Square Dance from a Mathematical Perspective

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Square dancing is quintessentially an American dance. It has ancient roots, especially in England and France, but its character was formed in the new world—among colonists and in the early history of the republic. It grew as the country grew, and like other institutions, it reflected the country's composition and currents.¹ What distinguishes the American square dance from its progenitors is the caller (more in a moment).

Readers of a certain age met square dancing in school as children. Readers whose age is not certain may have had little experience with the pastime. The second author of this column danced only occasionally growing up.² The first author has danced with Tech Squares, the square and round dance club of MIT, since 2015.³

In the last half-century, a new tradition has formed within square dancing: challenge square dance. While it shares the basic movements and structure of its predecessors, it emphasizes the caller's ability to challenge the dancers by showing them unfamiliar consequences of familiar rules. Its construction is almost mathematical, involving geometry, algebra, analysis, and logic. Its execution has elements in common with solving a puzzle or escape room. Square dancers now find themselves in hexagons and other figures.⁴ It is mathematical art. We're going to tell you about it.

Basic Structure

There are several square dance traditions still being danced today. In this column, we focus on modern Western square dance, which was standardized in the 1970s by the CALLERLAB organization.⁵ Below, we shall refer to modern Western square dance as “square dance” without qualification.

Square dance is primarily a social dance: it's done for the enjoyment of the act of dancing and interacting with other dancers, without aspects of a performance or a competition. Generally, each dancer interacts with seven other dancers, thereby forming a single square of dancers; during a dance, there may be many squares, but they will not interact with each other. The name “square” comes from the starting position, which looks like this:

This is a column about the mathematical structures that give us pleasure. Usefulness is irrelevant. Significance, depth, even truth are optional. If something appears in this column, it's because it's intriguing, or lovely, or just fun. Moreover, it is so intended.

¹For example, enslaved people were recruited early to supply music [2, 3].

²This was mostly when he was especially shy around girls. Square dancing helped him by making it absolutely clear what he was supposed to do and when he was supposed to do it.

³Although sadly, we have had to take an extended hiatus during the Covid-19 pandemic.

⁴And where, but in square dancing, will you find polygons with only two sides?

⁵You can check out their website at <https://www.callerlab.org/>.

Starting position for a square dance.

This diagram, and others like it that we shall use below, represents a view from directly above the square. Each small square is a dancer, and the black dots show the dancers' facing directions. Not explicitly shown in this diagram is the fact that each pair of adjacent dancers are holding hands: as a general rule, two adjacent dancers will always take hands with each other, as long as they are facing in the same or opposite directions. These diagrams are useful, because square dance actions generally involve only walking around the floor and taking hands with dancers when you are adjacent to them, and so the entire state of the square is captured by such an overhead view.

Square dance is a "called" dance, that is, it is neither improvised by the dancers nor learned by them ahead of time. Rather, the dance is communicated to the dancers as they are dancing. This communication originates from one person, the caller, who verbally announces (usually over a microphone/speaker system) what dance actions the dancers should perform a few seconds before they will need to begin them (i.e., while they are completing the previous dance action).⁶ Until the mid-twentieth century, the number of calls a dancer needed to know was reasonably small, but as interest grew, calls began to multiply. This was the start of what became "challenge square dancing."

Because of the growing complexity, standardization of vocabulary and syntax was needed. CALLERLAB was founded in 1974 to manage this. Dancers and callers now learn calls—each of which is a pair comprising a short phrase (suitable for communication during a dance) and a definition of a dance action (which can be quite complex). Dances are held at different "dance programs," each of which corresponds to a different vocabulary, and hence a different amount of memorization required. To go to a dance held at Mainstream, one of the smaller programs, one need learn only 90 calls; at the other extreme, dances held at the C4 program might require dancers to be prepared to execute any of over 700 different calls!

Some Example Calls

For the purposes of illustration, it will be useful to have some concrete examples of square dance calls. When illustrating calls, we shall usually show only the dancers that are directly involved in the call, rather than the full complement of eight.⁷

Trade. This call is performed by two dancers who are holding hands before the call starts. Each dancer walks along a half-circle, centered on their starting handhold, to take the position of the other dancer while ending with each facing in the opposite direction from which they started.

If the two dancers are facing in the same direction, their circular paths overlap, so they must both step to their left to make space for each other; this is an instance of an overarching rule of square dance, the right-shoulder rule, which says that if two dancers would need theoretically to occupy the same spot momentarily, instead they "pass right shoulders," i.e., step to the left so that their right shoulders are close to each other, then step back onto their original paths.

Before Trade

Halfway through

Done

If they are facing in opposite directions, their paths do not overlap.

Before Trade

Halfway through

Done

Swing Thru. Those who are holding right hands with each other Trade. Then those who are holding left hands with each other Trade. This illustrates an important idea: a call's definition can use other calls. In this way, a call can have a high complexity but a short definition, so it is still easily learnable (once you are familiar with the calls it depends on).

⁶Callers may have evolved from "dancing masters." Dancing masters instructed colonists in the sequence and steps of quadrilles. Colonists with no instruction used callers to keep dancers moving properly. In Southern states, enslaved people often served as callers.

⁷Or later, with hexagons, twelve.

Pass Thru. This call is performed by two dancers facing each other. They walk past each other, passing right shoulders (in accordance with the right-shoulder rule), to end on each other's spots. No turning occurs.

Pass the Ocean. This call is performed by two laterally adjacent pairs of facing dancers. Everyone Pass Thru, then turn one-fourth inward⁸ (toward the person next to you), then do half of a Pass Thru, to end holding right hands with the dancer originally next to you.

More Complex Calls

When thinking about computation in mathematics, we often need to require that programs behave correctly on an infinite number of possible inputs to prevent “hard-coded” solutions that don’t capture or engage with the problem we’re interested in. For example, the most efficient program to print out the first 100 prime numbers might be, essentially, a list of those numbers, which could also be the case for any other list of 100 numbers. In contrast, a program that must, on input n , produce the n th prime number will necessarily contain some computation that depends in a nontrivial way on the meaning of “prime number.”

Concepts

One can apply this analysis to square dance: so far, as we have described it, the “ideal” dancer might simply have a very large lookup table mapping calls and starting positions to movements.⁹ However, “concepts” prevent such an approach from working even in theory: they make the set of instructions one might hear from the caller infinite, so you really do have to think while dancing!

A concept is a function from calls to calls (or, by abuse of language, a name given to such a function). It may not be defined for all calls, and the output call may have a set of legal starting positions that is different from the input call’s set.

Some concepts only look at the motion of the dancers, such as *Stable* [call]: “Move as though you were doing [call], but don’t turn.” Other concepts look at how the call is defined, such as *Yoyo* [call]: “Replace the first instance of *Trade* in the definition of [call] with *Cast 3/4*, then execute the result.”

Some concepts use the call multiple times, such as *Crazy* [call]: “Each side of the square does [call], then the centers do [call], then each side of the square does [call], then the centers do [call].” Other concepts use only part of the call, such as *Finish* [call]: “Delete the first part from the definition of [call], then execute the result.”

Some concepts have the dancers combine with each other to form a “multiperson dancer,” such as *As Couples* [call]: “Pretend each couple (pair of laterally adjacent dancers facing the same way) has ‘fused’ together into a single dancer and do [call]; the dancers in each couple don’t move or turn relative to each other.”

⁸Square dance’s default unit of angle is the revolution, so this means a one-fourth revolution, that is, $\pi/2$ radians.

⁹In theory, there might be an infinite number of starting positions, given the call *Press Ahead*: “Step forward by one spot.” But in practice, the distance between any dancer and the center of the square is sharply bounded from above.

Before
As Couples Swing Thru

After couples Trade
by the right

After couples Trade
by the left (done)

Other concepts have the dancers work with imaginary dancers (traditionally called phantoms), such as *Split Phantom Lines* [call]: “Assume that you are facing a line of imaginary dancers, and do [call] working with them.”¹⁰

Before Split Phantom
Lines Pass the Ocean

After Pass Thru

After
1/2 Pass Thru
(done)

Dancers imagine
these phantoms

After 1/4 Turn In

Ignore phantoms when the
Split Phantom Lines call is
over—but don’t “squeeze
out” the gap

Metaconcepts

A metaconcept is usually a function from concepts to concepts, but the term can occasionally refer to functions with more diverse types, e.g., a function that takes a call and a concept and produces a concept. One metaconcept is *Echo* [concept] [call]: “Do [concept] [call], then do [call].” Another group of metaconcepts apply a concept only to a specific part or fraction of a call, such as *Initially* [concept] [call]: “Do [call], but apply [concept] to the first part,” and similarly, *Secondly*, *Thirdly*, and *Finally*.

Dancing as Puzzle-Solving

At some square dances, there is an understanding that one goal, in addition to having fun socializing and enjoying the physicality of the dance, is putting one’s ability to follow the caller’s instructions to the test. This ethos is the essence of “challenge dancing.”

One kind of puzzliness that the caller can use to challenge the dancers is found in concepts (and, more rarely, calls) that require the dancers to deduce information not provided by the caller. This information must be unambiguously deducible, but it is not always immediately obvious.

Here is a typical example, the concept *Funny* [call]: “As many dancers as possible without ending on the same spot, do your part of [call]; others do nothing.” This requires thinking about where the other dancers would end up after doing their part of the call in order to decide whether doing your part would be legal.

Another example is the *Offset Triple Boxes* concept, which creates phantoms and groups dancers together depending on where the farthest-from-the-center dancers are positioned; the dancers in the center need to look to the farthest ends to figure out how they are grouped with other dancers and phantoms.

Dancing Together

A square of eight dancers is somewhat analogous to a collection of networked computers maintaining a finite state machine (the state of the machine being the formation). In each case, the members communicate to prevent and resolve disagreements about the state. And in each case, the members can make mistakes, so they can’t be trusted entirely by the other members!

Communication can include giving verbal reminders about the definition—in some cases, simply stating the definition; in other cases, it’s anything that reminds you how to do an easily messed-up part of a call. If the [call] takes too long in *Catch* [call]: “Square Thru to a Wave; do [call]; Step and Fold,” some dancers might forget that the Step and Fold at the end is still required; people often say, “But there’s a catch!” to remind them to do it.

In addition to these verbal hints (and explicit verbal discussion of disagreements, e.g., “Shouldn’t you be Trading

¹⁰A more precise definition would be, “Divide the center 4×4 grid of spots into two 2×4 rectangles such that the short axis of each rectangle is parallel to your facing direction, and assume that there is a phantom in every unoccupied spot. Treat the 2×4 rectangle you are in as the entire formation, and do the call in that formation.”

with me?”), there is physical communication. Slapping hands (like a high-five) with another dancer can communicate a momentary agreement about a formation—“yes, the two of us are adjacent in a line here.” For calls that have dancers Pass Thru with those who are diagonal to them rather than directly in front of them, pointing and snapping at the person with whom you want to Pass Thru is common to make sure everyone agrees on the pairing.

All of this means that the members of a square of eight are capable together of more than any of them could do alone—which is what makes dancing concepts with phantoms (like Split Phantom Lines, above) so tricky, since the feedback is no longer there. Sometimes, to challenge themselves (or because the number of attendees at a dance is nonzero modulo 8), dancers will square up with some phantoms and some real dancers, keeping the same phantoms the whole way through.¹¹

Resolving Dances

The caller usually “resolves” the square at the end of each sequence by returning everyone to their initial position. Figuring out how to resolve a square from an arbitrary position has been likened to unscrambling a Rubik’s cube; the various calls play the role of the various possible twists of the cube. However, there’s an important difference from the Rubik’s cube: not every call is legal from every position. So while the Rubik’s cube is commonly modeled mathematically as a group, square dance makes more sense to model as a groupoid: a set of positions and, for each pair of positions, a set of (reversible) paths from one to the other.

In fact, this groupoid is analogous to the fundamental groupoids of spaces considered in homotopy theory; in particular, the space in question would be the Cartesian product of the “position” space—the configuration space of four distinguishable nonoverlapping points in $\mathbb{R} \setminus \{(0, 0)\}$, the punctured plane—and the “direction” space $(S^1)^4$. There are four rather than eight factors here because of the rotational symmetry of the dancers. This is also why $(0, 0)$ is excluded—that would require two dancers to occupy the same point. A call defines a set of paths (with distinct starting points) in this groupoid, one path for each legal starting position of the call. Of course, many “dance actions” in this groupoid are not realizable by a single call.

Squares with Two Sides

The rotational symmetry in square dancing—dancers that are 180 degrees apart (called “opposites”) are executing the same moves—suggests that a square dance could actually be carried out with just four dancers on two sides of a square. All the information about where the dancers are is contained in the positions of just four of the dancers (no

two of whom can be opposites), and they can “squeeze out the missing half” in order to keep the required movements continuous. In effect, this is a polygon with only two sides, a bigon.

This description of bigons is sufficient—all square dance calls can be carried out by just a pair of couples—but algebraic topology is useful, not just for bigons but for more adventurous structures. Call the punctured plane F (for the dance Floor). For simplicity, we shall ignore facing directions; usually, these are recoverable as the tangents to the dancers’ paths (i.e., dancers normally walk forward), but the interested reader can extend the discussion below to treat facing direction as well as position. The group generated by the 180-degree rotation (which is $\mathbb{Z}_2$ as an abstract group) acts freely on F , and the quotient space $F/\mathbb{Z}_2$ is in fact homeomorphic to F . Making this identification, the quotient map $F \rightarrow F/\mathbb{Z}_2$ can be viewed as a continuous map $p : F \rightarrow F$. One possible realization of this map, in polar coordinates, is as $(r, \theta) \mapsto (r, 2\theta)$. This is a covering map, which implies that there is a close connection between paths in the quotient space and paths in the original space—which is interesting because walking around on paths is what dancers do!

In a bigon dance, the four dancers imagine eight imaginary dancers occupying a square, and each of them chooses one to “follow.” They, the real dancers, move according to the following rule: whenever the imaginary dancer you are following is at a position $x \in F$, you must be at the position $p(x) \in F$. (For this to work, no two of the imagined dancers being followed can be opposites.) Following this rule, bigon dance is “call-compatible” with squares: a bigon can interpret and dance any sequence of calls that would be legal to call to a regular square.

In practice, dancers don’t actually imagine a square but rather learn to “see” how the calls work in the bigon geometry, using notions like “bigon straight lines” (the image of a straight line under p ; learning to see these helps understand the bigon formations) and “overachieving” (the angle that you rotate around the center of a bigon during a call is always double what it would be in a square). Also, physical constraints require using a somewhat different function p from the one we have given, but it is homotopic to this one.

Bigon dancing is fun both as a challenge and as something to do when there are $4 \bmod 8$ dancers in the room and everyone wants to dance without phantoms. Also, unlike many square dance games, it doesn’t require the caller to pay any attention to it; as long as they keep to symmetric choreography (which is very common), the bigon dancers can do their thing fully compatibly with anything the regular squares do. It is not too hard to see that when the imagined square resolves, the bigon will resolve as well: if an imagined square dancer is at their home position x_0 , then $p(x_0)$ is by definition the home position of the corresponding bigon dancer.

¹¹But if a concept also introduces phantoms, it’s important to distinguish between the “real” phantoms that last for the whole dance and the “temporary” phantoms for that concept.

A bigon and a (symmetric) square contain the same information, so rather than define bigons in terms of squares, one could take the opposite perspective and define square dance in terms of bigon dance. It works out to this rule: whenever there is a bigon dancer at position $y \in F$, there should be two square dancers at the two preimage positions $p^{-1}(y) \subset F$. There are always exactly two preimages, and because p is a covering map, as the bigon dancer moves from y_0 to y_1 via some path, it is unambiguous how the preimage dancers in the square should move to stay over the path; this is the “homotopy lifting property” of a covering map. So “dividing square dance by two and multiplying the result by two” has gotten us back to square dance, as it should.

Squares with Six Sides

We didn’t need any symmetry present to multiply bigon dance by 2, so what if we tried multiplying it by 3 instead? It turns that out this works, and it gives something totally unexpected—a generalization of square dancing to hexagons. We follow the algebra for bigons but replace the twofold covering map p with a threefold covering map $q : F \rightarrow F$, defined by $(r, \theta) \mapsto (r, 3\theta)$. Now there are three hexagon preimage dancers per bigon dancer—12 hexagon dancers in total. For comparison, here are the starting positions of a regular square and a hexagon.

Hexagons, like bigons, are call-compatible with squares. Here is the call Pass the Ocean, first in a square:

(and as before, we are showing only the dancers involved in the call) and then in the hexagon:

The homotopy lifting property ensures that the paths for the dancers to follow are still well defined, but now an interesting complication occurs: a hexagon won’t always resolve when a square would. When a bigon dancer reaches their home position in the bigon, the preimage hex dancers will be at the three preimages of that home position in the hexagon, but they may have been permuted—they each may have originally started at a different preimage of that home position.

We shall state the answer without proof: suppose a dance (viewed as a very long path for each dancer to follow) resolves in a bigon, i.e., each bigon dancer ends where they started. Then it will also resolve in a square if and only if each bigon dancer’s winding number about the center of the bigon is $0 \pmod 2$; and it will resolve in a hexagon if and only if each bigon dancer’s winding number about the center of the bigon is $0 \pmod 3$. Usually, callers resolve by paying attention to the squares, so a bigon dancing alongside the squares will end up “double-resolved”—resolved with each bigon dancer’s winding number equal to $0 \pmod 2$.

Hexagons similarly can be danced “in the back of the hall” without the caller needing to know about it; since the winding number’s value mod 2 implies nothing about it mod 3, however, a hexagon has no guarantee of resolving if it dances choreography that will resolve in a square. This observation (and the question of why it resolves sometimes and not others) is what originally led to this mathematical analysis of bigons and hexagons. The two were originally conceived of without any explicit discussion of algebraic topology, and are typically taught without the above mathematical description, just by describing the effects of the hexagon geometry (lines bend outward, rotations around the center are multiplied by $2/3$ compared to a square).

For an introduction to hexagons in this style (warning: familiarity with some calls is presupposed) and a discussion of the history of hexes, see [1]. For lovely animations of many, many calls in different geometries (hexagon, bigon), see [4] (look under Settings/Special Geometry).

More?

We hope that you have found this discussion interesting. There is so much more than we could cover in this column. If you are interested in learning more, please contact the authors, or look for your local square dance club. (Some clubs may still be on hiatus due to the pandemic.)

Acknowledgments

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