“Not Just an Idle Game”: The Story of Higher-Dimensional Versions of the Poincaré Fundamental Group

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Part of the title of this article is taken from writings of Albert Einstein (1879–1955) in the correspondence published in [42]:¹

... the following questions must burningly interest me as a disciple of science: What goal will be reached by the science to which I am dedicating myself? What is essential and what is based only on the accidents of development? ... Concepts which have proved useful for ordering things easily assume so great an authority over us, that we forget their terrestrial origin and accept them as unalterable facts. ... It is therefore not just an idle game to exercise our ability to analyse familiar concepts, and to demonstrate the conditions on which their justification and usefulness depend, and the way in which these developed, little by little

This quotation is about science rather than mathematics, and it is well known, for example in physics, that there are still fundamental questions, such as the nature of dark matter, to answer. There should be awareness in mathematics that there are still some basic questions that have failed to be pursued for decades; thus we need to think also of educational methods of encouraging their pursuit.

In particular, encouragements or discouragements are personal, and may turn out not to be correct. That is a standard hazard of research.

This is a story of two related projects. The first one dates from the early twentieth century. It was known that if X is

path connected, then the abelianized fundamental group $\pi_1(X, x)$ is isomorphic to the first homology group $H_1(X)$. The abelian homology groups $H_n(X)$ have been defined in all dimensions $n \geq 0$. Since the fundamental group was useful in many geometric problems, this led to the project of trying to generalize to higher dimensions the Poincaré fundamental group $\pi_1(X, x)$. This project was generally abandoned in favor of the study of the homotopy groups of pointed spaces, whose first public definition was given by Eduard Čech.

The second project dates from 1965 and involves the fact that the standard method of computing the fundamental group is by means of a result called van Kampen's theorem. However, this theorem is restricted to unions of two path-connected spaces with path-connected intersections. The author's paper [4] concerns the generalization to groupoids using the idea of a set of base points, and it was stated in the introduction to [4] that this theorem could be generalized to all dimensions.

That statement turned out to be premature. But it was the later possibilities for making reality of the intuitions of “algebraic inverse to subdivision” (see diagram (3)) and “commutative cube” (see Figure 1), in addition to various collaborations detailed in [15], that indicated relations with the earlier project.

Homotopy Groups at the ICM Zurich, 1932

Why am I considering the ancient 1932 International Congress of Mathematicians? Surely, we have advanced since

¹More information on most of the people mentioned in this article may be found at the website <https://mathshistory.st-andrews.ac.uk/Biographies/>.

then, and the basic ideas have surely long been totally sorted.

Many mathematicians, especially Alexander Grothendieck (1928–2014), have shown us that basic ideas can be looked at again and in some cases renewed.

The first main theme with which I am concerned in this paper is little discussed today, but it is stated in [39, p. 98]. It involves the introduction by the respected topologist Eduard Čech (1893–1960) of the homotopy groups $\pi_n(X, x)$ of a pointed space (X, x) , which he proved to be abelian for $n > 1$.

But it was argued that these groups were inappropriate for what was a key theme at the time, the development of higher-dimensional versions of the fundamental group $\pi_1(X, x)$ of a pointed space as defined by Henri Poincaré (1854–1912) [31]. In many of the known applications of the fundamental group in complex analysis and differential equations, the largely nonabelian nature of the fundamental group was a key factor.

Because of this abelian property of higher homotopy groups, Čech was persuaded by Heinz Hopf (1894–1971) to withdraw his paper, so that only a small paragraph appeared in the proceedings [24]. However, it was known at the time that the abelian homology groups $H_n(X)$ are well defined for every space X and that if X is path connected, then $H_1(X)$ is isomorphic to the abelianized fundamental group $\pi_1(X, x)$.

Indeed, Pavel Sergeevich Aleksandrov (1896–1982) was reported to have exclaimed, “But my dear Čech, how can they be anything but the homology groups?”²

REMARK 1 It should be useful to give here for $n = 2$ an intuitive argument that Čech might have used for the abelian nature of the homotopy groups $\pi_n(X, x)$, $n > 1$. It is possible to represent every $g \in G = \pi_2(X, x)$ by a map $a : I^2 \rightarrow X$ that is constant with value x outside a “small” square, say J , contained in I^2 .

Another such class $b \in G$ may be similarly represented by a map with b constant outside a small square K in I^2 and such that J does not meet K . Now we can see a clear difference between the cases $n = 2$ and $n = 1$. In the former case, we can choose J, K small enough and separated so that the maps a, b may be deformed in their classes so that J, K are interchanged.

This method is not possible if $n = 1$, since in that case, I is an interval, and one of J, K is to the left or right of the other. Thus it can seem that any expectations of higher-dimensional versions of the fundamental group were unrealizable. Nowadays, this argument would be put in the form of what is called the Eckmann–Hilton interchange

result, that in a set with two monoid structures each of which is a morphism for the other, the two structures coincide and are abelian.

In 1968, Eldon Dyer (1934–1997), a topologist at CUNY, told me that Hopf had told him in 1966 that the history of homotopy theory showed the danger of certain individuals being regarded as the “kings” of a subject and thus key in deciding its direction. There is considerable truth in this point; cf. [1].

Also at the 1932 Zurich ICM was Witold Hurewicz (1904–1956); his publication of two notes [38] shed light on the relation of the homotopy groups to homology groups, which stimulated interest in these homotopy groups. With the growing study of the complications of the homotopy groups of spheres,³ which became seen as a major problem in algebraic topology, cf. [51], the idea of generalization of the nonabelian fundamental group was disregarded, and it became easier to think of “space” and the “space with base point” necessary to define the homotopy groups as essentially synonymous—that was my experience up to 1965.

However, it can be argued that Aleksandrov and Hopf were correct in suggesting that the abelian homotopy groups were not what one would really like for a higher-dimensional generalization of the fundamental group! That does not mean that such “higher homotopy groups” would be without interest; nor does it mean that the search for nonabelian higher-dimensional generalizations of the fundamental groups should be completely abandoned.

Determining Fundamental Groups

One reason for this interest in fundamental groups was their known use in important questions relating complex analysis, covering spaces, integration, and group theory [31]. Herbert Seifert (1907–1996) proved useful relations between simplicial complexes and fundamental groups [50], and a paper by Egbert van Kampen (1908–1942) [41] stated a general result that could be applied to the complement in a 3-manifold of an algebraic curve. A modern proof for the case of path-connected intersections of families of open sets was given by Richard H. Crowell (1928–2006) in [25] following lectures of Ralph Fox (1913–1973). That result is often called van Kampen’s theorem (VKT), and there are many excellent examples of applications of it in expositions of algebraic topology.

The usual statement of VKT for a fundamental group is as follows:

THEOREM 1 (Van Kampen’s theorem) *Let the space X be the union of open sets U, V with intersection W and assume that U, V, W are path connected. Let $x \in W$. Then the*

²I heard of this comment in Tbilisi in 1987 from George Chogoshvili (1914–1998), whose doctoral supervisor was Aleksandrov. Compare also [1]. An irony is that the 1931 paper [37] already gave a counterexample to this statement, describing what is now known as the “Hopf map” $S^3 \rightarrow S^2$; this map is nontrivial homotopically, but it is trivial homologically. It also has very interesting relations to other aspects of algebra and geometry; cf. [39, Chapter 20]. Aleksandrov and Hopf were two of the most respected topologists of their time. Their standing is shown by their mentions in [39] and the invitation by Solomon Lefschetz (1884–1972) for them to spend the academic year 1926 in Princeton.

³Which can be seen by a web search on this topic.

following diagram of fundamental groups and morphisms induced by inclusions is a pushout square of groups:

$$\begin{array}{ccc} \pi_1(W, s) & \xrightarrow{j} & \pi_1(V, x) \\ i \downarrow & & \downarrow k \\ \pi_1(U, x) & \xrightarrow{h} & \pi_1(X, x) \end{array} \quad (1)$$

Note that a pushout square of groups is defined entirely in terms of the notion of morphisms of groups: using the diagram (1), the definition says that if G is any group and $f : \pi_1(U, x) \rightarrow G$, $g : \pi_1(V, x) \rightarrow G$ are morphisms of groups such that $fi = gj$, then there is a unique morphism of groups $\phi : \pi_1(X, x) \rightarrow G$ such that

$$\phi h = f, \quad \phi k = g.$$

This property is called the universal property of a pushout, and proving it is called verifying the universal property. It is often convenient that such a verification need not involve a particular construction of the pushout, nor a proof that all pushouts of morphisms of groups exist. See also [16].

The limitation to path-connected spaces and intersections in Theorem 1 is also very restrictive; cf. [60].⁴ Because of the connectivity condition on W , this standard version of van Kampen's theorem for the fundamental group of a pointed space does not compute the fundamental group of the circle,⁵ which is, after all, the basic example in topology; the standard treatments instead make a detour into a small part of covering space theory by introducing the winding number of the map $p : \mathbb{R} \rightarrow S^1$, $t \mapsto e^{2\pi i t}$, from the reals to the circle, which goes back to Poincaré in the 1890s.

From Groups to Groupoids

I became involved with groupoids in the years since 1965.

A groupoid is defined in modern terms as a small category in which every morphism is an isomorphism. It can be thought of as a group with many identities, or more formally, an algebraic structure with partial algebraic operations, as considered by Philip J. Higgins (1924–2015) in [33]. I like to define higher-dimensional algebra as the study of partial algebraic structures in which the domains of the algebraic operations are defined by geometric conditions.

Groupoids were defined by Heinrich Brandt (1886–1954) in 1926 [3] for extending to the quaternary case work

of Carl Friedrich Gauss (1777–1815) on compositions of binary quadratic forms; the use of groupoids in topology was initiated by Kurt Reidemeister (1893–1971) in his 1932 book [48].

The simplest nontrivial example of a groupoid is the groupoid, call it $\mathcal{I}$, that has two objects 0, 1 and only one nontrivial arrow $\iota : 0 \rightarrow 1$, and hence also $\iota^{-1} : 1 \rightarrow 0$. This groupoid looks trivial, but it is in fact the basic transition operator. (It is also, with its element ι , a generator for the category of groupoids, in analogy to how the integers $\mathbb{Z}$ with the element 1 form a generator for the category of groups. Neglecting $\mathcal{I}$ is analogous to the long-term neglect of zero in European mathematics.)

The use of the fundamental groupoid $\pi_1(X)$ of a space X , defined in terms of homotopy classes rel endpoints of paths $x \rightarrow y$ in X , was a commonplace by the 1960s. Students find it easy to see the idea of a path as a journey, not necessarily a return journey.⁶

I was led to Higgins's paper [34] on groupoids for its work on free groups. I noticed that he used pushouts of groupoids and so decided to insert in the book I was writing in 1965 an exercise on van Kampen's theorem for the fundamental groupoid $\pi_1(X)$. Then I thought I had better write out a proof. When I had done so, it seemed so much better than my previous attempts that I decided to explore the relevance of groupoids.

It was still annoying that I could not deduce the fundamental group of the circle! I then realized that we were in a “Goldilocks situation”: one base point was too small; taking the whole space was too large; but for the circle, taking two base points was just right! So we needed a definition of the fundamental groupoid $\pi_1(X, S)$ for a set S of base points chosen according to the geometry of the situation; see the paper [4]⁷ and all editions of [5] as well as [35].⁸

The new statement of Theorem 1 then replaces x by a set S meeting each path component of W , and we get a pushout of groupoids instead of groups. A form of this theorem for unions of a family of more than two open sets is given in [21]. To apply this, one needs to learn how to calculate with groupoids; cf. [34].

Constructions applied to the category $\mathbf{Gpd}_I$, in which all object maps are identities on the set I , are very similar to those in the category of groups. But morphisms of groupoids also allow for identifications in dimension 0. Thus a pair of morphisms of groupoids $a, b : A \rightarrow B$ has a coequalizer morphism $c : B \rightarrow C$ with a universal property for morphisms of a groupoid $f : B \rightarrow G$ such that $fa = fb$. To start this, we have first to coequalize on objects, forming $v : \text{Ob}(B) \rightarrow Y$ and a new groupoid morphism $u : B \rightarrow C$

⁴Comments by Grothendieck on this restriction, using the word “obstinate,” are quoted extensively in [8]; see also [32, Section 2].

⁵The argument for this is that if $S^1 = U \cup V$ is the union of two open path-connected sets, then $U \cap V$ has at least two path components.

⁶For a definition of rel endpoints, see Appendix 2 below.

⁷Corollary 3.8 of that paper seems to cover the most general formula stated in [41], namely for a fundamental group of the union of two spaces whose intersection is not path connected. However, Grothendieck has argued that such reductions to a group presentation may not increase understanding.

⁸For a discussion on this issue of many base points, see <https://mathoverflow.net/questions/40945/>.

such that the diagram in which $D(S)$ is the functor giving the discrete groupoid on a set S is a pushout of groupoids:

$$\begin{array}{ccc} D(\text{Ob}(B)) & \xrightarrow{D(v)} & D(Y) \\ \downarrow & & \downarrow \\ B & \xrightarrow{u} & C \end{array} \quad (2)$$

The construction of u in terms of v , B , and normal forms for elements of C generalizes those of free groups, free products of groups, and free groupoids, and may be found in [5, 35]. A nice application of this and the general VKT is that if X is the union of open simply connected spaces, then $\pi_1(X, B)$ is a free groupoid. The relation with bifibered categories is spelled out in [15, Appendix B3].

An inspiring conversation with George W. Mackey (1916–2006) in 1967 at a Swansea British Mathematical Colloquium, where I gave an invited talk on the fundamental groupoid, informed me of the notion of virtual groups, cf. [44, 47], and their relation to groupoids. That led me to study the extensive work of Charles Ehresmann (1905–1979) and his school, all showing that the idea of groupoid had much wider significance than I had suspected; cf. [10]. See also more recent work on, for example, Lie groupoids, Conway groupoids, groupoids and physics.

However, the texts on algebraic topology that give a many-pointed VKT (as published in 1967 in [4]) are currently (as far as I am aware) [5, 15, 61]; it is also in [35].

From Groupoids to Higher Groupoids

As we have shown, “higher-dimensional groups” are just abelian groups. However, this is no longer so for “higher-dimensional groupoids” [22, 23].

It seemed to me in 1965 that some of the arguments for VKT generalized to higher dimensions, and this was prematurely claimed as a theorem in [4].⁹

One of these arguments comes under the theme or slogan of “algebraic inverses to subdivision”:

$$\begin{array}{ccc} \boxed{} & \longleftrightarrow & \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array} \end{array} \quad (3)$$

From left to right gives subdivision. From right to left should give composition. What we need for higher-dimensional nonabelian local-to-global problems is “algebraic inverses to subdivision.”

Figure 1. “Composing” five faces of a cube.

This aspect is clearly more easily treated by cubical methods than the standard simplicial methods or the more recent “globular” ones.¹⁰

One part of the proof of VKT for the fundamental group or groupoid, namely the uniqueness of a universal morphism, is more easily expressed in terms of the double groupoid $\square G$ of commutative squares in a group or groupoid G , which I first saw defined in [27]. The essence of its use is as follows. Consider a diagram of morphisms in a groupoid:

$$\begin{array}{c} b \\ \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array} \\ 1 \end{array} \quad (4)$$

Suppose each individual square is commutative, and the two vertical outside edges are identities. Then we easily deduce that $a = b$.¹¹

For the next dimension, we therefore expect to need to know what a “commutative cube” is:

$$\begin{array}{c} \bullet \xrightarrow{\quad} \bullet \\ \swarrow \quad \searrow \quad \uparrow \quad \downarrow \\ \bullet \xrightarrow{\quad} \bullet \xrightarrow{\quad} \bullet \\ \downarrow \quad \uparrow \quad \downarrow \quad \uparrow \\ \bullet \xrightarrow{\quad} \bullet \end{array} \quad (5)$$

and this is expected to be in a double groupoid.¹² We want the “composed faces” to commute! What can this mean?

We might say that the “top” face is the “composite” of the other faces; so fold them flat to give the left-hand diagram of Figure 1, where the dotted lines show adjacent edges of a “cut.”¹³ We indicate how to glue these edges back together in the right-hand diagram of this figure by

⁹It could be more accurately called, “There are ideas for a proof in search of a theorem.”

¹⁰Cubical subdivisions are easily expressed using a matrix notation [15, 13.1.10]. The globular geometry is explained in, for example, [11].

¹¹The disarmingly simple higher-dimensional version of this argument is [15, 13.7.5].

¹²More explanation is given in [12] of how this may be given in a “double groupoid” of squares G in which the horizontal edges G_h and vertical edges G_v come from the same groupoid, i.e., $G_h = G_v$.

¹³To avoid adding an arrow to every edge of these diagrams, we adopt the convention that edges are directed from top to bottom and from left to right; this is why b^{-1} and c^{-1} appear but not a^{-1} , d^{-1} .

means of extra squares, which are a new kind of “degeneracy.”

Thus if we write the standard double groupoid identities in dimension 2 as

$$\square = ||,$$

where a solid line indicates a constant edge, then the new types of square with commutative boundaries are written¹⁴

$$\begin{array}{c} \lrcorner \quad \llcorner \quad \lrcorner \quad \llcorner \end{array}.$$

These new kinds of degeneracies were called connections in [23], because of a relation to path-connections in differential geometry. In a formal sense, and in all dimensions, they are constructed from the two functions $\max, \min : \{0, 1\} \rightarrow \{0, 1\}$.

A basic intuition for the proof of a 2-dimensional van Kampen’s theorem was also that the well-defined composition of commutative cubes in any of the three possible directions is also commutative; so this has to be proved once a full definition is set up and then generalized to all dimensions.

It is explained in [12, Section 8] how the use of these connections as an extra form of degeneracies for the traditional theory of cubical sets remedied some key deficiencies of the cubical, in contrast to the standard, simplicial theory, deficiencies that had been known since 1955; the wider use of such enhanced cubical methods also allowed more convenient control than standard methods of homotopies and higher homotopies (because of the rule $I^m \times I^n \cong I^{m+n}$). It of course maintained the cubically allowed “algebraic inverses to subdivision,” and so possibilities for higher van Kampen’s theorems; this is explained starting in dimensions 1, 2 in [15, Part I], and continuing in Parts II, III in all dimensions; this book gives what amounts to a rewrite of much traditional singular simplicial and cellular algebraic topology; cf. [12].

The Influence of 1941–1949 Work of J. H. C. Whitehead

The relation of the fundamental group to aspects of geometric group theory was an important feature of the work of Poincaré. The relation of various versions of homotopy groups to group theory was an important feature of the work of J. H. C. (Henry) Whitehead (1904–1960), my dissertation advisor in the period 1957–1959. His paper [56] is considered basic in homotopy theory, and the use of the word “combinatorial” in its title indicates its links with combinatorial group theory. His paper [57] is less well known, but is the basis of [45], which describes homotopy

3-types (now called 2-types) in terms of the algebra of crossed modules.¹⁵

I overheard Whitehead tell John Milnor (b. 1931) in 1958 that the early homotopy theorists were fascinated by the operations of the fundamental group on the higher homotopy groups, and also by the problem of computing the latter, preferably with this action.

Whitehead was able by very hard work and study to look at an area and seek out major problems. One of his aims in the late 1930s was to discover whether the Tietze transformations of combinatorial group theory could be somehow extended to higher dimensions. His main method of such extension was envisaged as “expansions” and “collapses” in a simplicial complex; he also wrestled with the problem of simplifying such complexes into some kind of “membrane complex”; in his work after the war, this became codified in [56] as the notion of CW-complex, and his work on generalizing Tietze transformations became a key part of algebraic K -theory.

He was very concerned with the work of Reidemeister on the relations between simplicial complexes and presentations of groups, and methods of finding appropriate geometric models of group constructions, particularly generators and relations, and possible higher analogues. It was only after the war that the topological notion of adding a 2-cell, as compared to adding a relation, was gradually codified through the notion of adjunction space [55]. This gives a useful method of constructing a space Y as an identification space $B \cup_f X$ of $X \sqcup B$ given a space X with an inclusion $i : A \rightarrow X$ and a map $f : A \rightarrow B$, yielding a map $g : X \rightarrow Y$. This definition, which allows for constructing continuous functions from Y , was background to the notion of CW-complex in [56]. A basic account of adjunction spaces and their use in homotopy theory is in [5].

It was gradually realized, cf. [54], that for a pair (X, A) of pointed spaces, i.e., a space X , subspace A of X , and point $x \in A$, there is an exact sequence of groups that ends with

$$\pi_2(X, x) \rightarrow \pi_2(X, A, x) \xrightarrow{\delta} 1(A, x) \xrightarrow{i} 1(X, x) \rightarrow 0, \quad ((6))$$

where $\pi_2(X, A, x)$ is the second relative homotopy group, defined as homotopy classes rel vertices of maps $(I^2, J, V) \rightarrow (X, A, x)$, where V are the vertices of the unit square I^2 , and J consists of all edges except one, say $\partial_1^- I^2$; with this choice, the composition of such classes is taken in direction 2, as in

$$x \overline{\left\| \begin{array}{c} X \\ A \end{array} \right\| \left\| \begin{array}{c} X \\ A \end{array} \right\|} - \frac{x}{A} - \overline{\left\| \begin{array}{c} X \\ A \end{array} \right\| \left\| \begin{array}{c} X \\ A \end{array} \right\|} x \xrightarrow[1]{2} \quad (7)$$

The groups on either side of $\xrightarrow{\delta}$ in the sequence (6) are in general nonabelian. Whitehead saw that there is an

¹⁴An advantage of this conceptual or analogical notation over a more traditional or logical notation is that large diagrams involving these operations can be evaluated by eye, as in [15, p. 188]; cf. [59] for the importance of a conceptual approach. However, such calculations in yet higher dimensions could require appropriate computer programs.

¹⁵For a discussion of crossed modules and compositions of labeled squares, see Appendix 1 below.

operation $(m, p) \mapsto m^p$ of $P = \pi_1(A, x)$ on $M = \pi_2(X, A, x)$ such that $\delta(m^p) = p^{-1}(\delta m)p$ for all $p \in P$, $m \in M$. In a footnote of [53, p. 422], he also stated a rule equivalent to $m^{-1}nm = n^{\delta m}$, $m, n \in M$. The standard proof of this rule uses a 2-dimensional argument, which can be shown in the following diagram, in which $a = \delta m$, $b = \delta n$:

$$\begin{array}{|c|c|c|} \hline -2m & \square & m \\ \hline || & n & || \\ \hline \end{array} \begin{array}{c} \xrightarrow{1} \\ \downarrow 1 \end{array} \quad (8)$$

$a^{-1} \quad b \quad a$

Here the double lines indicate constant paths, while $||$ and $\square$ denote a vertical and double identity respectively. He later introduced the term crossed module for this structure, which has an important place in our story.

Whitehead realized that to calculate $\pi_2(X, x)$, we were really in the business of calculating the group morphism δ of (6), and that such a calculation should involve the crossed module structure. So one of his strengths, as I see it, was that he was always on the lookout for the controlling underlying structure. In particular, [57, Section 16] sets up the notion of free crossed module and uses geometric methods from [53] to show how such can be obtained by attaching cells to a space. This theorem on free crossed modules, which is sometimes quoted but seldom proved in even advanced texts on algebraic topology, was a direct stimulus to the work on higher homotopy groupoids.

As explained in [12, Section 8], Higgins and I agreed in 1974 that this result on free crossed modules was a rare, if not the only, example of a universal nonabelian property in 2-dimensional homotopy theory. So if our conjectured but not yet formulated theory was to be any good, it should have Whitehead's theorem as a corollary. But that theorem was about second *relative* homotopy groups. Therefore, we also should look at a relative situation, say $S \subseteq A \subseteq X$, where S is a set of base points and X is a space.

There is then a simple way of getting what looks like a putative homotopy double groupoid from this situation: instead of the necessarily "1-dimensional composition" indicated in diagram (8), we should take the unit square $I^2 = I \times I$, where I is the unit interval $[0, 1]$, with E as its set of edges and V as its set of vertices. We should consider the set $R_2(X, A, S)$ of maps $(I^2, E, V) \rightarrow (X, A, S)$, and then take homotopy classes relative to the vertices V of such maps to form, say, $\rho_2(X, A, S)$. It was this last set that was now fairly easily shown to have the structure of a double groupoid over $\pi_1(A, S)$ with connections, and so to be a 2-dimensional version of the fundamental groupoid on a set of base points.

I need to explain the term *connections*, as introduced in dimension 2 in [23]. It arose from the desire to construct

examples of double groupoids other than the previously defined $\square G$ of commutative squares in a groupoid G .

Whitehead proved in [54] that the boundary $\delta : \pi_2(X, A, x) \rightarrow \pi_1(A, x)$ and an action of the group $\pi_1(A, x)$ on the group $\pi_2(X, A, x)$ have the structure of a crossed module. He also proved in [57, Section 16] what we call Whitehead's free crossed module theorem, that if X is formed from A by attaching 2-cells, then this crossed module is free on the characteristic maps of the attaching 2-cells; this topological model of adding relations to a group is sometimes stated but rarely proved in texts on algebraic topology. Later, an exposition of the proof as written out in [57, Section 16] was published as [7]; it uses methods of knot theory (Wirtinger presentation) developed in the 1930s and of transversality (developed further in the 1960s), but taken from the papers [52, 54]. The result was earlier put in the far wider context of a 2-dimensional van Kampen-type theorem in [14].

The notion of crossed module occurs in other algebraic contexts, cf. [43], which refers also to 1962 work of the algebraic number theorist Albrecht Fröhlich (1916–2001) on nonabelian homological algebra, and more recently in, for example, [40, 46].

The definition of this crossed module in (8) involves choosing which vertex should be the base point of the square and which edges of the square should map to the base point a , so that the remaining edge maps into A . However, it is a good principle to reduce, preferably completely, the number of choices used in basic definitions (though such choices are likely in developing consequences of the definitions). The paper [14], submitted in 1975, defined for a triple of spaces $\mathbf{X} = (X, A, S)$ such that $S \subseteq A \subseteq X$ a structure $\rho(\mathbf{X})$. This consisted in dimension 0 of S (as a set); in dimension 1, of $\pi_1(A, S)$; and in dimension 2, of homotopy classes relative to the vertices of maps $(I^2, E, V) \rightarrow (X, A, S)$, where E, V are the spaces of edges and vertices of the standard square:¹⁶

$$\begin{array}{ccc} S & -A- & S \\ | & X & | \\ S & -A- & S \end{array} \begin{array}{c} \xrightarrow{2} \\ \downarrow 1 \end{array} \quad (9)$$

This definition makes no choice of preferred direction. It is fairly easy and direct to prove that $\rho(\mathbf{X})$ may be given the structure of a double groupoid with connection¹⁷ containing a copy of the double groupoid $\square \pi_1(A, S)$. That is, the proofs of the required properties of $\rho(\mathbf{X})$ to make it a 2-dimensional version of the fundamental group as sought in the 1930s are fairly easy but not entirely trivial. The longer task, 1965–1974, was formulating the "correct" concepts (in the face of prejudice from some referees and editors). The proof of the corresponding van Kampen's theorem allows a nonabelian result in dimension 2 that vastly generalizes the work of [57, Section 16]. For example, it gives a result when

¹⁶In that paper, it was assumed that each loop in S is contractible in A , but this later proved too restrictive on A , and so homotopies fixed on the vertices of I^2 , and in general I^n , were used in [15].

¹⁷Or alternatively, of a double groupoid with thin structure [15, p. 163].

X is formed from A by attaching a cone on B , Whitehead's case occurring when B is a wedge of circles. See also [15, Chapter 5] for many other explicit homotopical excision examples, some using the GAP system.

Here is an example in which we use more than one base point. Let X be the space $S^2 \vee [0, 1]$, where 0 is identified with N , the north pole of the 2-sphere. Let $S = \{0, 1\}$ as a subset of X . We know that $\pi_2(X, 0) \cong \mathbb{Z}$, and it is easy to deduce that $\pi_2(X, S)$ is the free $\mathcal{I} = \pi_1([0, 1], S)$ -module on one generator. Now we can use the relevant 2-dimensional van Kampen's theorem to show that $\pi_2(S^2 \vee S^1, N)$ is the free $\mathbb{Z}$ -module on one generator.

Note also that it is easy to think of generalizations to higher dimensions of diagram (9), namely to the filtered spaces of [15], following the lead of [2].¹⁸

General Considerations

The paper [12, Section 2] argues that one difficulty in obtaining such strict higher structures, and so theorems on colimits rather than homotopy colimits, is the difficulty of working with bare topological spaces, that is, topological spaces with no other structure (the term “bare” comes from [32, Section 5]).

The argument is the practical one that in order to calculate an invariant of a space, one needs some information on that space: that information will have a particular algebraic or geometric structure that must be used. Because of the variety of convex sets in dimensions higher than 1, there is a variety also of potentially relevant higher algebraic structures. It turns out that some of these structures are nontrivially equivalent and can be described as either broad or narrow.

The broad ones are elegant and symmetric, and useful for conjecturing and proving theorems; the narrow ones are useful for calculating and relating to classical methods; the nontrivial equivalence allows one to get the best use of both. An example in [15] is the treatment of cellular methods, using filtered spaces and the related algebraic structures of crossed complexes (narrow), and cubical ω -groupoids (broad).

This use of structured spaces is one explanation why the account in [15] can, in the tradition of homotopy theory, use and calculate with strict rather than lax, i.e., up to homotopy, algebraic structures. In comparison, the paper [17] gives a strict result for all Hausdorff spaces, but so far has given no useful consequences.

There is a raft of other papers following Grothendieck in exploiting the idea of using lax structures involving homotopies, homotopies of homotopies, etc., in a simplicial context; this involves seeing the simplicial singular complex $S^A(X)$ of a space X as a form of ∞ -groupoid. These ideas have had some famous applications.

To get nearer to a fully nonabelian theory, we so far have only the use of n -cubes of pointed spaces, as in

[9, 18, 19]. It is this restriction to pointed spaces that is a kind of anomaly and was strongly criticized by Grothendieck as not suitable for modeling in algebraic geometry. However, the paper [29] gives an application to a well-known problem in homotopy theory, namely determining the first nonvanishing homotopy group of an n -ad. Also, the nonabelian tensor product of groups from [18] has become a flourishing topic in group theory (and analogously for Lie algebras); a bibliography 1952–2009 has 175 items.¹⁹

The title of the paper [6] was also intended to stimulate the intuition that higher-dimensional geometry requires higher-dimensional algebra, and so to encourage nonrigid argument on the forms that the latter could and should take. Perhaps the early seminar of Einstein [28] could be helpful in this.

It is now a commonplace that the further development of related higher structures is important for mathematics and particularly for applications in physics.²⁰ Note that the mathematical notion of group is deemed fundamental to the idea of symmetry, whose implications range far and wide. The bijections of a set S form a group $\text{Aut}(S)$. The automorphisms $\text{Aut}(G)$ of a group G form part of a crossed module $\chi : G \rightarrow \text{Aut}(G)$. The automorphisms of a crossed module form part of a crossed square [13]. These structures of set, group, crossed module, crossed square, are related to homotopy n -types for $n = 0, 1, 2, 3$.

The use in texts on algebraic topology of sets of base points for fundamental groupoids seems currently restricted to [5, 15, 61].

The argument over Čech's seminar at the 1932 ICM seems now able to be resolved through this development of groupoid and higher groupoid work, and he surely deserves credit for the first presentation on higher homotopy groups, as reported in [1, 24, 39].

Another way of putting our initial quotation from Einstein is that one should be wary of received wisdom.

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¹⁸That paper, as does [57], uses the term group system for what is later called a reduced crossed complex, i.e., one with a single base point.

¹⁹See <http://www.groupoids.org.uk/nonabtens.html>.

²⁰This assertion is supported by a web search on “higher structures in mathematics.”

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Appendix 1: Crossed Modules and Compositions of Labeled Squares

In this section, we give a glimpse of some of the calculations needed to show that the axioms of a crossed module indeed work to give rise to a double groupoid.

To obtain an easy example of a double groupoid, start with a group P and consider the set $\square P$ of commuting squares in P , i.e., quadruples $(a_b^c d)$ such that $ab = cd$. Every well-defined composition of commuting squares is commutative; for example, $ab = cd$ and $dg = ef$ implies $abg = cef$. So it is easy to see that $\square P$ forms the structure of a double groupoid.

In homotopy theory we do not expect all squares of morphisms to commute. So it is sensible to consider a subgroup, say M , of P and squares that commute up to an element of M . There are many choices to make here; suppose we adopt the convention that we consider squares

$$\begin{array}{ccc} & g & \\ h \swarrow & & \searrow a \\ & m & \\ k \swarrow & & \searrow \\ & & \end{array} \quad \begin{array}{c} \rightarrow 2 \\ \downarrow 1 \end{array} \quad (10)$$

in which $a, g, b, k \in P$, $m \in M$, and $k^{-1}b^{-1}ga = m$. So we are starting with the bottom right-hand corner as “base point,” and going clockwise around the square. You quickly find that for a composition of such squares to work, you need the subgroup M to be normal in P .

In homotopy theory you expect many ways of making a boundary commute. So it seems sensible to replace a subgroup M of P by a morphism $\mu : M \rightarrow P$. It also seems sensible to replace the group P by a groupoid. What then should be the conditions on μ ? Convenient ones turned out to be a groupoid version of a notion envisaged in a footnote of the paper [53, p. 422]; in [54], this structure was called a crossed module, and it was further developed in [57].

DEFINITION 1 A morphism $\mu : M \rightarrow P$ of groupoids is called a *crossed module* if μ is the identity on objects; M is discrete, i.e., $M(x, y)$ is empty for $x \neq y$; P operates on the right of the group M , $(m, p) \mapsto m^p$; and the following rules,

in addition to those of an operation, are satisfied for all $m, n \in M$, $p \in P$:

$$\text{CM1. } \mu(m^p) = p^{-1}\mu(m)p;$$

$$\text{CM2. } n^{-1}mn = m^{\mu n}$$

We begin with a crossed module $\mu : M \rightarrow P$, where P is a groupoid with object set S and M is a discrete groupoid consisting of groups $M(s)$, $s \in S$, on which P operates.

We then form the elements of our double groupoid in dimension 0, namely the elements of S ; in dimension 1, the elements of P ; and in dimension 2, the quintuples consisting of one element of M and four elements of P whose geometry forms a square as in the left-hand diagram of (11) and $\mu(n) = k^{-1}b^{-1}ga$.

Further, we define such a filled square to be thin if $n = 1$. Thus the thin elements form a special kind of commutative square.

We try a horizontal composition

$$\begin{array}{ccc} & g & \\ h \swarrow & & \searrow a \\ & n & \\ k \swarrow & & \searrow \\ & & \end{array} \circ_2 \begin{array}{ccc} & c & \\ a \swarrow & & \searrow d \\ & m & \\ b \swarrow & & \searrow \\ & & \end{array} = \begin{array}{ccc} & gc & \\ h \swarrow & & \searrow d \\ & \alpha & \\ kb \swarrow & & \searrow \\ & & \end{array} \quad \begin{array}{c} \rightarrow 2 \\ \downarrow 1 \end{array} \quad (11)$$

assuming that gc, kb are defined, and a vertical composition

$$\begin{array}{ccc} & g & \\ f \swarrow & & \searrow e \\ & u & \\ c \swarrow & & \searrow \\ & & \end{array} \circ_1 \begin{array}{ccc} & c & \\ a \swarrow & & \searrow d \\ & m & \\ h \swarrow & & \searrow \\ & & \end{array} = \begin{array}{ccc} & g & \\ fa \swarrow & & \searrow ed \\ & \beta & \\ h \swarrow & & \searrow \\ & & \end{array} \quad \begin{array}{c} \rightarrow 2 \\ \downarrow 1 \end{array} \quad (12)$$

assuming that fa, cd are defined. The problem is to give values for α, β and to prove in each case that the square fits the definition. In fact, we find that $\alpha = (n^b)m$ and $\beta = mu^d$ will do the trick; these calculations strongly use the two rules for crossed modules.

I won't give the argument here, since it is not difficult and quite fun and can be found in [15, pp. 176–178], including a full proof, which again needs both axioms CM1 and CM2 of the interchange law for $\circ_1, \circ_2$.

The thin elements are related to a functor from the category CM of crossed modules to the category DGT of double groupoids with thin structure, which is an equivalence of categories [23, 36].

These two categories play different roles, and in the language of [12], they are called narrow and broad algebraic structures respectively; the narrow category is used for calculation and relation to traditional terms such as, in our case, homotopy groups. The broad category is used for expressive work, such as formulating higher-dimensional composition, conjectures, and proofs. The equivalence between the two categories, which may entail a number of somewhat arbitrary choices, enables us to use whichever is

convenient for the job at hand, often without worrying about the details of the proof of the equivalence. This is especially important in the case of dimensions greater than 2. Such a use of equivalent categories to provide different types of tools for research should perhaps be considered part of the “methodology” of mathematics [20].

The discussion on [15, p. 163] relates crossed modules to double groupoids with thin structure; the category of those has the advantage of being Yoneda invariant and so can be repeated internally in any category with finite limits. Wider applications of the concept of crossed module are also shown in, for example, [46], and applications of double groupoids with thin structure appear in [30]. Perhaps this distinction between narrow and broad, which becomes even more stark in higher dimensions, is relevant to Einstein’s old discussion in [28].

I think it is fair to say that the higher SVKTs would not have been formulated, let alone proved, in a narrow category, cf. [12], while the formulation in the novel broad category was initially felt by one editor to be “an embarrassment.”²¹ The fact that an SVKT is essentially a colimit theorem implies, of course, that it should give precise algebraic calculations, not obtainable by means of, say, exact sequences or spectral sequences. The connectivity conditions for the use of the SVKT also limit its applicability; the 2-dimensional theorem enables some computations of homotopy 2-types, but as has been known since the 1940s, in terms of group theory considerations, computation of a morphism does not necessarily imply computation of its kernel.

Thus the search initiated by the early topologists has a resolution: precise higher-dimensional versions of the Poincaré fundamental group, using higher groupoids, exist in all dimensions, as described in [15]. Whitehead’s work on free crossed modules [15, p. 235] is seen as a special case of a result on “inducing” a new crossed module $f_*M \rightarrow Q$ from a crossed module $\mu : M \rightarrow P$ by a morphism $f : P \rightarrow Q$ of groups (or groupoids), and this result itself is a special case of a van Kampen-type theorem that includes classical theorems such as the relative Hurewicz theorem. The important point is that the theory allows calculations and applications not previously possible.

Those particular structures, however, model only a limited range of homotopy types. There is another theory for pointed spaces due to Jean-Louis Loday (1946–2012), which is proved in [18] also to have a higher SVKT. It has yielded a range of new applications [9, 29], but its current limitation to pointed spaces makes it less suitable for other areas, such as algebraic geometry.

Note that Whitehead’s paper [58] is a sequel to [56, 57] and other earlier papers, but the treatment is new.

There is also a large literature on models using lax higher homotopy groupoids, stimulated by ideas of Grothendieck; see [15, p. xiv].

Appendix 2: The Fundamental Groupoid of a Space

Since the topic of the fundamental groupoid of a space is not entirely standard, we give here a brief exposition, following roughly the books [26] and [5].

We first define the path category $P(X)$ of a topological space X . Its set of objects is the points of the space X .

Its elements are for all $r \geq 0$, all maps $a : [0, r] \rightarrow X$. Such a map is thought of as a journey from $a(0)$ to $a(r)$, of length r . If b is a path of length $s \geq 0$ starting at $a(r)$, then the path $a + b$ of length $r + s$ is given by $a(t)$ if $0 \leq t \leq r$ and by $b(t - r)$ if $r \leq t$. This addition is associative and makes $P(X)$ into a (small) category. (One needs to make a decision as to whether this addition is from $a(0) \rightarrow b(r + s)$ or the other way round!)

A homotopy $H : [0, r] \times [0, 1] \rightarrow X$ of two paths a, b of length r is a map $H : [0, r] \times [0, 1] \rightarrow X$ such that $H(t, 0) = a(t)$, $H(t, 1) = b(t)$, $t \in [0, 1]$. The homotopy H is said to be rel endpoints if $H(0, t)$ is constant in t , as is $H(r, t)$, and is then called a homotopy $H : a \simeq b$.

We next say that paths a, b from x to y are equivalent if there are constant paths r', s' such that $a + r', b + s'$ are defined and homotopic rel endpoints. It is proved, for example, in [5, Chapter 6] that this is an equivalence relation and that the equivalence classes form a groupoid called the fundamental groupoid $\pi_1(X)$ of the space X . One of the advantages of this approach is that the formulas for certain verifications needed are simple. Another is that students are used to the idea of journeys of different lengths.

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²¹We refer here to the comment in [49, p. 48] in reply to the question, “What can you prove with exterior algebra that you cannot prove without it?” Gian-Carlo Rota retorts, “Exterior algebra is not meant to prove old facts, it is meant to disclose a new world. Disclosing new worlds is as worthwhile a mathematical enterprise as proving old conjectures.”

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