The Origin of the Bernoulli Numbers: Mathematics in Basel and Edo in the Early Eighteenth Century

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Years Ago features essays by historians and mathematicians that take us back in time. Whether addressing special topics or general trends, individual mathematicians or "schools," the idea is always the same: to shed new light on the mathematics of the past.

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The Bernoulli numbers were named after the Swiss mathematician Jacob Bernoulli (1654–1705; Figure 1), whose posthumous book *Ars Conjectandi* (1713) demonstrated the calculation of sums of integer powers.[1] The same sequences of numbers were also published in the Japanese capital, Edo (today Tokyo), also as a posthumous publication, by Takakazu Seki (?–1708; Figure 2), appearing one year prior to Bernoulli's book, and so these numbers might just as well have become known as Seki numbers. This paper introduces the parallel development of mathematics in Basel, Switzerland, and Edo, Japan, and highlights the global conditions that allowed for multiple origins of mathematical discoveries.

It is of particular interest to look at the case of the Bernoulli numbers, because both Bernoulli and Seki had taken steps to extend algebraic methods in similar ways while working in two places remote from each other that had no exchange of mathematical knowledge. It is thus an early example in modern mathematics in which a particular technical advancement was made independently and simultaneously in two different parts of the world. The concept of "circulation of knowledge" cannot be applied in this case as an explanation. Introducing the two stories together, this study contributes to a reconstruction of the history of mathematics that moves away from the Euro-centric view; it proposes that we take into account the parallel trajectories of European and Asian mathematics in the early eighteenth century, thereby producing a more balanced narrative of a global history of mathematics.

The Bernoulli Numbers

Let us begin by describing the Bernoulli numbers. For specialists, the Bernoulli numbers are commonly defined as the coefficients of x in the expansion of the function $x/\left(1-e^{-x}\right)$ in powers of x:

$$\frac{x}{1-e^{-x}}=\frac{x}{2}\left(\coth\frac{x}{2}+1\right)=\sum_{k=0}^{\infty}\frac{B_k x^k}{k!},$$

where B_k is the kth Bernoulli number.[2]

But in the early eighteenth century, the number e that came to be known as Euler's constant had not yet come into existence [7]. Let us turn, therefore, to the original formulation, in which the Bernoulli numbers arise as coefficients in the sum of the first n consecutive kth powers. Here are the first thirteen Bernoulli numbers:

[1] Jacob Bernoulli's name appears in the literature with several different spellings. This article follows the way he signed his name: Jacob. A portrait with his signature appears as the frontispiece of [4].

[2] For more on the Bernoulli numbers, see the Bernoulli number page, https://www.bernoulli.org, accessed July 23, 2020. There are thousands of research papers related to the Bernoulli numbers; see, for example, Karl Dilcher's website https://www.mscs.dal.ca/ dilcher/bernoulli.html, accessed July 23, 2020.

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Figure 1. Portrait of Jacob Bernoulli (1686, around 32 years old), painted by his brother Nikolaus (1662–1716).

Figure 2. A portrait of Takakazu Seki (d. 1708). (Courtesy of the Ichinoseki City Museum.)

$$B_0 = 1,\ B_1 = \frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30},$$
$$B_5 = 0,\ B_6 = \frac{1}{42},\ B_7 = 0,\ B_8 = -\frac{1}{30},$$
$$B_9 = 0,\ B_{10} = \frac{5}{66},\ B_{11} = 0,\ B_{12} = -\frac{691}{2730}.$$

Starting from the Taylor series, the Euler–Maclaurin formula, and the Riemann zeta function, the importance and usefulness of the Bernoulli numbers have been widely acknowledged among contemporary mathematicians. As for its history, historian of mathematics Janet Beery has provided this chronological summary of the development of the formula for sums of integer powers [3]:

> Formulas for sums of integer powers were first given in generalizable form in the West by Thomas Harriot (c. 1560–1621) of England. At about the same time, Johann Faulhaber (1580–1635) of Germany gave formulas for these sums up to the 17th power, far higher than anyone before him, but he did not make clear how to generalize them. Pierre de Fermat (1601–1665) often is credited with the discovery of formulas for sums of integer powers, but his fellow French mathematician Blaise Pascal (1623–1662) gave the formulas much more explicitly. The Swiss mathematician Jakob Bernoulli (1654–1705) is perhaps best and most deservedly known for presenting formulas for sums of integer powers to the European mathematical community. His was the most useful and generalizable formulation to date because he gave by far the most explicit and succinct instructions for finding the coefficients of the formulas.

In Beery's paper, the developments related to sums of powers are listed in chronological order, and the history is dominated by some "big names" such as Harriot, Fermat, Pascal, and Bernoulli. The geographical scope of this approach, however, is limited to Europe, and the simultaneous development of mathematics outside of Europe has been entirely left out of the picture.

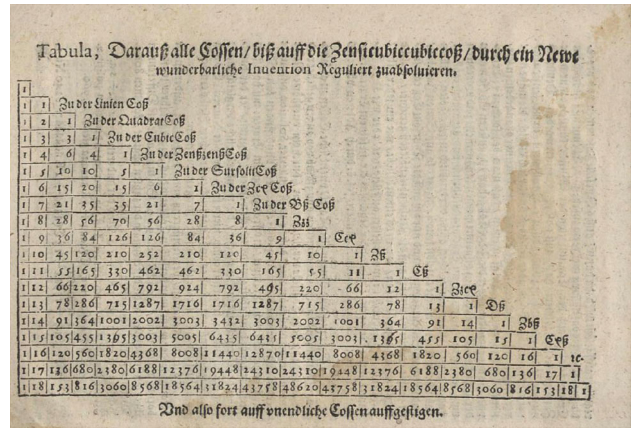
Jacob Bernoulli's *Ars Conjectandi* (1713)

We begin with a review of the discovery of the Bernoulli numbers in Basel, Switzerland.

Jacob's father expected his son, who had been given a theological education, to become a minister in Basel. Two events in Jacob's life, however, changed his career path. First, he traveled to France in 1677. At the age of twenty-three, Jacob studied with the followers of Descartes and was fascinated by mathematics and astronomy. Second, he was transfixed by the appearance of the great comet of 1680. Jacob believed that there must be a way to predict the occurrence of comets. Deeply absorbed in the study of mathematics and astronomy, he decided to abandon the clergy, following the motto *Invito Patre, Sidera verso* (against my father's will, I will turn to the stars). He then embarked on his second European trip, this time to the Netherlands and England, acquainting himself with important scientific intellectuals, including Robert Hooke and Robert Boyle.[3]

[3]For a biographical sketch of Jacob Bernoulli, see [30, pp. 248–250] or [25, pp. 3–5].

Figure 3. Copperplate engraving of Basel, Switzerland, Bernoulli's hometown, and the Rhine River, by Mattäus Merian (1593–1650).



Tabula, Darauß alle Cossen/ biß auff die Zensicubiccubiccoß/ durch ein Newe wunderbarliche Inuention Reguliert zuabsoluieren.

1																			
1	1	Zu der Linien Coß																	
1	2	1	Zu der Quadrat Coß																
1	3	3	1	Zu der Cubic Coß															
1	4	6	4	1	Zu der Zenßzenß Coß														
1	5	10	10	5	1	Zu der Sursolit Coß													
1	6	15	20	15	6	1	Zu der Zce Coß												
1	7	21	35	35	21	7	1	Zu der Bß Coß											
1	8	28	56	70	56	28	8	1	Zzz										
1	9	36	84	126	126	84	36	9	1	Cce									
1	10	45	120	210	252	210	120	45	10	1	Aß								
1	11	55	165	330	462	462	330	165	55	11	1	Cß							
1	12	66	220	465	792	924	792	495	220	66	12	1	Zzce						
1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1	Dß					
1	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	14	1	Zbß				
1	15	105	455	1365	3003	5005	6435	6435	5005	3003	1365	455	105	15	1	Ceß			
1	16	120	560	1820	4368	8008	11440	12870	11440	8008	4368	1820	560	120	16	1	ꝛc.		
1	17	136	680	2380	6188	12376	19448	24310	24310	19448	12376	6188	2380	680	136	17	1		
1	18	153	816	3060	8568	18564	31824	43758	48620	43758	31824	18564	8568	3060	816	153	18	1	

Vnd also fort auff vnendliche Cossen auffgestigen.

Figure 4. Table in the appendix of Faulhaber's *Academia Algebræ*, which the author claims is a *wunderbarliche Invention* (amazing invention).

Upon his return to Basel (see Figure 3) in 1682, Jacob began lecturing in experimental physics at the University of Basel, and in 1687, he became a professor of mathematics there. Although the university itself, failing to attract many students, was not thriving,[4] Jacob's talent was widely recognized in Europe, mainly thanks to contacts outside of Switzerland.[5] He continued to correspond with mathematicians whom he had met, and one of his most frequent contacts was Gottfried Wilhelm Leibniz (1646–1716).[6] Through correspondence, Jacob and his brother Johann Bernoulli (1667–1748) studied Leibniz's papers. As a result, they were up to date on the recent development of Leibniz's calculus, which was at the cutting edge of the mathematics of the time.[7]

The idea of calculating the sum of integer powers did not begin with Jacob Bernoulli. As Beery has noted, there were mathematicians prior to Jacob[8] who had studied the sum over n, n^2, n^3, $\ldots$, n^7. The German mathematician Johann Faulhaber (1580–1635) in particular had already tried to compute the sums of odd powers in his 1631 *Academia Algebræ* [11].[9]

Faulhaber analyzed the pattern that appears in the sums of integer powers, and he presented a table of binomial coefficients in his book that showed the result of his calculations on these sums. In the table (Figure 4) there appears the word *Coß* in relation to the integer power.[10] For Faulhaber, the word meant *unbekannte, aber auch Gleichungslehre* (unknowns, but also the theory of equations) [26, p. 51].

Faulhaber was a "cossist," one who adopted symbols and words to express mathematical concepts. The term derives from the Italian word *cosa*, meaning the "thing" that mathematicians used to represent an unknown quantity, so that a cossist might be seen as roughly equivalent to a contemporary algebraist [14, p. 51]. By the fifteenth century, the practice of solving algebraic equations came to be described in Germany as "the cossick art." For example, the abbreviations $\mathfrak{R}$, $\mathfrak{Z}$, and $\mathfrak{C}$, written in elaborate Gothic script, replaced the words *res. zensus* (or *census*) and *cubus*, and they were combined to express $\mathfrak{ZZ}$ for the fourth power, $\mathfrak{RZZ}$ for the fifth, $\mathfrak{ZZZ}$ for the sixth, $\mathfrak{CZZ}$ for the seventh, and so on [29, p. 58]. Subsequently, the cossists in the German-speaking countries, such as Michael Stifel (1487–1567), Johannes Scheubel (1494–1570), and Christoff Rudolff (1499–1545), adopted the use of symbols for arithmetic operations such as those for equality and extraction of roots [8].

Assigning the new symbols was not an easy task, but they were very helpful in solving some of the more complicated problems in algebra. By Faulhaber's time, the standardized symbols, ß, bß, cß, ... were used to represent fifth, seventh, eleventh powers, etc. [29, p. 38]. In adopting them, Faulhaber became a notable cossist, using new symbols wherever applicable. Eventually, as Knuth asserts, Faulhaber "may well have carried out more computing than anybody else in Europe during the first half of the 17th century" [18, p. 2]. Faulhaber, for example, represented the formulas for sums of odd powers as shown in Table 1.[11]

What is novel in this table is Faulhaber's cossist approach—assigning symbols to a set of ideas. Here, he expresses the formulas in terms of a symbol N that

[4]According to [12, p. 35], enrollment had declined significantly, and the university was not as active as it had been in the previous century.

[5]During this period, France became a great power in Europe. Switzerland was independent of the Holy Roman Empire and became the Swiss Confederacy after the Treaty of Westphalia, which came into effect in 1648. Intellectuals often traveled in Europe and communicated via networks of correspondents.

[6]Some of the correspondence can be found in [5]. Ian Hacking also studies the correspondence in [13].

[7]Stillwell states that the brothers equaled Leibniz himself in the brilliance of their discoveries [30, p. 250].

[8]See Knuth's article [18, pp. 7–8]. Dubeau's paper [9, p. 596] includes key writings regarding sums of powers of integers.

[9]Though the title is in Latin, this book was written in German. See also [22] and [2].

[10]The word *Coß* appears in a 1524 work by Adam Ries (1492–1559). There, the word referred to unknown variables. For a more detailed study of the word *Coß*, see [17].

[11]Masanobu Kaneko pointed out to me that the coefficient 10 of N^2 in the sixth row of Table 1 is given incorrectly in [18] as 5.

Table 1. **Faulhaber's representation of the formulas for the sum of the first n odd powers in terms of $N = \frac{1}{2}(n^2 + n)$**

$$1^1 + 2^1 + \cdots + n^1 = N;$$
$$1^3 + 2^3 + \cdots + n^3 = N^2;$$
$$1^5 + 2^5 + \cdots + n^5 = \tfrac{1}{3}(4N^3 - N^2);$$
$$1^7 + 2^7 + \cdots + n^7 = \tfrac{1}{5}(12N^4 - 8N^3 + 2N^2);$$
$$1^9 + 2^9 + \cdots + n^9 = \tfrac{1}{5}(16N^5 - 20N^4 + 12N^3 - 3N^2);$$
$$1^{11} + 2^{11} + \cdots + n^{11} = \tfrac{1}{6}(32N^6 - 64N^5 + 68N^4 - 40N^3 + 10N^2);$$
$$1^{13} + 2^{13} + \cdots + n^{13} = \tfrac{1}{105}(960N^7 - 2800N^6 + 4592N^5 - 4720N^4 + 2764N^3 - 691N^2);$$
$$1^{15} + 2^{15} + \cdots + n^{15} = \tfrac{1}{12}(192N^8 - 768N^7 + 1792N^6 - 2816N^5 + 2872N^4 - 1680N^3 + 420N^2);$$
$$1^{17} + 2^{17} + \cdots + n^{17} = \tfrac{1}{45}(1280N^9 - 6720N^8 + 21120N^7 - 46880N^6 + 72912N^5 - 74220N^4 + 43404N^3 - 10851N^2).$$

represents $(n^2 + n)/2$, not with a lowercase n representing an odd number, as in all previous attempts.

Although Jacob Bernoulli studied Faulhaber's work, a more direct source of information on sums of powers of integers was the work of Ismaël Boulliau (1605–1694), a French astronomer and polymath.[12] Boulliau was an active correspondent, exchanging letters with contemporary astronomers, mathematicians, and philosophers, including Galileo, Mersenne, Huygens, and Fermat, as well as Faulhaber himself. The year in which Boulliau and Faulhaber discussed the sum of powers is unknown, but Boulliau published his *Opus novum ad arithmeticam infinitorum* (*Arithmetica Infinitorum* thereafter) in 1682 at the age of seventy-seven.[13] Here he devoted many pages to explaining a formula concerning sums of powers of integers.

Jacob Bernoulli began his computations by studying Boulliau, believing that there must be an easier and faster method. He made the table shown in Figure 5, called *Summae Potestatum* (sums of powers), stating:

> With the help of this table it took me less than half of a quarter of an hour to find that the tenth powers of the first 1000 numbers being added together will yield the sum
>
> $$91{,}409{,}924{,}241{,}424{,}243{,}424{,}241{,}924{,}242{,}500\,.$$
>
> From this it will become clear how useless was the work of Ismaël Boulliau spent on the compilation of his voluminous *Arithmetica Infinitorum* in which he did nothing more than compute with immense labor the sums of the first six powers, which is only a part of what we have accomplished in the space of a single page.[14]

Posthumous Publication of *Ars Conjectandi*

Jacob did not publish the Bernoulli numbers immediately; they appeared only in his book *Ars Conjectandi*, published in 1713, eight years after his death.

... Atque si porrò ad altiores gradatim potestates pergere, levique negotio sequentem adornare laterculum licet :

Summae Potestatum

$$\int n = \tfrac{1}{2}nn + \tfrac{1}{2}n$$
$$\int nn = \tfrac{1}{3}n^3 + \tfrac{1}{2}nn + \tfrac{1}{6}n$$
$$\int n^3 = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}nn$$
$$\int n^4 = \tfrac{1}{5}n^5 + \tfrac{1}{2}n^4 + \tfrac{1}{3}n^3 - \tfrac{1}{30}n$$
$$\int n^5 = \tfrac{1}{6}n^6 + \tfrac{1}{2}n^5 + \tfrac{5}{12}n^4 - \tfrac{1}{12}nn$$
$$\int n^6 = \tfrac{1}{7}n^7 + \tfrac{1}{2}n^6 + \tfrac{1}{2}n^5 - \tfrac{1}{6}n^3 + \tfrac{1}{42}n$$
$$\int n^7 = \tfrac{1}{8}n^8 + \tfrac{1}{2}n^7 + \tfrac{7}{12}n^6 - \tfrac{7}{24}n^4 + \tfrac{1}{12}nn$$
$$\int n^8 = \tfrac{1}{9}n^9 + \tfrac{1}{2}n^8 + \tfrac{2}{3}n^7 - \tfrac{7}{15}n^5 + \tfrac{2}{9}n^3 - \tfrac{1}{30}n$$
$$\int n^9 = \tfrac{1}{10}n^{10} + \tfrac{1}{2}n^9 + \tfrac{3}{4}n^8 - \tfrac{7}{10}n^6 + \tfrac{1}{2}n^4 - \tfrac{1}{12}nn$$
$$\int n^{10} = \tfrac{1}{11}n^{11} + \tfrac{1}{2}n^{10} + \tfrac{5}{6}n^9 - 1n^7 + 1n^5 - \tfrac{1}{2}n^3 + \tfrac{5}{66}n$$

Quin imò qui legem progressionis inibi attentuis ensperexit, eundem etiam continuare poterit absque his ratiociniorum ambabimus : Sumtâ enim c pro potestatis cujuslibet exponente, fit summa omnium n^c seu

$$\int n^c = \frac{1}{c+1}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c \cdot c-1 \cdot c-2}{2 \cdot 3 \cdot 4}Bn^{c-3}$$
$$+ \frac{c \cdot c-1 \cdot c-2 \cdot c-3 \cdot c-4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}Cn^{c-5}$$
$$+ \frac{c \cdot c-1 \cdot c-2 \cdot c-3 \cdot c-4 \cdot c-5 \cdot c-6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}Dn^{c-7} \cdots \text{\& ita deinceps,}$$

exponentem potestatis ipsius n continué minuendo binario, quosque perveniatur ad n vel nn. Literae capitales A, B, C, D & c. ordine denotant coëfficientes ultimorum terminorum pro $\int nn$, $\int n^4$, $\int n^6$, $\int n^8$, & c. nempe

$$A = \frac{1}{6}, B = -\frac{1}{30}, C = \frac{1}{42}, D = -\frac{1}{30}\,.$$

Figure 5. Page 97 of *Ars Conjectandi* (1713). The symbols do not correspond to modern notation. For example, the integral symbol means a sum (Σ). There is one error on this page. In the last term of the sum of ninth powers, the coefficient should read not $-\frac{1}{12}nn$ but $-\frac{3}{20}nn$. The second, fourth, sixth, and eighth Bernoulli numbers are given as $A = \frac{1}{6}$, $B = -\frac{1}{30}$, $C = \frac{1}{42}$, and $D = -\frac{1}{30}$.

[12] Boulliau published on a wide range of topics from the fields of mathematics, astronomy, theology, and history.

[13] He published his most influential book, *Astronomia philolaica*, in 1645, in which he claimed the "inverse-square law" for the calculation of intensity of illumination or gravitational force in inverse proportion to the square of the distance from the source. Additionally, he stated a hypothesis, known as the conical hypothesis, in which he assumed that the planetary orbits are circular. He was also a collaborator of Johannes Kepler (1571–1630).

[14] Translation by Jekuthiel Ginsburg in [28, p. 90]. The spellings of some words have been changed for consistency.

Ars Conjectandi was a study of the mathematics of probability.[15] Analyzing how chance operates in games and applying such methods to broader concepts such as "judgments in civil, moral, and economic affairs," the book focuses on developing the analytical tools needed to apply probability to real-life problems.[16] The section on the Bernoulli numbers (a segment in Part II of the book) was somewhat out of place, since it discussed a completely different topic, namely shorter and more efficient methods of calculating sums of powers.

As for the statement Jacob made regarding Ismaël Boulliau's method, it was not unusual to claim to have discovered a better method and brag about it. After all, Bernoulli lived in an era of priority disputes. Newton's followers and those of Leibniz fought over credit for the invention of calculus. Johann, Jacob's younger brother, was involved in this controversy, which became most intense in 1713.[17] Nor were Jacob and Johann bystanders. Between them, they initiated a fierce public feud that raged for decades prior to this. Historian Jeanne Peiffer traced the tensions leading up to their rivalry and determined that their "split became definite in the summer of 1694" [25, p. 12].[18] The rivalry between Jacob and Johann was public knowledge, as was the later split between Newton and Leibniz.

This rivalry between the brothers prevented the rapid dissemination of the work left by Jacob. After his death, the manuscript of *Ars Conjectandi* was not immediately published. According to Martin Mattmüller, of the Bernoulli-Euler-Zentrum at the University of Basel, the delay in its publication was rooted in the discord in the Bernoulli family, and he explains the course of events in the following way.[19]

Johann claimed that Jacob's heirs would not let him access the unpublished document, and he was also busy with the publication of his own discovery. Jacob's widow, Judith Stupanus, first passed the task of editing the unpublished work to their son, allowing him to bring the manuscript to the mathematician Pierre Varignon (1654–1722) in Paris. But that plan did not bear fruit. Jacob and Johann's nephew Nikolaus (1687–1759) had studied mathematics with Jacob at the University of Basel and become a mathematician. He went to Groningen, where Johann worked as a professional mathematician. Both Nikolaus and Johann were aware of Jacob's unpublished work, but both of them were kept away from it by the surviving members of Jacob's family.

In 1711, Nikolaus convinced Jacob's son not to delay publication further. Because the surviving members of Jacob's family still opposed Johann and Nikolaus having access to Jacob's manuscript, two editors were selected to take charge of compiling the work. They were not mathematicians: one, Franz Christ (1688–1744), was a recent recipient of a doctoral degree in law (Dr. jur. 1711), and the other was an unemployed minister, Samuel Bringolf (1678–?).[20] They had no training in mathematics, and thus they did not understand the contents. Finally, in March 1713, Varignon urged Johann and Nikolaus to help them; Nikolaus "reluctantly" compiled an errata list and added a preface [21, pp. 286–287].

For Jacob and for his mathematician contemporaries, the discovery of a general pattern for calculating sums of integer powers was of only passing interest. Jacob made no further use of it, nor did he claim that his method was an important contribution to mathematics. There was no widespread discussion of this sequence,[21] and the result was somewhat haphazardly inserted into a book on probability theory, which did not have an examination of the characteristics of integers as a primary goal.

Takakazu Seki and Mathematics in Japan

Takakazu Seki's situation was very different from Jacob Bernoulli's. Takakazu Seki was not a mathematician teaching at a university. Indeed, there were no higher academic institutions in Japan that resembled the contemporary universities of Western Europe. Mathematical education was textbook-based and pursued individually or in an informal study group.

Seki's birth year has been estimated as lying between 1640 and 1645. He succeeded to his adoptive father's job in Kōfu (today Yamanashi prefecture) and served as a member of the guards of the Kōfu domain from 1665. Seki then lived in Edo (Figure 6) with his wife. They had two daughters, but both died in childhood, and the couple adopted two sons [20, pp. 3–4]. Seki was promoted several times in his career. Most notably, when his master, Ienobu Tokugawa (1662–1712), was called to be the shogun-designate (the next head of the government) in 1703, Seki entered Edo Castle with him as his close vassal and was appointed chief accountant of the palace (*nando kumigashira*); see [16, p. 104] and [20, p. 5].

Seki belonged to the warrior class, serving his master in exchange for an annual salary.[22] Having a day job as an administrator of finance, Seki taught mathematics only to a handful of disciples in his spare time. His knowledge of mathematics came from the sources available within Japan. He did not travel abroad, for Japan was under a national

[15] Edith Dudley Sylla states in [6, p. xiii], "Not only did *Ars Conjectandi* prove rigorously the first limit theorem in probability, it also founded the field of mathematical probability *conceptually*."

[16] Jacob Bernoulli himself used these terms. See his letter to Leibniz of October 3, 1703, in [5].

[17] The Royal Society took Newton's side by publishing a report, *Commercium Epitolicum*, in 1713.

[18] See also [15, pp. 73–93]. Sepideh Alassi discusses the case of Jacob and Johann's priority dispute in discovering the *velaria curve* in [1, p. 150]. John Stillwell also presents different evidence and claims that the rivalry became open hostility in 1697 [30, p. 251].

[19] For details, see [21, pp. 286–288].

[20] I am grateful to Martin Mattmüller for this information. The source is a letter from Johann Bernoulli to Nikolaus dated July 15, 1712.

[21] Although it might be anticipated that Jacob's notebook *Meditationes* would contain a trace of his thoughts on the application of Bernoulli numbers, such is not the case.

[22] In Seki's era, the samurai class was the ruling class of society. There were no wars in Japan during Seki's lifetime.

Figure 6. The town Edo, where Seki lived. This painting depicts Edo in the late seventeenth to early eighteenth century. The capital of Japan was flourishing, and mathematics was studied privately. (Courtesy of Edo-Tokyo Museum.)

seclusion policy whereby travel outside the country was forbidden, and foreigners, except for some Dutch, Korean, and Chinese merchants, were not allowed to enter the country. The central government maintained this policy from 1639, at which time most Westerners were expelled from the Japanese archipelago. Even the foreign books that were not censored by the central government were unavailable, and the Dutch, who were allowed to reside in a designated area remote from Edo, were never more than twenty in number [16, p. 100]. Some Chinese mathematics books, however, were available to Seki. From the ancient mathematical canon, *Nine Chapters on the Mathematical Arts* (*Kyūshō sanjutsu*, first century or later), Seki read Yang Hui's *Methods of Mathematics* (*Yōki sanpō*, c. 1378), Zhu Shijie's *Introduction to Mathematics* (*Sangaku keimō*, 1299), and Cheng Dawei's *Systematic Treatise on Arithmetic* (*Sanpō tōsō*, 1592) [23, p. 119].

Among those Chinese mathematics texts, not only did Seki transcribe Yang Hui's *Methods of Mathematics*, but he also corrected some errors that he found in the original. He then copied the entire book in 1661 [16, p. 107].[23] Most notably, this thirteenth-century book contained the Chinese version of Pascal's triangle, which shows the binomial coefficients in the shape of an actual triangle, with one vertex at the top (see Figures 7, 8, 9).[24]

After learning the basics, Seki published one manuscript, answering the challenge questions posed by the mathematician Kazuyuki Sawaguchi (dates unknown) in 1671 in his book *Kokin sanpōki* (*Old and New Mathematics*). In Japan, mathematical problems were posed widely to an interested audience, and the race to solve them created clusters of local amateur mathematicians who studied problems together. In his lifetime, Seki published only one book containing answers to challenge questions, and his other works were either written in collaboration with his disciples or published posthumously in edited volumes by

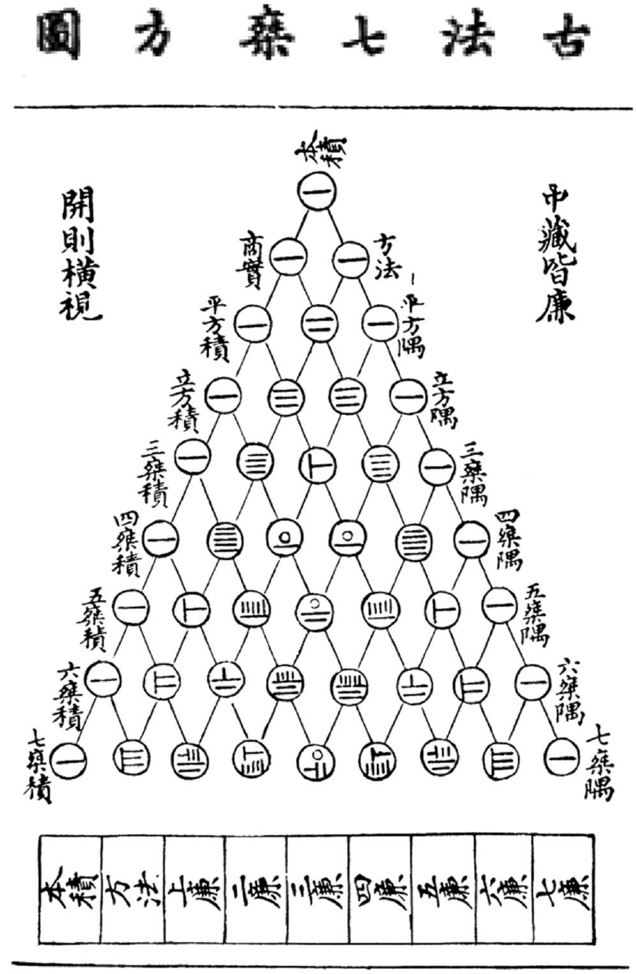


Figure 7. The chart in Yang Hui's *Methods of Mathematics* showing a triangular array of binomial coefficients. The numbers were written in rod numerals.

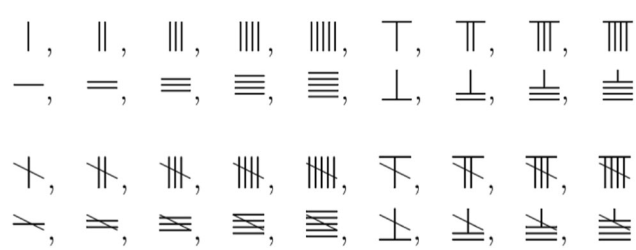


Figure 8. Positive and negative integers expressed by rods. The first row gives the units digit, from one to nine (left to right), and the second gives the tens place, representing $10, 20, \ldots, 90$ (left to right).

[23] Majima claims that there are two extant copies of Seki's work, one at the Shinminato Museum of Toyama (1661), and the other owned by Kiyoshi Yabuuchi (1673) [20, p. 14]. Seki was not the first to study Yang Hui's *Methods of Mathematics*, but it is worth noting that he transcribed and copied the text. His annotations were studied by his disciples. As a result, this Chinese textbook was widely circulated after Seki's death.

[24] For the history of Pascal's triangle in the Chinese context, see [19].

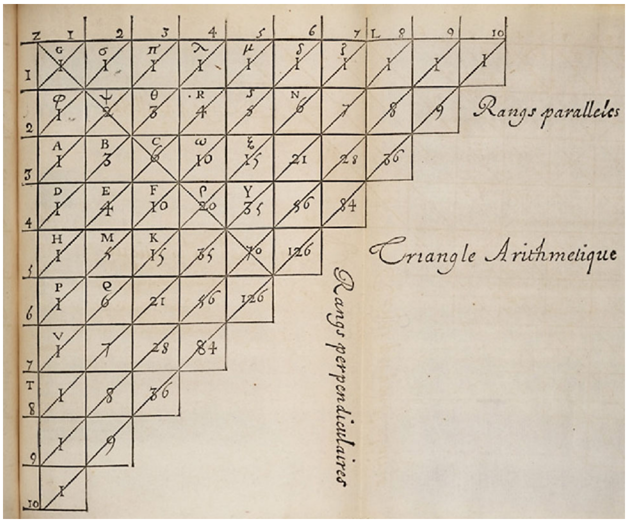


Figure 9. For comparison with Figure 7, Pascal's drawing displaying the binomial coefficients is pictured here; from *Traité du triangle arithmétique avec quelques autres petits traitez sur la mesme matière* (1665). (Courtesy of Cambridge University Digital Library.)

them.[25] Seki had two talented disciples who studied mathematics with him. They were brothers, both actively engaged in mathematical studies in the 1680s, and because of their contribution, mainly those of the younger brother, Katahiro Takebe (1664–1739), it is often difficult to draw a line between Seki's and Takabe's work.

The work by Seki that he published himself was *Mathematical Treatise Revealing the Hidden Meaning* (*Hatsubi sanpō*, 1674). In the preface, Seki set forth his views on mathematics as follows:

> Mathematics is practiced in excess in the world nowadays. One can no longer count the number of those who open their own school or publish their own textbook. Thus there is the *Treatise of Ancient and Modern Mathematics* [*kokon sanpōki*] where, without any consideration for the student, fifteen difficult problems are proposed. Since its publication, the mathematicians of this world take this textbook into their hands in vain, and suffer from not being able to elucidate its principle because it is too elevated. What is more, I have not yet seen any work that gave the answers to it. For my part, having formed long ago the wish to devote myself to this path, I could extricate its profound sense and compile the procedures and configurations. But I had put them aside, fearing to make them known. The students of my school unanimously begged me to print them in order to pass them on to a larger number, and they assured me that I had to bring some minimal help to future students. Hence, without thinking any more about my mediocrity, I complied with their request, and I gave the work the name *Hatsubi Sanpō* [*Mathematical Treatise Revealing the Hidden Meaning*]. The full text of it, involving the most refined details of method (*endan*), is long and overloaded, and including it would have obscured the discussion. For this reason I omit it here [16, p. 140].[26]

It is striking that Seki deliberately took out the "procedures and configurations" from his book and kept them secret, and he omitted his method. But the methods used to solve the problems in his book became controversial and drew criticism from other study groups. Katahiro Takebe then decided to publish annotations, presenting detailed steps to solve the problems, as *Commentaries of Operations in the Mathematical Treatise Revealing the Hidden Meaning* (1685). In Takebe's book, Seki's point of view in doing mathematics is said to be much clearer. The afterword of Takebe's book was written by Seki himself:[27]

> What is mathematics for? Either the problem is easy or difficult, we need to study it to know the proper methods for solving them. No matter how sophisticated the principle may be, it is unorthodox if the method is complicated.

Amateur mathematicians began to gather and form factions. Much like the "master institutions" existing in many other avocations, such as the tea ceremony and flower arrangement, these mathematics enthusiasts created study groups.[28] As a result of repeated competitions among those factions, the methods used to obtain Seki's answers in 1674 were eventually challenged, and Seki's disciples had to claim their authenticity while defending them as orthodox.[29]

Seki's posthumous publication *Compendium of Mathematics* (*Katsuyō sanpō*, 1712) was compiled by his disciples Yoshimasa Otaka (dates unknown) and Murahide Araki (1640–1718). By this time, Takebe was serving the shogun and had become, as his brother wrote, "a very busy official who was no longer in a position to go more thoroughly into [mathematics]" [16, p. 109]. The page where Seki tried to formulate the sum of powers was titled "sum of powers" (*dasekijutsu*), and this is the place where Seki expressed the formula for the sum of powers (Figures 10 and 11). Since he had often omitted the methods, he did not explain how he had obtained the numbers, which appear in the left-hand column.[30] How did this chart work and where do we see the Bernoulli numbers?

[25]The exact number of Seki's disciples as well as the ways he taught mathematics are unknown. For a list of Seki's works, see [27, pp. 2–3].
[26]The original can be found in [27, p. 105].
[27]The book's title was *Hatsubi sampō endan-genkai*, published by Katahiro Takebe. The "principle" is *li*, referring to the Confucian principle. It is not the same as the method (or *hō*), which means solving mathematical problems.
[28]For a cultural explanation as to why factions emerged in traditional Japanese mathematics (*wasan*), see "Master Institution—Wasan as a Hobby," available online at https://www.ndl.go.jp/math/e/s1/3.html (accessed January 2, 2021).
[29]The group in Kyoto, for example, became an immediate rival to the Edo-based mathematicians, including the Seki school.
[30]Ogawa explored the possible methods of obtaining the numbers, but there are no records [24].

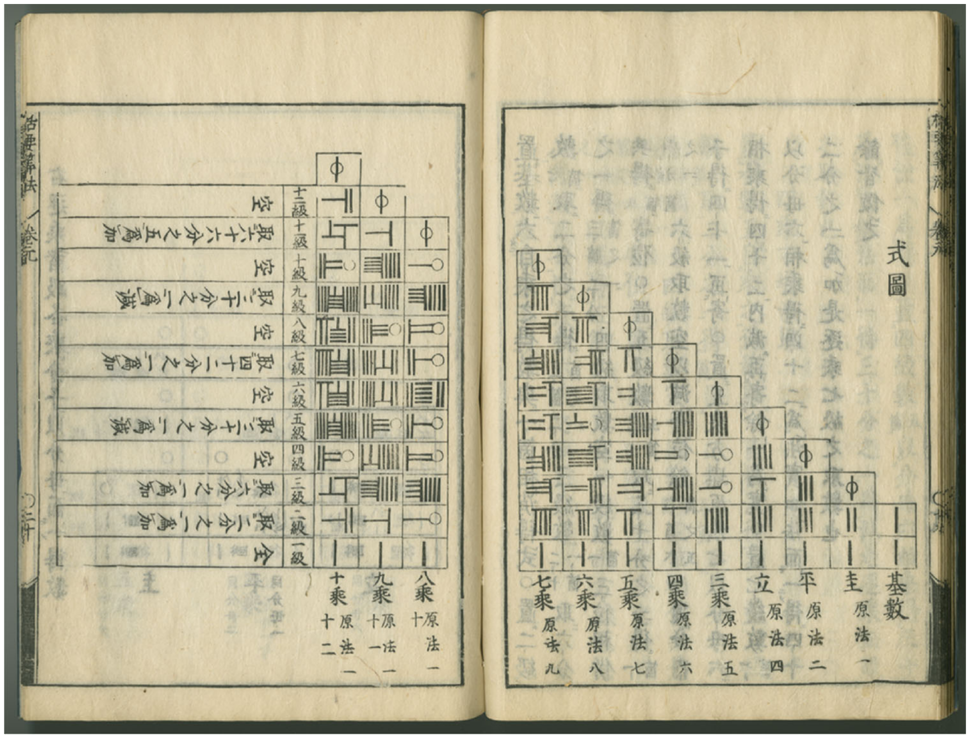


Figure 10. The table in *Compendium of Mathematics* (1712). The numbers were written using rod numerals.

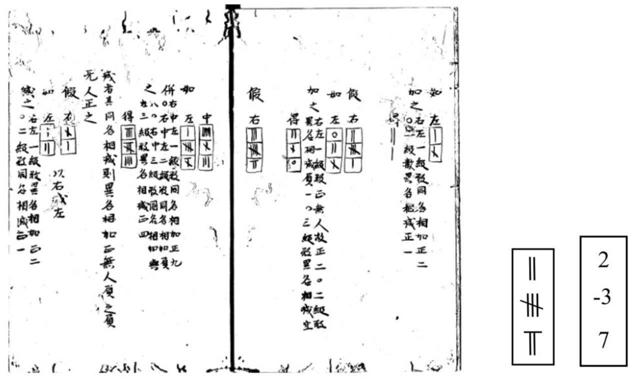
		Φ											
void	degree12	12	Φ										
take 5/66 and add	degree11	66	11	Φ									
void	degree10	220	55	10	Φ								
take 1/30 and subtract	degree 9	495	165	45	9	Φ							
void	degree 8	792	330	120	36	8	Φ						
take 1/42 and add	degree 7	992[c]	462	210	84	28	7	Φ					
void	degree 6	792	462	252	126	56	21	6	Φ				
take 1/30 and subtract	degree 5	495	330	210	126	70	35	15	5	Φ			
void	degree 4	220	165	120	84	56	35	20	10	4	Φ		
take 1/6 and add	degree 3	66	55	45	36	28	21	15	10	6	3	Φ	
take 1/2 and add	degree 2	12	11	10	9	8	7	6	5	4	3	2	1
all	degree 1	1	1	1	1	1	1	1	1	1	1	1	1
the number of multiplication		11	10	9	8	7	6	5	4	3	2	1	0
denominator		12	11	10	9	8	7	6	5	4	2[b]	1[a]	

Figure 11. Translation of Figure 10. Notes: (a) The number 1 should be 2. (b) The number 2 should be 3. (c) The number 992 should be 924.

In Figure 10 and its translation in Figure 11, the Bernoulli numbers appear in the left-hand columns:

$$1, +\frac{1}{2}, +\frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, 0.$$

The right part of this figure is "Pascal's triangle," that is, a list of the binomial coefficients. The second column from the right shows the numbers of repeated operation(s) of multiplication. One means the second power. Thus, this column shows the sum of second powers, i.e., $1^2 + 2^2 + \cdots + n^2$. Reading the column from the bottom as the numbers 1, 2, 0 yields $1 \times n^2 + 2 \times n + 0 \times 1 = n^2 + 2n$. But the left-hand column notes, "take $\frac{1}{2}$ and add," and thus an additional operation is needed for $2n$.



(a) page from the *Methods of Solving Implicit Problems* (b) translation

Figure 12. (a) A page from the *Methods of Solving Implicit Problems*. The polynomial $a_0 + a_1x + a_2x^2$ can be expressed on the counting board in the boxed symbols. (b) Representation of a polynomial with a translation into Arabic numerals. The symbols $2, -3, 7$ represent $2 - 3x + 7x^2$. Note that Japanese is read from top to bottom. (Courtesy of National Diet Library of Japan.)

This yields $n^2 + \frac{1}{2} \times 2n = n^2 + n$. Finally, the denominator of this should be 2, whence the final formula is $\frac{1}{2}\left(n^2 + n\right)$.

Similarly, repeating the multiplication seven times indicates a power of eight. For the sum of eighth powers, column 7 gives 1, 8, 28, 56, 70, 56, 28, 8. Thus, one can first derive the following formula:

$$1 \times n^8 + 8 \times n^7 + 28 \times n^6 + 56 \times n^5 + 70 \times n^4 + 56 \times n^3 + 28 \times n^2 + 8 \times n.$$

On including the operations as well as the Bernoulli numbers mentioned in the left-hand column, we have

$$\begin{aligned} &1 \times n^8 + \frac{1}{2} \times 8 \times n^7 + \frac{1}{6} \times 28 \times n^6 + 0 \times 56 \times n^5 \\ &\quad + \frac{-1}{30} \times 70 \times n^4 + 0 \times 56 \times n^3 + \frac{1}{42} \times 28 \times n^2 + 0 \times 8 \times n \\ &\quad = n^8 + 4n^7 + \frac{14}{3} \times n^6 - \frac{7}{3} \times n^4 + \frac{2}{3} \times n^2. \end{aligned}$$

Finally, the denominator is 8. Thus the formula for the sum of eighth powers is

$$\begin{aligned} &\frac{1}{8}\left(n^8 + 4n^7 + \frac{14}{3}n^6 - \frac{7}{3}n^4 + \frac{2}{3}n^2\right) \\ &\quad = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2. \end{aligned}$$

Expression of Zero and Counting Board Algebra

One notable feature of Seki's chart (Figure 10) is the descriptions of "zero." The place where one would usually write a zero in the contemporary notation was represented in three different ways. The first zero was on the number 1 at the top of each column. There is a circle written on top of each one. Placing the circle on top changes the number to zero. The second place where zero appears is at the beginning of the pattern called "base" (基). The word "base" actually meant zero, since it meant 1^0 (i.e., no multiplication has to be done). The row after the base begins with 1 (圭), and continues with 2 (平) and 3 (立), which record the number of multiplications done ($1^1, 1^2, 1^3$, respectively). The third type of zero in Seki's chart appears in the left-hand columns, written as "void" (空). It means that no operations are needed: neither addition nor subtraction. Instead of writing zero in those three places, Seki used the relevant Japanese expressions. All three are unique in their use in mathematics, demonstrating Seki's originality as well as mastery of his methods.

What made Seki try to calculate the sum of powers of integers in the first place? Since Seki himself decided not to write down a detailed description of his calculations, we do not know what motivated him to work on such sums. One thing is clear: Seki was interested in solving algebraic problems on the counting board. Just as the cossists in German-speaking countries began to use letters of the alphabet in their notation, Seki used Chinese characters to denote all of the coefficients [16, p. 187]. By doing so, he was encoding an algorithm for computing patterns, such as the length of an arc, which had never been done in the Chinese mathematical tradition [16, p. 187].

Thus, an important context in obtaining the Bernoulli numbers is Seki's use of rod numerals to solve algebraic equations. It was similar to the development we saw in Faulhaber's method. In the manuscript that Seki and his disciples had edited in 1683, Seki developed "counting board algebra" [23, p. 129]; see Figure 12. It is remarkable to see that Seki was writing the coefficients using counting rods and trying to find the solutions for "self-multiplication" (or integer powers).

Similarities and Originality

A comparison of Bernoulli's and Seki's methods shows a similarity in the endpoint of their calculations; Bernoulli stopped at +6/55, and Seki stopped one after +6/55. If they had continued their calculations, the numerators and denominators would have become large (the next Bernoulli number is −691/2730), and they knew the general pattern by completing their calculations up to +6/55. Other than that, their methods do not have much in common. The shape of Seki's chart (Figure 10) looks much more like that of Faulhaber's version (Figure 4) than the one in the Chinese text (Figure 7). Japanese mathematics is thought to be intuitively closer to Chinese than Western mathematics, but such an assumption does not apply to this case. Did Seki and Bernoulli know of Faulhaber's or Pascal's table of binomial coefficients?

Both Jacob and Johann Bernoulli knew of Faulhaber's work. Thus, it would not be a surprise if they had seen the table of binomial coefficients in Faulhaber's book (Figure 4). But according to the statistician A. W. F. Edwards, the Bernoullis had not seen Pascal's version [10, p. 123]. In 1695, Johann received a letter from Leibniz, stating:

> I have conceived then of a wonderful rule for the coefficients of the powers not only of the binomial $(x + y)$, but also of the trinomial $(x + y + z)$, in fact, of any polynomial; so that when given the power of

any degree say the tenth, and any term contained in it ... it should be possible to assign the coefficient [10, p. 113].

He received this letter on May 16, 1695, and replied with enthusiasm on June 8, stating, "It would be a pleasure to see your rule and it would be well to test whether they agree; yours is possibly simpler" [10, p. 113]. Soon thereafter, in 1698, the multinomial coefficient was published in England by the French mathematician Abraham de Moivre (1667–1754). Leibniz also drew Jacob's attention to the binomial coefficients in 1705, shortly before his death. But he was not aware of Pascal's work [10, p. 123]. Thus it was likely that the table in Faulhaber's manuscript was the reference to binomial coefficients available to the Bernoullis and that they did not see Pascal's version.

Did Seki have access to Faulhaber's 1631 book *Academia Algebræ*, written in German? Silke Wimmer-Zagier and Don Zagier investigated the possibility that Seki, Takebe, and other Japanese mathematicians had been influenced by the Dutch who had been permitted to reside in a designated area of Japan, but they found no evidence of this, let alone a book written by a German [32].[31]

Neither the Bernoulli numbers nor the methods of calculating the sum of integer powers were to be found in the Chinese books published up to Seki's time. Historian of Japanese mathematics Osamu Takenouchi [31] argues that Seki was initially unaware of a method for obtaining sums of powers, and he had therefore to calculate such sums meticulously by hand without the benefit of a formula. But at some point, Seki either discovered a pattern or read about it somewhere. Wimmer-Zagier and Zagier are correct in their assertion that Seki's mathematics made a "quantum jump" from what had been done before his time; his originality suggests a good probability that Seki discovered the Bernoulli numbers on his own.

The placement of words to solve algebraic equations was, however, similar to what the cossists had achieved in Europe. Therefore, what we can confirm at least is the following: computing sums of integer powers has been a common concern, and the Bernoulli numbers were destined to appear naturally in investigations of such sums. Although two centers of mathematical investigation in the world were thousands of miles apart with little communication between them, it was time for the Bernoulli numbers to appear in mathematical work as an outgrowth of the development of methods for solving algebraic equations. It is thus no accident that the Bernoulli numbers were discovered in two disparate places at around the same time; they were a natural discovery that arose in the process of investigating algebraic equations.

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REFERENCES

[1] Sepideh Alassi. Jacob Bernoulli's analysis of the *Funicularia* problem. *British Journal for the History of Mathematics* 35:2 (2020), 137–161.

[2] A. F. Beardon. Sums of powers of integers. *American Mathematical Monthly* 103:3 (1996), 201–213.

[3] Janet Beery. Sums of powers of positive integers. *Convergence* (July 2010), https://doi.org/10.4169/loci003284.

[31] The authors concluded that Seki and Takebe "had not had any direct contact with the mathematics which had been done in Europe between the closing of Japan in 1639 and the lifting of the partial ban on foreign books in 1720" [32, p. 291].

[4] Daniel Bernoulli. *Die gesammelten Werke der Mathematiker und Physiker der Familie Bernoulli*. Birkhäuser, 1989.

[5] Jacob Bernoulli. *Ars Conjectandi*: translations from James Bernoulli. Translated by Bing Sung. Harvard University Department of Statistics Technical Report No. 2. Harvard University Department of Statistics, 1966.

[6] Jacob Bernoulli. *The Art of Conjecturing*. Translated with an introduction and notes by Edith Dudley Sylla. Johns Hopkins University Press, 2006.

[7] Bill Casselman. Jacob Bernoulli's zoo. Available online at http://www.ams.org/publicoutreach/feature-column/fc-2018-02. Accessed July 23, 2020.

[8] Leo Corry. History of algebra. Available online at https://www.tau.ac.il/~corry/publications/articles/pdf/algebra. Accessed July 31, 2020.

[9] Francois Dubeau. Linear algebra and the sums of powers of integers. *Electronic Journal of Linear Algebra* 17 (2008), 577–596.

[10] A. W. F. Edwards. *Pascal's Arithmetical Triangle*. Johns Hopkins University Press, 2002.

[11] Johannes Faulhaber. *Academia Algebræ*, Augsburg 1631. A digital copy is available from the University of Dresden Digital Library and at the ETH Zurich website (https://doi.org/10.3931/e-rara-16627).

[12] Lionel Gossman. *Basel in the Age of Burckhardt: A Study in Unseasonable Ideas*. University of Chicago Press, 2000.

[13] Ian Hacking. Jacques Bernoulli's art of conjecturing. *British Journal for the Philosophy of Science* 22 (1971), 209–229.

[14] Julian Havil. *Gamma: Exploring Euler's Constant*. Princeton University Press, 2003.

[15] Hal Hellman. *Great Feuds in Mathematics: Ten of the Liveliest Disputes Ever*. John Wiley & Sons, 2006.

[16] Annick Horiuchi. *Japanese Mathematics in the Edo Period (1600–1868): A Study of the Works of Seki Takakazu (?–1708) and Takebe Katahiro (1664–1739)*, translated by Silke Wimmer-Zagier. Birkhäuser, 2000.

[17] Wolfgang Kaunzner and Hans Wußing, editors. *Coß*: Jubilaeumsedition. Vieweg+Teubner Verlag, 1992.

[18] Donald E. Knuth. Johann Faulhaber and sums of powers. *Mathematics of Computation* 61:203 (1993), 277–294.

[19] Lay-Yong Lam. The Chinese connection between the Pascal triangle and the solution of numerical equations of any degree. *Historia Mathematica* 7 (1980), 407–424.

[20] Hideyuki Majima. Seki Takakazu, his life and biography. In *Seki, Founder of Modern Mathematics in Japan: A Commemoration on His Tercentenary*, edited by Eberhard Knobloch, Hikosaburo Komatsu, and Dun Liu, pp. 3–20. Springer Japan, 2013.

[21] Martin Mattmüller. The difficult birth of stochastics: Jacob Bernoulli's *Ars Conjectandi* (1713). *Historia Mathematica* 41:3 (2014), 277–290.

[22] Torabi-Dashti Mohammad. Faulhaber's triangle. *College Mathematics Journal* 42:2 (2011), 96–97.

[23] Mitsuo Morimoto. The Suanxue Qimeng and its influence on Japanese mathematics. In *Seki, Founder of Modern Mathematics in Japan: A Commemoration on His Tercentenary*, edited by Eberhard Knobloch, Hikosaburo Komatsu, and Dun Liu, pp. 119–132. Springer Japan, 2013.

[24] Tsukane Ogawa. The Bernoulli numbers discovered by Seki Takakazu. *RIMS Kokyuroku* 1583 (2008), 1–18.

[25] Jeanne Peiffer. Jacob Bernoulli, teacher and rival of his brother Johann. *Electronic Journal for History of Probability and Statistics* 2:1 (2006), 1–22.

[26] Ivo Schneider. *Johannes Faulhaber: Rechenmeister in einer Welt des Umbruchs*, Vita Mathematica 7. Birkhäuser, 1993.

[27] Takakazu Seki. *Takakazu Seki's Collected Works, Edited with Explanations*. Osaka kyoiku tosho, 1974.

[28] David Eugene Smith. *A Source Book in Mathematics*. McGraw-Hill, 1929.

[29] Jacqueline Stedall. *A Discourse Concerning Algebra: English Algebra to 1685*. Oxford University Press, 2003.

[30] John Stillwell. *Mathematics and Its History*. Springer, 2002.

[31] Osamu Takenouchi. Seki Takakazu, hito to gyōseki. Available online at https://mathsoc.jp/meeting/kikaku/2008aki/2008_aki_takenouchi.pdf. Accessed August 5, 2020.

[32] Silke Wimmer-Zagier and Don Zagier. Some questions and observations around the mathematics of Seki Takakazu. In *Seki, Founder of Modern Mathematics in Japan: A Commemoration on His Tercentenary*, edited by Eberhard Knobloch, Hikosaburo Komatsu, and Dun Liu, pp. 275–297. Springer Japan, 2013.

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