Note

On Morley's Trisector Theorem

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Editor's Note: John Conway's simple and elegant proof of Morley's Theorem is legendary. That is, it's well known, and it's "out there," yet nowhere in print. We are pleased that he accepted our invitation to publish it, for the record, in The Mathematical Intelligencer. —M. S.

In their book *Geometry Revisited* Coxeter and Greitzer say

> One of the most surprising theorems in elementary geometry was discovered about 1904 by Frank Morley [...] Theorem: *The points of intersection of the adjacent trisectors of the angles of any triangle are the vertices of an equilateral triangle.*

The theorem was notorious throughout the 20th century as being difficult to prove. In the 21st century it has become easy! Here is the indisputably simplest

PROOF. Let the given triangle have angles $A = 3\alpha$, $B = 3\beta$, $C = 3\gamma$ and let $\theta+$ mean $\theta + 60°$ (and $+$ by itself mean $60°$). Then there exist seven triangles with angles

$$\alpha++, \beta, \gamma$$
$$\alpha, \beta++, \gamma$$
$$\alpha, \beta, \gamma++$$
$$\alpha, \beta+, \gamma+$$
$$\alpha+, \beta, \gamma+$$
$$\alpha+, \beta+, \gamma$$
$$+,\ +,\ +$$

because in each case these angles add to $(\alpha+\beta+\gamma)++ = 180°$. These triangles, so far determined only up to similarity, are illustrated in Figure 1. We can scale them so that the red lines in that figure all have the same length.

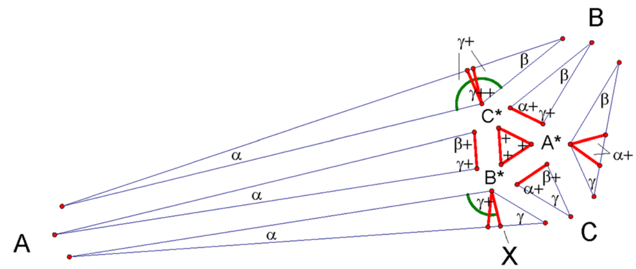

Figure 1. The seven puzzle-piece triangles.

The red lines from A^* to BC are the two lines through A^* that make angle $\alpha+$ with BC, and are the same length since they form an isosceles triangle. (I call drawing such lines "dropping non-perpendiculars of angle $\alpha+$".) If one of the angles of ABC is obtuse—as is C in the figure—then the two angles at the feet of these non-perpendiculars (here $\gamma+$) are *exterior* angles of the isosceles triangle they form, rather than interior ones, but this does not affect the proof.

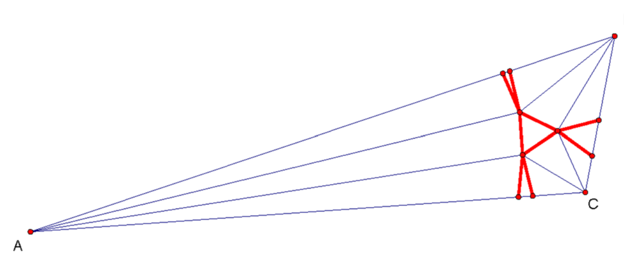

Figure 2. The assembled puzzle.

We shall show that these triangles fit together like a jigsaw puzzle to form a triangle with the required given angles $A = 3\alpha$, $B = 3\beta$, $C = 3\gamma$. For, the angles around A^*, B^*, C^* add up to $360°$; for example around C^* they are $\gamma++, \alpha+, +, \beta+$ totalling $(\alpha+\beta+\gamma)+++++ = 360°$. But also the two edges from A to B^*, say, must have the same length because the triangles AB^*C^* and AB^*X are congruent. (Their angles at A and B^* are α and $\gamma+$ and $B^*C = B^*X$, these being red lines.) Figure 2 shows the assembled puzzle that proves the theorem.

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