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Synchronization in networks of limit cycle oscillators

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Abstract. We investigate the synchronization behaviour of three different networks of nonlinearly coupled oscillators. Each network consists of several clusters of oscillators, and the clusters themselves consist of any number of oscillators. In each cluster the eigenfrequencies scatter around the cluster frequency (mean frequency). The coupling strength varies in each cluster, too. We analyze the synchronized states by means of the center manifold theorem. This enables us to calculate these states explicitly, and to prove their stability. Moreover we are able to determine frequency shifts caused by different coupling mechanisms. In a number of cases we calculate the synchronisation threshold explicitely. Numerical simulations illustrate our analytical results. In one of the three networks we have additionally analyzed a single cluster consisting of infinitely many oscillators, that is an oscillatory field. Again, the center manifold theorem enabled us to calculate the synchronized state explicitly and to prove its stability. Our results concerning the oscillatory field are in contradiction to Ermentrout's analysis [6].

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1. Introduction

A huge variety of dynamical phenomena in nature is caused by the nonlinear interactions of oscillators (cf. [1, 2, 3, 6, 7, 9, 11, 16, 17, 21, 24]). Every area of the natural sciences provides famous examples of oscillatory behaviour. We recall some of them: ultrashort laser pulses [11], wave propagation in the Belousov-Zhabotinski reaction [16], phase transitions in human hand movements [12], and spinal generators for locomotion [3].

Systems of weakly coupled limit cycle oscillators are quite popular. They not only provide suitable models in different areas of the natural sciences, moreover they allow us to consider the amplitude of the motion in a first approximation as constant in order to approximate the entire system by phase models [16]. This simplifies the analytical investigation decisively. Nevertheless even in this case it is often difficult to prove the stability of the synchronized states.

We investigate the synchronization behaviour of three different networks of nonlinearly coupled oscillators. From the physicist's point of view, it is often a crucial restriction to assume that the couplings are weak. Therefore we do not restrict our investigation to the case of weak coupling strength. Rather we analyze the total systems, consisting of phase and amplitude dynamics, too. By means of the center manifold theorem [15, 18] we are able to proof the stability of the synchronized states and to calculate these states explicitly. The center manifold theorem may be considered as a special case of the slaving principle of synergetics [10, 11]. After introducing a rotating coordinate system the center manifold theorem allows us to reduce the many degrees of freedom of the oscillatory system (cf. [15, 18, 10, 11]). Obviously the coupling mechanism is of great importance for the synchronization behaviour of an oscillatory network. In this paper we analyze the influence of different coupling mechanisms on the frequency of the synchronized state. To this end we explicitely calculate the shifts of the cluster frequency caused by different coupling mechanisms.

The three different models are analyzed one after the other. For each model we start with the analysis of the entire system. After this we investigate the respective phase model. The results of the analysis of the entire system and the results of the analysis of the phase model are compared in order to check the validity of the phase model. For model I we additionally investigate an oscillatory field.

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2. Model I

2.1. The total system

2.1.1. One cluster. Let us consider a single limit cycle oscillator obeying the equation

$$\dot{z} = (\alpha + i\omega)z - Kz^2z^*. \tag{1}$$

z is a complex variable. z^* denotes the complex conjugate of z. Inserting the hypothesis $z = r\,e^{i\psi}$ $(r, \psi \in \mathbf{R})$ into Eq. (1) provides us with evolution equations of amplitude r and phase ψ:

$$\dot{r} = \alpha r - Kr^3, \tag{2}$$

$$\dot{\psi} = \omega. \tag{3}$$

Obviously for negative α the amplitude vanishes. When α becomes positive the amplitude dynamics given by Eq. (2) undergoes a Hopf-bifurcation giving rise to a stable oscillation with frequency ω and amplitude $\sqrt{\alpha/K}$ (cf. [11]).

Equation (1) is a normal form, which means that the limit cycle dynamics of many and even more complicated oscillators can be transformed onto or can be approximated by the dynamics given by Eq. (1) [5, 14]. Applying the rotating wave approximation and the slowly varying wave approximation for instance to the van der Pool oscillator or to the neurophysiological HKB oscillator [12] we end up with Eqs. (2) and (3) [12]. Thus, Eq. (1) provides us with a suitable minimal model of a limit cycle oscillator.

In the present paper we analyze oscillators which are able to synchronize in phase due to their mutual continuous interactions. Therefore we are interested in couplings which minimize the oscillators' phase difference (modulo 2π). Before we turn to networks consisting of large numbers of oscillators let us first dwell on two oscillators with eigenfrequencies ω_1 and ω_2 which are described by the complex variables z_1, z_2 and coupled according to

$$\dot{z}_j = (\alpha + i\omega_j)z_j - \frac{K}{2}(z_j^2 z_j^* + z_j^2 z_k^*)$$

where $j, k = 1, 2$ and $k \neq j$. (4)

With the hypothesis

$$z_j = r_j e^{i\psi_j}\ (r, \psi \in \mathbf{R}) \tag{5}$$

we immediately obtain evolution equations for the oscillators' amplitudes (r_1, r_2) and phases (ψ_1, ψ_2):

$$\dot{r}_j = \alpha r_j - \frac{K}{2}[r_j^3 + r_j^2 r_k \cos(\psi_j - \psi_k)], \tag{6}$$

$$\dot{\psi}_j = \omega_j - \frac{K}{2} r_j r_k \sin(\psi_j - \psi_k), \tag{7}$$

where $j, k = 1, 2$ and $k \neq j$. According to Eq. (7) the cubic coupling term $z_j^2 z_k^*$ of the right hand side of Eq. (4) corresponds to a synchronizing coupling mechanism. This mechanism can be encountered in many pairs of self-synchronizing limit cycle oscillators (cf. [11]). If we apply the rotating wave approximation and the slowly varying wave approximation for instance to a pair of van der Pool oscillators or HKB oscillators with self-synchronizing interactions we end up with Eqs. (6) and (7) [12]. For this reason the cubic coupling term $z_j^2 z_k^*$ provides us with a suitable mechanism modelling continuously synchronizing interactions of limit cycle oscillators in the sense of the normal form theorem [5, 14] mentioned above.

We want to analyze the synchronization behaviour of networks consisting of many limit cycle oscillators. Extending the notion of self-synchronizing interactions we end up with the network model

$$\dot{z}_j = (\alpha + i\omega_j)z_j - \frac{K}{N} z_j^2 \sum_{k=1}^{N} z_k^*\ (j = 1, \dots, N), \tag{8}$$

with $\alpha > 0$, $K > 0$. The j-th oscillator is described by the complex variable z_j.

According to the normal form theorem [5, 14] networks consisting of even more complicated limit cycle oscillators can be approximated by the dynamics given by Eq. (8). Thus, Eq. (8) may be considered as a normal form in the class of the all-to-all coupled oscillatory networks.

With the hypothesis (5) we obtain for the amplitudes

$$\dot{r}_j = \alpha r_j - \frac{K}{N} r_j^2 \sum_{k=1}^{N} r_k \cos(\psi_j - \psi_k) \tag{9}$$

and for the phases

$$\dot{\psi}_j = \omega_j - \frac{K}{N} r_j \sum_{k=1}^{N} r_k \sin(\psi_j - \psi_k). \tag{10}$$

Now we introduce relative phases

$$\phi_j(t) = \psi_j(t) - \Omega t - \theta, \quad \theta = \text{const.}\ (j = 1, \dots, N), \tag{11}$$

where

$$\Omega = \frac{1}{N}\sum_{k=1}^{N} \omega_k \tag{12}$$

is called the cluster frequency. By means of deviations η_j from the cluster frequency the eigenfrequencies may be written as

$$\omega_j = \Omega + \eta_j. \tag{13}$$

We assume that

$$\|\boldsymbol{\eta}\| \ll \alpha \quad \text{and} \quad \|\boldsymbol{\eta}\| \ll \alpha^{3/4}K^{1/4}, \tag{14}$$

holds, where $\boldsymbol{\eta} = (\eta_1, \dots, \eta_N)^T$. For the amplitudes we make the hypothesis

$$r_j(t) = \sqrt{\frac{\alpha}{K}} + \xi_j(t), \tag{15}$$

because $\sqrt{\alpha/K}$ is the nontrivial stable stationary solution of Eq. (9) for $\eta_j = 0$ for all $j = 1, \dots, N$. Because of $\sum_{k=1}^{N} \dot{\phi}_k \equiv 0$ we choose $\theta = 1/N \sum_{k=1}^{N} \psi_k(0)$. This yields

$$\sum_{k=1}^{N} \phi_k \equiv 0. \tag{16}$$

According to Eq. (13) $\eta_1, \dots, \eta_N$ are constant frequency deviations. We exploit this fact by treating $\eta_1, \dots, \eta_N$ as variables with vanishing time derivative: $\dot{\eta}_j = 0$ for all $j = 1, \dots, N$.

As a result of this extension there are two types of variables: on the one hand the amplitude deviations ξ_j and relative phases ϕ_j, both rapidly changing, and on the other hand the constant frequency deviations η_j. Thus, there are two different time scales, one corresponding to the rapidly changing variables, the other one corresponding to the constant frequency deviations. This enables us to apply the center manifold theorem, where $\eta_1, \dots, \eta_N$ are the center modes. The latter may be considered as a special case of the order parameters known from synergetics [10, 11]. Likewise the stable modes correspond to the enslaved modes in synergetics [10, 11]. We briefly mention the connection with results in synergetics: Here the order parameter equation describes both the relaxation of the system towards the center manifold and its motion within it. In the present context we focus our attention on the motion on the center manifold.

We included the frequency deviations η_j as variables in order to apply the center manifold theorem. For two reasons the latter turns out to be a powerful tool for instance in comparison to linear perturbation theory:

1. The center manifold theorem provides us with a rigorous proof of the stability of the synchronized states under consideration (cf. [15, 18]).
2. Moreover we are able to expand the amplitude deviations ξ_j and relative phases ϕ_j in the synchronized states in powers of ξ_j. Obviously linear perturbation theory could not provide us with this nonlinear expansion. From this point of view our approach can be considered as a nonlinear perturbation theory.

In order to separate the linear parts of the center modes from the linear parts of the stable modes, we carry out the transformation

$$\varphi_j = \phi_j - \frac{1}{\alpha}\eta_j \quad \text{for} \quad j = 1, \dots, N. \tag{17}$$

Furthermore we introduce $\mathbf{x}_c = \boldsymbol{\eta}$ and $\mathbf{x}_h = (\xi_1, \dots, \xi_N, \varphi_1, \dots, \varphi_N)^T$. "c" stands for center modes, whereas "h" stands for hyperbolic (stable) modes.

With Eq. (16) and Eq. (17) the linearization (around zero) yields for the center modes $\dot{\mathbf{x}}_c = 0$. This is the reduced problem which in synergetics is called the order parameter equation [10, 11]. For the stable modes we obtain $\dot{\mathbf{x}}_h = \mathbf{B}_h + \mathbf{m}_h(\mathbf{x}_c, \mathbf{x}_h)$, with

$$\mathbf{B}_h = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & -\alpha\mathbf{I} \end{pmatrix}. \tag{18}$$

Here $\mathbf{A} = (a_{jk})$, with $a_{jk} = -\alpha(1 + 1/N)$ for $j = k$ and $-\alpha/N$ for $j \neq k$. $\mathbf{I}$ is the $N \times N$-identity matrix. $\mathbf{0}$ stands for the null matrix.

The eigenvalues of $\mathbf{A}$ lie in the intervall $[-2\alpha, -2\alpha/N]$. This follows from the theorem of Gerschgorin [20]. Therefore $\mathbf{B}_h$ has only negative eigenvalues, and $\mathbf{x}_h$ is the vector of the stable modes.

With $\mathbf{x} = (\mathbf{x}_c, \mathbf{x}_h)^T$ the nonlinear transformed system reads

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_h \end{pmatrix} \mathbf{x} + \begin{pmatrix} \mathbf{0} \\ \mathbf{m}_h(\mathbf{x}) \end{pmatrix} \equiv \mathbf{f}(\mathbf{x}). \tag{19}$$

We are now able to determine the center manifold. To this end we derive the map $\mathbf{h}$ which on the center manifold gives the stable modes as a function of the center modes: $\mathbf{x}_h = \mathbf{h}(\mathbf{x}_c)$. With $\dot{\mathbf{x}}_h = \mathbf{Dh}(\mathbf{x}_c)\,\dot{\mathbf{x}}_c$ we obtain $\mathbf{h}(\mathbf{x}_c) = -\mathbf{B}_h^{-1}\mathbf{m}_h(\mathbf{x}_c, \mathbf{h}(\mathbf{x}_c))$, where

$$\mathbf{B}_h^{-1} = \begin{pmatrix} \mathbf{A}^{-1} & 0 \\ 0 & -1/\alpha\,\mathbf{I} \end{pmatrix} \tag{20}$$

and $\mathbf{A}^{-1} = (a_{jk}^{-1})$ with $a_{jk}^{-1} = (-1 + 1/(2N))/\alpha$ if $j = k$ and $1/(2\alpha N)$ if $j \neq k$. The symmetry of the system is given by

$$\mathbf{T} = \begin{pmatrix} \mathbf{T}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_h \end{pmatrix} \quad \text{with} \quad \mathbf{T}_h = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix} \tag{21}$$

and $\mathbf{T}_c = -\mathbf{I}$ (cf. [19]). Therefore $\mathbf{h}(-\mathbf{x}_c) = \mathbf{T}_h\mathbf{h}(\mathbf{x}_c)$ holds. This enables us to determine the coefficients of $\mathbf{h}$ of lowest order. Hereby we expand the sine and the cosine terms according to Taylor. With this we obtain the stationary synchronized state. This state is stable. Moreover, due to the center manifold theorem it is a local attractor which our model approaches in an overdamped fashion [15, 18]. Transforming back to phases and amplitudes yields the synchronized state

$$\psi_j(t) = \Omega t + \theta + \frac{\eta_j}{\alpha} + \frac{1}{6N\alpha^3}\sum_{k=1}^{N}(\eta_j - \eta_k)^3 \underbrace{-\frac{1}{2N\alpha^3}\eta_j\sum_{k=1}^{N}(\eta_j - \eta_k)^2 + \frac{1}{2N^2\alpha^3}\sum_{k,l=1}^{N}\eta_k(\eta_k - \eta_l)^2}_{=I} + O(\|\eta\|^5) \tag{22}$$

$$r_j \equiv \sqrt{\frac{\alpha}{K}} + \frac{1}{2\alpha^{3/2}K^{1/2}}\eta_j^2 + O(\|\boldsymbol{\eta}\|^4). \tag{23}$$

The terms denoted by "I" do not appear in the formula of the synchronized state of the phase model. Simplifying Eq. (22) finally yields

$$\psi_j(t) = \Omega t + \theta + \frac{\eta_j}{\alpha} - \frac{\eta_j^3}{3\alpha^3} + \frac{1}{3N\alpha^3}\sum_{k=1}^{N}\eta_k^3 + O(\|\boldsymbol{\eta}\|^5). \tag{24}$$

The terms of the right hand sides of equations (24) and (23) find a clear interpretation. In Eq. (23) $\sqrt{\alpha/K}$ denotes the unperturbed limit cycle amplitude for $\eta_1, \dots, \eta_N = 0$. Neglecting terms of fourth order the deviation of the oscillator's amplitude from the unperturbed limit cycle only depends on its eigenfrequency deviation η_j. In Eq. (24) the term Ωt indicates that in the synchronized state all oscillators have the same frequency. The other terms contribute to a phase shift. θ is the part of this phase shift which is common to all oscillators, whereas the other terms may differ from oscillator to oscillator.

The deviations of the eigenfrequencies $\eta_1, \dots, \eta_N$ act as order parameters: they determine the relative phases

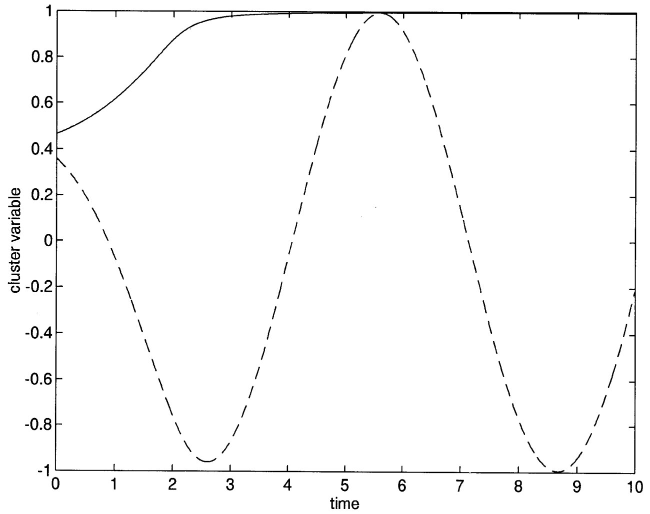

Fig. 1. Model I, total system, 10 oscillators. $\Omega = 1, K = \alpha = 0.5$. $\sigma(t)$ (solid line) and $\mathrm{Re}(Z(t))$ (dashed line). $\eta_1, \ldots, \eta_N$ are uniformly distributed in $[-0.1, 0.1]$

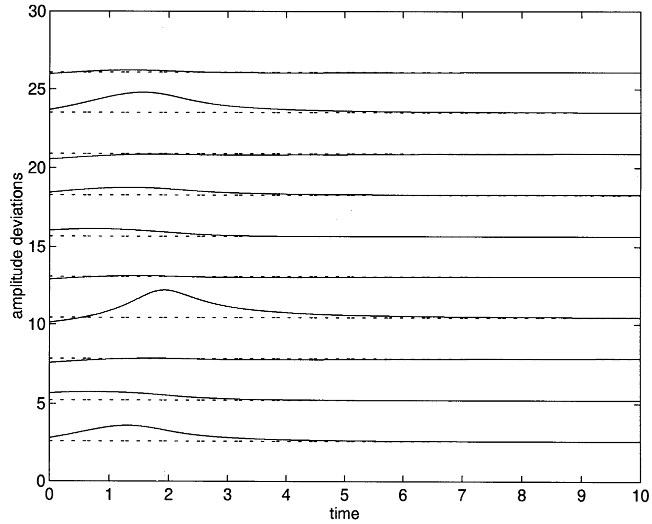

Fig. 2. Model I, total system, 10 oscillators. $\Omega = 1, K = \alpha = 0.5$. $\eta_1, \ldots, \eta_N$ are uniformly distributed in $[-0.1, 0.1]$. Amplitude deviations $\xi_j(t)$ with $j = 1, \ldots, 10$, and their "abscissas" (dotted lines)

and the deviations of the limit cycle amplitudes. Note that the assumption (14) is necessary in order to apply the center manifold theorem. To get an impression of how the oscillators synchronize, we integrate the system (9), (10) numerically for $K = \alpha = 0.5$ and $\Omega = 1$. $\eta_1, \ldots, \eta_N$ are uniformly distributed in $[-0.1, 0.1]$. The initial values of the phases are uniformly distributed in $[0, 2\pi]$. The initial values of the amplitude deviations $\xi_j(0)$ are uniformly distributed in $[0, 2]$. We introduce the cluster variable $Z(t)$, the cluster amplitude $\sigma(t)$ and the cluster phase $\Phi(t)$ by putting

$$Z(t) = \sigma(t)\, e^{i\Phi(t)} = \frac{1}{10} \sum_{k=1}^{10} e^{i\psi_j(t))}. \tag{25}$$

The evolution of the cluster amplitude is shown in Fig. 1, and the evolution of the amplitude deviations $\xi_1, \ldots, \xi_N$ in Fig. 2. Although the magnitude of the initial amplitude deviations is large compared to K and α the oscillators synchronize rapidly.

2.1.2. Several clusters. In this section we investigate the interaction of several groups (clusters) of oscillators. Certainly the behaviour of a system like that may be arbitrarily complex. Therefore we restrict our analysis to the case when it is possible to separate the clusters by means of averaging theorems [9, 13]. In other words, we approximate the behaviour of interacting clusters by the behaviour of noninteracting clusters.

Our model consists of n clusters with the cluster frequencies $\Omega_1, \ldots, \Omega_n$. I_ν is the index set of the ν-th cluster: $I_\nu = \{M_{\nu-1}, \ldots, M_\nu\}$, where $M_0 = 0$ and $M_n = N$. $N_\nu = M_\nu - M_{\nu-1}$ denotes the number of oscillators of the ν-th cluster, and $n_\nu = N_\nu / N$.

$$\Omega_\nu = \frac{1}{N_\nu} \sum_{k \in I_\nu} \omega_k \quad (\nu = 1, \ldots, n) \tag{26}$$

is the cluster frequency of the ν-th cluster. We introduce the smallness parameter ε by putting

$$\omega_j = \Omega_\nu + \varepsilon \eta_j \quad \text{for} \quad j \in I_\nu. \tag{27}$$

We will comment on the size of ε below. Let us assume that the coupling strength by which the oscillators of the ν-th cluster influence the j-th oscillator is given by $K_\nu \varepsilon$. We introduced the parameter ε in order to point out that we assume that the frequency deviations and the coupling strength are of the same magnitude.

Below it will turn out that we can apply the averaging theorem only if

$$|\Omega_\nu - \Omega_m| \gg \varepsilon \quad \text{for} \quad \nu \neq m \tag{28}$$

holds. We assume that the cluster frequencies and the differences of the cluster frequencies are all of the same order of magnitude, so that the relations

$$\Omega_\nu = O(1) \quad \text{and} \quad |\Omega_\nu - \Omega_m| = O(1) \tag{29}$$

are fulfilled. From Eqs. (28) and (29) it immediately follows that

$$0 < \varepsilon \ll 1 \tag{30}$$

holds. Thus, ε is a smallness parameter which guarentees that the cluster frequencies are sufficiently far away from each other compared to the magnitude of the coupling strength. Below this will turn out to be a necessary condition for applying the averaging theorem.

With Eqs. (26), (27), and (28) our model reads

$$\dot{z}_j = (\varepsilon\alpha + i\omega_j) z_j - \frac{1}{N} z_j^2 \sum_{\nu=1}^{n} \varepsilon K_\nu \sum_{k \in I_\nu} z_k^*, \tag{31}$$

for $j = 1, \ldots, N$. Hereby $\alpha > 0$, and $K_\nu > 0$ for all $\nu = 1, \ldots, n$.

First we choose the hypothesis $z_j = r_j e^{i\psi_j}$ with real r_j and ψ_j. After this we introduce amplitude deviations

ξ_j by means of the hypothesis

$$r_j(t) = R + \xi_j(t) \quad \text{with} \quad R = \sqrt{\frac{\alpha}{\sum_{\nu=1}^{n} K_\nu n_\nu}}. \tag{32}$$

Next we turn to relative phases by putting

$$\phi_j(t) = \psi_j(t) - \Omega_m t - \theta_m, \quad \theta_m = \text{konst.} \quad (j \in I_m). \tag{33}$$

With $\mathbf{u} = (\xi_1, \ldots, \xi_N, \phi_1, \ldots, \phi_N)^T$ the transformed system may concisely be written as

$$\dot{\mathbf{u}} = \varepsilon \mathbf{f}(\mathbf{u}, t). \tag{34}$$

$\mathbf{f}(\mathbf{u}, t)$ ist smooth and bounded. A little calculation shows that $\mathbf{f}(\mathbf{u}, t)$ contains two types of trigonometric terms: $\sin(\phi_j - \phi_k)$ if the j-th and the k-th oscillator belong to the same cluster, and $\sin[\phi_j - \phi_k + (\Omega_m - \Omega_\nu)\,t + \theta_m - \theta_\nu]$ if the j-th/k-th oscillator belongs to the m-th/ν-th cluster. Obviously $\mathbf{f}(\mathbf{u}, t)$ is T-periodic with the non-trivial period $T > 0$ iff the ratios of the cluster frequencies Ω_ν/Ω_μ are integers.

Thus, according to our assumptions (27), (28), (29), and (30) there are two time scale: As a result of the smallness parameter ε in Eq. (34) the system's dynamics changes on a slow time scale (cf. [9], theorem 4.1.1). This slow evolution is perturbed by a rapid periodic oscillation which is due to the time dependent sine terms on the right hand side of Eq. (34) (cf. [9], theorem 4.1.1). The perturbing rapid periodic oscillation which is a result of the interactions of different clusters can be averaged out by means of the averaging theorem of Guckenheimer und Holmes ([9], theorem 4.1.1). Thus, we approximate the dynamics of n interacting clusters by the dynamics of n separate clusters. Mathematically this means that we approximate the system's dynamics by means of the autonomous differential equation

$$\dot{\mathbf{U}} = \varepsilon \mathbf{F}(\mathbf{U}) = \varepsilon \frac{1}{T} \int_0^T \mathbf{f}(\mathbf{U}, t)\, dt, \tag{35}$$

where $\mathbf{U} = (\bar{\xi}_1, \ldots, \bar{\xi}_N, \ldots, \bar{\phi}_1, \ldots, \bar{\phi}_N)^T$ is the vector of the averaged variables.

Note that **averaging weakens the coupling strength**. The effective coupling constant in the m-th cluster is $K_m^{\text{eff}} = K_m N_m / N = K_m n_m < K_m$, because $n_m < 1$. Obviously not only the coupling strength but also the cluster's size determines whether a cluster will synchronize or not.

Let us consider the case when the effective coupling is strong enough, i.e. for $j \in I_m$ we assume that

$$|\eta_j| \ll \alpha \quad \text{and} \quad |\eta_j| \ll \alpha^{3/4} (K_m^{\text{eff}})^{1/4} \tag{36}$$

hold. With the results of Sect. 2.1.1 we immediately obtain the stable synchronized state of the averaged system:

$$\bar{\psi}_j(t) = \Omega_m t + \theta_m + \frac{\eta_j}{\alpha} - \frac{\eta_j^3}{3\alpha^3} + \frac{1}{3N\alpha^3} \sum_{k \in I_m} \eta_k^3 + O(\|\boldsymbol{\eta}\|^5) \tag{37}$$

and

$$\bar{r}_j = R + \bar{\xi}_j \equiv \sqrt{\frac{\alpha}{\sum_{\nu=1}^{n} K_\nu^{\text{eff}}}} + \frac{1}{2\alpha^{3/2} (K_m^{\text{eff}})^{1/2}} \eta_j^2 + O(\|\boldsymbol{\eta}\|^4) \tag{38}$$

for $j \in I_m$. The interpretation of the right hand sides of equations (37) and (38) is straightforward. Due to the term $\Omega_m t$ in (37) all oscillators of the m-th cluster have the same frequency, namely Ω_m. The other terms cause a constant phase shift. θ_m is a phase shift which is common to all oscillators of the m-th cluster. The other terms contribute to a phase shift which may be different for every oscillator of the m-th cluster. Note that these terms only depend on the eigenfrequency deviations of oscillators of the m-th cluster. Thus, the averaged phases of the m-th cluster are not influenced by the other clusters (cf. Eq. (24)). Comparing equation (38) with (23) shows that the interaction of the different oscillatory clusters determines the averaged amplitudes. R, the limit cycle amplitude for vanishing eigenfrequency deviations, depends on the sum of the effective coupling constants. By means of the effective strength K_m^{eff} in the denominator of the second term even the deviation of the oscillator's amplitude is influenced by all other clusters.

The solution $\mathbf{U}(t)$ of the averaged system approximates the solution $\mathbf{u}(t)$ of Eq. (34). If $\|\mathbf{u}(0) - \mathbf{U}(0)\| = O(\varepsilon)$ then $\|\mathbf{u}(t) - \mathbf{U}(t)\| = O(\varepsilon)$ on a time scale $t \sim 1/\varepsilon$ ([9], theorem 4.1.1). $\mathbf{U}(t)$ approximates $\mathbf{u}(t)$ even for $t \in [0, \infty)$. This follows from an averaging theorem of Hale ([13], theorem 3.2).

If the ratio of the cluster frequencies is not an integer, the averaging can also be applied if $\mathbf{f}(\mathbf{u}, t)$ in Eq. (34) is almost periodic ([13], theorem 3.1).

We can apply the averaging theorems only if the coupling strength is weak. In the case of strong coupling this approach is no longer succesful: the clusters disturb each other too much, as we shall see below.

2.1.3. *Repulsive coupling.*

In this section we change the kind of coupling. Suppose we have two clusters. Within the clusters we do not change the coupling constants. Rather we change the sign of the coupling constants between both clusters. So, every oscillator is coupled with $K > 0$ to the oscillators of its own cluster, and with $-K$ to the oscillators of the other cluster. The latter is called repulsive coupling (cf. [16]).

With this our oscillatory network consists of two repulsively coupled clusters. For $(j = 1, \ldots, M)$ the system reads

$$\dot{z}_j = (\alpha + i\omega_j)\, z_j - \frac{K}{N} z_j^2 \sum_{k=1}^{M} z_k^* + \frac{K}{N} z_j^2 \sum_{k=M+1}^{N} z_k^*, \tag{39}$$

and for $(j = M+1, \ldots, N)$

$$\dot{z}_j = (\alpha + i\omega_j)\, z_j + \frac{K}{N} z_j^2 \sum_{k=1}^{M} z_k^* - \frac{K}{N} z_j^2 \sum_{k=M+1}^{N} z_k^*. \tag{40}$$

We make the usual hypothesis $z_j = r_j e^{i\psi_j}$, and turn to relative phases by introducing

$$\phi_j(t) = \psi_j(t) - \Omega t - \theta \quad \text{for} \quad j = 1, \ldots, M \tag{41}$$

$$\phi_j(t) = \psi_j(t) - \Omega t - \theta - \pi \qquad \text{for} \quad j = 1, \ldots, M, \tag{42}$$

where $\theta = \text{konst.}$ For all oscillators we put $r_j(t) = \sqrt{\alpha/K} + \xi_j(t)$. The eigenfrequencies are $\omega_j = \Omega + \eta_j$. We assume that Eq. (14) holds. With the results of section 2.1.1 the stable synchronized state finally reads:

$$\psi_j(t) = \Omega t + \theta + \frac{\eta_j}{\alpha} - \frac{\eta_j^3}{3\alpha^3} + \frac{1}{3N\alpha^3}\sum_{k=1}^{N}\eta_k^3 + O(\|\boldsymbol{\eta}\|^5). \quad (43)$$

for $j = 1, \ldots, M$, and

$$\psi_j(t) = \Omega t + \theta + \pi + \frac{\eta_j}{\alpha} - \frac{\eta_j^3}{3\alpha^3} + \frac{1}{3N\alpha^3}\sum_{k=1}^{N}\eta_k^3 + O(\|\boldsymbol{\eta}\|^5). \quad (44)$$

for $j = M+1, \ldots, N$, and

$$r_j \equiv \sqrt{\frac{\alpha}{K}} + \frac{1}{2\alpha^{3/2}K^{1/2}}\eta_j^2 + O(\|\boldsymbol{\eta}\|^4), \quad (45)$$

for $j = 1, \ldots, N$. Comparing equations (43) and (44) with equation (24) we immediately see that the two clusters are synchronized in antiphase. From equations (45) and (23) it follows that the antiphase synchronization does not perturb the oscillators' amplitudes.

In the case of the phase model the analysis is exactly the same: by introducing relative phases in the way we have done it above, we end up with the synchronized antiphase state.

2.1.4. Shift of the cluster frequency. Up to now the cluster frequency has been the mean value of the eigenfrequencies as well as the frequency of the synchronized state. In this section we analyze an extended oscillatory network which shows a different behaviour: the frequency of the synchronized state differs from the cluster frequency, which is the mean of the cluster's eigenfrequencies.

Let us first consider a single limit cycle oscillator with the evolution equation

$$\dot{z} = (\alpha + i\omega)z - K(1 + i\beta)z^2 z^*. \quad (46)$$

When the sign of α turns from negative to positive, this oscillator undergoes a Hopf bifurcation. Hereby the frequency shift $-\alpha\beta$ occurs due to the complex coefficient in front of the cubic term (see e.g. [11]). An analogous phenomenon occurs if N oscillators are nonlinearly coupled with complex coupling coefficients. To show this, we investigate the oscillatory system

$$\dot{z}_j = (\alpha + i\beta)z_j - \frac{K}{N}(1 + i\beta)z_j^2\sum_{k=1}^{N} z_k^*, \quad (47)$$

with $j = 1, \ldots, N$ and $\alpha > 0$. With $z_j = r_j e^{i\psi_j}$ we get

$$\dot{r}_j = \alpha r_j - \frac{K}{N}r_j^2\sum_{k=1}^{N} r_k(\cos(\psi_j - \psi_k) - \beta\sin(\psi_j - \psi_k)) \quad (48)$$

$$\dot{\psi}_j = \omega_j - \frac{K}{N}r_j\sum_{k=1}^{N} r_k(\sin(\psi_j - \psi_k) + \beta\cos(\psi_j - \psi_k)). \quad (49)$$

$\Omega = 1/N\sum_{k=1}^{N}\omega_k$ is the cluster frequency. For the eigenfrequencies we write $\omega_j = \Omega + \eta_j$. We turn to relative phases by means of

$$\phi_j(t) = \psi_j(t) - \Omega' t - \theta \quad \text{with} \quad \Omega' = \Omega - \triangle, \quad (50)$$

and $\theta = \text{const}$. Below it will turn out how to choose $\triangle$ appropriately in order to simplify our analysis decisively. Furthermore we put $r_j(t) = \sqrt{\alpha/K} + \xi_j(t)$. In order to avoid a clumsy analytical investigation, we treat $\eta_1, \ldots, \eta_N$ as well as β as variables by putting $\dot{\eta}_j = 0$ for $j = 1, \ldots, N$, and $\dot{\beta} = 0$.

We introduce a further variable by putting

$$y = \sum_{k=1}^{N}\phi_k. \quad (51)$$

Next we make the transformation

$$\varphi_j = \phi_j - \frac{\eta_j}{\alpha} - \frac{y}{N} \quad \text{for} \quad j = 1, \ldots, N. \quad (52)$$

By putting $\triangle = \alpha\beta$, the linear part of $\dot{y}$ vanishes. So, with

$$\mathbf{x}_c = (y_1\beta, \eta_1, \ldots, \eta_N)^T \quad \text{and} \quad \mathbf{x}_h = (\xi_1, \ldots, \xi_N, \varphi_1, \ldots, \varphi_N)^T \quad (53)$$

our transformed system is described by

$$\dot{\mathbf{x}}_c = \mathbf{m}_c(\mathbf{x}_c, \mathbf{x}_h) \quad (54)$$

$$\dot{\mathbf{x}}_h = \mathbf{B}_h\mathbf{x}_h + \mathbf{m}_h(\mathbf{x}_c, \mathbf{x}_h), \quad (55)$$

where $\mathbf{m}_c$ and $\mathbf{m}_h$ only contain nonlinear terms, and $\mathbf{B}_h$ is the matrix of Sect. 2.1.1. $\mathbf{h}$ is determined quite analogously to Sect. 2.1.1. With this we arrive at the order parameter equation

$$\dot{\mathbf{x}}_c = \mathbf{m}_c(\mathbf{x}_c, \mathbf{h}(\mathbf{x}_c)). \quad (56)$$

The equations for $\eta_1, \ldots, \eta_N$ and β are trivial, and for y we obtain

$$y(t) = \left[\beta\left(1 - \frac{1}{\alpha}\right)\sum_{k=1}^{N}\eta_k^2 + O(\|\boldsymbol{\eta}'\|^4)\right]t + y(0). \quad (57)$$

The center manifold theorem is only valid if the magnitude of all variables is small compared to $\max\{\mathrm{re}(\lambda); \lambda \in \mathrm{spec}(-(K\mathbf{I} + \varepsilon\mathbf{A}))\}$. After some time $y(t)$ violates this condition. Therefore we identify 0 and 2π. That means, we solve our system on the $(N+1)$-torus. Next we transform the time according to $\tau = t/\mu$, with $\mu\lambda_{\max} \gg 2\pi$. So, for all t the phase variables stay in a neighbourhood of zero where the center manifold theorem is valid.

Therefore the synchronized state is a local attractor. Transforming back to phases and amplitudes we arrive at

$$\psi_j(t) = \Omega(\beta)t + \theta' + \frac{1}{\alpha}\eta_j - \frac{1}{3\alpha^3}\eta_j^3 + \frac{1}{3\alpha^3 N}\sum_{k=1}^{N}\eta_k^3 - \frac{\beta^2}{\alpha}\eta_j - \frac{\beta}{\alpha^2}\eta_j^2 + \frac{\beta}{\alpha^2 N}\sum_{k=1}^{N}\eta_k^2 + O(\|\boldsymbol{\eta}'\|^5), \quad (58)$$

with $\theta' = \theta + y(0)/N = 1/N\sum_{k=1}^{N}\psi_k(0)$ and

$$\Omega(\beta) = \Omega - \alpha\beta + \frac{\beta}{N}\left(1 - \frac{1}{\alpha}\right)\sum_{k=1}^{N}\eta_k^2 + O(\|\boldsymbol{\eta}'\|^4). \quad (59)$$

We have used the abbreviation $\boldsymbol{\eta}' = (\beta, \eta_1, \ldots, \eta_N)^T$. For the amplitudes we obtain

$$r_j \equiv \sqrt{\frac{\alpha}{K}} + \frac{\beta}{\sqrt{\alpha K}}\eta_j + \frac{1}{2\alpha^{3/2}K^{1/2}}\eta_j^2 + O(\|\boldsymbol{\eta}'\|^4). \quad (60)$$

Note that the center manifold theorem can only be applied if

$$\|\boldsymbol{\eta}'\| \ll \alpha \quad \text{and} \quad \|\boldsymbol{\eta}'\| \ll \alpha^{3/4}K^{1/4} \tag{61}$$

holds. The right hand side of (58) shows that all oscillators have the same frequency which is shifted against Ω according to equation (59). The other terms of the right hand side of equation (58) contribute to a phase shift. The cosine coupling term in (49) and (48) does not only cause a frequency shift according to Eq. (59). Comparing Eq. (58) with Eq. (24) shows that the cosine coupling also induces a phase shift. By means of the Eqs. (60) and (23) we immediately see that the unperturbed limit cycle amplitude (i.e. the amplitude for $\eta_1, \ldots, \eta_N$) is not influenced by the cosine coupling term in Eqs. (49) and (48). Nevertheless the second term of the right hand side of Eq. (60) describes how the cosine coupling constant β contributes to the amplitude deviation.

2.2. Phase model

2.2.1. One cluster. According to Eq. (15) the amplitudes may be written as $r_j(t) = \sqrt{\alpha/K} + \xi_j(t)$, where $\sqrt{\alpha/K}$ is the constant amplitude of the limit cycle, and $\xi_j(t)$ is the deviation of the limit cycle. If the oscillators are weakly coupled, we are able to neglect the amplitude deviations $\xi_j(t)$ in a first approximation. This has been proven to be correct in the case of weakly coupled oscillators which are characterized by the prescription $0 < K \ll 1$ [16]. Thus, for small (and fixed) K we put $\xi_j = 0$ for all oscillators. As a consequence of this all oscillators have the same, time independent amplitude

$$r_j = \sqrt{\alpha/K} = R = \text{const} \quad \text{for all} \quad j = 1, \ldots, N. \tag{62}$$

We insert Eq. (62) into Eq. (10). For the sake of brevity we additionally put

$$\alpha = KR^2 \to K. \tag{63}$$

With this we obtain our phase model

$$\dot{\psi}_j = \omega_j - \frac{K}{N}\sum_{k=1}^{N} \sin(\psi_j - \psi_k) \quad (j = 1, \ldots, N). \tag{64}$$

Note that $K > 0$. Up to now the coupling coefficients between oscillators of a single cluster have been identical. In some cases this may be an appropriate idealization. But considering for instance systems in physics and biology, it would be desirable to have coupling coefficients scattering around a mean. Therefore in this section we investigate an extended model:

$$\dot{\psi}_j = \omega_j - \frac{1}{N}\sum_{k=1}^{N} (K + \varepsilon K_{jk}) \sin(\psi_j - \psi_k)$$

$$\text{for} \quad j = 1, \ldots, N. \tag{65}$$

In order to simplify our investigation we make some assumptions concerning the coupling constants. In the context of oscillatory networks it is often assumed that the coupling constants are symmetrical, i.e. $K_{jk} = K_{kj}$ holds. Nevertheless in several cases this assumption is a rather rigid restriction. Consider for example interacting oscillatory neurons. According to physiology the interactions are far from being symmetrical. Therefore we do not assume that the coupling constants are symmetric. According to the notion of scattering coupling constants we rather assume that

$$\sum_{k=1}^{N} K_{jk} = \sum_{k=1}^{N} K_{kj} = 0 \tag{66}$$

holds for $j = 1, \ldots, N$. We introduce relative phases by means of the transformation (11). Obviously $\sum_{j=1}^{N}\dot{\phi}_j \not\equiv 0$ holds. Therefore we introduce a further variable

$$y = \sum_{k=1}^{N} \phi_k. \tag{67}$$

We extend the system by including y and $\eta_1, \ldots, \eta_N$ as variables, where

$$\dot{y} = 0 \quad \text{and} \quad \dot{\eta}_j = 0 \quad \text{for all} \quad j = 1, \ldots, N \tag{68}$$

holds. As a result of this extension we are able to apply the center manifold theorem (cf. remarks on page 2.1.1). The vector of the center modes is $\mathbf{x}_c = (y, \eta_1, \ldots, \eta_\mathrm{N})^T$, whereas the vector of the stable modes reads

$$\mathbf{x}_h = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} - (K\mathbf{I} + \varepsilon\mathbf{A})^{-1}\left(\begin{pmatrix} \eta_1 \\ \vdots \\ \eta_N \end{pmatrix} + \frac{K}{N}\begin{pmatrix} y \\ \vdots \\ y \end{pmatrix}\right). \tag{69}$$

This transformation separates the center part from the stable part. The transformed system reads

$$\dot{\mathbf{x}}_c = \mathbf{m}_c(\mathbf{x}_c, \mathbf{x}_h) \tag{70}$$

$$\dot{\mathbf{x}}_h = -(K\mathbf{I} + \varepsilon\mathbf{A})\,\mathbf{x}_h + \mathbf{m}_h(\mathbf{x}_c, \mathbf{x}_h). \tag{71}$$

$\mathbf{m}_h$ and $\mathbf{m}_c$ contain only nonlinear terms. We assume that the matrix $-(K\mathbf{I} + \varepsilon\mathbf{A})$ has only eigenvalues with negative real part. Nevertheless ε may be of the order of K. By means of the theorem of Gershgorin [20] for a given $\mathbf{A}$ we may check the "permitted" size of ε. Let $0 < \varepsilon \ll K$. From this theorem it follows that $-(K\mathbf{I} + \varepsilon\mathbf{A})$ only has eigenvalues with negative real part. In this case it is easy to determine

$$(K\mathbf{I} + \varepsilon\mathbf{A})^{-1} = \frac{1}{K}\mathbf{I} + \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{K^{\nu+1}}\mathbf{A}^\nu\varepsilon^\nu. \tag{72}$$

On the center manifold $\mathbf{x}_h = \mathbf{h}(\mathbf{x}_c; \varepsilon)$ holds. Making use of Eq. (72), we determine $\mathbf{h}$ as shown above up to the terms of third order (cf. Sect. 2.1.1) by means of

$$\mathbf{h}(\mathbf{x}_c; \varepsilon) = (K\mathbf{I} + \varepsilon\mathbf{A})^{-1}\mathbf{m}_h(\mathbf{x}_c, \mathbf{h}(\mathbf{x}_c; \varepsilon)) + O(\|\mathbf{x}_c\|^5). \tag{73}$$

With this we are able to solve the order parameter equation (the reduced problem):

$$\dot{\mathbf{x}}_c = \mathbf{m}_c(\mathbf{x}_c, \mathbf{h}(\mathbf{x}_c; \varepsilon)). \tag{74}$$

The equations for $\eta_1, \ldots, \eta_N$ are trivial, and for y we obtain

$$y(t) = \left[\frac{\varepsilon}{2NK^3} \sum_{j,k=1}^{N} (K_{kj} - K_{jk})\, \eta_j^2 \eta_k + O(\|\mathbf{x}_c\|^5 + \|\mathbf{x}_c\|^3 \varepsilon^2)\right] t + y(0). \tag{75}$$

We solve our system on the $(N+1)$-torus (cf. Sect. 2.1.4). The time transformation from Sect. 2.1.4 shows, that for all t the center manifold theorem is valid. Therefore the synchronized state is a local attractor. It reads

$$\begin{aligned}\psi_j(t) = {} & \Omega^* t + \theta' + \frac{\eta_j}{K} - \frac{\varepsilon}{NK^2} \sum_{k=1}^{N} K_{jk}\eta_k \\ & + \frac{1}{6NK^3} \sum_{k=1}^{N} (\eta_j - \eta_k)^3 + \frac{\varepsilon}{2N^2K^4} \sum_{k,l=1}^{N} \\ & \times (K_{kl} - K_{jl})(\eta_j - \eta_k)^2 \eta_l \\ & + \frac{\varepsilon}{6NK^4} \sum_{k=1}^{N} K_{jk}(\eta_j - \eta_k)^3 - \frac{\varepsilon}{6N^2K^4} \\ & \times \sum_{k,l=1}^{N} K_{jk}(\eta_k - \eta_l)^3 + O(\|\eta\|^5 + \|\eta\|^3 \varepsilon^2),\end{aligned} \tag{76}$$

with θ' as in Sect. 2.1.4.

$$\Omega^* = \Omega + \frac{\varepsilon}{2N^2K^3} \sum_{k,l=1}^{N} (K_{kl} - K_{lk}) \eta_l^2 \eta_k + O(\|\boldsymbol{\eta}\|^5 + \|\boldsymbol{\eta}\|^3 \varepsilon^2) \tag{77}$$

is the renormalized cluster frequency. The first term of the right hand side of equation (76) shows that in the synchronized state Ω^* is the frequency of all oscillators. According to equation (77) the magnitude of ε and the symmetries of the matrix (K_{jk}) and the vector $(\eta_1, \ldots, \eta_N)$ determine how much the renormalized cluster frequency is shifted against the mean of the eigenfrequencies. Note that the frequency shift crucially depends on the symmetry of the coupling matrix (K_{jk}). All terms on the right hand side of Eq. (76) except for the first one determine a constant phase shift. Obviously the coupling deviations (K_{jk}) cause a phase shift, too. Moreover they determine whether Ω^* differs from the cluster frequency Ω or not. If $K_{jk} = K_{kj}$ for all $j, k = 1, \ldots, N$ we get $\Omega = \Omega^*$. The renormalization of the cluster frequency is due to the loss of symmetry of the coupling deviations K_{jk}.

Note that we have assumed that

$$\|\boldsymbol{\eta}\| \ll K \quad \text{and} \quad 0 < \varepsilon \ll K \tag{78}$$

hold.

The analysis presented in this section may also be applied to the total system with coupling constants $K + \varepsilon K_{jk}$.

If we assume $K_{jk} = K_{kj}$ for all $j, k = 1, \ldots, N$ instead of Eq. (66), we do not have to introduce y. Therefore the analysis is simpler, and we just want to present the result here. With $0 < \varepsilon \ll K$ we finally obtain the stable synchronized state

$$\begin{aligned}\psi_j(t) = {} & \Omega t + \theta + \frac{1}{K}\eta_j - \frac{\varepsilon}{K^2} \sum_{k=1}^{N} a_{jk}\eta_k + \frac{1}{6NK^3} \\ & \sum_{k=1}^{N} (\eta_j - \eta_k)^3 - \frac{\varepsilon}{2NK^4} \sum_{k,l=1}^{N} (a_{kl} - a_{jl})(\eta_j - \eta_k)^2 \eta_l \\ & + \frac{\varepsilon}{6NK^4} \sum_{k=1}^{N} K_{jk}(\eta_j - \eta_k)^3 - \frac{\varepsilon}{6NK^4} \sum_{k,l=1}^{N} a_{jk}(\eta_j - \eta_k)^3 \\ & + O(\|\boldsymbol{\eta}\|^5) + O(\|\boldsymbol{\eta}\|^3 \varepsilon^2).\end{aligned} \tag{79}$$

for $j = 1, \ldots, N$. With $a_{jk} = -\alpha(1 + 1/N)$ for $j = k$ and $a_{jk} = -\alpha/N$ for $j \neq k$. Comparing Eq. (79) with Eqs. (76) and (77) shows that the symmetric coupling deviations (K_{jk}) only cause phase shifts. No frequency shifts occur.

2.2.2 Several clusters. With the notations of section 2.1.2 the phase model of (31) is

$$\dot{\psi}_j = \omega_j - \frac{1}{N} \sum_{\nu=1}^{n} \varepsilon K_{m\nu} \sum_{k \in I_\nu} \sin(\psi_j - \psi_k). \tag{80}$$

for $j \in I_m$. As usual we make the hypothesis (33). As in Sect. 2.1.2 we can separate the clusters if the ratio of the cluster frequencies is an integer (with [9, 13]). We can even separate them if the vector field is an almost periodic function (with [13]). As we have already seen in Sect. 2.1.2 averaging weakens the coupling strength. If we want to apply the averaging theorems of [9] and [13] the condition $|\Omega_\nu - \Omega_m| \gg \varepsilon$ has to be fullfilled.

If the cluster frequencies come closer together, it is no longer possible to separate the clusters by means of averaging. To show this, we have integrated the system

$$\dot{\psi}_j = \omega_j - \frac{K}{20} \sum_{k=1}^{20} \sin(\psi_j - \psi_k) \quad (j = 1, \ldots, 20) \tag{81}$$

numerically. Ω_1 is the cluster frequency of the oscillators $1, \ldots, 10$, whereas Ω_2 is the cluster frequency of the oscillators $11, \ldots, 20$. $\eta_1, \ldots, \eta_N$ are uniformly distributed in the intervall $[-0.1, 0.1]$. We choose $K = 0.3$ and $\Omega_1 = 0.5$. The initial values of the phases are uniformly distributed in $[0, 2\pi]$. We introduce

$$\sigma_1(t) e^{i\Phi_1(t)} = \frac{1}{10} \sum_{k=1}^{10} e^{i\psi_k(t)} \text{ and } \sigma_2(t) e^{i\Phi_2(t)} = \frac{1}{10} \sum_{k=11}^{20} e^{i\psi_k(t)}. \tag{82}$$

The cluster amplitudes $\sigma_1(t)$ and $\sigma_2(t)$ are plotted for three different values of Ω_2. For $\Omega_2 = 4$ there is a weak disturbation of the stable synchronized state (Fig. 3). Lowering Ω_2 increases the amplitude and the period of the oscillations of the cluster amplitudes. The weakly perturbed stable synchronization vanishes more and more. For $\Omega_2 = 0.77$ both clusters are no longer synchronized in a stable way (Fig. 4). They strongly disturb each other. If Ω_2 approaches Ω_1 even more, both clusters melt and form one big cluster (Fig. 5).

2.2.3. Continuum of oscillators. In physics modeling by means of field theory has often been very succesful. Therefore we are interested whether an oscillatory field shows

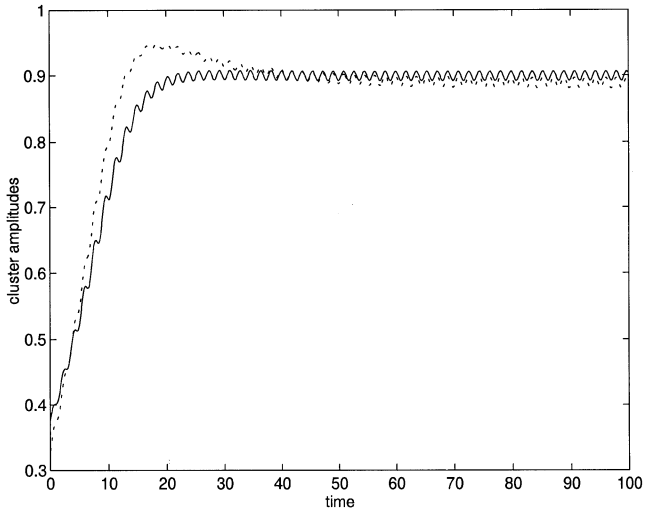

Fig. 3. Phase model I, cluster 1: $\Omega_1 = 0.5$ (solid line) scluster 2: $\Omega_2 = 4$ (dotted line)

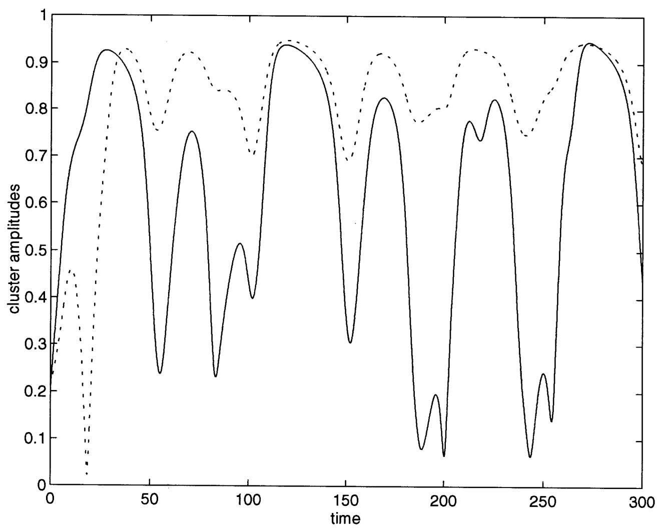

Fig. 4. Phase model I, cluster 1: $\Omega_1 = 0.5$ (solid line), cluster 2: $\Omega_2 = 0.77$ (dotted line)

a qualitatively different behaviour compared to the dynamical behaviour of finitely many oscillators. To this end we replace the discrete system (64) by the partial differential equation. This yields

$$\frac{\partial \psi(x,t)}{\partial t} = \omega(x) - \frac{K}{2\pi} \int_0^{2\pi} \sin(\psi(x,t) - \psi(\xi,t))\, d\xi. \tag{83}$$

$\psi(x,t)$ is now the phase field belonging to a field of weakly coupled oscillators. We will analyze eq. (83) for periodic boundary conditions, i.e. $\psi(0,\cdot) \equiv \psi(2\pi,\cdot)$. With the cluster frequency

$$\Omega = \frac{1}{2\pi} \int_0^{2\pi} \omega(x)\, dx \tag{84}$$

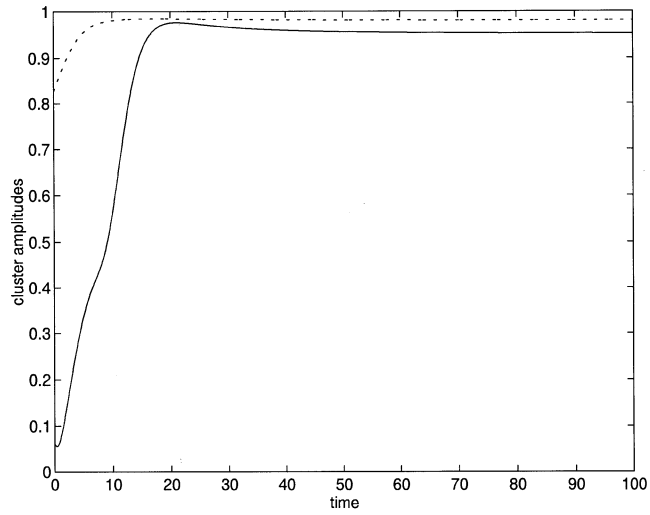

Fig. 5. Phase model I, cluster 1: $\Omega_1 = 0.5$ (solid line), cluster 2: $\Omega_2 = 0.7$ (dotted line)

the eigenfrequency field is $\omega(x) = \Omega + \eta(x)$. We assume that

$$\|\eta\|_{H^1([0,2\pi])} \ll K \tag{85}$$

holds. $\|\cdot\|_{H^1([0,2\pi])}$ is the norm in the Sobolev-space $H^1([0,2\pi])$ (cf. [22]).

The transformation $\phi(x,t) = \psi(x,t) - \Omega t - \theta$, with $\theta \equiv$ const. yields

$$\frac{\partial \phi(x,t)}{\partial t} = \eta(x) - \frac{K}{2\pi} \int_0^{2\pi} \sin(\phi(x,t) - \phi(\xi,t))\, d\xi \tag{86}$$

with periodic boundary conditions $\phi(0,\cdot) \equiv \phi(2\pi,\cdot)$. These boundary conditions enable us to make a Fourier transformation. We are able to apply the center manifold theorem if we expand the system by putting $\partial \hat{\eta}(k)/\partial t = 0$ for all $k \in \mathbf{Z} \backslash \{0\}$) (cf. remarks on page 2.1.1). Next we separate the center modes from the stable modes with the transformation

$$\hat{\varphi}(k,t) = \hat{\phi}(k,t) - \frac{\hat{\eta}(k)}{K} \quad \text{for} \quad k \in \mathbf{Z} \backslash \{0\}. \tag{87}$$

$$\hat{\mathbf{x}}_c = (\ldots, \hat{\eta}(2), \hat{\eta}(1), \hat{\eta}(-1), \hat{\eta}(-2), \ldots)^T \tag{88}$$

and

$$\hat{\mathbf{x}}_h(t) = (\ldots, \varphi(2,t), \varphi(1,t), \varphi(-1,t), \hat{\varphi}(-2,t), \ldots)^T \tag{89}$$

are the center and the stable modes, respectively. The existence of the center manifold is proven in the appendix. We now calculate the coefficients of this center manifold. The stable part of the linearized system may be written in the form

$$\frac{\partial \hat{\varphi}(k,t)}{\partial t} = -K\hat{\varphi}(k,t) + \hat{m}_k(\hat{\mathbf{x}}_c, \hat{\mathbf{x}}_h), \tag{90}$$

where $\hat{m}_k$ only contains nonlinear terms. We are now interested in the map $\hat{\mathbf{h}}(\hat{\mathbf{x}}_c) = \hat{\mathbf{x}}_h$ of the center manifold.

Differentiating this equation with respect to time finally yields for all $k \in \mathbf{Z}\backslash\{0\}$

$$\hat{h}_k(\hat{\mathbf{x}}_c) = \frac{1}{K}\hat{m}_k(\hat{\mathbf{x}}_c, \hat{\mathbf{h}}(\hat{\mathbf{x}}_c)) \tag{91}$$

with

$$\hat{\mathbf{h}}(\hat{\mathbf{x}}_c) = (\ldots, \hat{h}_2(\hat{\mathbf{x}}_c), \hat{h}_1(\hat{\mathbf{x}}_c), \hat{h}_{-1}(\hat{\mathbf{x}}_c), \hat{h}_{-2}(\hat{\mathbf{x}}_c), \ldots)^T. \tag{92}$$

This enables us to determine the coefficients of lowest order. On the center manifold there is $\hat{\mathbf{x}}_h = \hat{\mathbf{h}}(\hat{\mathbf{x}}_c)$, and therefore

$$\varphi(x) = \sum_{k \in \mathbf{Z}\backslash\{0\}} \hat{h}_k(\hat{\mathbf{x}}_c)e^{ikx} \tag{93}$$

holds.

Transforming back to the phase field we immediately obtain the synchronized state

$$\psi(x,t) = \Omega t + \theta + \eta(x)/K + \frac{1}{12\pi K^3}\int_0^{2\pi}(\eta(x) - \eta(\xi))^3\,d\xi + O(\|\eta\|^5_{H^1([0,2\pi])}). \tag{94}$$

This state is stable. Moreover due to the center manifold theorem it is a local attractor. Obviously Eq. (94) is the analogue of Eq. (76) for $\varepsilon = 0$. So, in the parameter range we have explored (cf. Eqs. (78) and (85)), finitely many oscillators as well as an oscillatory field show the same synchronization behaviour.

2.2.4. Time dependent cluster frequency. Considering the oscillatory neuronal activity in the brain (cf. [4, 8]), we became interested in a synchronized cluster of oscillators with time dependent cluster frequency. Therefore we analyze Eq. (64) for time dependent eigenfrequencies. The model reads

$$\dot{\psi}_j = \omega_j(t) - \frac{K}{N}\sum_{k=1}^{N}\sin(\psi_j - \psi_k) \quad (j = 1, \ldots, N), \tag{95}$$

with $\omega_j(t) = \Omega + \eta_j + f(t)$. $f(t)$ models the time dependence of the cluster frequency. In order to apply the center manifold theorem we assume that $\|\boldsymbol{\eta}\| \ll K$ holds. We make the hypothesis $\psi_j(t) = \tilde{\psi}_j(t) + p(t)$ for $j = 1, \ldots, N$, where $\tilde{\psi}_j(t)$ is a solution of Eq. (64). With the results of section 2.1.1 we immediately obtain the synchronized state, which is a local attractor:

$$\psi_j(t) = \int_{t_0}^{t} f(\xi)\,d\xi + \Omega t + \theta + \frac{\eta_j}{K} + \frac{1}{6NK^3}\sum_{k=1}^{N}(\eta_j - \eta_k)^3 + O(\|\boldsymbol{\eta}\|^5) \tag{96}$$

for $j = 1, \ldots, N$. Comparing Eq. (96) with Eqs. (76) and (77) (for $\varepsilon = 0$ and $K_{jk} = 0$) shows that $f(t)$ causes a time dependent shift of the cluster frequency.

2.2.5. Shift of the cluster frequency. In Sect. 2.1.4 we have analyzed a shift of the cluster frequency caused by the imaginary part of the coupling constants. Considering Eq. (49) we can immediately write down the analogous problem for the phase model:

$$\dot{\psi}_j = \omega_j - \frac{K}{N}\sum_{k=1}^{N}(\sin(\psi_j - \psi_k) + \beta\cos(\psi_j - \psi_k)) \tag{97}$$

for $j = 1, \ldots, N$. With $\Omega = 1/N\sum_{k=1}^{N}\omega_k$ we put $\omega_j = \Omega + \eta_j$. We make the transformation

$$\phi_j(t) = \psi_j(t) - \Omega' t - \theta \quad \text{with} \quad \theta = \text{konst}. \tag{98}$$

Moreover we introduce $y = \sum_{k=1}^{N}\phi_k$. y is a center mode if we choose $\Omega' = \Omega - \beta K$. In order to separate the center part from the stable part we make the transformation

$$\varphi_j = \phi_j - \frac{\eta_j}{K} - \frac{y}{N}. \tag{99}$$

The map of the center manifold is determined as shown above. It is an easy task to solve the order parameter equation. Next, we identify 0 and 2π, i.e. we solve the system on the $(N + 1)$ -torus. According to the center manifold theorem the synchronized state is a local attractor. It reads

$$\psi_j(t) = \Omega(\beta)t + \theta' + \frac{\eta_j}{K} + \frac{\beta}{2NK^2}\sum_{k=1}^{N}(\eta_j - \eta_k)^2 + \frac{1}{6NK^3}\sum_{k=1}^{N}(\eta_j - \eta_k)^3 + O(\|\boldsymbol{\eta}\|^4), \tag{100}$$

where $\theta' = \theta + y(0) = 1/N\sum_{k=1}^{N}\psi_k(0)$.

$$\Omega(\beta) = \Omega - \beta K + \frac{\beta}{NK}\sum_{k=1}^{N}\eta_k^2 + O(\|\boldsymbol{\eta}\|^4) \tag{101}$$

is the shifted cluster frequency. Note that we assume that $\|\eta\| \ll K$ holds. In contrast to Sect. 2.1.4 β needs not to be small. Apart from that Eqs. (100) and (101) have a similar structure compared to Eqs. (58) and (59).

3. Model II

In model I stable synchronized clusters with different cluster frequencies can only exist, if their cluster frequencies fulfill the condition

$$|\Omega_\nu - \Omega_\mu| \gg \varepsilon \quad \text{for} \quad \nu \neq \mu, \tag{102}$$

where ε is the magnitude of the coupling strength (cf. Sects. 2.1.2 and 2.2.2). In this case we are able to separate the different clusters by means of averaging.

Obviously this is a significant restriction if we are interested in an oscillatory network, which allows for the stable synchronization of clusters with frequencies that are arbitrarily close to each other. This motivates us to investigate our models II and III. The latter is analyzed in the next section. Both allow for stable synchronized clusters with arbitrarily close cluster frequencies, even in the case of high coupling strength. The above mentioned condition has no longer to be fullfilled. In both models the state of several synchronized clusters, all having different cluster frequencies, is nothing but a stable fixed point, moreover a local attractor. Therefore the different synchronized clusters do not disturb each other.

3.1. Total system

Before we turn to model II we recall some notations of Sect. 2.1.2. The cluster frequency of the ν-th cluster is

$$\Omega_\nu = f_\nu \Omega = \sum_{k \in I_\nu} \frac{\omega_k}{N_\nu} \quad \text{with} \quad \omega_k = f_\nu \Omega + \eta_k. \tag{103}$$

With this, model II reads (in polar coordinates)

$$\dot{r}_j = \alpha r_j - \frac{K}{N} \sum_{\nu=1}^{n} \frac{1}{f_\nu} r_j^2 \sum_{k \in I_\nu} r_k \cos(f_\nu \psi_j - f_m \psi_k) \tag{104}$$

$$\dot{\psi}_j = \omega_j - \frac{K}{N} \sum_{\nu=1}^{n} \frac{1}{f_\nu} r_j \sum_{k \in I_\nu} r_k \sin(f_\nu \psi_j - f_m \psi_k) \tag{105}$$

for $j \in I_m$. K and α are positive constants. In order to investigate the well-synchronized state, we assume that

$$\|\eta\| \ll \alpha \quad \text{and} \quad \|\boldsymbol{\eta}\| \ll \alpha^{3/4} K^{1/2} \tag{106}$$

hold. We introduce

$$\phi_j(t) = \psi_j(t) - f_\nu \Omega t - f_\nu \theta \quad \text{for} \quad j \in I_\nu \text{ with } \theta = \text{const}, \tag{107}$$

$$r_j = R + \xi_j \quad \text{with} \quad R = \sqrt{\frac{\alpha}{K \sum_{\nu=1}^{n} n_\nu / f_\nu}}, \tag{108}$$

because R is the limit cycle solution for $\eta_1, \dots, \eta_N = 0$. Putting

$$\theta = \frac{1}{N} \sum_{\nu=1}^{n} \frac{1}{f_\nu} \sum_{k \in I_\nu} \psi_k(0) \quad \text{yields} \quad \sum_{\nu=1}^{n} \frac{1}{f_\nu} \sum_{k \in I_\nu} \phi_j \equiv 0. \tag{109}$$

Below this will turn out to be important for the spectrum of the linear operator. In order to apply the center manifold theorem we include $\eta_1, \dots, \eta_N$ as variables with vanishing time derivative (cf. remarks in Sect. 2.1.1). By means of introducing

$$\varphi_j = \phi_j - \frac{\eta_j}{KR^2} \tag{110}$$

we separate the center part from the stable part. We denote the center modes by $\mathbf{x}_c = (\eta_1, \dots, \eta_N)^T$ and the stable modes by $\mathbf{x}_h = (\xi_1, \dots, \xi_N, \varphi_1, \dots, \varphi_N)^T$. With these notations the transformed system reads

$$\dot{\mathbf{x}}_c = 0 \quad \text{and} \quad \dot{\mathbf{x}}_h = \mathbf{B}_h \mathbf{x}_h + \mathbf{m}_h(\mathbf{x}_c, \mathbf{x}_h) \text{ with} \tag{111}$$

$$\mathbf{B}_h = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & -KR^2 \mathbf{I} \end{pmatrix}. \tag{112}$$

$\mathbf{I}$ denotes the $N \times N$-identity matrix, and $\mathbf{A} = (a_{jk})$ with $a_{jk} = -\alpha - KR^2/(Nf_\nu)$ for $j = k \in I_\nu$ and $= -KR^2/(Nf_\nu)$ else. All eigenvalues of A have negative real part (proof by means of the theorem of Gershgorin (see e.g. [20])).

Differentiating $\mathbf{x}_h = \mathbf{h}(\mathbf{x}_c)$ with respect to time, we finally get

$$\mathbf{h}(\mathbf{x}_c) = -\mathbf{B}_h^{-1} \mathbf{m}_h(\mathbf{x}_c, \mathbf{h}(\mathbf{x}_c)), \tag{113}$$

which serves for determinig $\mathbf{h}$ up to terms of third order. We are interested in an oscillatory network which consists of many oscillators. Calculating $\mathbf{A}^{-1}$ is not straightforward. Therefore we restrict our analysis to networks consisting of many oscillators, i.e. we put $\varepsilon = 1/N$. With a little calculation we get

$$\mathbf{A}^{-1} = -\frac{1}{\alpha} \mathbf{I} - \sum_{\nu=1}^{\infty} \frac{1}{\alpha^{\nu+1} N^\nu} \mathbf{L}^\nu, \tag{114}$$

with $\mathbf{L} = (l_{jk})$ and $l_{jk} = -KR^2/f_\nu$ for $k \in I_\nu$. With this we finally obtain the synchronized state. For $j \in I_m$ and with the abbreviations

$$\gamma = \sum_{\nu=1}^{n} \frac{n_\nu}{f_\nu}, \quad \beta = \sum_{\nu=1}^{n} n_\nu f_\nu \tag{115}$$

the synchronized state reads

$$\begin{aligned} r_j = R + \xi_j \equiv \sqrt{\frac{\alpha}{K\gamma}} &+ \frac{\gamma^{1/2}}{2\alpha^{3/2} K^{1/2} N} \sum_{\nu=1}^{n} \frac{1}{f_\nu} \\ &\sum_{k \in I_\nu} (f_\nu \eta_j - f_m \eta_k)^2 - \frac{1}{\alpha^{3/2} K^{1/2} N \gamma^{1/2}} \sum_{\nu=1}^{n} \frac{1}{f_\nu} \sum_{k \in I_\nu} \eta_k^2 \\ &+ O(\|\boldsymbol{\eta}\|^4 \varepsilon + \|\boldsymbol{\eta}\|^2 \varepsilon^2). \end{aligned} \tag{116}$$

$$\begin{aligned} \psi_j(t) = \Omega_m t + f_m \theta + \frac{\gamma}{\alpha} \eta_j + \frac{\gamma^2 \beta}{2\alpha^3} \eta_j^3 \\ + \frac{\gamma^2 \beta}{2\alpha^3 N} \eta_j \sum_{k=1}^{N} \eta_k^2 + \frac{\gamma^3}{6N\alpha^3} \sum_{\nu=1}^{n} \frac{1}{f_\nu} \sum_{k \in I_\nu} (f_\nu \eta_j - f_m \eta_k)^3 \\ - \frac{f_m \gamma^2 \beta}{2\alpha^3 N} \sum_{\nu=1}^{n} \frac{1}{f_\nu} \sum_{k \in I_\nu} \eta_k^3 + O(\|\boldsymbol{\eta}\|^4 \varepsilon + \|\boldsymbol{\eta}\|^2 \varepsilon^2). \end{aligned} \tag{117}$$

The interpretation of equation (116) is straightforword. $\sqrt{\alpha/(K\gamma)}$ denotes the unperturbed limit cycle amplitude (for $\eta_1, \dots, \eta_N = 0$). The other terms on the right hand side of this equation contribute to a deviation of this limit cycle amplitude which is caused by the eigenfrequencies' deviations $\eta_1, \dots, \eta_N$. Note that the limit cycle amplitude as well as its deviation is influenced by the interactions of all clusters (cf. Eqs. (115) and (103)). All oscillators of the m-th cluster have the frequency Ω_m according to the first term on the right hand side of (117). The other terms on the right hand side give rise to a constant phase shift.

3.2. Phase model

Carrying out the adiabatic elimination of the amplitudes and putting $KR^2 \to K$ in Eq. (105), the phase approximation of model II is

$$\dot{\psi}_j = \Omega_m + \eta_j - \frac{K}{N} \sum_{\nu=1}^{n} \frac{1}{f_\nu} \sum_{k \in I_\nu} \sin(f_\nu \psi_j - f_m \psi_k) \; (j \in I_m). \tag{118}$$

As in the former section we turn to relative phases. With Eq. (109) and the transformation

$$\varphi_j = \phi_j - \frac{\eta_j}{K} \tag{119}$$

a little calculation yields the stable synchronized state

$$\psi_j(t) = \Omega_m t + f_m\theta + \frac{\eta_j}{K} + \frac{1}{6NK^3}\sum_{\nu=1}^{n}\frac{1}{f_\nu}\sum_{k\in I_\nu}(f_\nu\eta_j - f_m\eta_k)^3 + O(\|\boldsymbol{\eta}\|^5) \quad (120)$$

for $j \in I_m$. Equation (120) has a similar structur compared to equation (117). According to the center manifold theorem the synchronized state is a local attractor. With regard to the synchronization frequency the clusters do not disturb each other. If the cluster frequency is time dependent, the problem is easily solved by means of the hypothesis we have used in Sect. 2.2.4.

4. Model III

In model III (as in model II) synchronized clusters coexist although their frequencies may be close. This is due to the fact that the synchronized state is a stable fixed point. We do not need any averaging procedure in order to separate the different clusters. As in model II the synchronized state is a local attractor. We use the same notations as in Sect. 3. A simple transformation enables us to use the results of Sect. 2.

4.1. Total system

The third network is given by

$$\dot{z}_j = \left(\alpha + i\frac{\omega_j}{f_m}\right)z_j - \frac{K}{N}z_j^2\sum_{k=1}^{N}z_k^* \quad (j \in I_m). \quad (121)$$

K and α are positive constants. With $z_j = r_j\exp(i\psi_j/f_m)$ we obtain model III in polar coordinates:

$$\dot{r}_j = \alpha r_j - \frac{K}{N}r_j^2\sum_{\nu=1}^{n}\sum_{k\in I_\nu}r_k\cos\left(\frac{\psi_j}{f_m} - \frac{\psi_k}{f_\nu}\right) \quad (122)$$

$$\dot{\psi}_j = \omega_j - \frac{Kf_m}{N}r_j\sum_{\nu=1}^{n}\sum_{k\in I_\nu}r_k\sin\left(\frac{\psi_j}{f_m} - \frac{\psi_k}{f_\nu}\right) \quad (123)$$

for $j \in I_m$. By putting

$$\tilde{\psi}_j = \frac{\psi_j}{f_m},\ \tilde{\omega}_j = \frac{\omega_j}{f_m},\ \tilde{\eta}_j = \frac{\eta_j}{f_m} \quad (j \in I_m) \quad (124)$$

we arrive at model I, which we have analyzed in Sect. 2.1.1. Therefore we can immediately write down the stable synchronized state. For $j \in I_m$ we obtain

$$\psi_j(t) = \Omega_m t + f_m\theta + \frac{\eta_j}{\alpha} - \frac{\eta_j^3}{3\alpha^3} + \frac{1}{3N\alpha^3}\sum_{\nu=1}^{n}\sum_{k\in I_\nu}\eta_k^3 + O(\|\hat{\boldsymbol{\eta}}\|^5) \quad (125)$$

$$r_j \equiv \sqrt{\frac{\alpha}{K}} + \frac{1}{2\alpha^{3/2}K^{1/2}f_m^2}\eta_j^2 + O(\|\hat{\boldsymbol{\eta}}\|^4), \quad (126)$$

with $\hat{\boldsymbol{\eta}} = (\tilde{\eta}_1, \ldots, \tilde{\eta}_N)^T$. Note that we have made the assumption

$$\|\hat{\boldsymbol{\eta}}\| \ll \alpha \quad \text{and} \quad \|\hat{\boldsymbol{\eta}}\| \ll \alpha^{3/4}K^{1/4}. \quad (127)$$

According to equation (125) the unperturbed limit cycle amplitude, $\sqrt{\alpha/K}$, is not influenced by parameters which are related to the mutual interaction of the clusters (cf. Eq. (116)). Note that Eq. (126) has a similar structure compared to Eq. (23). According to Eq. (125) all oscillators of the m-th cluster have the cluster frequency Ω_m. The constant phase shift is similar to that one in Eq. (24).

4.1.1. Repulsive coupling. In order to investigate repulsively coupled clusters of different frequencies, we have to introduce some notations. For every cluster frequency Ω_ν there are now two groups of oscillators. Their index sets are denoted by I_ν^1 and I_ν^2 respectively. On the whole there are two groups of oscillators: $I^1 = \bigcup_{\nu=1}^{n} I_\nu^1$ and $I^2 = \bigcup_{\nu=1}^{n} I_\nu^2$. An oscillator of I^μ is coupled with the other oscillators of I^μ with the coupling constant K. With the oscillators of the other group it is coupled repulsively with $-K$. The other notations are as in the former section.

With this our system reads

$$\dot{z}_j = \left(\alpha + i\frac{\omega_j}{f_m}\right)z_j - \frac{K}{N}z_j^2\sum_{k\in I^1}z_k^* + \frac{K}{N}z_j^2\sum_{k\in I^2}z_k^* \quad (j \in I_m^1) \quad (128)$$

$$\dot{z}_j = \left(\alpha + i\frac{\omega_j}{f_m}\right)z_j + \frac{K}{N}z_j^2\sum_{k\in I^1}z_k^* - \frac{K}{N}z_j^2\sum_{k\in I^2}z_k^* \ (j \in I_m^2). \quad (129)$$

We make the hypothesis $z_j = r_j(\exp(i\psi_j/f_m)$ for $j \in I_m^1 \cup I_m^2$. By means of introducing

$$\tilde{\psi}_j = \frac{\psi_j}{f_m},\ \tilde{\omega}_j = \frac{\omega_j}{f_m},\ \tilde{\eta}_j = \frac{\eta_j}{f_m} \quad \text{for} \quad j \in I_m^1 \cup I_m^2, \quad (130)$$

$$\tilde{\phi}_j = \tilde{\psi}_j - \Omega t - \theta \quad \text{for} \quad j \in I^1,$$
$$\tilde{\phi}_j = \tilde{\psi}_j - \Omega t - \theta - \pi \quad \text{for} \quad j \in I^2, \quad (131)$$

$$r_j(t) = \sqrt{\frac{\alpha}{K}} + \xi_j(t) \quad \text{for all } j \quad (132)$$

we transform model III to model I. We make use of the results of Sect. 2.1.1. Transforming back, immediately gives us the stable synchronized state

$$r_j \equiv \sqrt{\frac{\alpha}{K}} + \frac{1}{2\alpha^{3/2}K^{1/2}f_m^2}\eta_j^2 + O(\|\hat{\boldsymbol{\eta}}\|^4) \quad (133)$$

$$\psi_j(t) = \Omega_m t + f_m\theta + u_{jm} \quad \text{for} \quad j \in I_m^1 \quad (134)$$

$$\psi_j(t) = \Omega_m t + f_m\theta + f_m\pi + u_{jm} \quad \text{for} \quad j \in I_m^2, \quad (135)$$

where

$$u_{jm} = \frac{\eta_j}{\alpha} - \frac{\eta_j^3}{3\alpha^3} + \frac{1}{3N\alpha^3}\sum_{\nu=1}^{n}\sum_{k\in I_\nu^1\cup I_\nu^2}\left(\frac{\eta_k}{f_\nu}\right)^3 + O(\|\boldsymbol{\eta}\|^5). \quad (136)$$

Note that we have assumed that $\|\hat{\boldsymbol{\eta}}\| \ll K$ holds. Comparing Eq. (133) with Eq. (126) clearly shows that the anti-phase synchronization does not influence the oscillators' amplitudes in the synchronized state. From Eqs. (134) and (135) we immediately read off that both clusters with the cluster frequency Ω_m have the constant phase difference $f_m\pi$ (cf. Eq. (125)).

Repulsive coupling in the phase model is analyzed in an analogous way.

4.1.2. Shift of the cluster frequency. In model I we have already seen that the imaginary part of the coupling constants causes a shift of the cluster frequency. Model III exhibits the same phenomenon. To show this we investigate

$$\dot{z}_j = \left(\alpha + i\frac{\omega_j}{f_m}\right) z_j - \frac{K}{N}(1+i\beta) z_j^2 \sum_{k=1}^{N} z_k^* \quad (j \in I_m), \tag{137}$$

with $\alpha > 0$, and β real. The hypothesis $z_j = r_j \exp(i\psi_j/f_m)$ for $j \in I_m$ and the transformation (124) brings us to the extended model I. We profit by the results from Sect. 2.1.4. The stable synchronized state, which is a local attractor, is

$$\begin{aligned}\psi_j(t) &= \Omega_m(\beta)\, t + \theta'_m + \frac{\eta_j}{\alpha} - \frac{1}{3\alpha^3 f_m^2}\eta_j^3 \\ &+ \frac{f_m}{3\alpha^3 N}\sum_{v=1}^{n}\sum_{k\in I_v}\left(\frac{\eta_k}{f_v}\right)^3 - \frac{\beta^2}{\alpha}\eta_j - \frac{\beta}{\alpha^2 f_m}\eta_j^2 \\ &+ \frac{\beta f_m}{N\alpha^2}\sum_{v=1}^{n}\sum_{k\in I_v}\left(\frac{\eta_k}{f_v}\right)^2 + O(\|\boldsymbol{\eta}''\|^5)\end{aligned} \tag{138}$$

for $j \in I_m$, where $\theta_m{}' = (\sum_{k=1}^{N} \psi_k(0))\, f_m/N$.

$$\begin{aligned}\Omega_m(\beta) &= \Omega_m - f_m\alpha\beta + \frac{f_m\beta}{N}\left(1 - \frac{1}{\alpha}\right) \\ &\times \sum_{v=1}^{n}\sum_{k\in I_v}\left(\frac{\eta_k}{f_v}\right)^2 + O(\|\boldsymbol{\eta}''\|^5)\end{aligned} \tag{139}$$

is the shifted frequency of the m-th cluster. Equations (138) and (139) have similar structures compared to (58) and (59). Note that the higher the cluster frequency, the higher the shift of the cluster frequency.

The amplitudes in the synchronized state are

$$r_j \equiv \sqrt{\frac{\alpha}{K}} + \frac{\beta}{f_m\sqrt{\alpha K}}\eta_j + \frac{1}{2\alpha^{3/2}K^{1/2}f_m^2}\eta_j^2 + O(\|\boldsymbol{\eta}''\|^5) \tag{140}$$

for $j \in I_m$, where $\boldsymbol{\eta}'' = (\beta, \tilde{\eta}_1, \ldots, \tilde{\eta}_N)^T$. The structure of (140) is well known to us from (60). The deviations from the unperturbed limit cycle amplitude remarkably increase with decreasing cluster frequency, i.e. with decreasing f_m. Note our assumption

$$\|\boldsymbol{\eta}''\| \ll \alpha \text{ and } \|\boldsymbol{\eta}''\| \ll \alpha^{3/4}K^{1/4}. \tag{141}$$

4.2. Phase model

4.2.1. One cluster. After the adiabatic elimination of the amplitudes, and after putting $KR^2 \to K + \varepsilon K_{jk}$, we obtain the system

$$\dot{\psi}_j = \omega_j - \frac{f_m}{N}\sum_{v=1}^{n}(K + \varepsilon K_{jk})\sum_{k\in I_v}\sin\left(\frac{\psi_j}{f_m} - \frac{\psi_k}{f_v}\right) \text{ for } j \in I_m. \tag{142}$$

We assume that $0 < \varepsilon \ll 1$, $0 < \varepsilon \ll K$, $\|\hat{\boldsymbol{\eta}}\| \ll K$, and (66) hold. The transformation (124) enables us to make use of the results of Sect. 2.2.1. In the stable synchronized state for $j \in I_m$ there is

$$\begin{aligned}\psi_j(t) &= \Omega_m^* t + f_m\theta' + \frac{1}{K}\eta_j \\ &- \frac{\varepsilon f_m}{NK^2}\sum_{v=1}^{n}\frac{1}{f_v}\sum_{k\in I_v}K_{jk}\eta_k + \frac{f_m}{6NK^3}\sum_{v=1}^{n}\sum_{k\in I_v}\left(\frac{\eta_j}{f_m} - \frac{\eta_k}{f_v}\right)^3 \\ &+ \frac{\varepsilon f_m}{2N^2K^4}\sum_{v,\mu=1}^{n}\frac{1}{f_\mu}\sum_{\substack{k\in I_v\\ l\in I_\mu}}(K_{kl} - K_{jl})\left(\frac{\eta_j}{f_m} - \frac{\eta_k}{f_v}\right)^2\eta_l \\ &+ \frac{\varepsilon f_m}{6NK^4}\sum_{v=1}^{n}\sum_{k\in I_v}K_{jk}\left(\frac{\eta_j}{f_m} - \frac{\eta_k}{f_v}\right)^3 \\ &- \frac{\varepsilon}{6N^2K^4}\sum_{v,\mu=1}^{n}\sum_{\substack{k\in I_v\\ l\in I_\mu}}K_{jk}\left(\frac{\eta_k}{f_v} - \frac{\eta_l}{f_\mu}\right)^3 \\ &+ O(\|\hat{\boldsymbol{\eta}}\|^5 + \varepsilon^2\|\hat{\boldsymbol{\eta}}\|^3),\end{aligned} \tag{143}$$

with the renormalized frequency of the m-th cluster

$$\begin{aligned}\Omega_m^* &= \Omega_m + \frac{f_m\varepsilon}{2N^2K^3}\sum_{v,\mu=1}^{n}\frac{1}{f_v^2 f_\mu}\sum_{\substack{k\in I_v\\ l\in I_\mu}}(K_{lk} - K_{kl})\eta_k^2\eta_l \\ &+ O\left(\|\hat{\boldsymbol{\eta}}\|^5 + \varepsilon^2\|\hat{\boldsymbol{\eta}}\|^3\right),\end{aligned} \tag{144}$$

and $\theta' = 1/N\sum_{k=1}^{N}\psi_k(0)$. Equation (143) corresponds to (76), whereas (144) corresponds to (77). There are n different synchronized clusters with cluster frequencies $\Omega_1^*, \ldots, \Omega_n^*$. The clusters' interaction contributes to the frequency shifts (cf. Eq. (144)) as well as to the constant phase shifts (cf. Eq. (143)).

For symmetrical couplings $K_{jk} = K_{kj}$ (instead of (66)), the transformation (124) brings us back to model I, and the problem is solved, too.

If the cluster frequency is time dependent, i.e. if we put $\Omega_m \to \Omega_m + f_m f(t)$, we just have to add $f_m\int_{t_0}^{t} f(\xi)\, d\xi$ to the right hand side of Eq. (143) (cf. Sect. 2.2.4).

4.2.2. Shift of the cluster frequency. The extended phase model reads (with $KR^2 \to K$)

$$\begin{aligned}\dot{\psi}_j &= \omega_j - \frac{Kf_m}{N}\sum_{v=1}^{n}\sum_{k\in I_v}\left(\sin\left(\frac{\psi_j}{f_m} - \frac{\psi_k}{f_v}\right)\right. \\ &\left. + \beta\cos\left(\frac{\psi_j}{f_m} - \frac{\psi_k}{f_v}\right)\right)\end{aligned} \tag{145}$$

for $j \in I_m$. The transformation (124) enables us to make use of the results of Sect. 2.2.5. The stable synchronized state is

$$\psi_j(t) = \Omega_m(\beta)t + \theta'_m + \frac{\eta_j}{K} + \frac{\beta f_m}{2NK^2}\sum_{v=1}^{n}\sum_{k\in I_v}\left(\frac{\eta_j}{f_m} - \frac{\eta_k}{f_v}\right)^2 + \frac{f_m}{6NK^3}\sum_{v=1}^{n}\sum_{k\in I_v}\left(\frac{\eta_j}{f_m} - \frac{\eta_k}{f_v}\right)^3 + O(\|\hat{\boldsymbol{\eta}}\|^5) \quad \text{for} \quad j \in I_m, \tag{146}$$

where $\theta'_m = (\sum_{k=1}^{N}\psi_k(0))f_m/N$. The shifted cluster frequency is

$$\Omega_m(\beta) = f_m(\Omega - \beta K) + \frac{f_m\beta}{NK}\sum_{v=1}^{n}\sum_{k\in I_v}\left(\frac{\eta_k}{f_v}\right)^2 + O(\|\hat{\boldsymbol{\eta}}\|^5). \tag{147}$$

Equations (146) and (147) correspond to equations (100) and (101). Again, the clusters' mutual interaction gives rise to frequency (cf. Eq. (147)) and phase shifts (146)). According to (147) the frequency shift increases with increasing cluster frequency.

5. Synchronization thresholds

For the time being we define the synchronization threshold K_{crit} as the coupling strength which has to be exceeded in order to cause a synchronized state.

Let us for example consider Eq. (64). If we introduce relative phases in the system (64) with the transformation (11), the transformed system has the potential

$$V(\phi_1, \dots, \phi_N) = -\sum_{k=1}^{N}\eta_k\phi_k - \frac{K}{N}\sum_{v=1}^{N}\sum_{\substack{k=1\\k>v}}^{N}\cos(\phi_v - \phi_k). \tag{148}$$

For $K > K_{crit}$ this potential has a minimum.

In this section we only analyze the phase models. First we investigate model I and II. At the end we turn to model III. In the whole section we let $\eta > 0$.

5.1. Model I and II

5.1.1. Two oscillators. We first consider model I (Eq. (64) with $N = 2$). The two oscillators have the eigenfrequencies $\Omega + \eta/2$ and $\Omega - \eta/2$. A little calculation shows that $K_{crit} = \eta$. This still holds if we have N oscillators with the eigenfrequency $\Omega + \eta/2$ and N oscillators with the eigenfrequency $\Omega - \eta/2$ [23]. Obviously in the phase model II for two oscillators we get $K_{crit} = 0$.

5.1.2. Three oscillators. In this section we analyze three coupled oscillators with the eigenfrequencies $\omega_1 = \Omega$, $\omega_2 = f\Omega + \eta$, $\omega_3 = f\Omega - \eta$, where f is a constant real parameter. If $f \neq 1$ this is the phase model II, and if $f = 1$ this is the phase model I. Introducing relative phases in the usual way, we obtain

$$\dot{\phi}_1 = -\frac{K}{3f}\sin(f\phi_1 - \phi_2) - \frac{K}{3f}\sin(f\phi_1 - \phi_3)$$

$$\dot{\phi}_2 = \eta - \frac{K}{3}\sin(\phi_2 - f\phi_1) - \frac{K}{3f}\sin(f\phi_2 - f\phi_3)$$

$$\dot{\phi}_3 = -\eta - \frac{K}{3}\sin(\phi_3 - f\phi_1) - \frac{K}{3f}\sin(f\phi_3 - f\phi_2). \tag{149}$$

We are looking for stable fixed points. Because of the symmetry of the system for the fixed point we make the hypothesis $\phi_1 = a$, $\phi_2 = fa + c$, $\phi_3 = fa - c$, where a and c are real constants. If we identify 0 and 2π, a causes a SO(2)-symmetry of the fixed point. We eliminate this by means of the transformation $\varphi_1 = f\phi_1 - \phi_2$, $\varphi_2 = f\phi_1 - \phi_3$. This yields

$$\dot{\varphi}_1 = -\eta - \frac{2K}{3}\sin\varphi_1 - \frac{K}{3}\sin\varphi_2 + \frac{K}{3f}\sin(f\varphi_2)\cos(f\varphi_1) - \frac{K}{3f}\cos(f\varphi_2)\sin(f\varphi_1) \tag{150}$$

$$\dot{\varphi}_2 = \eta - \frac{K}{3}\sin\varphi_1 - \frac{2K}{3}\sin\varphi_2 + \frac{K}{3f}\sin(f\varphi_1)\cos(f\varphi_2) - \frac{K}{3f}\cos(f\varphi_1)\sin(f\varphi_2). \tag{151}$$

The fixed point is $(\varphi_1, \varphi_2) = (-c, c)$, with c still to be determined. To this end we put $\dot{\varphi}_1 = \dot{\varphi}_2 = 0$. This gives us

$$\frac{3\eta}{K} = \sin c + \frac{1}{f}\sin(2fc) =: g(c;r). \tag{152}$$

Given f, our task is to find, whether there exists a solution c. But furthermore we want to know whether this solution is stable or not. Therefore we have to make a linear stability analysis of the fixed point. To this end we put $\varphi_1 = -c + \xi_1$ and $\varphi_2 = c + \xi_2$. With $\mathbf{x} = (\xi_1, \xi_2)^T$ the linearised equation is of the form $\dot{\mathbf{x}} = \mathbf{A}(c;f)\mathbf{x}$. With a little calculation we get the eigenvalues of $\mathbf{A}(c;f)$: They are

$$\lambda_{1,2}(c;\mathrm{f}) = -\frac{K}{3}(2\cos c + \cos(2fc)) \pm \frac{K}{3}|\cos(2fc) - \cos c| \tag{153}$$

for $0 < f < \infty$, and

$$\lambda_{1,2}(c;\infty) = -\frac{2K}{3}\cos c \pm \frac{K}{3}|\cos c| \tag{154}$$

in the limit $f \to \infty$. For some values of f we are able to discuss this problem analytically. For this purpose we use suitable trigonometric formulas.

1. $f \to \infty$: From Eq. (154) it follows that $K_{\mathrm{crit}}(\infty) = 3\eta$. If $K > K_{\mathrm{crit}}(\infty)$ we have $c \in [0, \pi/2[$, where $[$ denotes the open interval to the right.

2. $f = 3/2$: In this case we obtain $g(c;3/2) = \sin c + 2/3\sin(3c)$ and

$$\lambda_1\left(c;\frac{3}{2}\right) = \begin{cases} -K\cos c: & c \in \left[0,\frac{\pi}{2}\right]\cup\left[\frac{3\pi}{2},2\pi\right] \\ -\frac{K}{3}(\cos c + 2\cos(3c)): & c \in \left[\frac{\pi}{2},\frac{3\pi}{2}\right] \end{cases} \tag{155}$$

$$\lambda_2\left(c;\frac{3}{2}\right) = \begin{cases} -\frac{K}{3}(\cos c + 2\cos(3c)): & c \in \left[0,\frac{\pi}{2}\right]\cup\left[\frac{3\pi}{2},2\pi\right] \\ -K\cos c: & c \in \left[\frac{\pi}{2},\frac{3\pi}{2}\right]. \end{cases} \tag{156}$$

Plotting these functions, it is easy to see that $K_{\mathrm{crit}}(3/2) = 3\eta/\gamma_{3/2} \approx 2.45\eta$, with $\gamma_{3/2} = \max\{g(c;3/2)\ , c \text{ real}\}$. In the synchronized state there is $c \in [0, \zeta_{3/2}[$, where $\cos\zeta_{3/2} = \sqrt{5/8}$, $\zeta_{3/2} \approx 0.659$.

3. $f = 1$: We obtain $g(c;1) = \sin c + \sin(2c)$ and

$$\lambda_1(c;1) = \begin{cases} -K\cos c: & c \in \left[\frac{2\pi}{3},\frac{4\pi}{3}\right] \\ -\frac{K}{3}(\cos c + 2\cos(2c)): & c \in \left[0,\frac{2\pi}{3}\right]\cup\left[\frac{4\pi}{3},2\pi\right] \end{cases} \tag{157}$$

$$\lambda_2(c;1) = \begin{cases} -\frac{K}{3}(\cos c + 2\cos(2c)): & c \in \left[\frac{2\pi}{3},\frac{4\pi}{3}\right] \\ -K\cos c: & c \in \left[0,\frac{2\pi}{3}\right]\cup\left[\frac{4\pi}{3},2\pi\right]. \end{cases} \tag{158}$$

By plotting these functions we get $K_{\mathrm{crit}} = 3\eta/\gamma_1 \approx 1.70\eta$, with $\gamma_1 = \max\{g(c;1), c \text{ real}\}$. In the synchronized state $c \in [0, \zeta_1[$holds, where $\cos\zeta_1 = (\sqrt{33} - 1)/8$ and $\zeta_1 \approx 0.936$.

4. $f = 1/2$: In this case $g(c;1/2) = 3\sin c$ and $\lambda_{1,2}(c;1/2) = -K\cos c$. Therefore $K_{\mathrm{crit}} = \eta$ and $c \in [0, \pi/2[$.

5. $f = 1/4$: Now $g(c;1/4) = \sin c + 4\sin(c/2)$ and

$$\lambda_1\left(c;\frac{1}{4}\right) = \begin{cases} -K\cos c: & c \in [0,c_1]\cup[c_2,4\pi] \\ -\frac{K}{3}\left(\cos c + 2\cos\left(\frac{c}{2}\right)\right): & c \in [c_1,c_2] \end{cases} \tag{159}$$

$$\lambda_2\left(c;\frac{1}{4}\right) = \begin{cases} -\frac{K}{3}\left(\cos c + 2\cos\left(\frac{c}{2}\right)\right): & c \in [0,c_1]\cup[c_2,4\pi] \\ -K\cos c: & c \in [c_1,c_2], \end{cases} \tag{160}$$

with $\cos(c_1/2) = -1/2$, $c_1 \approx 4.189$ and $c_2 = 4\pi - c_1 \approx 8.378$. By plotting these functions we get $K_{crit}(1/4) = 3\eta/g(\pi/2;1/4) \approx 0.78\eta$, and $c \in [0, \pi/2[$.

6. $f = 1/6$: The system shows an interesting behaviour, because it does not have a single synchronization threshold. We get $g(c;1/6) = \sin c + 6\sin(c/3)$ and

$$\lambda_1\left(c;\frac{1}{6}\right) = \begin{cases} -K\cos c: & c \in \left[0,\frac{3\pi}{2}\right]\cup\left[\frac{9\pi}{2},6\pi\right] \\ -\frac{K}{3}\left(\cos c + 2\cos\left(\frac{c}{3}\right)\right): & c \in \left[\frac{3\pi}{2},\frac{9\pi}{2}\right] \end{cases} \tag{161}$$

$$\lambda_2\left(c;\frac{1}{6}\right) = \begin{cases} -\frac{K}{3}\left(\cos c + 2\cos\left(\frac{c}{3}\right)\right): & c \in \left[0,\frac{3\pi}{2}\right]\cup\left[\frac{9\pi}{2},6\pi\right] \\ -K\cos c: & c \in \left[\frac{3\pi}{2},\frac{9\pi}{2}\right]. \end{cases} \tag{162}$$

As is shown in Fig. 6, starting with $K = 0$ and increasing K, the synchronization becomes stable for the first time when $K_I < K < K_{II}$, with $K_I = 3/g(2\pi;1/6) \approx 0.58\eta$ and $K_{II} = 3/g(3\pi/2;1/6) = 0.6\eta$ (cf. Eq. (152)). In this case we have $c \in]3\pi/2, 2\pi[$. Increasing K makes the synchronization unstable again, but finally for $K > K_{III} = 3/g(\pi/2;1/6) = 0.75\eta$ the synchronization remains stable. Now $c \in [0, \pi/2]$.

The smaller f, the smaller is the synchronization threshold. This is not surprising, because f stands in the denominator of the coupling strength (cf. system (149)).

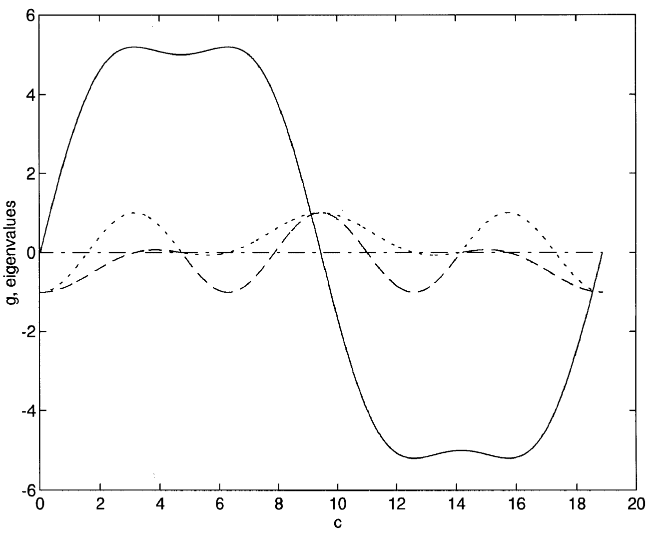

Fig. 6. $g(c;(1/6)$ (solid line), $\lambda_1(c;1/6)$ (dotted line), $\lambda_2(c;1/6)$ (dashed line), reference line (dashed and dotted line)

5.2. Model III

5.2.1. Two oscillators. Let us consider Eq. (142) for $\varepsilon = 0$, $N = 2$, $\omega_1 = f\Omega + \eta/2$ and $\omega_2 = f\Omega - \eta/2$. With the transformation (124) and the results of section 5.1 we obtain $K_{\mathrm{crit}} = \eta/f$.

5.2.2. Three oscillators. In Eq. (142) we put $\varepsilon = 0$, $N = 3$. The eigenfrequencies of the three oscillators are $\omega_1 = f_1\Omega$, $\omega_2 = f_2\Omega + \eta$, $\omega_3 = f_2\Omega - \eta$. With transformation (124) and the results of section 5.1 we get $K_{\mathrm{crit}} \approx 1.70\eta/f_2$.

The synchronization threshold decreases with increasing f_2, because f_2 stands in the numerator of the coupling strength.

5.3. Stepwise synchronization

We illustrate this sequence of synchronization and desynchronization by an example. Suppose there are two clusters. The first cluster consists of two oscillators with the eigenfrequencies

$$\omega_1 = \Omega_1 + \frac{\eta}{2} \quad \text{and} \quad \omega_2 = \Omega_1 - \frac{\eta}{2}. \tag{163}$$

The synchronization threshold of the first cluster is $K^1_{\mathrm{crit}} = \eta$. The eigenfrequencies of the second cluster are

$$\omega_3 = \Omega_2,\ \omega_4 = \Omega_2 + 0.7\eta \quad \text{and} \quad \omega_5 = \Omega_2 - 0.7\eta. \tag{164}$$

Their synchronization threshold is $K^2_{\mathrm{crit}} \approx 1.19\eta$. We suppose $|\Omega_1 - \Omega_2| \gg \eta$.

If all five oscillators are coupled according to Eq. (46) (phase model I), $K^1_{\mathrm{crit,eff}} = 5/2K^1_{\mathrm{crit}} = 2.5\eta$ and $K^2_{\mathrm{crit,eff}} = 5/3K^2_{\mathrm{crit}} \approx 1.98\eta$ are the effective synchronization thresholds of the two clusters respectively. Note that $K^1_{\mathrm{crit}} < K^2_{\mathrm{crit}}$, but $K^1_{\mathrm{crit,eff}} > K^2_{\mathrm{crit,eff}}$. $0 \leqslant K < 1.98\eta$: All of the oscillators are desynchronized. $1.98\eta < K < 2.5\eta$: The first two oscillators are synchronized. $2.5\eta < K \ll |\Omega_1 - \Omega_2|$: Both clusters are synchronized separately.

Further increasing of the coupling will destroy the synchronization. And finally, when K exceeds the synchronization threshold of all five oscillators, all of them will join into a single synchronized cluster.

We have focused on synchronized states. Nevertheless, concerning the dynamical patterns, the desynchronized states are very interesting, too. For example in the desynchronized states near to the synchronization threshold intermittency phenomena occur (cf. [9]).

Let us turn towards the phase model I (cf. Eq. (64)). We suppose that there are several clusters. The eigenfrequencies within a single cluster are assumed to be close compared to the distances of the cluster frequencies. We start with $K = 0$ and increase the coupling strength. How do the clusters behave?

If K exceeds the effective synchronization threshold of a single cluster, this cluster will synchronize and behave like a single giant oscillator. The synchronization is weakly perturbed by the influence of the other clusters. If we further increase the coupling strength, the single synchronized clusters will perturb each other more and more, and finally the synchronization of the single clusters will be destroyed (cf. Sect. 2.2.2).

Finally the coupling strength will exceed the synchronization threshold of groups of clusters. This will cause the corresponding clusters to join into single synchronized clusters, respectively.

These different synchronized groups of clusters will perturb each other more and more when the coupling strength is further increased. Therefore the synchronization of these clusters will finally vanish.

This stepwise synchronization and desynchronization of increasing groups of clusters will find an end if the coupling exceeds the synchronization threshold of all oscillators. This will force all of them to join into a single giant cluster.

For this sketch we have assumed that averaging may be carried out.

6. Discussion

Several authors have already analyzed synchronization processes in oscillatory networks (cf. e.g. [2, 6, 16, 24]). Nevertheless still there are many open questions.

In order to study the impact of an external field on synchronization processes Christiansen et al. investigated a large pool of coupled oscillators in the presence of a modulated external field [2]. They were able to show that in their model phase locking of the oscillator community to the harmonics of the frequency of the external field is associated with a complete loss of coherence between the oscillators. This was a result of random distributed pinning phases which were introduced as a disordering element. In contrast to Christiansen et al. in the present paper we investigate self-synchronization. Thus, in our model there is no external field which influences the synchronization process.

There are different types of coupling mechanisms between oscillators. For instance, oscillators may synchronize due to pulselike interactions. Tsodyks et al. investigated globally coupled oscillators with pulse interactions [24]. They showed that in their system the completely phase-locked state is unstable to weak disorder. As a result of a small degree of inhomogeneity they observed two subpopulations of oscillators exhibiting different dynamical behaviour: one that is phase locked and another one that consists of aperiodic oscillators.

In contrast to Tsodyks et al. we investigated a network of oscillators which are continuously interacting. For the first time a rigorous analytical investigation of the stability of synchronized clusters of continuously interacting oscillators is presented in this paper. Our approach essentially relies on the center manifold theorem [15, 18]. The latter may be considered as a special case of the slaving principle of synergetics [11, 10].

In the physical world synchronized states of an oscillatory network can only be observed if they are stable with respect to perturbing fluctuations. Therefore it is not sufficient to prove the existence of synchronized solutions. Moreover it is indispensable to check the stability of the synchronized states. Up to now the drawback of many investigations of synchronization phenomena in oscillatory networks is the lack of a rigorous stability proof. For instance Ermentrout [6] and Kuramoto [16] determined synchronization thresholds. Note that the notion of synchronization thresholds in their investigation implies that when the coupling strength exceeds this threshold a synchronized solution *exists*. In this case all oscillators (in [6]) or a small group of oscillators (in [16]) are synchronized. Without a stability proof it remains unclear whether the synchronized state is physically relevant (i.e. stable) or not.

The center manifold theorem enables us to prove the stability of the synchronized states under consideration. It turns out that the latter are local attractors, and getting synchronized means that the oscillatory network approaches this local attractor in an overdamped fashion. Moreover we are able to calculate the phases of all oscillators explicitely. Therefore we can detect frequency shifts which are caused by different coupling mechanisms.

Parts of our results are in contradiction to Ermentrout's analysis [6]. In [6] Ermentrout analyzed the patial differential equation

$$\frac{\partial \psi(x,t)}{\partial t} = \omega(x) - K \int_0^1 \sin(\psi(x,t) - \psi(\xi,t))\, d\xi. \tag{165}$$

Except for the range of the values of x this equation is exactly the same as our Eq. (83). Rescaling the range of the x-values merely implies rescaling the coupling strength K. Ermentrout has analyzed the *existence* of synchronized states of the oscillatory field, which are of the form

$$\psi(x,t) = \Omega t + \phi(x), \tag{166}$$

where Ω is the cluster frequency. Note that ϕ is not time dependent. This is the very reason why Ermentrout is not able to investigate the stability of the synchronized states. He remarks that numerical simulations indicate that all stable solutions of Eq. (165) are of the form

$$\sin \phi(x) = c\,\eta(x), \tag{167}$$

where c is a constant (cf. Remark after Prop. 1 in ([6])). The hypothesis (167) is quite enticing for technical reasons. It simplifies the equation for the stationary synchronized state remarkably. Ermentrout's analysis is based on this hypothesis (167). Nevertheless in section 2.2.3 we have proven rigorously that in a reasonable parameter range (cf. condition (85)) the stable synchronized state $\phi(x)$ does not only depend on $\eta(x)$. According to Eq. (94) $\phi(x)$ also depends on $\eta(\xi)$, where $\xi \neq x$. Thus, hypothesis (167) is wrong for the "well-synchronized" parameter range in which we analyzed Eq. (165) rigorously.

In this paper we investigated different coupling mechanisms. The latter do not only cause frequency shifts. Furthermore they determine whether synchronized clusters of different frequencies mutually perturb each other (model I) or tolerate each other (models II and III).

Many authors have prefered to restrict their analysis to the case of weak coupling strength (cf. [3, 6, 7, 16, 17, 21]). In this case it is allowed to consider the amplitudes in a first approximation as beeing constant [16]. This simplifies the analysis significantly. Nonetheless it is of great interest to know whether an increase of the coupling strength changes the behaviour of the synchronized clusters qualitatively. Phase transitions might for instance occur, revealing totally different dynamics. Thus, in this paper we investigated the synchronization behaviour of our models in the case of strong coupling strength, too. Again the center manifold theorem turned out to be a powerful tool providing us with rigorous stability proofs of the synchronized states. Our analysis of the total systems basically revealed the same results as for the respective phase models. This confirms the quality of the approximation by phase models for the networks which we analyzed in this paper.

In the case of three coupled oscilltors we were able to determine the synchronization threshold, i.e. the critical coupling strength related to stable synchronization. Applying averaging arguments enabled us to sketch how distinct synchronized clusters merge in one giant synchronized cluster.

7. Appendix

We prove that the partial differential equation (86) has a center manifold. Our proof is quite analogous to the one of Kelley and Pliss [15, 18]. The only difference is, that we need a suitable function space (cf. [22]). With

$$\mathbf{u}(x,t) = \begin{pmatrix} \eta(x) \\ \varphi(x,t) \end{pmatrix} \quad \text{and} \quad \mathbf{M}(\mathbf{u}) = \begin{pmatrix} 0 \\ m(\mathbf{u}) \end{pmatrix} \tag{168}$$

the extended transformed system may be written in the form

$$\frac{\partial \mathbf{u}}{\partial t} = \begin{pmatrix} 0 & 0 \\ 0 & -K \end{pmatrix} \mathbf{u} + \mathbf{M}(\mathbf{u}). \tag{169}$$

Note that $\|\eta\|_{H^1([0,2\pi])} \ll K$ holds.

1. function space: Let $I = [0, 2\pi]$. We choose $\mathbf{u} \in X_\delta$, where $0 < 2\delta < K$ and $X_\delta = \{\mathbf{u}: \mathrm{I} \times \mathbf{R} \to \mathbf{R}^2; \hat{\mathbf{u}}(k,t) \in C(\mathbf{R}, \mathbf{C}^2)\ (k \in \mathbf{Z}), \|\mathbf{u}\|_\delta < \infty\}$, with $\|\mathbf{u}\|_\delta = \sup\{\exp(-\delta|t|)\|\mathbf{u}(.,t)\|_{H^1(I)}\}$.

2. localization of the nonlinearity: We localize the nonlinearity by putting $\mathbf{M}(\mathbf{u}) \to \mathbf{M}_\varepsilon(\mathbf{u}) = \mathbf{M}(\mathbf{u})\, \chi(\|\mathbf{u}\|_{H^1(I)}/\varepsilon^2)$, with $0 < \varepsilon \ll 1$, χ smooth, and $\chi(\xi) = 1$ for $0 \leq \xi \leq 1$, $= 0$ for $2 \leq \xi$. $\mathbf{M}_\varepsilon$ is Lipschitz continuous with the Lipschitz constant $\mathrm{Lip}(\mathbf{M}_\varepsilon) = O(\varepsilon^2)$, because the vector field is odd.

3. linear part: For $\mathbf{u}(0) = \mathbf{u}_0 \in X_\delta$ and

$$\mathbf{g}(x,t) = \begin{pmatrix} 0 \\ g(x,t) \end{pmatrix} \in X_\delta \tag{170}$$

the equation

$$\frac{\partial \mathbf{u}}{\partial t} = \begin{pmatrix} 0 & 0 \\ 0 & -K \end{pmatrix} \mathbf{u} + \mathbf{g}(x,t) \tag{171}$$

has a unique solution in X_δ. Proof by means of a Fourier transformation: existence and uniqueness for the single Fourier coefficient follow from the theory of ordinary differential equations. The rest is shown by $\|\mathbf{u}\|_\delta \leq C\|\mathbf{g}\|_\delta$.

Let us denote the unique solution of (171) by $\mathbf{u} = \mathbf{f}(\mathbf{u}_0, \mathbf{g})$.

4. contraction: We consider the Nemitskii-Operator $\mathbf{N}$: $X_\delta \to X_\delta$, $\mathbf{u} \mapsto \mathbf{M}_\varepsilon(\mathbf{u})$. $\mathbf{N}$ is Lipschitz continuous, because $\mathbf{M}_\varepsilon$ is Lipschitz continuous. Solving (169) with the

localized nonlinearity is equivalent to solving $\mathbf{u} = \mathbf{f}(\mathbf{u}_0, \mathbf{N}(\mathbf{u})) =: \mathbf{S}(\mathbf{u}_0, \mathbf{u})$. $\mathbf{S}(\mathbf{u}_0, .)$ is a contraction in X_δ for all $\mathbf{u}_0 \in X_\delta$, because

$$\|\mathbf{S}(\mathbf{u}_0, \mathbf{u}) - \mathbf{S}(\mathbf{u}_0, \tilde{\mathbf{u}})\|\delta \overset{3.}{\leq} C\|\mathbf{N}(\mathbf{u}) - \mathbf{N}(\tilde{\mathbf{u}})\|_\delta$$

$$\leq C' \operatorname{Lip}(\mathbf{M}_\varepsilon)\|\mathbf{u} - \tilde{\mathbf{u}}\|_\delta < 1 \quad (172)$$

for ε small enough. According to Banach's fixed point theorem for all $\mathbf{u}_0 \in X_\delta$ there is a unique fixed point $\mathbf{u} = \mathbf{u}(\mathbf{u}_0)$.

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