Note

Euler's Wonderful Insight

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Since the zeros of $\sin x$ are $0, \pm\pi, \pm 2\pi, \ldots$, we can factorize $\sin x$:

$$\sin x = ax(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)\ldots$$

and using $\lim_{x\to 0} \frac{\sin x}{x} = 1$ we can determine the value of a:

$$\begin{aligned}\sin x &= x \cdot \left(1-\frac{x}{\pi}\right)\cdot\left(1+\frac{x}{\pi}\right)\cdot\left(1-\frac{x}{2\pi}\right)\cdot\left(1+\frac{x}{2\pi}\right)\ldots\\ &= x\cdot\left(1-\frac{x^2}{\pi^2}\right)\cdot\left(1-\frac{x^2}{(2\pi)^2}\right)\cdot\left(1-\frac{x^2}{(3\pi)^2}\right)\ldots\end{aligned}$$

If we take $x = \frac{\pi}{2}$ in this formula, we get

$$\frac{2}{\pi} = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\ldots \qquad \text{(Wallis's formula)}$$

Taking the logarithm of both sides, we find:

$$\ln\sin x = \ln x + \ln\left(1-\frac{x}{\pi}\right)+\ln\left(1+\frac{x}{\pi}\right)+\ln\left(1-\frac{x}{2\pi}\right)+\ldots$$

We then take the derivative:

$$\frac{\cos x}{\sin x} = \frac{1}{x} - \frac{1}{\pi - x} + \frac{1}{\pi+x} - \frac{1}{2\pi - x} + \frac{1}{2\pi + x} - \frac{1}{3\pi - x} + \ldots$$

For $x = \frac{\pi}{4}$ this leads to

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots \qquad \text{(the Gregory-Leibniz series)}$$

$$\begin{aligned}\frac{4}{\pi} - 1 &= \frac{1-\frac{\pi}{4}}{\frac{\pi}{4}} = \frac{\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\ldots}{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots}\\ &= \frac{\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\ldots}{2\cdot(\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\ldots)+3\cdot(\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots)}\\ &= \frac{1}{2+3\cdot\frac{\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots}{\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\ldots}}\\ &= \frac{1}{2+9\cdot\frac{\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots}{1-\frac{3}{5}+\frac{3}{7}-\frac{3}{9}+\ldots}}\\ &= \frac{1}{2+9\cdot\frac{\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots}{2\cdot(\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots)+5\cdot(\frac{1}{7}-\frac{1}{9}+\ldots)}}\\ &= \ldots\\ &= \frac{1}{2+9\frac{1}{2+25\frac{1}{2+49\frac{1}{2+\ldots}}}}\end{aligned}$$

Hence

$$\frac{4}{\pi} = 1 + \frac{1^2}{2+\frac{3^2}{2+\frac{5^2}{2+\ldots}}} \qquad \text{(Lord Brouncker's continued fraction)}$$

If we expand the right-hand side, we get something of the form $x + c_1x^3 + c_2x^5 + \ldots$

The coefficient of x^3 is

$$c_1 = -\frac{1}{\pi^2} - \frac{1}{(2\pi)^2} - \frac{1}{(3\pi)^2} - \frac{1}{(4\pi)^2} - \ldots$$

Its value can be obtained by calculating the third derivative of both sides:

$$D^3 \sin x = 6c_1 + 60c_2x^2 + \ldots$$

With $x = 0$ we find that $-1 = 6c_1$. Hence, after rearranging:

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \qquad \text{(Euler's Basel formula)}$$

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