \mathcal{C})$.*

Proof For each $(N, v, \mathcal{C}) \in \mathcal{CG}^N$, $i \in C_k \in \mathcal{C}$, it follows from Eq.(3) and Eq.(4) that,

$$\begin{aligned}
&\psi_i(N, v, \mathcal{C}) \\
&= \frac{1}{|\Pi_{N,\mathcal{C}}|} \sum_{\pi_c \in \Pi_{N,\mathcal{C}}} \psi_i^{\pi_c}(N, v, \mathcal{C}) \\
&= \frac{1}{|\Pi_{N,\mathcal{C}}|} \left(\sum_{\pi_c \in \Pi_{N,\mathcal{C}}} \frac{M_i^{\pi_c}(C_k)}{|p^{\pi_c}(C_k, i)|} + \sum_{\substack{\pi_c \in \Pi_{N,\mathcal{C}}: \\ \pi_c(i) \neq |p^{\pi_c}(N, C_k)| + |C_k|}} \beta_i^{\pi_c} + \sum_{\substack{\pi_c \in \Pi_{N,\mathcal{C}}: \\ \pi_c(i) \neq |p^{\pi_c}(N, C_k)| + 1}} \alpha_i^{\pi_c} \right. \\
&\qquad \left. + \sum_{\substack{\pi_c \in \Pi_{N,\mathcal{C}}: \\ \pi_c(i) = |p^{\pi_c}(N, C_k)| + |C_k|}} \gamma_i^{\pi_c} \right) \\
&= \underbrace{\frac{1}{|\Pi_{N,\mathcal{C}}|} \sum_{\pi_c \in \Pi_{N,\mathcal{C}}} \frac{M_i^{\pi_c}(C_k)}{|p^{\pi_c}(C_k, i)|} + \sum_{\substack{\pi_c \in \Pi_{N,\mathcal{C}}: \\ \pi_c(i) \neq |p^{\pi_c}(N, C_k)| + |C_k|}} \frac{1}{|\Pi_{N,\mathcal{C}}|} \sum_{z=\pi_c(i)+1}^{|p^{\pi_c}(N,C_k)|+|C_k|} \frac{M^{\pi_c}_{\pi_c^{-1}(z)}(C_k)}{|p^{\pi_c}(C_k, \pi_c^{-1}(z))|}}_{\text{Part I}} \\
&\quad + \underbrace{\sum_{\substack{\pi_c \in \Pi_{N,\mathcal{C}}: \\ \pi_c(i) \neq |p^{\pi_c}(N, C_k)| + 1}} \frac{1}{|\Pi_{N,\mathcal{C}}|} \sum_{r=|p^{\pi_c}(N,C_k)|+1}^{\pi_c(i)-1} \frac{M^{\pi_c}_{\pi_c^{-1}(r)}(N) - M^{\pi_c}_{\pi_c^{-1}(r)}(C_k)}{|p^{\pi_c}(N, C_k)| + |C_k| - r}}_{\text{Part II(1)}} \\
&\quad + \underbrace{\sum_{\substack{\pi_c \in \Pi_{N,\mathcal{C}}: \\ \pi_c(i) = |p^{\pi_c}(N, C_k)| + |C_k|}} \frac{1}{|\Pi_{N,\mathcal{C}}|} (M_i^{\pi_c}(N) - M_i^{\pi_c}(C_k) - v(p^{\pi_c}(N, C_k)))}_{\text{Part II(2)}}
\end{aligned}$$

Next, we will show the Part I is consistent with player i's payoff which results from the internal bargaining according to the subgame $(C_k, v_{|C_k})$, while Part II, i.e., the sum of Part $II(1)$ and Part $II(2)$, coincides with the surplus that player i can obtain due to the union's collective bargaining.

Let us focus on Part I first. For every $S \subseteq C_k$ such that $j \in S$, it is worth noting that there are $m! \prod_{p \neq k} |C_p|!(s-1)!(|C_k|-s)!$ consistent permutations for which player j is a successor of the players in $S \setminus j$ and the players in $C_k \setminus S$ are the successors of player j. Hence, it means that for each consistent permutation $\pi_c \in \Pi_{N,\mathcal{C}}$ such that $p^{\pi_c}(C_k, j) = S$ the player j's marginal contribution with respect to the union C_k is given by $v(S) - v(S \setminus j)$.

$$
\begin{aligned}
&\text{Part I} \\
&= \frac{1}{|\Pi_{N,\mathcal{C}}|} \sum_{\pi_c \in \Pi_{N,\mathcal{C}}} \frac{v(p^{\pi_c}(C_k, i)) - v(p^{\pi_c}(C_k, i) \setminus i)}{|p^{\pi_c}(C_k, i)|} + \sum_{\substack{\pi_c \in \Pi_{N,\mathcal{C}}:\\ \pi_c(i) \neq |p^{\pi_c}(N, C_k)| + |C_k|}} \Bigg(\frac{1}{|\Pi_{N,\mathcal{C}}|} \\
&\sum_{z=\pi_c(i)+1}^{|p^{\pi_c}(N,C_k)|+|C_k|} \frac{v(p^{\pi_c}(C_k, \pi_c^{-1}(z))) - v(p^{\pi_c}(C_k, \pi_c^{-1}(z)) \setminus \pi_c^{-1}(z))}{|p^{\pi_c}(C_k, \pi_c^{-1}(z))|} \Bigg) \\
&= \frac{(s-1)!(|C_k|-s)!}{|C_k|!} \left(\sum_{S \subseteq C_k : i \in S} \frac{v(S) - v(S \setminus i)}{s} + \sum_{j \in C_k \setminus i} \sum_{\substack{S \subseteq C_k :\\ \{i,j\} \subseteq S}} \frac{v(S) - v(S \setminus j)}{s} \right) \\
&= \frac{(s-1)!(|C_k|-s)!}{|C_k|!} \left(\sum_{S \subseteq C_k : i \in S} \frac{v(S) - v(S \setminus i)}{s} + \sum_{\substack{S \subseteq C_k :\\ i \in S, s \geq 2}} \sum_{j \in S \setminus i} \frac{v(S) - v(S \setminus j)}{s} \right) \\
&= \sum_{S \subseteq C_k : i \in S} \frac{(s-1)!(|C_k|-s)!}{|C_k|!} \sum_{j \in S} \frac{v(S) - v(S \setminus j)}{s} \\
&= Sol_i(C_k, v_{|C_k})
\end{aligned}
$$

Then, for Part II, we look at Part $II(1)$ and Part $II(2)$ separately.

Part II(1)

$$= \sum_{\substack{\pi_c \in \Pi_{N,\mathcal{C}}: \\ \pi_c(i) \neq |p^{\pi_c}(N,C_k)|+1}} \frac{1}{|\Pi_{N,\mathcal{C}}|} \sum_{r=|p^{\pi_c}(N,C_k)|+1}^{\pi_c(i)-1} \Bigg(\frac{v(p^{\pi_c}(N,\pi_c^{-1}(r))) - v(p^{\pi_c}(N,\pi_c^{-1}(r)) \setminus \pi_c^{-1}(r))}{|p^{\pi_c}(N,C_k)| + |C_k| - r}$$

$$- \frac{v(p^{\pi_c}(C_k,\pi_c^{-1}(r))) - v(p^{\pi_c}(C_k,\pi_c^{-1}(r)) \setminus \pi_c^{-1}(r))}{|p^{\pi_c}(N,C_k)| + |C_k| - r} \Bigg)$$

$$= \sum_{Q \subseteq M\setminus k} \sum_{j \in C_k \setminus i} \sum_{\substack{S \subseteq C_k: \\ i \notin S, j \in S}} \frac{q!(m-q-1)!}{m!} \cdot \frac{(s-1)!(|C_k|-s)!}{|C_k|!} \cdot$$

$$\left(\frac{v(\cup_{h\in Q} C_h \cup S) - v(\cup_{h\in Q} C_h \cup S \setminus j)}{|C_k| - s} - \frac{(v(S) - v(S \setminus j))}{|C_k| - s} \right)$$

$$= \sum_{Q \subseteq M\setminus k} \sum_{\substack{S \subseteq C_k: \\ i \notin S}} \frac{q!(m-q-1)!}{m!} \cdot \frac{(s-1)!(|C_k|-s-1)!}{|C_k|!} \cdot s \cdot (v(\cup_{h\in Q} C_h \cup S) - v(S))$$

$$- \sum_{Q \subseteq M\setminus k} \sum_{\substack{T \subseteq C_k: i \notin T, \\ t < |C_k|-1}} \sum_{\substack{j \in C_k \setminus i: \\ j \notin T}} \frac{q!(m-q-1)!}{m!} \cdot \frac{t!(|C_k|-t-2)!}{|C_k|!} \cdot (v(\cup_{h\in Q} C_h \cup T) - v(T))$$

$$= \sum_{Q \subseteq M\setminus k} \sum_{\substack{S \subseteq C_k: \\ i \notin S}} \frac{q!(m-q-1)!}{m!} \cdot \frac{s!(|C_k|-s-1)!}{|C_k|!} \cdot (v(\cup_{h\in Q} C_h \cup S) - v(S))$$

$$- \sum_{Q \subseteq M\setminus k} \sum_{\substack{T \subseteq C_k: \\ i \notin T,\ t < |C_k|-1}} \frac{q!(m-q-1)!}{m!} \cdot \frac{t!(|C_k|-t-1)!}{|C_k|!} \cdot (v(\cup_{h\in Q} C_h \cup T) - v(T))$$

$$= \sum_{Q \subseteq M\setminus k} \frac{q!(m-q-1)!}{m!} \cdot \frac{1}{|C_k|} \cdot (v(\cup_{h\in Q} C_h \cup C_k \setminus i) - v(C_k \setminus i)).$$

The second equality comes from the fact that, for any coalition $\cup_{h\in Q} C_h \cup S$ where $Q \subseteq M \setminus k$, and $j \in S \subseteq C_k$, there are $q!(m-q-1)! \prod_{p\neq k} |C_p|!(s-1)!(|C_k|-s)!$ permutations for which the predecessors of the union C_k consist of the players in $\cup_{h\in Q} C_h$, and player $j \in S$ is both a successor of the players in $S \setminus j$ and a predecessor of the players in $C_k \setminus S$. Hence, for each permutation such that $p^{\pi_c}(N, C_k) = \cup_{h\in Q} C_h$, $p^{\pi_c}(C_k, j) = S$ and $\pi_c(j) = |\cup_{h\in Q} C_h| + s$, the marginal contributions of player $j \in S \subseteq C_k$ with respect to N and C_k are given by $v(\cup_{h\in Q} C_h \cup S) - (\cup_{h\in Q} C_h \cup S\setminus j)$ and $v(S) - v(S\setminus j)$ respectively. For Part *II*(2), observing that there are $q!(m-q-1)! \prod_{p\neq k} |C_p|!(|C_k|-1)!$ permutations where player $i \in C_k$ is the last entrant among her union members and players in $\cup_{h\in Q} C_h$ are the predecessors of the union C_k, we have

$$
\begin{aligned}
&\text{Part II(2)}\\
&= \sum_{\substack{\pi_c \in \Pi_{N,\mathcal{C}}:\\ \pi_c(i) = |p^{\pi_c}(N, C_k)| + |C_k|}} \frac{1}{|\Pi_{N,\mathcal{C}}|}\Big((v(p^{\pi_c}(N,i)) - v(p^{\pi_c}(N,i)\setminus i))\\
&\quad - (v(p^{\pi_c}(C_k,i)) - v(p^{\pi_c}(C_k,i)\setminus i)) - v(p^{\pi_c}(N,C_k))\Big)\\
&= \sum_{Q\subseteq M\setminus k} \frac{q!(m-q-1)!}{m!}\cdot\frac{1}{|C_k|}\cdot\Big(v(\cup_{h\in Q}C_h \cup C_k) - v(\cup_{h\in Q}C_h \cup C_k \setminus i)\\
&\quad - (v(C_k) - v(C_k\setminus i)) - v(\cup_{h\in Q}C_h)\Big)
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
\text{Part II} &= \text{Part II(1)} + \text{Part II(2)}\\
&= \sum_{Q\subseteq M\setminus k} \frac{q!(m-q-1)!}{m!}\cdot\frac{1}{|C_k|}\cdot\big(v(\cup_{h\in Q}C_h\cup C_k) - v(\cup_{h\in Q}C_h) - v(C_k)\big)\\
&= \frac{1}{|C_k|}\left\{\sum_{Q\subseteq M\setminus k}\frac{q!(m-q-1)!}{m!}\cdot\big(v(\cup_{h\in Q}C_h\cup C_k) - v(\cup_{h\in Q}C_h)\big) - v(C_k)\right\}\\
&= \frac{1}{|C_k|}\left\{\left(\sum_{Q\subseteq M\setminus k}\frac{q!(m-q-1)!}{m!}\cdot(v^{\mathcal{C}}(Q\cup k) - v^{\mathcal{C}}(Q))\right) - v(C_k)\right\}\\
&= \frac{Sh_k(M, v^{\mathcal{C}}) - v(C_k)}{|C_k|}
\end{aligned}
$$

Putting all this together, itis immediate that

$$
\psi_i(N, v, \mathcal{C}) = Sol_i(C_k, v_{|C_k}) + \frac{Sh_k(M, v^{\mathcal{C}}) - v(C_k)}{|C_k|} = TSS_i(N, v, \mathcal{C}),
$$

which completes the proof. □

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References

Alonso-Meijide JM, Costa J, García-Jurado I, Gonalves-Dosantos JC (2020) On egalitarian values for cooperative games with a priori unions. TOP 28:672–688
Aumann RJ, Dreze J (1974) Cooperative games with coalition structures. Int J Game Theory 3:217–237
Banzhaf JF (1965) Weighted voting does not work: a mathematical analysis. Rutgers Law Rev 19:317–343
Béal S, Remila E, Solal P (2017) Axiomatization and implementation of a class of solidarity values for TU-games. Theory Decis 83(1):61–94
Borkotokey S, Choudhury D, Gogoi L, Kumar R (2020) Group contributions in TU-games: a class of k-lateral Shapley values. Eur J Oper Res 286:637–648
Calvo E, Gutiérrez E (2010) Solidarity in games with a coalition structure. Math Soc Sci 60:196–203
Calvo E, Gutiérrez E (2013) The Shapley-solidarity value for games with a coalition structure. Int Game Theory Rev 15(01):1350002
Calvo E, Lasaga J, Winter E (1996) The principle of balanced contributions and hierarchies of cooperation. Math Soc Sci 31:171–182
Casas-Méndez B, García-Jurado I, van den Nouweland A, Vázquez-Brage M (2003) An extension of the τ-value to games with coalition structures. Eur J Oper Res 148:494–513
Driessen TSH, Funaki Y (1991) Coincidence of and collinearity between game theoretic solutions. OR Spectrum 13:15–30
Harsanyi JC (1963) A simplified bargaining model for the n-person cooperative game. Int Econ Rev 4(3):194–220
Hu X (2020) The weighted Shapley-egalitarian value for cooperative games with a coalition structure. TOP 28:193–212
Hu X, Li D (2018) A new axiomatization of the Shapley-solidarity value for games with a coalition structure. Oper Res Lett 46(2):163–167
Kamijo Y (2009) A two-step Shapley value in a cooperative game with a coalition structure. Int Game Theory Rev 11(2):207–214
Kamijo Y (2013) The collective value: a new solution for games with coalition structures. TOP 21(3):572–589
Myerson RB (1980) Conference structures and fair allocation rules. Int J Game Theory 9:169–182
Nowak AS, Radzik T (1994) A solidarity value for n-person transferable utility games. Int J Game Theory 23(1):43–48
Owen G (1977) Values of games with a priori unions. In: Henn R, Moeschlin O (eds) Mathematical Economics and Game Theory. Springer, Berlin, pp 76–88
Owen G (1982) Modifcation of the Banzhaf-Coleman index for games with a priori unions. In: Holler MJ (ed) Power, voting and voting power. Physica, Heidelberg, pp 232–238
Shapley LS (1953a) Additive and non-additive set functions. Ph.D. thesis, Princeton University, Princeton
Shapley LS (1953) A value for n-person games. In: Kuhn HW, Tucker AW (eds) Contributions to the Theory of Games, vol II. Princeton University Press, Princeton, pp 307–317
Sprumont Y (1990) Population monotonic allocation schemes for cooperative games with transferable utility. Games Econom Behav 2(4):378–394
Tijs S (1981) Bounds for the core and the τ-value. In: Moeschlin O, Pallaschke D (eds) Game Theory and Mathematical Economics. North-Holland, Amsterdam
van den Brink R, Funaki Y (2015) Implementation and axiomatization of discounted Shapley values. Soc Choice Welf 45:329–344
Xu G, Dai H, Hou D, Sun H (2016) A-potential function and a non-cooperative foundation for the solidarity value. Oper Res Lett 44:86–91
Young HP (1985) Monotonic solutions of cooperative games. Int J Game Theory 14:65–72
Zou R, Xu G, Li W, Hu X (2020) A coalitional compromised solution for cooperative games. Soc Choice Welf 55:735–758