



Wonder Cubes: Theme and Variations

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Theme: A magic square is an $n \times n$ square subdivided into n^2 cells inscribed with disjoint integers arranged in such a way that the sums of the integers in all the rows and all the columns and the two diagonals are equal to the same constant, which is called the magic constant. A famous example, known as the Lo Shu square,¹ is shown in Figure 1.

The magic constant of the Lo Shu magic square is 15, which happens to be the number of days in each of the 24 cycles (or terms) of the Chinese solar year. The Lo Shu magic square is mentioned in a Chinese legend dated several centuries before the Common Era. Since then, and throughout the following centuries, magic squares have been studied by hobbyists, astrologers, alchemists, and mathematicians. They have been used as talismans and amulets and have stirred the imagination of recreational mathematicians.

Magic squares have been generalized in many ways, such as multidimensional, with words or geometric figures instead of numbers, and with numerous rows and columns. Indeed, there are more than three hundred papers in the literature of recreational mathematics on the subject of magic squares. In this paper, we will generalize magic squares in a new way, forming what we shall call a wonder cube. We shall consider 3×3 squares only and set six such squares with the same magic constant on the six faces of a cube. The squares will be chosen in such a way to create magic properties linking neighboring squares (SameSum links).

The Wonder Cube

We begin with some notation and definitions.

Definition 1.

1. A magic square is *positive* if all its entries are positive. For example, the magic square shown in Figure 1 is positive.
2. The *magic constant* is the constant to which each row, each column, and the two diagonals of the magic square sum. For example, the magic constant of the magic square in Figure 1 is 15.
3. Two magic squares are *equivalent* if one can be derived from the other by rotation or reflection.
4. The *pivot* is the integer located at the center of the magic square. For example, the pivot of the magic square in Figure 1 is 5.

This column is a place for those bits of contagious mathematics that travel from person to person in the community, because they are so elegant, surprising, or appealing that one has an urge to pass them on. Contributions are most welcome.

Submissions should be uploaded to <http://tmin.edmgr.com> or sent directly to Sophie Morier-Genoud (sophie.morier-genoud@univ-reims.fr) or Valentin Ovsienko (valentin.ovsienko@univ-reims.fr).

¹For more on this, see the article “Lo Shu Square” in Wikipedia.

4	9	2
3	5	7
8	1	6

Figure 1. The Lo Shu magic square.

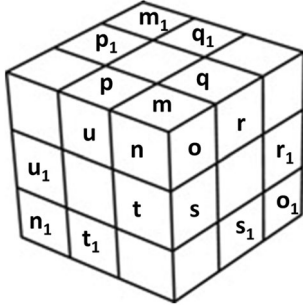


Figure 2. Groups of integers that show SameSum magic links on a cube. m, n, o : corner triplet; $m_1n_1o_1$: associated corner triplet; p, q, r, s, t, u : corner ring 6-tuple; $p_1, q_1, r_1, s_1, t_1, u_1$: associated corner ring 6-tuple.

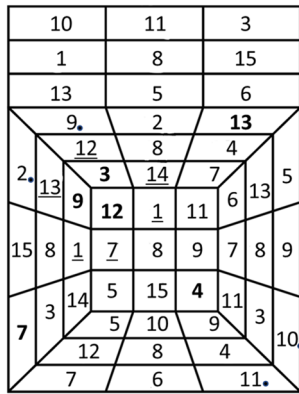


Figure 3. Unfolded wonder cube with pivot 8. The exemplary associated triplet of (9, 12, 3), shown in boldface, is (7, 4, 13), also in boldface. The corner ring 6-tuple of (9, 12, 3), (1, 13, 7, 1, 12, 14), is underlined. The exemplary complementary pair 4-tuple (2, 9, 10, 11) is marked with dots.

- An *associated pair of integers* comprises the two integers on either side of the pivot on a line passing through the pivot. There are four such pairs in every magic square. For example, the pairs (2, 8) and (3, 7) are each an associated pair of integers in the magic square shown in Figure 1.
- A *half-diagonal* is a pair of integers located on one of the lines parallel to one of the diagonal lines of three integers

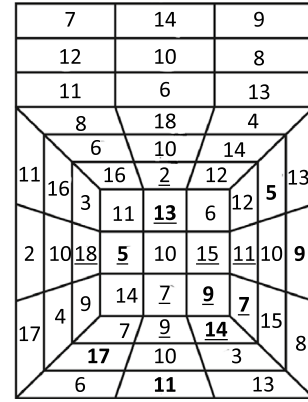


Figure 4. Unfolded wonder cube with pivot 10. The exemplary associated corner ring 6-tuple for (9, 7, 14) is (17, 11, 5, 13, 5, 9), all shown in boldface. Two exemplary midedge hoop 4-tuples are (13, 5, 7, 15) and (9, 11, 2, 18), underlined.

$c - b$	$c + a + b$	$c - a$
$c - a + b$	c	$c + a - b$
$c + a$	$c - a - b$	$c + b$

Figure 5. Eduard Lucas's formula.

passing through the pivot. For example, the pairs (9, 7), (7, 1), (1, 3), and (3, 9) in Figure 1 are half-diagonals.

Introducing Wonder Cubes

Figure 2 shows groups of integers on a cube that represent SameSum magic links in a wonder cube. Note that in most cases, we represent cubes in planar form, so as to retain as much of their structure as possible, by unfolding the cube and stretching squares into trapezoids or rectangles. Figure 3 shows such an unfolded cube. The 3×3 square in the middle represents the bottom face of the cube, the four trapezoids around it represent the four side faces of the cube around the bottom face, and the top rectangle represents the top face of the cube.

Definition 2. A *corner triplet* is a triplet of integers located around the corner of a cube having six inequivalent magic squares inscribed on its six faces. For example, the integers in the locations m, n , and o in Figure 2 form a corner triplet.

Definition 3. A *wonder cube*, as exemplified in Figures 3 and 4, is a cube inscribed with six magic squares on its six faces having the following properties:

1. The six magic squares are positive and inequivalent and have the same pivot: c .
2. The eight corner triplets of the cube sum to the same magic constant ($3c$) as the magic constant of the six magic squares.

For example, in the wonder cubes shown in Figures 3 and 4, the pivots are equal respectively to 8 and 10, and the eight corner triplets of the two wonder cubes sum respectively to 24 and 30.

A wonder cube is different from a magic cube: It should be noted that magic cubes have been studied in the literature,² where they have been defined as a three-dimensional analogue of magic squares, or a cube subdivided into $3 \times 3 \times 3$ cells inscribed with twenty-seven different integers arranged in such a way that the three integers in every row, in every column, in every pillar, and in the four main space diagonals sum to the same magic constant. It is clear that the magic cube is completely different from the wonder cube. The wonder cube is, in fact, a two-dimensional object consisting of six magic two-dimensional squares inscribed on its six faces and includes fifty-four not necessarily distinct integers. So the notion of wonder cubes is a new and unique notion.

A Formula for Generating 3×3 Magic Squares

We will consider in this paper 3×3 magic squares only. For such squares, a mathematician by the name of Eduard Lucas developed in the nineteenth century a general formula, shown in Figure 5. He proved that every 3×3 positive magic square satisfies this formula or can be derived by means of this formula, where the following inequalities must hold: For positive squares, one must have $0 < a < b < c - a$, with b not equal to $2a$. The restrictions imposed on the three parameters a, b, c are dictated by the requirement that all nine integers included in a magic square be positive and distinct. Thus, if one were to allow $b = 2a$, then the second entry in the last column of Figure 5, $c + a - b$, would become $c + a - 2a = c - a$, which is equal to the first entry in that column, violating the rule of distinct integers. And if the inequality $b < c - a$ did not hold, then the bottom entry of the middle column would represent a nonpositive integer.

In addition to the fact that the integers in the square are disjoint, it follows from the formula in Figure 5 that the integers have the following properties: There are three numbers (a, b, c in Figure 5) that determine the integers in the magic square: c , the pivot, is a free parameter, and a and b are bounded by c , which is the average of all the integers in the square and is located in the center cell of the square. The magic constant is equal to $3c$. The maximal integer in the square is bounded by $2c - 1$.

It is clear that all magic squares with the same pivot have the same magic constant, and every pair of associated integers sum to $2c$. In addition, all magic squares satisfy the

“half-diagonal property,” namely, that the sum of the two integers in a half-diagonal pair is an even integer equal to twice the value of the corner integer located on the other side of the main diagonal that is parallel to the half-diagonal. This property can be verified easily in Figure 5, where the half-diagonal formed by the integers in the middle of the upper row and the middle of the leftmost column ($c + a + b, c - a + b$) sum to $2(c + b)$, which is twice the value of the bottom right-hand corner.

Determining the Number of Inequivalent Magic Squares

We can now prove the following result.

Theorem 1. For positive magic squares whose pivot c is odd, the number of inequivalent magic squares is

$$\frac{c-1}{2} \cdot \frac{c-3}{2} - \left\{ \frac{c-1}{3} \right\},$$

where $\{x\}$ is the greatest integer less than or equal to x . If c is even, then this number is

$$\left(\frac{c}{2} - 1 \right)^2 - \left\{ \frac{c-1}{3} \right\}.$$

Proof. We prove the case with an even c for positive magic squares. The other case is proved in a similar way. It follows from Lucas’s formula that the parameters a and b that determine the square together with c must satisfy the following inequalities: $a < b < c - a$ and $b \neq 2a$. The first condition requires that $a + b < c$, so we must find how many pairs (a, b) satisfy this condition. Clearly, if $a = 1$, then b can assume one of the values $2, 3, \dots, c - 2$, and there are therefore $c - 3$ pairs (a, b) with $a = 1$ that satisfy the first condition.

In the same way, we find that the number of pairs with $a = 2$ satisfying this condition is $c - 5$. Then for $a = 3$, the number of pairs is $c - 7$, etc. So the total number of such pairs is equal to the sum of the arithmetic series $1, 3, 5, \dots, c - 3$. Using the well-known formula for finding the sum of this series, we find that the number of such pairs, which is equal to the number of inequivalent magic squares with pivot equal to c , is $(c/2 - 1)^2$, as required. We still have to remove from this count the pairs $(a, 2a)$. It is easy to see that the number of such pairs with $a + 2a = 3a < c$ is equal to the greatest integer that is less than or equal to $c/3$. This completes the proof. \square

As an example, we can find by applying the above formula that the number of inequivalent magic squares with c equal to 10 is 13.

The number of possible inequivalent magic squares with the same pivot on a cube: It follows from the above formulas that the number of cubes having their six faces inscribed with inequivalent magic squares having the same pivot c can be determined as follows.

²For an introduction to magic squares and magic cubes and suggestions for further reading, see the Wikipedia articles on those and related topics as well as the Wolfram MathWorld article on magic cubes, available at <https://mathworld.wolfram.com/MagicCube.html>, and the website “Magic Squares, Magic Stars & Other Patterns,” available at <http://recmath.org/Magic%20Squares/>.

Let $M(c)$ be the number of inequivalent magic squares whose pivot is equal to c . Then the number of cubes as above is calculated as follows: Choose one of the $M(c)$ squares for the first face of the cube, then choose one of the remaining $M(c) - 1$ squares for the second face, etc. Then every chosen square can be set in one of its eight equivalent forms (four rotations plus four rotations after reflection). All in all, this amounts to

$$M(c) \times (M(c) - 1) \times \dots \times (M(c) - 6) \times (8^6).$$

It can be shown using the above formula that for c as small as 10, the number of such cubes exceeds two trillion, so if we want to find a cube with special magic properties in addition to the magic properties of the magic squares inscribed on its faces, an exhaustive search is impractical.

Minimal pivot of magic squares for wonder cubes:

To create a wonder cube with inequivalent positive magic squares inscribed on its six faces, we shall require at least six inequivalent positive magic squares. By Theorem 1, we shall therefore need a pivot of at least 8.

It also follows from Theorem 1 that the number of positive inequivalent magic squares with pivot equal to 8 or more grows quadratically with the value of the pivot. So when the value of the pivot grows, the number of cubes that satisfy the first condition of Definition 3 for wonder cubes also grows exponentially, thus increasing the probability that there exist wonder cubes with a large pivot. Some examples of wonder cubes with various pivots are shown in the sequel. However, we do not have an algorithm that enables the creation of wonder cubes for a given sufficiently large pivot.

Wonder Properties of Wonder Cubes

Two wonder cubes are shown schematically in Figures 3 and 4. Let us consider the cube in Figure 3. The pivot of each of the magic squares on its faces is equal to 8, so that the magic constant of each of the squares is 24. This cube is, in fact, an example of the kind of cubes we want to consider in this article. In addition to the fact that its six faces are inscribed with inequivalent magic squares, notice that its eight corner triplets sum to the magic constant of the magic squares on its faces. We defined such cubes as wonder cubes. As we describe and show in the theorem below, they deserve the epithet “wonder” because of the numerous wondrous SameSum properties they exhibit.

We need some additional definitions before proving the main property of wonder cubes in Theorem 2.

Definition 4.

1. An *associated corner triplet* is a triplet of integers that are associated with the three integers in a corner triplet. For example, (m_1, n_1, o_1) is the associated corner triplet of (m, n, o) in Figure 2, and the associated corner triplet of the corner triplet $(9, 12, 3)$, shown in boldface in Figure 3, is $(7, 4, 13)$, also in boldface.

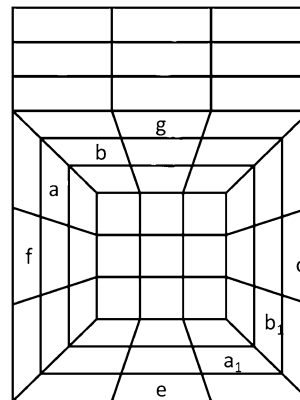


Figure 6. Schematic presentation of a wonder cube. A complementary pair of integers 4-tuple is $(a, b), (a_1, b_1)$. A midedge hoop 4-tuple is (g, d, e, f) .

2. A *corner ring 6-tuple* is a ring of six integers formed by the six integers inscribed in the three half-diagonals set around a corner integer. For example, the corner ring 6-tuple of the corner triplet $(9, 12, 3)$, in boldface in Figure 3, is $(1, 13, 7, 1, 12, 14)$, underlined in the figure.
3. An *associated corner ring 6-tuple* is a 6-tuple of integers that are associated with the six integers in a corner ring 6-tuple. For example, for $(9, 7, 14)$ in Figure 4, the associated ring 6-tuple is $(5, 13, 5, 9, 17, 11)$, shown in boldface.
4. A *midedge hoop 4-tuple* is the 4-tuple formed by the four integers located in the middle of four horizontal edges of the cube wrapped around the cube at the same level for any orientation of the cube in space. For example, two midedge hoop 4-tuples in Figure 4 are $(13, 5, 6, 16)$ and $(9, 11, 2, 18)$.
5. *Complementary pairs of integer 4-tuples* include two pairs of integers located at the same level (top, middle, or bottom), on both sides of two parallel edges of the cube, edges that are not bordering the same face, for any orientation of the cube in space; (a, b) and (a_1, b_1) in Figure 6. For example, $(2, 9)$ and $(10, 11)$ are a complementary pair 4-tuple in Figure 3, marked with a dot.

The eight corner triplets of a wonder cube whose pivot is equal to c sum to $3c$ by definition. The next theorem states that wonder cubes have many more SameSum properties.

Theorem 2.

1. *The eight associated corner triplets of a wonder cube sum to $3c$.*
2. *The eight corner ring 6-tuple and the eight associated corner ring 6-tuples of a wonder cube whose pivot is equal to c sum to $6c$.*

3. The eighteen complementary pairs of integer 4-tuples of a wonder cube whose pivot is equal to c sum to $4c$.
4. The nine midedge hoop 4-tuples of a wonder cube whose pivot is equal to c sum to $4c$.

Proof. 1. As follows from Lucas's formula, every pair of associated integers sum to $2c$. Therefore, the three integers in a corner triplet together with the three integers in the associated corner triplet sum to $6c$. Since the three integers in the corner triplet sum to $3c$, it follows that the three integers in the associated corner triplet also sum to $3c$.

2. By the half-diagonal property, the three half-diagonals around a corner that form a corner ring 6-tuple sum to twice the value of the corresponding associated corner triplet, which sums to $3c$ by assertion 1 above. Therefore, the corresponding ring 6-tuple sums to $6c$. Using a similar argument to that used in proving assertion 1 above, we can prove that the associated ring 6-tuple also sums to $6c$.

3. It is easy to see that for every orientation of the cube in space, two complementary pairs of integers at the top level or the bottom level of the cube can be augmented to two corner triplets located on both sides of a top-or-bottom corresponding diagonal of the top-or-bottom face of the cube. Now, the two top or the two bottom corner triplets sum to $6c$ together, and the added two integers sum to $2c$, since they are associated corner integers, implying that the two complementary pairs sum to $6c - 2c = 4c$.

4. For every orientation of the cube in space, the four rows around the top or the bottom of the cube sum together to $4 \times 3c = 12c$. Since those rows are edges of magic squares with the same pivot c , the twelve integers on those four rows can be separated into three sets. One set consists of the four integers included in the midedge hoop 4-tuple. The other two sets are two complementary pairs of integer 4-tuples located around the two pairs of opposite corners of the square made by the four edges. So we have the following count: The twelve integers in the four rows around the top or bottom of the cube sum to $12c$. The eight integers forming the two complementary pairs of 4-tuples sum to $8c$, as proved above, implying that the remaining midedge 4-tuple sums to $12c - 8c = 4c$, as required. \square

SameSum tuple groups: It follows from Lucas's formula that with a pivot equal to c , a cube that satisfies the first property of a wonder cube in Definition 3 includes the following groups of SameSum tuples:

- four associated pairs that sum to $2c$ each, on each of its faces, a total of $4 \times 6 = 24$ pairs;
- eight triplets (for three rows, three columns, and two diagonals) that sum to $3c$ each, on each of its faces, a total of $6 \times 8 = 48$ triplets.

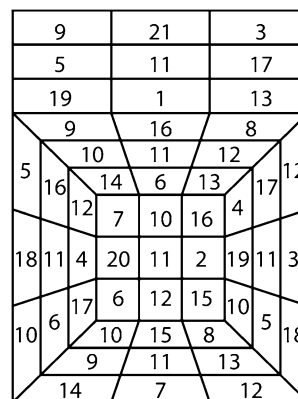


Figure 7. Unfolded wonder cube with pivot 11.

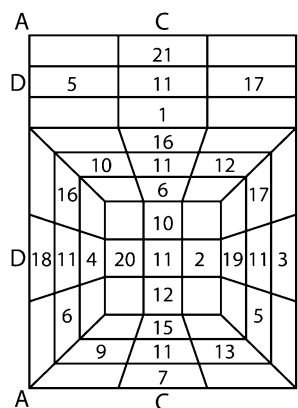


Figure 8. Cross-shaped wonder cube with pivot 11.

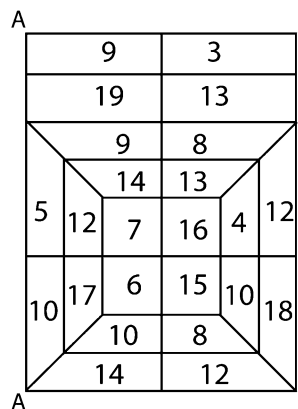


Figure 9. Unfolded $2 \times 2 \times 3$ wonder cube with pivot (hidden) 11.

This yields a total of $48 + 24 = 72$ SameSum tuples.

It follows from Theorem 2 that a wonder cube has many more SameSum tuples that link between the faces of the cube:

- 16 corner triplets and their associated triplets that sum to $3c$ each;
- 18 complementary pairs of 4-tuples that sum to $4c$ each;
- 9 midedge hoop 4-tuples that sum to $4c$ each;
- 16 corner ring 6-tuples and their associated corner ring 6-tuples that sum to $6c$ each.

Which yields a total of 59 SameSum tuples.

It follows, therefore, that wonder cubes whose pivot is equal to c include 131 SameSum tuples over its 54, not necessarily distinct, integers inscribed on its six faces.

Number of wonder cubes for a fixed pivot: It follows from Theorem 2 that the eight corners of the cube have the same SameSum properties: every corner of the cube is linked to a corner triplet and an associated corner triplet that sum to $3c$ each, and a corner ring 6-tuple and an associated corner ring 6-tuple that sum to $6c$ each. Also, every set of four rows or columns that wrap around the cube, in any of its orientations along one of the three dimensions of space, have the same SameSum properties: every such set of four rows or columns includes two complementary pairs of integer 4-tuples and one midedge hoop 4-tuple that sum to $4c$ each.

We may ask now how many wonder cubes there are for a fixed pivot c . This question is open and will be further considered.

However, if we do not fix the pivot, then we can obtain an unbounded number of wonder cubes. The wonder cube whose pivot is equal to 8 is shown in Figure 3. As previously mentioned, there is no wonder cube whose pivot is less than or equal to 8. But for every integer $q \geq 9$, if we add the integer $q - 8$ to all the integers included in the wonder cube whose pivot is equal to 8, we get a wonder cube whose pivot is equal to q . So we can create a wonder cube with pivot equal to any integer $q \geq 9$. Figure 7 shows an additional wonder cube whose pivot is equal to 11.

Variations

The SameSum Family

In this section of the paper we will show how to split a wonder cube into two solid complementary bodies in such a way that the wonder properties of the cube are inherited and distributed between the two bodies. This can be done in several ways, and we will show one of them.

Decomposition of a Wonder Cube into Derived Bodies

This decomposition is illustrated in Figures 7, 8, and 9. Given a wonder cube with pivot equal to c , e.g., $c = 11$ in Figure 7, remove first all 24 corner cells from the cube, as in Figure 8, resulting in a cube whose six faces are

cross-shaped. As an alternative, remove from the wonder cube's six faces the middle rows and the middle columns, as in Figure 9, resulting in a cube whose faces are 2×2 squares. We will call such a cube a $2 \times 2 \times 3$ cube. We show below that the wonder properties of the wonder cube are distributed between these two solid bodies and that all the integers on the wonder cube are distributed between these two bodies.

We will consider now these two bodies one by one.

The Cube with Cross-Shaped Faces

It is easy to verify that this cube inherits the following properties from the wonder cube source:

1. the six midedge hoop 4-tuples that sum to $4c$ ($= 44$ in Figure 8);
2. the six complementary pairs of integer 4-tuples that sum to $4c$;
3. the eight corner ring 6-tuples that sum to $6c$.
4. the eight associated corner ring 6-tuples that sum to $6c$.
5. the twelve arms of the six crosses that each sum to $3c$.

It is possible to reconstruct the source wonder cube from this derived cube using the half-diagonal property, which is also inherited by the derived cube. By this property, every missing corner in the derived cube is equal to half the sum of the two integers forming the half-diagonal on the other side of the main diagonal facing the corner.

Cross-shaped wonder cubes can be defined independently and not as a derived cube. In order to do this, we need first the following definition: for a given pivot c , let (a, b) be a pair of positive integers such that $a < b$, $a + b < c$, and b is not equal to $2a$. We will denote by (a, b, L) (where L stands for Lucas) the 3×3 cross-shaped form whose horizontal arm is $(c - a + b, c, c + a - b)$ and whose vertical arm is $(c + a + b, c, c - a - b)$. Note that both arms sum to $3c$ each. Also, the left and right integers of the horizontal arm as well as the top and bottom integers of the vertical arm sum to $2c$; i.e., these two integers in each arm are associated.

Definition 5. A C -wonder cube is a cube with the following properties:

1. An (a, b, L) cross-shaped form or a form equivalent to it is inscribed on the i th face of the cube, $i = 1, \dots, 6$, and the pairs of integers (a_i, b_i) , $i = 1, \dots, 6$, are distinct.
2. The eight corner ring 6-tuples of the cube sum to $6c$ each.

We can now prove the following:

Proposition 1. C -wonder cubes have the same properties 1 through 5 that hold for cross-shaped cubes derived from a wonder cube by deleting from it its eight corner triplets.

Proof. Properties 3 and 5 follow from the definitions. Property 4 follows from the fact that the three half-diagonals that

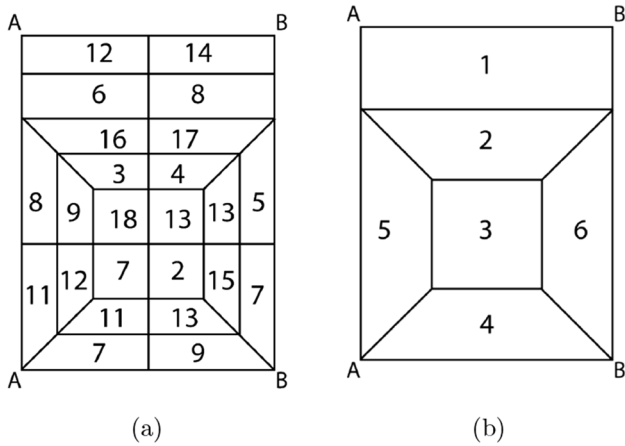


Figure 10. (a) A $2 \times 2 \times 3$ unfolded wonder cube that cannot be extended to a $3 \times 3 \times 3$ cube. (b) Location map to the various 2×2 squares of the cube in (a).

form a corner ring 6-tuple sum to the same value as the three associated half-diagonals that form an associated corner ring 6-tuple.

To prove properties 1 and 2, we proceed as follows: The cross-shaped forms on every face of the cube fit, by definition, the middle rows and columns in the Lucas formula, or in an equivalent formula. Therefore, by extending the cross-shaped cube by inserting at its corners half the sum of the corresponding half-diagonals, we get a cube whose faces are inscribed with magic squares that fit the Lucas formula, whose pivot is c , and whose corner triplets sum to $3c$, since the half-diagonals corresponding to the corner triplets sum to $6c$. So the resulting cube is a wonder cube, and the cross-shaped cube we started with is a derived cross-shaped cube from the extended wonder cube. This implies that the derived cross-shaped cube has properties 1 and 2. \square

Corollary 1. *The cross-shaped cube derived from a wonder cube is a C-wonder cube.*

The $2 \times 2 \times 3$ Derived Cube

It is well known that it is impossible to find four distinct positive integers arranged in a 2×2 square such that the two rows, the two columns, and the two diagonals sum to the same constant. Indeed, at most only one of the three pairs can sum to the same constant. We will consider here the case that the two diagonals sum to the same constant, which will be denoted by $2c$, where c is a hidden integer. One can define $2 \times 2 \times 3$ wonder cubes independently of $3 \times 3 \times 3$ wonder cubes as follows.

Definition 6. A $2 \times 2 \times 3$ wonder cube is a cube whose six faces are inscribed with 2×2 squares having the following properties:

- Every square includes four distinct positive integers.
- No two of the six squares are equivalent.
- All 12 diagonals on the squares sum to the same constant, an even integer to be denoted by $2c$, where c is a (hidden) pivot.
- Each of the eight corner triplets of the cube sums to $3c$.

A $2 \times 2 \times 3$ wonder cube whose hidden pivot is equal to 11 is shown in Figure 9. One would expect that every $2 \times 2 \times 3$ wonder cube could be expanded into a $3 \times 3 \times 3$ wonder cube by inserting proper missing integers into the rows and columns of the 2×2 squares so as to transform them into 3×3 magic squares. This is not the case. Consider, for example, the cube shown in Figure 10. The locations of the squares we refer to are indicated by the numbers 1 through 6 in the diagram of Figure 10(b). To expand the 2×2 square at location 1 into a magic square, we must add in the middle of the first column the integer 12 so that the column will sum to $3c = 30$, but this will create a column with two equal integers. Also, the 2×2 square at location 2 cannot be expanded into a magic 3×3 square, since the two integers in its first row sum to more than $3c = 30$, so we will have to add in this row a negative integer in order to get a row that sums to $3c = 30$.

One can prove now, in the same way as was proved for $3 \times 3 \times 3$ wonder cubes, that $2 \times 2 \times 3$ wonder cubes have the following properties:

- Its eight dual corner triplets sum to $3c$.
- Its twelve complementary pair 4-tuples sum to $4c$.

Remark 1. It follows from the above considerations that the SameSum tuples that exist in the $3 \times 3 \times 3$ wonder cube in Figure 7 are distributed among the cubes shown in Figures 8 and 9.

It also follows from the above considerations that the set of $2 \times 2 \times 3$ wonder cubes induced by $3 \times 3 \times 3$ wonder cubes is a subset of the set of $2 \times 2 \times 3$ wonder cubes.

Assessing the sizes of the above two sets and that of the set of $3 \times 3 \times 3$ wonder cubes are open problems.

The Star of David

We will consider now the SameSum properties of stars of David. It has been shown [1, 2] that it is possible to attach the integers 1 to 12 to the six vertices of the two triangles and the six intersections of the triangles of a star of David in such a way that the sum of the four integers along any of the six edges of the star sum to the same constant, to be called the magic constant of the star, which is 26. It is easy to prove, and left to the reader, that it is impossible to construct a star of David inscribed with twelve distinct

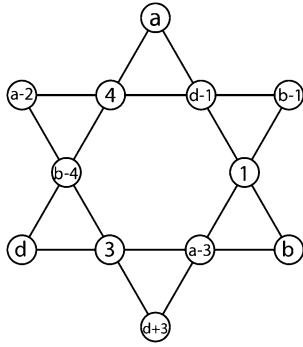


Figure 11. A formula for creating stars of David with magic constant equal to 26 or greater than or equal to 28.

positive integers whose magic constant is less than 26. We now specify this property of stars of David:

Property 1. A star of David can be inscribed with twelve distinct positive integers on the six vertices of its two triangles and on the six intersections of the two triangles in such a way that the four integers along each of the six edges of the two triangles sum to the same constant (the magic constant of the star).

Definition 7. A magic star of David is a star of David that satisfies Property 1.

However, Property 1 is not the only property of magic stars of David. Another property, implied by Property 1, is the following.

Property 2. The three integers inscribed on the vertices of one of the two triangles of a star of David sum to the same constant as the three integers inscribed on the vertices of the second triangle of the star.

To prove this property, we notice that by Property 1, the 12 integers inscribed along the three edges of each triangle sum to $3c$, where c is the magic constant of the star. This sum includes twice the sum of the three integers inscribed on the vertices of the triangle, since every vertex belongs to two edges, plus the six integers inscribed on the intersection of the triangles. Therefore, removing from the total sum the sum of the six integers at the intersection of the triangles, we get twice the sum of the integers at the three vertices of the triangle. This holds for both triangles, implying that the sums of the three integers at the vertices of the two triangles are equal to the same constant, $3c$ minus the sum of the six integers at the intersection of the two triangles. However, this constant depends on the sum of the six integers at the intersection of the triangles and is not necessarily equal to c .

This brings us to another property, which is an extension of Property 2:

Property 3. The three integers inscribed on the vertices of one of the two triangles sum to the same constant as the three

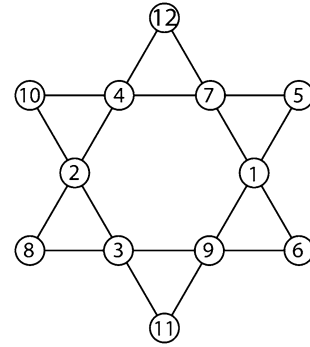


Figure 12. Star of David with magic constant equal to 26.

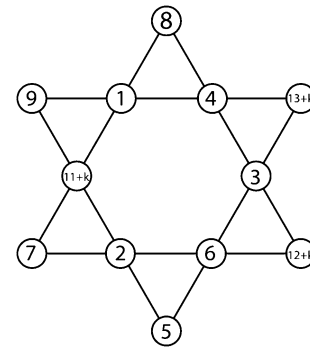


Figure 13. A formula for constructing magic stars of David with magic constant equal to any integer greater than 26.

integers inscribed on the vertices of the second triangle of the star, which is the magic constant of the star.

We can now define magic⁺ stars of David.

Definition 8. Magic⁺ stars of David are stars of David that in addition to Property 1, also satisfy Property 3.

Based on Properties 1, 2 and 3, we can prove the following.

Proposition 2. A magic⁺ star of David has, in addition to Properties 1, 2, and 3, the following additional two properties:

Property 4. The six integers inscribed on the intersection of the triangles of a star of David sum to the magic constant of the star.

Property 5. An integer inscribed on a vertex of a triangle of a star of David is equal to the sum of the two integers inscribed in the middle of the edge opposite the vertex.

Proof of Property 4. We use an argument similar to that used for proving Property 2. Choosing one of the two triangles forming the star, we can write the following equation with regard to the chosen triangle: $3c = 2c$ plus the sum of the six integers inscribed on the intersection of the two triangles, where c is the

magic constant of the star. By Properties 1 and 2, $3c$ equals the sum of the 12 integers inscribed on the edges of the triangle, and by Property 3, $2c$ equals twice the sum of the three integers inscribed on the vertices of the triangle. Therefore, the sum of the six integers inscribed at the intersection of the two triangles is equal to $3c - 2c = c$, which equals the sum of the six integers inscribed at the intersection of the two triangles. \square

Proof of Property 5. Choosing one of the two triangles forming the star, we will denote by a, b, d the integers at the vertices of this triangle, and we will denote by p and q the integers at the intersection of the two triangles located on the edge (a, b) . Then by Properties 1 and 2, the integers a, p, q, d sum to c . By Property 3, the integers a, b, d also sum to c . Therefore, $a + b + d = a + d + p + q$, and so $d = p + q$, as required. \square

Remark 2. Notice that Properties 4 and 5 hold for magic⁺ stars of David, but they do not necessarily hold for all magic stars of David.

In Figure 11 we provide a formula for constructing magic⁺ stars of David with magic constant equal to 26 or greater than or equal to 28.

It can be shown that the five properties mentioned above hold for every magic⁺ star of David generated by the formula shown in Figure 11. An example of such a magic⁺ star of David is shown in Figure 12, in which a, b, d are substituted by 12, 6, 8, generating a magic⁺ star of David with magic constant equal to 26.

To see that the formula in Figure 11 can generate magic⁺ stars with any magic constant greater than or equal to 28, we observe the following: In the formula in Figure 11, substitute for a, b, d the integers 10, 6, $12 + k$, where k is a nonnegative integer. It is left to the reader to verify that this generates a magic⁺ star of David whose magic constant is equal to $28 + k$, and the twelve integers inscribed on it are positive and distinct.

As for a magic⁺ star of David with magic constant equal to 27, another formula for generating magic⁺ stars of David is shown in Figure 13. Choosing $k = 0$ results in a magic⁺ star of David with magic constant equal to 27. When k is set to any value greater than or equal to -1 in the formula shown in Figure 13, the resulting magic⁺ star of David has magic constant equal to $27 + k$. The formula in Figure 13, however, generates a unique magic⁺ star of David for every k , while the formula in Figure 11 depends on the three parameters a, b, d , which for every value of the magic constant can be chosen in several ways, so that the integers inscribed in the resulting star of David can have a personal, traditional, or national significance.

Remark 3.

1. The formula in Figure 11 is not unique, and there may exist other equivalent formulas.
2. For a given magic constant, there may be many corresponding magic⁺ stars of David generated by the formula in Figure 11.

3. It can be shown that it is impossible to find three distinct positive integers a, b, d that will generate, by the formula in Figure 11, a magic⁺ star of David with a magic constant equal to 27.
4. For the magic constant 26, the corresponding star of David must include the 12 consecutive numbers from 1 to 12.
5. For magic constant equal to 26, a magic⁺ star of David is included in the internet paper on magic stars by Mutsumi Suzuki [2]. That star of David is different from the one shown in Figure 12 and was constructed by a different procedure.
6. Observe that the formula in Figure 11 generates magic⁺ stars of David with magic constant equal to $a + b + d$ for any positive integers a, b, d . The three integers inscribed at the vertices of each of its two triangles also sum to $a + b + d$, so that such magic stars of David generated by this formula satisfy Properties 1, 2, and 3, as required by magic⁺ stars of David. In the same way, the magic⁺ stars of David generated by the formula in Figure 13 have magic constant equal to $27 + k$, where k is an integer greater than or equal to -1 , and the integers inscribed at the three vertices of each of the two triangles sum to the same integer $27 + k$, so that Properties 1, 2, and 3 are satisfied by the magic⁺ stars of David so generated.

Platonic Dual Solids

A Platonic solid is a regular convex polyhedron having the following properties:

1. All of its faces are identical regular convex polygons.
2. The same number of faces meet at every vertex of the polyhedron.

The Platonic solids and their properties have been studied extensively, in particular, their combinatorial, geometric, algebraic, and symmetric properties. It has been proved in several ways that there are only five polyhedra that have the above properties. The numbers of faces, vertices, and edges of the five Platonic solids are listed in Table 1.

Definition 9. Two polyhedra are said to be dual to each other if one can be mapped into the other by interchanging faces and vertices.

It is well known that the dual of a Platonic solid is another Platonic solid, and in fact, the cube is dual to the octahedron, the dodecahedron is dual to the icosahedron, and the tetrahedron is self-dual; that is, its dual is another tetrahedron.

The Platonic solids were known and studied by the ancient Greeks. They were prominent in the philosophy of Plato and are mentioned in his dialogue *Timaeus*. Platonic solids were associated with the classical elements (water,

Table 1. The five Platonic solids

	Faces	Vertices	Edges	Shape of Faces
Tetrahedron	4	4	6	equilateral triangles
Cube	6	8	12	squares
Octahedron	8	6	12	equilateral triangles
Dodecahedron	12	20	30	regular pentagons
Icosahedron	20	12	30	equilateral triangles

fire, earth, and air) and are also described in the *Elements* of Euclid. In the sixteenth century, the astronomer Johannes Kepler associated them with the then known extraterrestrial planets, and in modern times, they were associated with the electron shell in what is known as the moon model. The Platonic solids are sketched in Figure 14.

We will expand here on the property of duality between pairs of Platonic solids and introduce a new type of duality, called arithmetic duality, in which there is a special one-to-one “magic” correspondence between the vertices of one solid and the faces of its dual.

This correspondence is created by inscribing positive integers on the faces of each of the two pairs of dual Platonic solids in such a way such that the following conditions are satisfied:

1. The integers inscribed on every face of any of the two dual Platonic solids sum to the same constant.
2. For each vertex of one of the two dual solids, there is a face of the dual solid such that the sum of the integers around the vertex in the first solid is equal to the sum of the integers inscribed on the corresponding face of the dual solid.

The above conditions create a one-to-one correspondence between the vertices of one solid to the faces of its dual. As for the tetrahedron, which is self-dual, a similar procedure will create a one-to-one correspondence between its four vertices and its four faces.

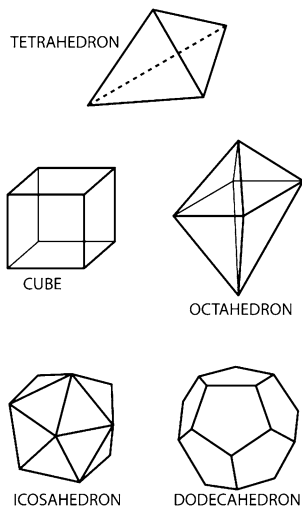


Figure 14. The five Platonic solids.

We shall now introduce our arithmetic generalization of the geometric duality of the Platonic solids. We begin with the tetrahedron, continue with the cube–octahedron pair, and finish with the dodecahedron–icosahedron pair.

The Arithmetic Duality of the Self-Dual Tetrahedron

Figure 15 includes an unfolded tetrahedron with three integers inscribed in each of its three faces. To retrieve the full tetrahedron, we have to raise the three external triangles and join the three vertices labeled A. The self-duality of the tetrahedron is suggested by the fact that the number of its faces is equal to the number of its vertices. Observe the following properties of this arrangement:

- (a) The three integers in each triangle as well as the three integers around each vertex are disjoint, and each such triplet of integers sums to 36.

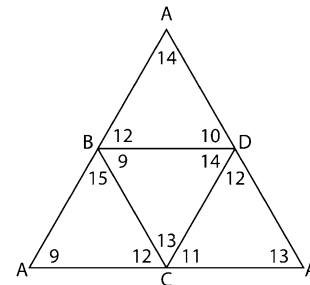


Figure 15. Unfolded numbered tetrahedron with (hidden) magic constant 12.

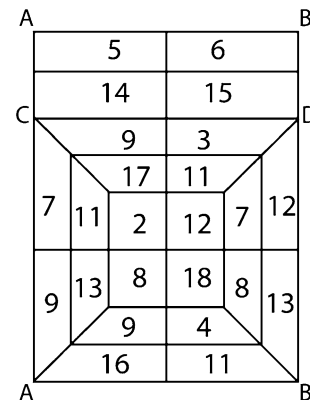


Figure 16. A $2 \times 2 \times 3$ unfolded and numbered wonder cube.

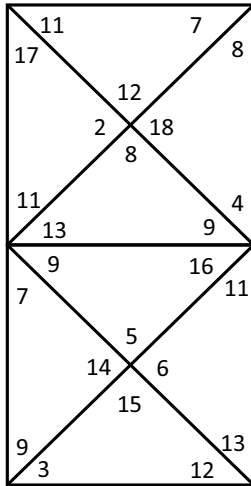


Figure 17. An unfolded octahedron that is dual to the cube in Figure 16.

- (b) For each of the four triangles, there is a vertex such that the three integers in the triangle are the same as the three integers around this vertex. Notice that property (b) establishes a one-to-one correspondence between the faces and the vertices of the self-dual tetrahedron. For example, the three integers in the triangle ABC (9, 15, 12) are the same as the three integers around the vertex B , and those three integers sum to 36. This establishes the correspondence between the triangle ABC and the vertex B .

In this way, an arithmetic one-to-one correspondence is established between the vertices and the faces of the tetrahedron, showing its self-duality. Also, the distinct triplets inscribed in the four triangles distinguish the two triangles from each other.

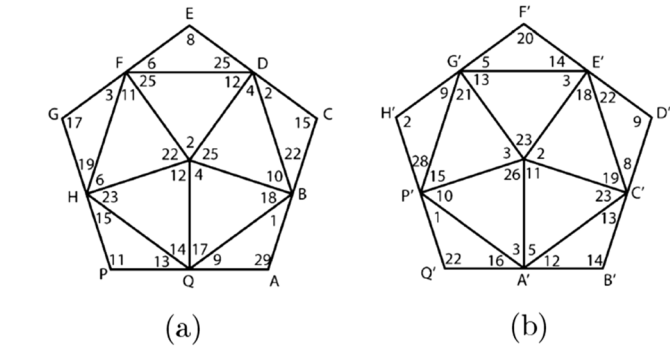
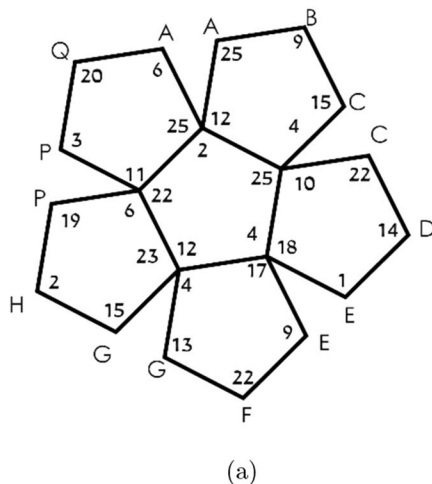


Figure 19. A numbered and unfolded icosahedron split into two parts.

Notice also that this choice of the twelve numbers inscribed on the faces of the tetrahedron is not unique, and there may be other sets of twelve positive integers that achieve the same goal.

The Arithmetic Duality of the Cube–Octahedron Dual Pair

Figure 16 shows an unfolded $2 \times 2 \times 3$ wonder cube with (hidden) pivot equal to 10. There are 24 integers inscribed on the cube, four integers on each of its six faces. Figure 17 shows an unfolded octahedron.

There are 24 integers inscribed on its eight triangular faces, three on each of its eight faces. The integers inscribed on its faces are exactly the same as those inscribed on the cube including their multiplicities. Observe the following:

- (a) The four integers on every face of the cube are disjoint and sum to 40. The three integers around each corner of the cube are disjoint and sum to 30.

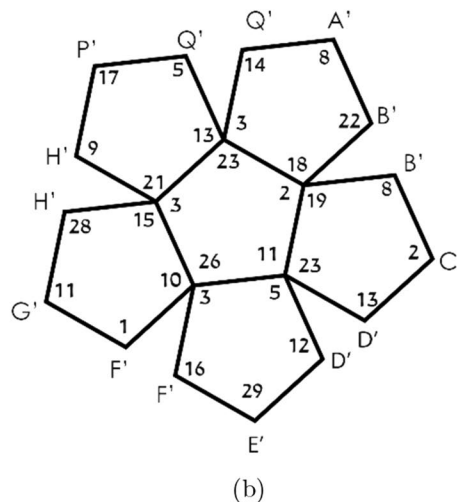


Figure 18. A numbered and unfolded dodecahedron split into two parts.

- (b) The three integers on any face of the octahedron are disjoint and sum to 30. Each set of four integers around one of its eight vertices is disjoint and sums to 40.
- (c) Every face of the cube has a corresponding vertex on the octahedron, and the four integers on the face of the cube are the same as the four integers around the vertex of the octahedron.
- (d) Every vertex of the cube has a corresponding face on the octahedron, and the three integers around the vertex of the cube are the same as the three integers on the face of the octahedron.

For example, the four integers on the face $ABCD$ of the cube (5, 6, 14, 15) in Figure 16 are the same as the four integers around the vertex C of the octahedron. The three integers around the vertex C of the cube in Figure 16 (14, 9, 7) are the same as the three integers on the face COB of the octahedron in Figure 17.

A one-to-one correspondence between the cube and the octahedron is established in this way, thus creating an arithmetic concrete duality between the two solids. Notice that the integers inscribed in the two solids to induce their arithmetic duality are not unique.

The Arithmetic Duality of the Dodecahedron–Icosahedron Dual Pair

Figures 18(a) and 18(b) show an unfolded dodecahedron split into two parts including six pentagons each.

To get the full dodecahedron, one must bend the five pentagons around the central pentagon of the dodecahedron's two parts and then join the three vertices labeled A , the three vertices labeled B , etc. The dodecahedron is inscribed with 60 integers, five integers on each vertex of each pentagon. It is easy to verify that the five integers in every pentagon are positive and distinct and sum to 65 ($= 5 \times 13$). The three integers around any one of the 20 vertices are positive and distinct and sum to 39 ($= 3 \times 13$). One can create an icosahedron that is arithmetically dual to this dodecahedron. Such an icosahedron is shown in Figure 19, split into two unfolded parts. To get the full icosahedron, one has to join the vertices A, B, C , etc., in Figure 19(a) to the corresponding vertices A', B', C' , etc., and restore the solid form of the icosahedron.

The full icosahedron has 60 integers inscribed in it that are identical, including multiplicities, to the 60 integers inscribed in the dodecahedron. Every face of the icosahedron includes three integers that sum to 39 and are the same as the three integers around a corresponding vertex of the dodecahedron. And the five integers around every vertex

of the icosahedron sum to 65 and are identical to the five integers in a corresponding pentagon of the dodecahedron. An arithmetic duality is established in this way between the two solids. For example, in Figure 19b, the five integers around the central vertex (23, 2, 11, 26, 3) sum to 65 and are the same as the five integers in the central pentagon in Figure 18b. Also, the three integers in the rightmost triangle of Figure 19b (11, 23, 5) are distinct and positive and sum to 39 and are the same as the three integers around the lower rightmost vertex of the central pentagon in Figure 18b.

Open Problems

1. As we remarked in the section defining the wonder cubes, the number of wonder cubes with a variable pivot is unbounded. However, a formula for the number of wonder cubes with the same pivot is an open problem.
2. Find a way to generalize wonder cubes to cubes with inscribed $4 \times 4, 5 \times 5$, etc., magic squares on their faces.
3. Find an algorithm for creating $3 \times 3 \times 3$ wonder cubes.
4. Find an algorithm for generating $2 \times 2 \times 3$ wonder cubes.

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