Letter to the Editors

The Volume of a Reversible Tetrahedron

Martin Lukarevski

In his recent article "Tetrahedra with Congruent Facet Pairs" in this journal [1], Daniel A. Klain defines a reversible tetrahedron as one having two pairs of congruent facets. For such a tetrahedron T with edge lengths a, b, c, a, b, d and facets $f_1 \cong f_2 = (a, b, c)$, $f_3 \cong f_4 = (a, b, d)$, he gives a nice factorization of its volume $V(T)$ by the formula

$$144V(T)^2 = \left(cd + a^2 - b^2\right)\left(cd - a^2 + b^2\right) \cdot \left(2a^2 + 2b^2 - c^2 - d^2\right). \tag{1}$$

The class of more regular isosceles tetrahedra with all four facets congruent with edges a, b, c has the beautiful factorization

$$72V(T)^2 = \left(a^2 + b^2 - c^2\right)\left(a^2 - b^2 + c^2\right)\left(-a^2 + b^2 + c^2\right),$$

akin to Heron's formula for the area of a triangle.

Here we present a more elementary proof of (1) than that given in [1] in which the trapezoidal law from geometry and unique factorization domains from algebra are used. Since an isosceles trapezoid is a cyclic quadrilateral, the trapezoid law $b^2 = a^2 + cd$ given in the article for an isosceles trapezoid with sides a, c, d, where c and d are parallel and b is the diagonal, follows directly from Ptolemy's theorem for a cyclic quadrilateral, which states that $ef = ac + bd$, where e, f are the diagonals.

Now let $T = ABCD$ be a general tetrahedron with squared lengths of the edges $p = |AB|^2$, $q = |BC|^2$, $r = |CA|^2$, $u = |DC|^2$, $v = |DA|^2$, $w = |DB|^2$. Then the volume of T is given by the Cayley–Menger determinant [2, pp. 285–289]

$$288V(T)^2 = \begin{vmatrix} 0 & p & q & r & 1 \\ p & 0 & w & v & 1 \\ q & w & 0 & u & 1 \\ r & v & u & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}.$$

Martin Lukarevski, Department of Mathematics and Statistics, University "Goce Delcev" - Stip, Stip, North Macedonia. E-mail: martin.lukarevski@ugd.edu.mk

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Expansion of the determinant, whereby products of the opposite edges are extracted, yields

$$\begin{aligned}144V(T)^2 &= pu(-p + q + r - u + v + w) \\ &+ qv(p - q + r + u - v + w) + rw(p + q - r + u + v - w) \\ &- (pqw + qru + rpv + uvw).\end{aligned}$$

We apply this formula to a reversible tetrahedron with edge lengths a, b, c, a, b, d. Then $p = u = a^2$, $q = v = b^2$, $r = c^2$, and $w = d^2$ give for its volume

$$\begin{aligned}144V(T)^2 &= a^4\left(-2a^2 + 2b^2 + c^2 + d^2\right) \\ &+ b^4\left(2a^2 - 2b^2 + c^2 + d^2\right) + c^2d^2\left(2a^2 + 2b^2 - c^2 - d^2\right) \\ &- 2\left(a^2b^2c^2 + a^2b^2d^2\right).\end{aligned}$$

Rearranging the right hand side, we obtain

$$\begin{aligned}144V(T)^2 &= c^2d^2\left(2a^2 + 2b^2 - c^2 - d^2\right) - 2a^2b^2\left(c^2 + d^2\right) \\ &\quad + a^4\left(-2a^2 + 2b^2 + c^2 + d^2\right) + b^4\left(2a^2 - 2b^2 + c^2 + d^2\right) \\ &= c^2d^2\left(2a^2 + 2b^2 - c^2 - d^2\right) - 2a^2b^2\left(c^2 + d^2\right) \\ &\quad - a^4\left(2a^2 + 2b^2 - c^2 - d^2\right) - b^4\left(2a^2 + 2b^2 - c^2 - d^2\right) \\ &\quad + 4a^2b^2\left(a^2 + b^2\right) \\ &= c^2d^2\left(2a^2 + 2b^2 - c^2 - d^2\right) + 2a^2b^2\left(2a^2 + 2b^2 - c^2 - d^2\right) \\ &\quad - a^4\left(2a^2 + 2b^2 - c^2 - d^2\right) - b^4\left(2a^2 + 2b^2 - c^2 - d^2\right) \\ &= \left(2a^2 + 2b^2 - c^2 - d^2\right)\left(c^2d^2 + 2a^2b^2 - a^4 - b^4\right) \\ &= \left(2a^2 + 2b^2 - c^2 - d^2\right)\left(cd + a^2 - b^2\right)\left(cd - a^2 + b^2\right).\end{aligned}$$

The last line is the factorization of the volume (1).

References

[1] D. A. Klein. Tetrahedra with Congruent Facet Pairs. *Mathematical Intelligencer* 45:4 (2023), 251–255

[2] H. Dörrie. *100 Great Problems of Elementary Mathematics*. Dover, 1965.

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