Von Neumann Explains His Game Theory to Gödel, September 1940

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Noted logician Kurt Gödel was a regular visitor to Princeton's Institute for Advanced Study in the 1930s. He returned to Vienna from a visit in June 1939, against the advice of many, mainly to fetch his wife, Adele. At that time, his home country, Austria, had ceased to exist, and he was formally a German citizen, which meant that for the couple's journey to Princeton, this time for good, they would be unable to take a transatlantic steamer, because the British regularly screened out German citizens. He was, however, allowed to travel on the Trans-Siberian Railway, for the Soviet Union and Germany were allies, and both had diplomatic relations with the United States.

The Gödels departed in early January 1940, traveling through occupied Poland and the Baltic countries, the Finnish–Soviet Winter War raging just a few hundred kilometers away. No narration of the Gödels' more than two-week train ride to Soviet Vladivostok on the Pacific coast has been found; the only witnesses are stamps in the couple's passports. The journey continued through Japan and across the Pacific, followed by another long journey by train, from California to Princeton, where they arrived in March. Dawson [1] gives all the known facts about this journey.

Gödel was a strange character with some compulsive habits, among the foremost that of writing things down: what he thought, what he read, what he discussed with people. So much writing was possible because he had been taught shorthand in high school, and writing in shorthand is a habit that will stick once acquired, mainly because shorthand writers experience longhand writing as exasperatingly slow.

Except for his famous results on the field equations of general relativity from around 1950, Gödel did not publish any substantial new results after 1939, but he sure kept producing them! After his death, in 1978, literally tens of thousands of pages of shorthand notes were found. Their

detailed, systematic study has begun only recently, the main reason being that Gödel's archaic Gabelsberger system of shorthand was abandoned around 1925, only a few years after he had learned it.

When he arrived in Princeton in March 1940, Gödel was using four series of notebooks: the mathematical *Arbeitshefte*, or workbooks, in which he recorded his ongoing research, sometimes with just formulas, at other times in a more coherent way; the *Resultate Grundlagen* (Results on Foundations), in which he wrote down finished results in a clear way, with almost no cancellations or changes; the *Logik und Grundlagen* series, which mainly recorded his summaries of works of others; and finally, the *MaxPhil* series, in which he wrote down his mainly philosophical musings, often about the relevance of the formal work he was conducting.

While in Princeton, Gödel started a new type of notebook in which he wrote down discussions with colleagues, talks he attended, and the like. He called it "Excerpts notebook 7," in continuation of the six *Logik und Grundlagen* notebooks. The first entry is a "Tarski discussion 30./VIII. 1940," referring to the Polish logician Alfred Tarski, who had barely escaped from Poland the previous year. Most discussions recorded are with Tarski and with Gödel's Princeton colleagues John von Neumann and Paul Erdős.

Gödel and von Neumann

Gödel and von Neumann had first met at the famous September 1930 Königsberg conference on the foundations of mathematics. At the end of his talk about the completeness of the usual logical calculus with the connectives and the universal and existential quantifiers, Gödel mentioned briefly that this type of completeness would not hold for the kind of logic needed for higher mathematics, indeed not even for elementary arithmetic. Von Neumann is said to have pulled him aside for a fuller explanation. During that fall, Gödel was writing his famous paper on the incompleteness theorems and corresponded with von Neumann (Gödel's part of the exchange was found only recently, in the form of shorthand drafts; see [5]). Afterward, the two met in Vienna, which von Neumann used to visit in the summer and at Christmas. Thus, von Neumann was the first to know about Gödel's proof of the consistency of the axiom of choice and the continuum hypothesis (cf. [2]). From March 1940 on, they were colleagues at the Institute for Advanced Study until von Neumann's death, in 1957.

Von Neumann on Games

After the first discussion with Tarski, the new notebook has the title line "von Neumann 18./IX.40 Fuld Hall, in my

room, about games.” Von Neumann had published his famous paper [3] on two-person games in 1928, and in 1944, he would publish his book *Theory of Games and Economic Behavior* [4], written with Oskar Morgenstern, another Princeton scholar and friend of Gödel’s.

Von Neumann’s explanation to Gödel of game theory took place before he began his collaboration with Morgenstern. As told in [6], von Neumann held lectures on game theory in Seattle in the summer of 1940 and wrote his first paper (since the one in 1928) in October of that year. The idea of collaboration with Morgenstern began taking shape in 1941.

In the discussion, he explains to Gödel first his early work, then goes on to consider games with “coalitions.” The text occupies pages 9 to 11 of the notebook (document number 030078 of the Kurt Gödel Papers, reel 20 of the Gödel microfilm collection, frames 91 and 92, Gabelsberger shorthand), and is reproduced here:

Von Neumann on games

I

Definition. Let a game be taken in which there are only finitely many (N) methods of play (α_i). A real number a_{ik} is associated to each pair of methods of play α_i, α_k (i.e., player α wins a_{ik} , player β wins $-a_{ik}$).

A method of play α_n is called reasonable if $\min_k \alpha_{nk}$ is a max with respect to n .

$$\begin{aligned} \min_k \alpha_{nk} &\geq \min_k \alpha_{ik} \\ &\geq \max_i \min_k \alpha_{ik}, \text{ i.e.,} \\ \alpha_{nk} &\geq \max_i \min_k \alpha_{ik} \text{ for all } j, \text{ i.e., } \min_j \alpha_{nj} \geq \max_i \min_k \alpha_{ik} \\ \alpha_{nj} &\geq \alpha_{ij} \text{ for each } j \end{aligned}$$

Therefore ? $\max_i \min_k \alpha_{ik} \geq \min_k \max_i \alpha_{ik}$ (the converse holds in general)

One can prove the existence of such a “reasonable” n for board games, and for other games, one can associate a probability w_n to each n ($< N$, $\sum w_n = 1$).

We designate from now on by w, v a sequence of n numbers with the sum = 1. One can then define a new matrix b_{vw} that determines the expected value of a win for a player who changes the method statistically with the probability v , against the player who does the same with probability w . One can prove for b_{vw} that there exists always a reasonable method in the previous sense (because it is a bilinear form). (I is the content of an earlier work of von Neumann.)

II

With games of more than two players, there does not always exist any reasonable statistical system of play:

1. A player who stands alone. [Let one, e.g., consider the game with three persons in which each writes a number of another player on a piece of paper. If two have written down each other, they have won, otherwise undecided.] I.e., there exists to each statistical method of play of a player a statistical method of play of a coalition of the other two that will beat him (even if the game is completely

symmetric). So let us consider all the possible coalitions (into two parties). One can then associate the expected value of a win for each of the two parties. Question: When this association is given, to calculate what coalitions will be formed, and how the win is to be divided within the coalition (possibly also, that there will always develop a two-party game).

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