Note

One-Line Proof of the AM-GM Inequality

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The AM-GM inequality says $\frac{x_1+x_2+\cdots+x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$ (all $x_i > 0$). This is perhaps the best known and most useful nontrivial inequality in mathematics, with a large number of interesting proofs in the literature. A very clever proof of this inequality, which appeared of late in [1], has given me motivation to communicate a very short proof of it, which I used to present to Iran's Olympiad team, preparing for the IMO, several years ago.

Let me, before going any further, give a generalization of the AM-GM inequality in the case $n = 2$. It is trivial to see that, whenever $0 < a \leq b$ and $x > 0$, $a \leq x \leq b$ if and only if $a + b \geq x + \frac{ab}{x}$. The latter inequality is stronger than the AM-GM inequality for $n = 2$ (indeed, just choose $x = \sqrt{ab}$). It also allows equality only for either $x = a$ or $x = b$.

Now the quick proof of the general case goes as follows. If $x_1 = x_2 = \cdots = x_n$, then we are done. If not, put $x_1 x_2 \cdots x_n = g^n$, and without loss we may assume $x_1 \leq x_i$ and $x_2 \geq x_i$ for all i, hence $x_1 < g < x_2$. $\left(\text{We have noted that } x_1 < g < x_2 \text{ if and only if } x_1 + x_2 > g + \frac{x_1 x_2}{g}.\right)$ Therefore $x_1 + x_2 + \cdots + x_n > g + \frac{x_1 x_2}{g} + x_3 + \cdots + x_n$; but the latter expression is $\geq g + (n-1)g = ng$, by the inductive hypothesis (noting that $\frac{x_1 x_2}{g} x_3 \cdots x_n = g^{n-1}$).

Remark The last part of the above proof shows that if not all x_i's are equal, then in fact we have $\frac{x_1+x_2+\cdots+x_n}{n} > \sqrt[n]{x_1 x_2 \cdots x_n}$, that is to say, the equality $\frac{x_1+x_2+\cdots+x_n}{n} = \sqrt[n]{x_1 x_2 \cdots x_n}$ holds only when $x_1 = x_2 = \cdots = x_n$. The latter fact does not follow automatically from most of other proofs of this inequality in the literature.

REFERENCE

[1] Michael D. Hirschhorn, "The AM-GM Inequality," *Mathematical Intelligencer* 29 (2007), nos. 4, 7.

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