Painting New Lines: Maximizing Color Difference in Metro Maps

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This column is a place for those bits of contagious mathematics that travel from person to person in the community, because they are so elegant, surprising, or appealing that one has an urge to pass them on.

Contributions are most welcome.

“Have you seen the new terminal on the *green*?”

“The *red* is overcrowded every morning at this time.”

“I live at the very end of *orange*, still OK with me: the escalator brings me right into the office.”

“I have to change to *purple* here. See you tomorrow!”

Sure, it is all about metro. The subway. The metropoli-
tan underground railway system. Each line has its
own color and each color is the name of a line. A
problem arises when another line is built: the old lines must
keep their colors, whereas the new line should have a color
that looks maximally different from the old ones. It
becomes even more complicated if there are two or more
new lines in a subway. Not only must the new colors be
unlike the old ones, but also they must differ from each
other as much as possible.

In this article we discuss a mathematical strategy to
choose these colors. By the way, what new colors would
you suggest for the real metro map in Figure 1?¹ For our
purposes this map is ideal: not only are there 14 colors in it,
but also there will be several new lines built in that subway
in the near future (according to [2], and the planned
extensions are in Figure 6). Let us use this map to illustrate
our strategy—yet the reader can apply the techniques to
whatever other map he or she likes (we invite the reader to
analyse the London metro map in Figure 2²).

Measuring the Color Difference

To maximize color differences—here, in metro maps—we
need a way to measure them and argue about colors
objectively. For this we describe them as tuples of num-
bers by using some coordinate system (or *color space*). For
example, the red, green, and blue (RGB) color space is
well known: in it every color is encoded by its RGB
components. Unfortunately, in the RGB space the Eucli-
dean distance is not proportional to the visible distance
that people perceive between colors. That is, the RGB
space is not *perceptually uniform*—which we would like.

In 1931, the Commission Internationale de l'Eclairage
(CIE) did measurements on the vision of hundreds of
humans. From those data, several color spaces were cre-
ated to make communication about colors easy and
unambiguous. In 1976, the CIE recommended the use of

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¹ Opened in 1935 with 11 kilometres of lines and 13 stations, the Moscow metro has 196 stations nowadays and the total route length of its lines is at least 327.5 km [5]. It is one of the world's most heavily used metro systems.

² The London underground system serves 270 stations and has 402 kilometres of track. Opened in 1863, it is the world's oldest underground railway network and is one of the largest [14].

Figure 1. It is likely that you have already recognised the Moscow metro here.

two approximately perceptually uniform color spaces and color-difference formulas. These spaces have become known by their officially recommended abbreviations CIELAB and CIELUV. In the former, a color is described by three coordinates: L (lightness), a (red-green scale), and b (yellow-blue scale).³ We will use this space to describe the colors of the Moscow metro map in terms of numbers: their

CIELAB coordinates can be found in Table 1; Figure 3 shows a stereographical plot of the colors of the Moscow metro in the CIELAB space.

The CIELAB space itself is not bounded; however, based on what one wants to measure, only part of it matters in practice (for example, only the colors that a printer can print or only the colors the human eye can see). Such a

³ The L -scale runs from 0 to 100; the parameters a and b typically have values in the range -150 to 150 . In this article we will round the coordinates of the colors to integers. The human eye can discriminate up to ten million colors, so it can distinguish even finer differences than integer steps.

Figure 2. The map of the London metro.

subset of colors is called a color *gamut*. The gamut of Adobe 98 in the CIELAB space can be seen in Figure 4. To keep the problem simple, we approximate this gamut by a polyhedron. Because the CIELAB space was supposed to be almost perceptually uniform, the Euclidean distance gives the CIE 1976 color difference formula (CIE76)

$$\Delta E_{ab} = \sqrt{(L_1 - L_2)^2 + (a_1 - a_2)^2 + (b_1 - b_2)^2}.$$

In fact, the CIELAB space is not *really* uniform. In particular, at high values of a and b , the far-too-simple CIE76 formula values color differences too strongly compared to the experimental results on color perception. This is why several new color-difference formulas were proposed, of which the newest is the CIEDE2000 formula⁴ (or ΔE_{00}).

Mathematical Formulation of the Problem

Given the CIELAB space equipped with the Euclidean metric—or even better, the CIEDE2000 color difference formula—and the color gamut Γ (which we chose to be a polyhedron), we consider a set of existing colors $P =$

$\{p_1, \dots, p_k\}$ within this gamut, represented by their CIE-LAB coordinates (e.g., see Table 1). If we want to add only one new line, this leads to the following maximin problem:

PROBLEM 1 (one new color). *How can we find an $x \in \Gamma$ such that*

$$\min_{p_i \in P} \|x - p_i\| \rightarrow \max_{x \in \Gamma} ?$$

That is, the minimal distance from the new color x to the existing ones must be as large as possible, provided that the color x stays inside or on the boundary of the gamut Γ .

If we want to add several new lines, each having its own new color, the maximin problem becomes this:

PROBLEM 2 (m new colors). *How can we find a collection of m points $X = \{x_1, \dots, x_m\}$ such that*

$$\min_{\substack{x_i, x_j \in X \\ i \neq j \\ p_k \in P}} \{\|x_i - x_j\|, \|x_i - p_k\|\} \rightarrow \max_{X \subset \Gamma} ?$$

⁴ In the CIEDE2000 distance, some weighting coefficients have to be chosen. We use the ratio $k_L^{-1} : k_c^{-1} : k_b^{-1} = \frac{1}{2} : 1 : 1$. This means that we consider the difference in chroma $c = \sqrt{a^2 + b^2}$ and hue $b = \arctan(b/a)$ to be more important than the difference in lightness L . For more information about this color difference formula, see [4].

Figure 3. A stereographic image of the colors of the Moscow metro map at their location in the CIELAB space. The L -axis is the vertical axis, the a -axis runs from southwest to northeast, and the b -axis from southeast to northwest. (To see a 3-dimensional picture, view the left picture with your left eye and the right picture with your right eye at the same time.)

Table 1. Coordinates of colors on the Moscow metro map. The names of the colors come from [8]. The average minimal distance is 16.3. Note that some of the minimal distances are the same; this happens when the two points are the closest to each other.

Line	Color	L	a	b	$\min \Delta E_{00}$
1	Red	52	74	53	29.1
2	Ocean green	56	-45	26	21.7
3	Cobalt	35	7	-43	19.4
4	Sky blue	61	-16	-42	6.3
5	Olive brown	41	6	27	20.4
6	Peach	76	24	67	14.5
7	Pinkish purple	43	64	-24	28.3
8	Light mustard	86	4	85	14.5
9	Light grey	71	0	-2	9.4
10	Greenish yellow	80	-26	68	17.8
11	Turquoise blue	56	-21	-29	6.3
Water	Pale blue	93	-8	-9	11.3
Background	White	100	0	0	9.4
Text	Black	13	2	0	20.4

That is, we maximize both the minimal distance in between the new colors $x_1, \dots, x_m$ and their minimal distance to the old colors. Alternatively, we could add the new colors $x_1, \dots, x_m$ one by one, using a method for solving Problem 1. However this does not always yield the optimal result.⁵

The Voronoi Diagram Method

Let us consider Problem 1. Because the CIELAB space is continuous (at least in principle), it is impossible to “try all points” in the gamut to find the one with the maximal

minimal distance. Therefore it would be much easier to have only a finite number of candidate points to check. This can be achieved by using Voronoi diagrams.

DEFINITION 1. Let $P = \{p_1, \dots, p_k\} \subset \mathbb{R}^3$ be a finite set (e.g., of old colors) such that $p_i \neq p_j$ for $i \neq j$. The *Voronoi polyhedron* of a point p_i is the region

$$V(p_i) := \{x \in \mathbb{R}^3 : \|x - p_i\| \leq \|x - p_j\| \text{ for all } j \neq i\}.$$

The set $\mathcal{V}(P) = \{V(p_1), \dots, V(p_k)\}$ is the (3-dimensional) *Voronoi diagram* generated by the set of points P .

It is easily shown that the sought-for point x maximizing the minimal distance in Problem 1 belongs to the intersection of

- either: three Voronoi planes, that is, three faces of Voronoi polyhedra,
- or: two Voronoi planes and a boundary plane of the gamut,
- or: a Voronoi plane and two boundary planes,
- or: three boundary planes.

This means that as soon as these candidate points are calculated,⁶ it only remains to compare their minimal distances to the existing points $p_i \in P$, and then to take the one whose distance is maximal; in fact, the resulting set is finite and reasonably small. This is the *CIE76 Voronoi method* of solving Problem 1 with respect to the Euclidean distance. In Table 2 we list the five new colors that we obtained, one by one, for the Moscow metro by using this method.

It is readily seen that the Voronoi method would provide the global optimal solution to maximin Problem 1—if there

⁵ Metro lines are built one after another, so choosing new colors one by one is quite logical. However, sometimes we know in advance that several new lines are being built.

⁶ This can be done by elementary linear algebra, that is, by calculating bisectors and intersections of planes. In Figure 5, a 2-dimensional example of a Voronoi diagram can be found.

Figure 4. The gamut of Adobe 98 in the CIELAB space. We approximate this gamut by a polyhedron by taking the corner points of this polyhedron close to the points $A, \dots, H$ in the figure.

were a perceptually uniform color space. This is not yet the case nowadays; therefore let us study how better estimates can be obtained.

Because the CIEDE2000 distance is much more accurate than the Euclidean distance, we can improve our method by selecting the optimal point from the candidate points by comparing the ΔE_{00} distances instead of the Euclidean distances ΔE_{ab} . This is the combined *CIE76-CIEDE2000 Voronoi method*. Of course, this method does not solve Problem 1 with respect to ΔE_{00} alone, because the candidate points are found by using ΔE_{ab} . However it is a fast method to obtain colors by using the CIEDE2000 distance. The five new colors obtained, one by one, for the Moscow metro by this method are given in Table 3.⁷ We see that this gives us much better solutions with respect to the CIEDE2000 distance than the Voronoi CIE76 method: indeed, the ΔE_{00} distance drops at a much slower rate.

The Simplex Method

We can also solve Problem 2 with respect to the Euclidean distance by using the simplex method. Let us consider here the case when the number of colors m is two, which gives us the following function to maximize,

$$M(x, y) = \min_{\substack{p_i \in P \\ x, y \in \Gamma}} \{ \|x - p_i\|, \|y - p_i\|, \|x - y\| \}.$$

We have $(x, y) \in \Gamma \times \Gamma \subset \mathbb{R}^6$. Let us rewrite our problem: first, put one extra parameter x_0 and the two vectors x, y together in $\tilde{x} = (x_0, x, y) \in \mathbb{R}^7$, and now maximize the function $M_0(\tilde{x}) = x_0$ over all $\tilde{x} \in \mathbb{R} \times \Gamma \times \Gamma$ satisfying the constraints

$$\|x - p_i\| - x_0 \geq 0 \quad \text{for each } p_i \in P, \quad (1a)$$

Figure 5. A 2-dimensional Voronoi diagram. Note that all edges of the Voronoi polygons are parts of the bisectors of lines connecting two points in the diagram.

Table 2. Colors we find one by one for the first five new lines in the Moscow metro by using the CIE76 Voronoi method, that is, by solving Problem 1 with respect to the Euclidean distance. The minimal distances that we indicate here take into account the points that came earlier into the table. Note that the minimal ΔE_{00} distance is then not necessarily decreasing.

Color	$\min \Delta E_{ab}$	$\min \Delta E_{00}$	L	a	b
Bright green	114.4	22.0	83	-138	91
Blue	87.0	19.8	33	80	-109
Sea green	67.7	13.1	85	-106	31
Cyan	65.3	21.9	87	-78	-21
Bright pink	56.1	14.4	60	100	-64

$$\|y - p_j\| - x_0 \geq 0 \quad \text{for each } p_j \in P, \quad (1b)$$

$$\|x - y\| - x_0 \geq 0. \quad (1c)$$

Of course, more constraints on (x, y) are used to delimit the set $\Gamma \times \Gamma$. Now we use the simplex method to maximize our function.

The gamut's boundary gives the linear constraints, and the nonlinear constraint functions are the three inequalities (1). Because the simplex method gives us only a local optimum, we apply it for a million random initial points $(x_0, x, y) \in \mathbb{R}^7$ (with $x_0 \in [0, 20]$ and $(x, y) \in \Gamma \times \Gamma$). These are local optima with respect to the ΔE_{ab} distance; let us now choose the one with maximal ΔE_{00} distance. This way, we obtain the two colors

Bright aqua $\bullet$ (84, -65, -12) and Baby pink $\bullet$ (86, 35, 7).

Their combined minimal CIEDE2000 distance to the old colors and each other is 22.7. This is more than the minimal

⁷ The minimal CIEDE2000 distances do not always decrease at each step; this is because the candidate points are selected by using the Euclidean distance, which does not necessarily give an optimum with respect to the CIEDE2000 distance. Therefore it is possible that, after adding a new color, one of the new candidate points is an even better solution than the point added before.

Table 3. Colors we find one by one for the first five new lines using the combined CIE76-CIEDE2000 Voronoi method. Note that the distances here are the CIEDE2000 distances, and they drop at a much slower rate than in Table 2.

Conclusion and Discussion

We conclude that the Voronoi diagram method (and its improvement by using both the CIE76 and CIEDE2000 distance functions) for adding new colors one by one gives solutions that are comparable with the ones produced by the nonsmooth optimization method for finding two or more new colors at once. This is true for the map of the Moscow metro under study. Other solution schemes are of course possible, for example the *Monte-Carlo* method: test-points $x_i \in \Gamma$ for the new color(s) are generated within the gamut at random, and the new color—or a tuple of new colors—is the maximin optimum over a sufficiently large number of such attempts. The *ballistic method* is realized as follows: first, suppose that each old color $p_i \in \Gamma$ and every

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new point x_j in the sought-for tuple carries a positive electric charge (so that they repulse by Coulomb's law). To prevent an escape of new points from the gamut Γ across its boundary, put a negligibly small but positive charge at the boundary's closest point to x_j for every j . By starting from random initial data $x_j(0) \in \Gamma$, the configurations $\{x_j(t)\}$ of charges evolve inside the polyhedron Γ , which, by assumption, is filled with a viscous medium so that the new points' motion slows down; sooner or later they are captured near the local minima of the potential (see [1] for details on both methods).

Finding several new colors is still not all that one needs in order to paint the new lines on a metro map. Clearly, it is the configuration of transfers from the old to new lines that must further be taken into account—to distribute the new colors between the new lines in such a way that at every transfer station, the colors of intersecting lines are maximally different.

Breaking News: An Online Poll

The 16.8-km long Kozhukhovskaya line with eight stations on it is scheduled to appear in the Moscow underground system in 2015–2016, officials say. The new line will connect the (south)east with the city centre, see Fig. 6: starting at the *ocean green*, that line continues along the *turquoise blue* toward *red*.

Two colors, to freely choose from, were offered to preregistered “active citizens” during an interactive poll, which was held from 20 October until 10 November 2014. Namely, *black* and *pink* were claimed as the only colors remaining from the gamut that were not yet used to paint lines on the Moscow metro map (see Figure 1). As many as 307,350 active citizens took part in the online poll, media report; 71% voted pink and 18% preferred black, whereas 7% trusted the choice to experts (such as designers, ethnographers, or psychologists). Finally, 4% of the respondents either made their own suggestions of the new color: turquoise, olive, coral, etc.—or even proposed to make that line on the map speckled (see [12, 13]).

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