Note

Visual Proof of Catalan's Identity for the Fibonacci Numbers

Nam Gu Heo

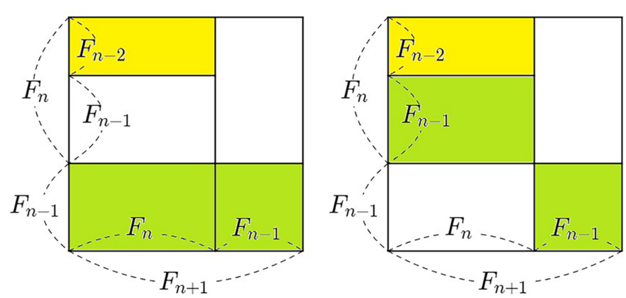
Catalan's identity. *Let F_n denote the nth Fibonacci number with $F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1}$. Then the identity $F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r}F_r^2$ holds for all positive integers.*

Lemma. *Let $f(n, r) = F_n^2 - F_{n+r}F_{n-r}$. Then $f(n-1, r) + f(n, r) = 0$.*

That is, $F_{n+r}F_{n-r} + F_{n+r-1}F_{n-r-1} = F_n^2 + F_{n-1}^2$.

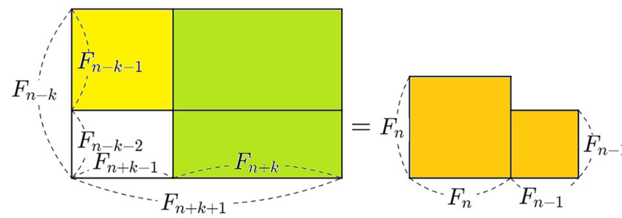
Proof. By induction.

When $r = 1$, we have $F_{n+1}F_{n-1} + F_nF_{n-2} = F_n^2 + F_{n-1}^2$.

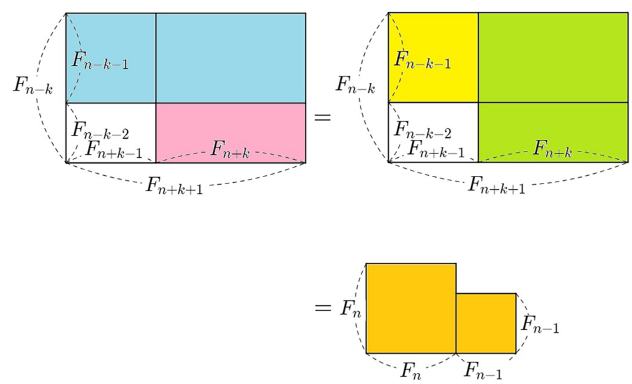


Suppose that $F_{n+r}F_{n-r} + F_{n+r-1}F_{n-r-1} = F_n^2 + F_{n-1}^2$ is true when $r = k$.

That is, $F_{n+k}F_{n-k} + F_{n+k-1}F_{n-k-1} = F_n^2 + F_{n-1}^2$.



When $r = k + 1$, we must show that $F_{n+k+1}F_{n-k-1} + F_{n+k}F_{n-k-2} = F_n^2 + F_{n-1}^2$.



Theorem (Catalan's identity). $F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r}F_r^2$.

Proof. Since $f(n, r) = -f(n-1, r)$, $f(r, r) = F_r^2$, it follows that $f(n, r) = (-1)^{n-r}F_r^2$. That is, $F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r}F_r^2$.

Nam Gu Heo, Department of Mathematics Education, Sunchon National University, Suncheon, Republic of Korea. E-mail: ngheo@scnu.ac.kr

https://doi.org/10.1007/s00283-024-10340-7