



# Visual Proof of Catalan's Identity for the Fibonacci Numbers

Nam Gu Heo

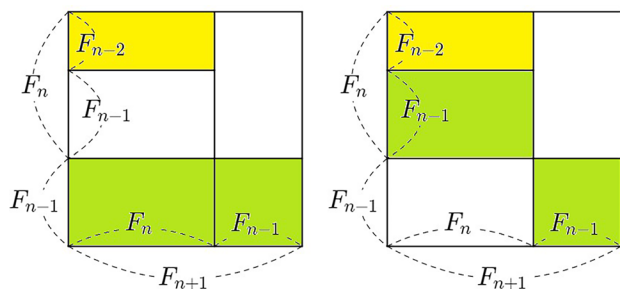
**C**atalan's identity. Let  $F_n$  denote the  $n$ th Fibonacci number with  $F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1}$ . Then the identity  $F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r}F_r^2$  holds for all positive integers.

**Lemma.** Let  $f(n, r) = F_n^2 - F_{n+r}F_{n-r}$ . Then  $f(n-1, r) + f(n, r) = 0$ .

That is,  $F_{n+r}F_{n-r} + F_{n+r-1}F_{n-r-1} = F_n^2 + F_{n-1}^2$ .

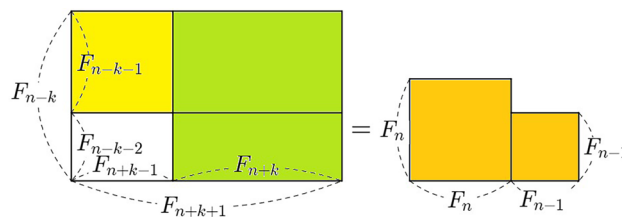
*Proof.* By induction.

When  $r = 1$ , we have  $F_{n+1}F_{n-1} + F_nF_{n-2} = F_n^2 + F_{n-1}^2$ .

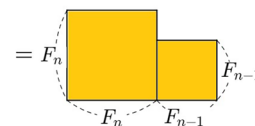
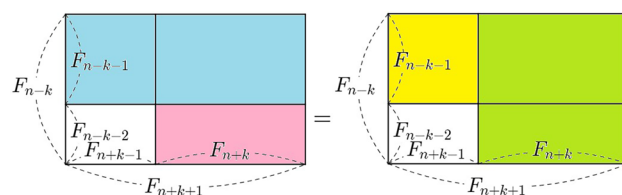


Suppose that  $F_{n+r}F_{n-r} + F_{n+r-1}F_{n-r-1} = F_n^2 + F_{n-1}^2$  is true when  $r = k$ .

That is,  $F_{n+k}F_{n-k} + F_{n+k-1}F_{n-k-1} = F_n^2 + F_{n-1}^2$ .



When  $r = k + 1$ , we must show that  $F_{n+k+1}F_{n-k-1} + F_{n+k}F_{n-k-2} = F_n^2 + F_{n-1}^2$ .



**Theorem (Catalan's identity).**  $F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r}F_r^2$ .

*Proof.* Since  $f(n, r) = -f(n-1, r)$ ,  $f(r, r) = F_r^2$ , it follows that  $f(n, r) = (-1)^{n-r}F_r^2$ . That is,  $F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r}F_r^2$ .

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