One Pythagoras for All Dimensions

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It is obvious that the Pythagorean theorem is equivalent to the following statement: the sum of the squares over the sides of a rectangle is equal to the sum of the squares over the diagonals (Fig. 1).

Diagonals in a box join opposite corners, as in Figure 2. It can be easily seen that the sum of squares over the edges of a box is equal to the sum of squares over the diagonals. In fact, the same statement holds true for all finite dimensions.

Theorem (Pythagorean theorem for all finite dimensions): The sum of the squares over the edges of any n -dimensional box is equal to the sum of the squares over the diagonals.

Proof. Any n -dimensional box B^n can be viewed as the topological product $[0, a_1] \times [0, a_2] \times \cdots \times [0, a_n]$ of intervals.

Figure 1. Pythagorean theorem: $2a^2 + 2b^2 = 2c^2$.

Figure 2. A 3-box and its 4 diagonals.

There are $n2^{n-1}$ edges in B^n , joining corners $(x_1, x_2, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_n)$ & $(x_1, x_2, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$, $i = 1, 2, \dots, n$, where each x_j is in the set $\{0, a_j\}$. Out of these, there are 2^{n-1} edges of length a_i , $i = 1, 2, \dots, n$. The sum of the squares over the edges is thus $2^{n-1}(a_1^2 + a_2^2 + \cdots + a_n^2)$.

Opposite corners in B^n are points of type $(x_1, x_2, \dots, x_n)$, $(y_1, y_2, \dots, y_n)$, where $\{x_i, y_i\} = \{0, a_i\}$ for each $i = 1, 2, \dots, n$. The length of each diagonal is $\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$. There are 2^n corners, thus 2^{n-1} diagonals. So, the sum of the squares of diagonals is $2^{n-1}(\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2})^2$, as desired.

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