Applied Mathematics & Optimization (2024) 89:55
https://doi.org/10.1007/s00245-024-10127-4

On the Existence of Global Weak Solutions to the 3D Electrically Conductive Rosensweig System and Their Convergence Towards Quasi-Equilibrium

A. Ndongmo Ngana[1] · **P. A. Razafimandimby**[2]

Accepted: 11 March 2024 / Published online: 8 April 2024

Abstract

In this article, we study an electrically conductive Rosensweig model for ferrofluids, whose Bloch–Torrey regularization was studied by Hamdache and Hamroun (Appl Math Optim 81(2):479–509, 2020). We mainly prove the global existence of weak solutions to the non-regularized model under a certain smallness condition on the electric conductivity. Hence, our result not only solves a problem that was left open by Hamdache and Hamroun, but it can also serve as a confirmation that ferrofluids are naturally poor conductors of electric current. The proof, which is interesting in itself, is quite involved and relies on the Helmohltz–Leray decomposition of the magnetic fields and the use of renormalized solutions for the magnetization. We also give a rigorous and detailed description of the convergence of the global weak solutions towards the quasi-equilibrium in the relaxation time limit regime $\tau \to 0$.

Keywords Ferrofluids · Navier–Stokes equations · Quasi-static Maxwell equations · Internal rotations · Relaxation time

1 Introduction

Ferrofluids were developed by NASA in the 1960s and consist of ferromagnetic particles suspended in a liquid carrier (water, oil, etc.). The particles suspended in a ferrofluid conform to Brownian motion, which means particles' movement is generally random. The ferrofluids also become strongly magnetized in the presence of a external magnetic field and can be controlled to flow via the positioning and strength of the applied field. Thus, the ferrofluid can be positioned very exactly. This property

✉ A. Ndongmo Ngana
ndongmoaristide@yahoo.fr

1 Department of Mathematics and Statistical Sciences, North-West University, Potchefstroom, South Africa

2 School of Mathematical Sciences, Dublin City University, Collins Avenue, Dublin D09W6Y4, Ireland

gives an impetus to numerous ferrofluid applications in industry, technology, engineering and medicine: pumping fuel in spacecraft without mechanical action, liquid seal around the computer hard-drives rotating shaft, improving the heat transfer in electrical transformers, carrying medications to exact locations within the body. We refer to [14, 17, 19, 20] for an overview of the physics and applications of ferrofluids.

There are two commonly accepted mathematical models for ferrofluids: the Rosensweig and the Shliomis models. The Rosensweig model describes the dynamics of ferrofluids with internal rotations and is seen as a generalization of the simpler Shliomis model. In this paper, we will mainly consider a Rosensweig model which governs the motion of electrically conductive ferrofluids (see [19]). The ferrofluid system fills a connected and bounded open subset $\mathcal{O} \subset \mathbb{R}^3$, with a smooth boundary $\partial\mathcal{O}$ (i.e., of class $\mathcal{C}^\infty$). The fluid carrier is assumed to be an incompressible Newtonian fluid. The model is described by the following system of partial differential equations:

$$\begin{aligned}
&\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p - \nu\Delta\boldsymbol{u} - \mu_0(\boldsymbol{M}\cdot\nabla)\boldsymbol{H} \\
&\quad = \mu_0(\nabla\times\boldsymbol{H})\times\boldsymbol{H} - \alpha\nabla\times(\nabla\times\boldsymbol{u} - 2\mathbf{w}), &(1.1a)\\
&\partial_t \mathbf{w} + (\boldsymbol{u}\cdot\nabla)\mathbf{w} - (\lambda_1+\lambda_2)\nabla\operatorname{div}\mathbf{w} - \lambda_1\Delta\mathbf{w} \\
&\quad = 2\alpha(\nabla\times\boldsymbol{u} - 2\mathbf{w}) + \mu_0\boldsymbol{M}\times\boldsymbol{H}, &(1.1b)\\
&\partial_t \boldsymbol{M} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{M} = \mathbf{w}\times\boldsymbol{M} - \frac{1}{\tau}(\boldsymbol{M} - \chi_0\boldsymbol{H}), &(1.1c)\\
&\partial_t \boldsymbol{B} + \frac{1}{\sigma}\nabla\times(\nabla\times\boldsymbol{H}) = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}), &(1.1d)\\
&\boldsymbol{B} = \mu_0(\boldsymbol{M}+\boldsymbol{H}), \quad \operatorname{div}\boldsymbol{B} = 0, &(1.1e)\\
&\operatorname{div}\boldsymbol{u} = 0. &(1.1f)
\end{aligned}$$

We call this system the electrically conductive Rosensweig equations (ECREs for short notation). The unknown functions are the velocity $\boldsymbol{u}$ of the fluid, its pressure p, the fluid internal rotation $\mathbf{w}$, the magnetization $\boldsymbol{M}$, and the magnetic field $\boldsymbol{H}$. Here, the viscosity coefficients ν, λ_1 and λ_2, the relaxation time τ, the magnetic susceptibility coefficient χ_0 and the electric conductivity σ are all positive and assumed to be constant. The constant μ_0 is a fundamental constant called magnetic permeability of the vacuum. The forcing term $\mu_0(\boldsymbol{M}\cdot\nabla)\boldsymbol{H}$ in the linear momentum equation is the so-called Kelvin force. The term $\mu_0\boldsymbol{M}$ represents the vector moment per unit volume. The equation $\operatorname{div}\boldsymbol{B} = 0$ (cf. (1.1e)) is the Maxwell equation for the magnetic induction $\boldsymbol{B} = \mu_0(\boldsymbol{M}+\boldsymbol{H})$ in $\mathcal{O}$ and $\boldsymbol{B} = \mu_0\boldsymbol{H}$ outside $\mathcal{O}$, where the magnetization $\boldsymbol{M}$ vanishes.

We endow problem (1.1a)–(1.1f) with the following boundary conditions

$$\begin{aligned}
\boldsymbol{u} = \mathbf{w} = 0, &\quad \text{on } (0,T)\times\partial\mathcal{O},\\
\operatorname{div}\boldsymbol{M} = 0, \quad \boldsymbol{H}\times\boldsymbol{n} = \boldsymbol{M}\times\boldsymbol{n} = 0, &\quad \text{on } (0,T)\times\partial\mathcal{O},
\end{aligned} \tag{1.2}$$

where $\partial\mathcal{O}$ is the boundary of $\mathcal{O}$ and $\boldsymbol{n}$ is its outward normal.

The initial condition is given by

$$\begin{aligned}&(\boldsymbol{u}, \mathrm{w}, \boldsymbol{M}, \boldsymbol{H})(t=0) = (\boldsymbol{u}_0, \mathrm{w}_0, \boldsymbol{M}_0, \boldsymbol{H}_0) \quad \text{in} \quad \mathcal{O},\\ &\operatorname{div} \boldsymbol{u}_0 = \operatorname{div}(\boldsymbol{M}_0 + \boldsymbol{H}_0) = 0 \quad \text{in} \quad \mathcal{O},\end{aligned} \tag{1.3}$$

where $(\boldsymbol{u}_0, \mathrm{w}_0, \boldsymbol{M}_0, \boldsymbol{H}_0)$ is a given initial data.

Compared to the classical models introduced in [17], the present model, which was introduced in [13, 19], presents three novelties. Firstly, the Navier–Stokes equations contain an additional volume force, which is the Lorentz force $\mu_0(\nabla \times \boldsymbol{H}) \times \boldsymbol{H}$. Secondly, the magnetic field $\boldsymbol{H}$ satisfies the quasi-static Maxwell equations (1.1d) instead of the magnetostatic ones. Thirdly, the effective magnetizing field $\boldsymbol{H}$ is considered in the case where there are both currents and magnetic charges.

Despite the numerous ferrofluids applications, the mathematical analysis of the ferrofluids models is still quite recent. The mathematical analysis of the ferrofluids started with the investigation of weak solutions of the Bloch–Torrey regularization of the either electrically non-conductive Shliomis or Rosensweig models (see, e.g., [1, 2, 6]). Later on, results on local existence of strong solutions to the non-regularized models were established in [3, 4]. It is only recently that the global existence of weak solutions of the non-regularized and electrically non-conductive model was established in [17]. Since then, the electrically non-conductive ferrofluids models has been the subject of intensive mathematical analysis which has generated several important results (see, e.g., [7, 21, 22, 28] and references therein). As far as the electrically conductive ferrofluids models are concerned, the analysis is very recent. The existence of global weak solutions and local strong solutions of the conductive ferrofluids were only proved in the recent papers [5, 13].

Since the paper [13] deals only with the global weak solution of the Bloch–Torrey regularization of an electrically ferrofluids model, there remain many unsolved mathematical questions for the non-regularized electrically conductive ferrofluids models. We will address two examples of these open problems in this paper. In fact, we will mainly show that under a smallness constraint on the electric conductivity σ, the ECREs (1.1) has at least one global weak solution (see Theorem 2.4). This constraint on σ confirms the fact that ferrofluids are naturally very poor conductor of electric current or dielectric.

In the presence of magnetic field, the ferrofluid relaxation time, which is the average time needed by the ferrofluid to recover an equilibrium state once perturbed, is of order $\tau \simeq 10^{-9}$. This motivates us to give a rigorous and detailed description of the behavior of the global weak solutions in the relaxation time limit regime $\tau \to 0$ (cf. Theorem 4.2). The proof of these results are difficult even in the case of non-conductive ferrofluids. The main difficulties being the non-parabolicity of the magnetization equation (1.1c) and the irregularity of the Kelvin force $\mu_0(\boldsymbol{M} \cdot \nabla)\boldsymbol{H}$ on one hand, and the nonlinear couplings among the velocity, pressure, magnetic field, magnetization field on the other hand. In order to overcome these difficulties, we use the approximation of the problem by the Bloch–Torrey regularization problem studied in [13] and employ the notion of renormalized weak solution, as was done in [17]. The latter approach is motivated by the fact that (1.1c) is a transport equation with linear perturbations. Note that in [17], the magnetic field $\boldsymbol{H}$ solves the magnetostatic equations

$$\begin{cases} \operatorname{div} \boldsymbol{H} &= -\operatorname{div} \boldsymbol{M}, \\ \operatorname{curl} \boldsymbol{H} &= 0. \end{cases} \tag{1.4}$$

Thus, it can be expressed as the gradient of a potential, which is essential to deal with the irregularity of the Kelvin force and to obtain strong convergence in the passage to the limit. In contrast to [17], our magnetic field $\boldsymbol{H}$ is not the gradient of a potential because it solves the quasi-static Maxwell equations (1.1d). In order to obtain the right convergence for the passage to the limit, we decompose $\boldsymbol{H}$ as the sum of a curl-free and a divergence-free fields and establish several non-trivial estimates for these fields and their time derivatives. This is an interesting technique in itself and may help in further analysis of the electrically conductive ferrofluids models.

We close this introduction with a presentation of the layout of the present paper. In Sect. 2, we fix the frequently used notation in the manuscript and the mathematical tools used in the analysis. We also introduce the notion of weak solutions we shall work with in this work and state in Theorem 2.4 our existence result. The existence proof is inspired by the Diperna–Lions theory of renormalized solutions [8, 10, 15], and it is part of Sect. 3. Section 4 is devoted to the zero limit of the relaxation time to Problem (1.1a)–(1.3). We report in Sect. 5 a general conclusion drawn from the presented research and outline future directions.

2 Preliminaries

2.1 Notation, Functions Spaces, and Auxiliary Results

Throughout the paper, we will use the notation $Q_t := (0,t) \times \mathcal{O}$ for every $t \in (0,T]$ and set $\Sigma := (0,T) \times \partial\mathcal{O}$. Next we introduce some notations and background following the mathematical theory of hydrodynamic equations such as Navier–Stokes equations or Rosensweig equations. For any $q \in [1,\infty)$ and $s \in \mathbb{R}$, we denote by $L^q(\mathcal{O})$ and $W^{s,q}(\mathcal{O})$ the usual Lebesgue and Sobolev spaces of scalar functions, respectively. When $q = 2$, we write $W^{s,2}(\mathcal{O}) = H^s(\mathcal{O})$. We denote by $H_0^1(\mathcal{O})$ the closure of $\mathcal{C}_0^\infty(\mathcal{O})$ in $H^1(\mathcal{O})$ and by $H^{-1}(\mathcal{O})$ its dual space. We use the notations $\mathbb{L}^q(\mathcal{O})$, $\mathbb{W}^{s,q}(\mathcal{O})$, $\mathbb{H}^s(\mathcal{O})$, to denote the spaces $L^q(\mathcal{O})^3$, $[W^{s,q}(\mathcal{O})]^3$, $[H^s(\mathcal{O})]^3$, respectively. We use the same notations $\|\cdot\|$ and $(\cdot,\cdot)$ to denote the norms and the scalar products of the Hilbert spaces $L^2(\mathcal{O})$ and $\mathbb{L}^2(\mathcal{O})$. We also introduce the space

$$E_2(\mathcal{O}) := \{\boldsymbol{v} : \boldsymbol{v} = \nabla h,\ h \in L^2_{\mathrm{loc}}(\mathcal{O}),\ \nabla h \in \mathbb{L}^2(\mathcal{O})\},$$

equipped with the norm

$$\|\boldsymbol{v}\|_{E_2(\mathcal{O})} = \|\nabla h\|_{\mathbb{L}^2(\mathcal{O})} \quad \text{for } \boldsymbol{v} = \nabla h \in E_2(\mathcal{O}).$$

We also consider the space

$$E_3(\mathcal{O}) := \{\boldsymbol{v} : \boldsymbol{v} = \nabla h \in E_2(\mathcal{O}),\ \Delta h \in L^2(\mathcal{O})\}$$

equipped with the following norm

$$\|\boldsymbol{v}\|_{E_3(\mathcal{O})} := (\|\nabla h\|^2 + \|\Delta h\|^2)^{\frac{1}{2}} \quad \text{for} \quad \boldsymbol{v} = \nabla h \in E_3(\mathcal{O}).$$

Let X be a real Banach or Hilbert space with norm denoted by $\|\cdot\|_X$. The symbol $\langle\cdot,\cdot\rangle_{X',X}$ (or simply $\langle\cdot,\cdot\rangle$ if there is no confusion) will stand for the duality product between X and its dual X'. We let $\mathcal{C}_w(0,T;X)$ be the space of functions that are weakly continuous in time, i.e., if $u \in \mathcal{C}_w(0,T;X)$, then for any $s \to t$,

$$\langle u^*, u(s)t\rangle_{X',X} \to \langle u^*, u(t)\rangle_{X',X} \quad \forall u^* \in X'.$$

We will also use the symbol $\mathcal{C}_c^1([0,T);X)$ for the space of functions $\phi = \zeta_1\zeta_2$ with $\zeta_1 \in \mathcal{C}([0,T]) \cap \mathcal{C}^1(0,T)$, $\zeta_1(T) = 0$, and $\zeta_2 \in X$.

Notations for the Velocity Field $\boldsymbol{u}$

We denote by $\mathcal{V}$ the space of divergence free vector fields in $\mathcal{C}_0^\infty(\mathcal{O})$. Let

$$\begin{aligned} E(\mathcal{O}) &= \{\boldsymbol{v} \in \mathbb{L}^2(\mathcal{O}) : \operatorname{div} \boldsymbol{v} \in L^2(\mathcal{O})\},\\ H &= \text{the closure of } \ \mathcal{V} \ \text{ in } \ \mathbb{L}^2(\mathcal{O}),\\ V &= \text{the closure of } \ \mathcal{V} \ \text{ in } \ [H_0^1(\mathcal{O})]^3. \end{aligned}$$

The space $E(\mathcal{O})$ is a Hilbert space with a scalar product

$$(\boldsymbol{u},\boldsymbol{v})_{E(\mathcal{O})} = (\boldsymbol{u},\boldsymbol{v}) + (\operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{v}).$$

We endow the set H with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$ induced by $\mathbb{L}^2(\mathcal{O})$. The space H can also be characterized in the following way (see [27, Theorem I.1.4] or [26, Theorem 1.4])

$$H = \{\boldsymbol{u} \in E(\mathcal{O}) : \quad \operatorname{div}\boldsymbol{u} = 0 \text{ in } \mathcal{O}, \quad \boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\mathcal{O}} = 0\},$$

and also by (see [23, Theorem 1.4])

$$H = \{\boldsymbol{u} \in \mathbb{L}^2(\mathcal{O}) : \langle \boldsymbol{u}, \nabla h\rangle = 0 \quad \forall\ \nabla h \in (E_2(\mathcal{O}))'\}.$$

The space V has the following characterization (see [26, Theorem 1.6])

$$V = \{\boldsymbol{u} \in \mathbb{H}^1(\mathcal{O}) : \ \operatorname{div}\boldsymbol{u} = 0 \ \text{ in } \ \mathcal{O}, \quad \boldsymbol{u}|_{\partial\mathcal{O}} = 0\}.$$

The space V is a separable Hilbert space when endowed with the inner product and norm

$$(\boldsymbol{u}, \boldsymbol{v})_V := (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) = \int_{\mathcal{O}} \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \, \mathrm{d}x \quad \text{and} \quad \|\boldsymbol{u}\|_V = \|\nabla \boldsymbol{u}\|, \quad \text{with} \quad \boldsymbol{u} \in V.$$

Notations for the Magnetization Field M and the Magnetic Field H

The following is an abridged version of notations and preliminaries of the paper [13]. We introduce the following Hilbert spaces

$$\begin{aligned} H_{\boldsymbol{n}} &= \{\boldsymbol{M} \in \mathbb{L}^2(\mathcal{O});\ \boldsymbol{M} \times \boldsymbol{n} = 0 \text{ on } \partial\mathcal{O}\}, \\ V_1 &= \{\boldsymbol{M} \in \mathbb{H}^1(\mathcal{O});\ \boldsymbol{M} \times \boldsymbol{n} = 0 \text{ on } \partial\mathcal{O}\}, \\ V^1_{\mathrm{div}} &= \{\boldsymbol{B} \in V_1 :\ \operatorname{div} \boldsymbol{B} = 0 \text{ in } \mathcal{O}\}. \end{aligned}$$

The spaces $H_{\boldsymbol{n}}$ and H are equipped with the $\mathbb{L}^2(\mathcal{O})$-norm, and the spaces V, V_1 and V^1_{div} are equipped with the norm inhereted from $\mathbb{H}^1(\mathcal{O})$. Hence, V^1_{div} is a closed subset of V_1. Furthermore, the space V_1 admits the following orthogonal decomposition (see [13]):

$$\begin{aligned} &V_1 = V^1_{\mathrm{div}} \oplus \mathcal{H}, \\ &\mathcal{H} = \{w : w = \nabla\phi,\ \phi \in H^1_0(\mathcal{O}) \cap H^2(\mathcal{O})\}. \end{aligned}$$

Hereafter, we set

$$\mathbb{H} = H \times \mathbb{L}^2(\mathcal{O}) \times H_{\boldsymbol{n}} \times H_{\boldsymbol{n}},\ \ \mathbb{V} = V \times [H^1_0(\mathcal{O})]^3 \times V_1 \times V_1.$$

By identifying H with its dual through the Riesz isomorphism, we see that

$$V \hookrightarrow H \cong H' \hookrightarrow V',$$

is a Gelfand triple, i.e. both embeddings are continuous and have dense images. Moreover, the embeddings are also compact. We will use the notations $\mathbb{H}^s_{\mathrm{div}}(\mathcal{O}) := \mathbb{H}^s(\mathcal{O}) \cap V$ and $\mathbb{H}^s_0(\mathcal{O}) := [H^1_0(\mathcal{O})]^3 \cap \mathbb{H}^s(\mathcal{O})$, $s \geqslant 1$, to denote the space of divergence-free functions in $\mathbb{H}^s(\mathcal{O})$ with vanishing trace, and the space of functions in $\mathbb{H}^s(\mathcal{O})$ with vanishing trace, respectively. Their dual spaces are denoted by $\mathbb{H}^{-s}_{\mathrm{div}}(\mathcal{O})$ and $\mathbb{H}^{-s}_0(\mathcal{O})$, respectively.
We recall the classical result: there exists a positive constant $C > 0$ such that for all $\boldsymbol{M}_1 \in V_1$, we have

$$\|\nabla \boldsymbol{M}_1\|^2 \leqslant C(\|\boldsymbol{M}_1\|^2 + \|\operatorname{curl} \boldsymbol{M}_1\|^2 + \|\operatorname{div} \boldsymbol{M}_1\|^2), \tag{2.1}$$

which provides V_1 and V^1_{div} with the equivalent norm associated to the inner product defined by

$$[\boldsymbol{M}; \boldsymbol{M}_1] = (\boldsymbol{M}, \boldsymbol{M}_1) + (\operatorname{curl} \boldsymbol{M}, \operatorname{curl} \boldsymbol{M}_1) + (\operatorname{div} \boldsymbol{M}, \operatorname{div} \boldsymbol{M}_1),\ \ \boldsymbol{M}, \boldsymbol{M}_1 \in V_1.$$

We recall the following result taken from [25].

Lemma 2.1 *Let X, Y be two Banach spaces such that $X \hookrightarrow Y$ and $Y' \hookrightarrow X'$ densely. Then,* $L^\infty(0, T; X) \cap \mathcal{C}_w([0, T]; Y) \hookrightarrow \mathcal{C}_w([0, T]; X)$.

2.2 Main Assumptions

With the aim of solving Problem (1.1)–(1.3), we make the following assumption on the initial data

$$(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{M}_0, \boldsymbol{H}_0) \in \mathbb{H}$$
$$\operatorname{div} \boldsymbol{u}_0 = 0, \quad \operatorname{div}(\boldsymbol{M}_0 + \boldsymbol{H}_0) = 0 \quad \text{in} \quad \mathcal{O}. \tag{2.2}$$

General Agreement

Throughout the paper, the symbol C will denote a generic positive constant that does not depend on the data, but can depend on $\mathcal{O}$, T, and the physical parameters appearing in the equations. Any further dependence will be explicitly pointed out when necessary. In particular, the notation C_T, C_ε, or $C(T, \ldots)$ denotes a positive constant which explicitly depends on the quantities T or ε or $C(T, \ldots)$.

2.3 Existence of Weak Solutions

We now define the notion of weak solutions to the problem (1.1)–(1.3).

Definition 2.2 A quadriplet of functions $(\boldsymbol{u}, \mathbf{w}, \boldsymbol{M}, \boldsymbol{H})$ is a weak solution to the ECREs (1.1)–(1.3) if:

(i) The functions $\boldsymbol{u}, \mathbf{w}, \boldsymbol{M}, \boldsymbol{H}$ satisfy

$$\begin{gathered}
(\boldsymbol{u}, \mathbf{w}, \boldsymbol{M}, \boldsymbol{H}) \in L^\infty(0, T; \mathbb{H}), \\
\boldsymbol{u} \in L^2(0, T; V) \cap \mathcal{C}_w([0, T]; H), \\
\mathbf{w} \in L^2(0, T; [H_0^1(\mathcal{O})]^3) \cap \mathcal{C}_w([0, T]; \mathbb{L}^2(\mathcal{O})), \\
\boldsymbol{M} \in \mathcal{C}_w([0, T]; \mathbb{L}^2(\mathcal{O})), \quad \boldsymbol{H} \in \mathcal{C}_w([0, T]; \mathbb{L}^2(\mathcal{O})), \\
\operatorname{div}(\boldsymbol{H} + \boldsymbol{M}) = 0, \quad \text{a.e. in} \quad Q_T.
\end{gathered}$$

(ii) The following equations hold

$$
\begin{aligned}
&-\int_{Q_T} \boldsymbol{u}\cdot\partial_t \boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t-\int_{Q_T}(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{u}\,\mathrm{d}x\,\mathrm{d}t+\nu\int_{Q_T}\nabla\boldsymbol{u}:\nabla\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\mu_0\int_{Q_T}[(\boldsymbol{M}+\boldsymbol{H})\cdot\nabla]\boldsymbol{v}\cdot\boldsymbol{H}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_{\mathcal{O}}\boldsymbol{u}_0\cdot\boldsymbol{v}(0)\,\mathrm{d}x-\alpha\int_{Q_T}(\operatorname{curl}\boldsymbol{u}-2\mathbf{w})\cdot\operatorname{curl}\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t,\\
&-\int_{Q_T}\mathbf{w}\cdot\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t-\int_{Q_T}(\boldsymbol{u}\cdot\nabla)\psi\cdot\mathbf{w}\,\mathrm{d}x\,\mathrm{d}t+(\lambda_1+\lambda_2)\int_{Q_T}\operatorname{div}\mathbf{w}\operatorname{div}\psi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\lambda_1\int_{Q_T}\nabla\mathbf{w}:\nabla\psi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_{\mathcal{O}}\mathbf{w}_0\cdot\psi(0)\,\mathrm{d}x+2\alpha\int_{Q_T}\boldsymbol{u}\cdot\operatorname{curl}\psi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-4\alpha\int_{Q_T}\mathbf{w}\cdot\psi\,\mathrm{d}x\,\mathrm{d}t+\mu_0\int_{Q_T}(\boldsymbol{M}\times\boldsymbol{H})\cdot\psi\,\mathrm{d}x\,\mathrm{d}t,\\
&-\int_{Q_T}\boldsymbol{M}\cdot\partial_t\psi_1\,\mathrm{d}x\,\mathrm{d}t-\int_{Q_T}(\boldsymbol{u}\cdot\nabla)\psi_1\cdot\boldsymbol{M}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_{\mathcal{O}}\boldsymbol{M}_0\cdot\psi_1(0)\,\mathrm{d}x+\int_{Q_T}(\mathbf{w}\times\boldsymbol{M})\cdot\psi_1\,\mathrm{d}x\,\mathrm{d}t-\frac{1}{\tau}\int_{Q_T}(\boldsymbol{M}-\chi_0\boldsymbol{H})\cdot\psi_1\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{2.3}
$$

and

$$
\begin{aligned}
&-\int_{Q_T}\boldsymbol{M}\cdot\partial_t(\nabla\phi)\,\mathrm{d}x\,\mathrm{d}t-\int_{Q_T}(\boldsymbol{u}\cdot\nabla)\nabla\phi\cdot\boldsymbol{M}\,\mathrm{d}x\,\mathrm{d}t-\int_{Q_T}\operatorname{div}\boldsymbol{M}\Delta\phi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_{\mathcal{O}}\boldsymbol{M}_0\cdot\nabla\phi(0)\,\mathrm{d}x+\int_{Q_T}(\mathbf{w}\times\boldsymbol{M})\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-\frac{1}{\tau}\int_{Q_T}(\boldsymbol{M}-\chi_0\boldsymbol{H})\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t,\\
&\quad-\mu_0\int_{Q_T}(\boldsymbol{M}+\boldsymbol{H})\cdot\partial_t\psi_1\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{\sigma}\int_{Q_T}\operatorname{curl}\boldsymbol{H}\cdot\operatorname{curl}\psi_1\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\mu_0\int_{\mathcal{O}}(\boldsymbol{M}_0+\boldsymbol{H}_0)\cdot\psi_1\,\mathrm{d}x-\mu_0\int_{Q_T}((\boldsymbol{M}+\boldsymbol{H})\cdot\nabla)\psi_1\cdot\boldsymbol{u}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\mu_0\int_{Q_T}(\boldsymbol{u}\cdot\nabla)\psi_1\cdot(\boldsymbol{M}+\boldsymbol{H})\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
\tag{2.4}
$$

where the test functions are $\boldsymbol{v}\in\mathcal{C}_c^1([0,T);V\cap\mathbb{H}^3(\mathcal{O}))$, $\psi\in\mathcal{C}_c^1([0,T);[H_0^1(\mathcal{O})]^3\cap\mathbb{H}^2(\mathcal{O}))$, $\psi_1+\nabla\phi=\zeta\zeta_1+\zeta\nabla\zeta_2\in\mathcal{C}_c^1([0,T);[V_{\text{div}}^1\cap\mathbb{H}^2(\mathcal{O})]\oplus\mathcal{H})$, with $\zeta\in\mathcal{C}([0,T])\cap\mathcal{C}^1(0,T)$, $\zeta(T)=0$, $\zeta_1\in V_{\text{div}}^1\cap\mathbb{H}^2(\mathcal{O})$ and $\nabla\zeta_2\in\mathcal{H}$.

(iii) We also require that $(\boldsymbol{u}, \mathbf{w}, \boldsymbol{M}, \boldsymbol{H})$ is right-continuous at $t = 0$, i.e., as $t \to 0^+$

$$\begin{aligned} \boldsymbol{u}(t) &\to \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \mathbb{L}^2(\mathcal{O}), \\ \boldsymbol{M}(t) &\to \boldsymbol{M}(0) = \boldsymbol{M}_0 \quad \text{in } \mathbb{L}^2(\mathcal{O}), \\ \mathbf{w}(t) &\to \mathbf{w}(0) = \mathbf{w}_0 \quad \text{in } \mathbb{L}^2(\mathcal{O}), \\ \boldsymbol{H}(t) &\to \boldsymbol{H}(0) = \boldsymbol{H}_0 \quad \text{in } \mathbb{L}^2(\mathcal{O}). \end{aligned}$$

We state the following remarks with regard to the above definition.

Remark 2.3 (1) We should note that Eqs. (2.3)–(2.4) are the weak formulations of Eqs. (1.1a)–(1.1d).

(2) We also note that since $\operatorname{div}(\boldsymbol{M} + \boldsymbol{H}) = 0$ in $\mathcal{O}$, we have

$$\mu_0 \int_{\mathcal{O}} \partial_t(\boldsymbol{M} + \boldsymbol{H}) \cdot \nabla\phi \, \mathrm{d}x = -\mu_0 \int_{\mathcal{O}} (\boldsymbol{M} + \boldsymbol{H}) \cdot \zeta' \nabla\zeta_2 \, \mathrm{d}x = 0.$$

With the above definition in mind, we are now ready to formulate our first main result.

Theorem 2.4 *Let $\mathcal{O} \subset \mathbb{R}^3$ be a simply connected bounded domain of class $\mathcal{C}^\infty$, and T be a fixed positive time. Suppose that $\mu_0 = \sigma = 1$ or $\sigma\mu_0^2 > 2$. Assume also that (2.2) holds. Then, there exists a weak solution $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{M}, \boldsymbol{H})$ to system (1.1)–(1.3) in the sense of Definition 2.2. Moreover, $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{M}, \boldsymbol{H})$ satisfies the following inequality*

$$\begin{aligned} &\frac{1}{2}\mathcal{E}_{\text{tot}}(\boldsymbol{u}(t), \boldsymbol{w}(t), \boldsymbol{M}(t), \boldsymbol{H}(t)) + \int_0^t [\nu\|\nabla\boldsymbol{u}(s)\|^2 + (\lambda_1 + \lambda_2)\|\operatorname{div}\boldsymbol{w}(s)\|^2] \, \mathrm{d}s \\ &\quad + \int_0^t [\lambda_1\|\nabla\boldsymbol{w}(s)\|^2 + \alpha\|\operatorname{curl}\boldsymbol{u}(s) - 2\boldsymbol{w}(s)\|^2] \, \mathrm{d}s \\ &\quad + \int_0^t \left[\frac{1}{\tau}\|\boldsymbol{M}(s)\|^2 + \frac{\mu_0\chi_0}{\tau}\|\boldsymbol{H}(s)\|^2 + \frac{1}{\sigma}\|\operatorname{curl}\boldsymbol{H}(s)\|^2\right] \mathrm{d}s \\ &\leqslant \frac{1}{2}\mathcal{E}_{\text{tot}}(\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{M}_0, \boldsymbol{H}_0) + \frac{\mu_0 + \chi_0}{\tau} \int_{Q_t} \boldsymbol{H}(s) \cdot \boldsymbol{M}(s) \, \mathrm{d}x \, \mathrm{d}s, \quad \forall t \in [0, T]. \end{aligned} \tag{2.5}$$

Here $\mathcal{E}_{\text{tot}}(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{M}, \boldsymbol{H}) = \|\boldsymbol{u}\|^2 + \|\boldsymbol{w}\|^2 + \|\boldsymbol{M}\|^2 + \mu_0\|\boldsymbol{H}\|^2$.

Furthermore, the function $\boldsymbol{H}$ admits the decomposition $\boldsymbol{H} = \boldsymbol{H}_a + \boldsymbol{H}_d$ with the functions $\boldsymbol{H}_a$ and $\boldsymbol{H}_d$ satisfying:

- $\boldsymbol{H}_a \in L^\infty(0, T; H) \cap L^2(0, T; V)$ *and*

$$\begin{aligned} \operatorname{div}\boldsymbol{H}_a &= 0 && \text{in } \mathcal{O}, \\ \operatorname{curl}\boldsymbol{H}_a &= \operatorname{curl}\boldsymbol{H} && \text{in } \mathcal{O}, \end{aligned}$$

- $\boldsymbol{H}_d \in L^\infty(0, T; [E_2(\mathcal{O})]^3)$, *i.e., there exists $\varphi_d \in L^\infty(0, T; H^1(\mathcal{O}))$ such that*

$$\boldsymbol{H}_d = \nabla\varphi_d \in L^\infty(0, T; \mathbb{L}^2(\mathcal{O})).$$

Moreover, $\boldsymbol{H}_d$ *solves*

$$\begin{aligned} \operatorname{curl} \boldsymbol{H}_d &= 0 \quad \text{in } \mathcal{O}, \\ \operatorname{div} \boldsymbol{H}_d &= -\operatorname{div} \boldsymbol{M} \quad \text{in } \mathcal{O}, \\ \boldsymbol{H}_d \cdot \boldsymbol{n} &= \frac{\partial \varphi_d}{\partial \boldsymbol{n}} = -\boldsymbol{M} \cdot \boldsymbol{n} \text{ on } \partial\mathcal{O}. \end{aligned} \tag{2.6}$$

Furthermore, for all $\psi \in H^1(\mathcal{O})$,

$$\int_{\mathcal{O}} \nabla\varphi_d \cdot \nabla\psi \, \mathrm{d}x = \int_{\mathcal{O}} (\boldsymbol{H} - \boldsymbol{H}_a) \cdot \nabla\psi \, \mathrm{d}x = \int_{\mathcal{O}} \boldsymbol{H} \cdot \nabla\psi \, \mathrm{d}x = -\int_{\mathcal{O}} \boldsymbol{M} \cdot \nabla\psi \, \mathrm{d}x.$$

Renormalized Solutions

We now state the following important lemmas. In particular, Lemma 2.6 will play a crucial role in the proof of Proposition 3.7 below. Indeed, it will be used to derive the strong convergence of the sequence $\boldsymbol{M}^\gamma$, $\gamma \in (0, 1)$, cf. (3.80).
Roughly speaking, the Eq. (1.1c) for $\boldsymbol{M}$ is a transport equation. In order to overcome the lack of regularity of $\boldsymbol{M}$, we need to use the regularization technique developed by DiPerna–Lions [8, 10, 15] so as to "renormalize" the magnetization field $\boldsymbol{M}$ as follows.

Lemma 2.5 *Let* $\boldsymbol{M} \in L^\infty(0, T; \mathbb{L}^2(\mathcal{O}))$ *be a weak distributional solution of*

$$\partial_t \boldsymbol{M} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{M} = \boldsymbol{w} \times \boldsymbol{M} - \frac{1}{\tau}(\boldsymbol{M} - \chi_0 \boldsymbol{H})$$

for a given $\boldsymbol{u} \in L^\infty(0, T; H) \cap L^2(0, T; V)$, $\boldsymbol{w} \in L^\infty(0, T; \mathbb{L}^2(\mathcal{O})) \cap L^2(0, T; [H_0^1(\mathcal{O})]^3)$, *and* $\boldsymbol{H} \in L^\infty(0, T; \mathbb{L}^2(\mathcal{O}))$. *Then the components* $\boldsymbol{M}_i$ *of* $\boldsymbol{M}$ *satisfy, for any* $b \in \mathcal{C}^1(\mathbb{R})$ *with* $b'(\cdot)$ *bounded and* $b(0) = 0$,

$$\partial_t b(\boldsymbol{M}_i) + \boldsymbol{u} \cdot \nabla b(\boldsymbol{M}_i) = b'(\boldsymbol{M}_i)(\boldsymbol{w} \times \boldsymbol{M})_i - \frac{1}{\tau} b'(\boldsymbol{M}_i)(\boldsymbol{M}_i - \chi_0 \boldsymbol{H}_i)$$

in the sense of distributions. Here $(\boldsymbol{w} \times \boldsymbol{M})_i$ *and* $\boldsymbol{H}_i$ *denote the* i^{th} *component of the vectors* $\boldsymbol{w} \times \boldsymbol{M}$ *and* $\boldsymbol{H}$, *respectively.*

Proof We refer to [17] for the proof. □

The following lemma easily follows from the previous one (see [17]).

Lemma 2.6 *Let* $\boldsymbol{M} \in L^\infty(0, T; H_{\boldsymbol{n}})$ *be a weak distributional solution of*

$$\partial_t \boldsymbol{M} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{M} = \boldsymbol{w} \times \boldsymbol{M} - \frac{1}{\tau}(\boldsymbol{M} - \chi_0 \boldsymbol{H})$$

for given $\boldsymbol{u} \in L^\infty(0, T; H) \cap L^2(0, T; V)$, $\boldsymbol{w} \in L^\infty(0, T; \mathbb{L}^2(\mathcal{O})) \cap L^2(0, T; [H_0^1(\mathcal{O})]^3)$, $\boldsymbol{H} \in L^\infty(0, T; H_{\boldsymbol{n}})$, *and initial data* $\boldsymbol{M}_0 \in \mathbb{L}^2(\mathcal{O})$. *Then* $\boldsymbol{M}$ *satisfies*

$$\frac{1}{2}\|\boldsymbol{M}(t)\|^2 = \frac{1}{2}\|\boldsymbol{M}_0\|^2 - \frac{1}{\tau}\int_0^t \|\boldsymbol{M}(s)\|^2 \,\mathrm{d}s + \frac{\chi_0}{\tau}\int_{Q_t} \boldsymbol{M}(s)\cdot\boldsymbol{H}(s)\,\mathrm{d}x\,\mathrm{d}s \quad \forall t \in [0, T]. \tag{2.7}$$

3 Existence of Global Weak Solutions to the ECREs

In this section, we prove the global existence of weak solutions to the ECREs, i.e., the problem (1.1)–(1.3). The proof is based on approximation of the ECREs by a family of systems of parabolic equations that admit global weak solutions. We then prove that the approximating sequences of solutions are compact in appropriate spaces and converge to a global weak solution of the ECREs. The main tools of the proof are Propositions 3.2 and 3.5 below, and the final step of the proof is given in Subsect. 3.3.

3.1 The Approximating Problems

We consider an approximation of the ECREs which is the Bloch–Torrey regularization of the ECREs called BT-ECREs. The BT-ECREs are obtained by adding the term $\gamma\nabla\times(\nabla\times\boldsymbol{M}) - \gamma\nabla\operatorname{div}\boldsymbol{M}$ on the left-hand side of Eq. (1.1c) for the magnetization $\boldsymbol{M}$, where $\gamma > 0$ is a small parameter. More precisely, the BT-ECREs are given by

$$\begin{aligned}
&\partial_t\boldsymbol{u}^\gamma + (\boldsymbol{u}^\gamma\cdot\nabla)\boldsymbol{u}^\gamma + \nabla p^\gamma - \nu\Delta\boldsymbol{u}^\gamma - \mu_0(\boldsymbol{M}^\gamma\cdot\nabla)\boldsymbol{H}^\gamma \\
&\quad = \mu_0(\nabla\times\boldsymbol{H}^\gamma)\times\boldsymbol{H}^\gamma - \alpha\nabla\times(\nabla\times\boldsymbol{u}^\gamma - 2\mathbf{w}^\gamma), &(3.1a)\\
&\partial_t\mathbf{w}^\gamma + (\boldsymbol{u}^\gamma\cdot\nabla)\mathbf{w}^\gamma - (\lambda_1+\lambda_2)\nabla\operatorname{div}\mathbf{w}^\gamma - \lambda_1\Delta\mathbf{w}^\gamma \\
&\quad = 2\alpha(\nabla\times\boldsymbol{u}^\gamma - 2\mathbf{w}^\gamma) + \mu_0\boldsymbol{M}^\gamma\times\boldsymbol{H}^\gamma, &(3.1b)\\
&\partial_t\boldsymbol{M}^\gamma + (\boldsymbol{u}^\gamma\cdot\nabla)\boldsymbol{M}^\gamma + \gamma\nabla\times(\nabla\times\boldsymbol{M}^\gamma) - \gamma\nabla\operatorname{div}\boldsymbol{M}^\gamma \\
&\quad = \mathbf{w}^\gamma\times\boldsymbol{M}^\gamma - \frac{1}{\tau}(\boldsymbol{M}^\gamma - \chi_0\boldsymbol{H}^\gamma), &(3.1c)\\
&\partial_t\boldsymbol{B}^\gamma + \frac{1}{\sigma}\nabla\times(\nabla\times\boldsymbol{H}^\gamma) = \nabla\times(\boldsymbol{u}^\gamma\times\boldsymbol{B}^\gamma), &(3.1d)\\
&\boldsymbol{B}^\gamma = \mu_0(\boldsymbol{M}^\gamma+\boldsymbol{H}^\gamma), \quad \operatorname{div}\boldsymbol{M}^\gamma = -\operatorname{div}\boldsymbol{H}^\gamma, &(3.1e)\\
&\operatorname{div}\boldsymbol{u}^\gamma = 0. &(3.1f)
\end{aligned}$$

We endow problem (3.1a)–(3.1f) with the following boundary conditions

$$\begin{aligned}
\boldsymbol{u}^\gamma = 0, \quad \mathbf{w}^\gamma = 0, \quad &\text{on} \quad \Sigma. \\
\operatorname{div}\boldsymbol{M}^\gamma = 0, \quad \boldsymbol{H}^\gamma\times\boldsymbol{n} = \boldsymbol{M}^\gamma\times\boldsymbol{n} = 0, \quad &\text{on} \quad \Sigma.
\end{aligned} \tag{3.2}$$

The initial condition is given by

$$\begin{aligned}
(\boldsymbol{u}^\gamma, \mathbf{w}^\gamma, \boldsymbol{M}^\gamma, \boldsymbol{H}^\gamma)(t=0) &= (\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{M}_0, \boldsymbol{H}_0) \text{ in } \mathcal{O}, \\
\operatorname{div}\boldsymbol{u}_0 &= 0, \quad \operatorname{div}(\boldsymbol{M}_0+\boldsymbol{H}_0) = 0, \quad \text{in } \mathcal{O}.
\end{aligned} \tag{3.3}$$

We now state the definition of weak solution of the BT-ECREs (3.1)–(3.3).

Definition 3.1 We say that $(\boldsymbol{u}^\gamma, \mathbf{w}^\gamma, \boldsymbol{M}^\gamma, \boldsymbol{H}^\gamma)$ is a global weak solution of the BT-ECREs (3.1)–(3.3) if the following conditions hold:

(i) The functions $\boldsymbol{u}^\gamma, \mathbf{w}^\gamma, \boldsymbol{M}^\gamma, \boldsymbol{H}^\gamma$ satisfy

$$\begin{aligned}&(\boldsymbol{u}^\gamma, \mathbf{w}^\gamma, \boldsymbol{M}^\gamma, \boldsymbol{H}^\gamma) \in L^\infty(0,T;\mathbb{H}) \cap L^2(0,T;\mathbb{V}),\\ &\operatorname{div}(\boldsymbol{H}^\gamma + \boldsymbol{M}^\gamma) = 0 \quad \text{a.e. in} \quad Q_T.\end{aligned}$$

(ii) The Eqs. (3.1a)–(3.1d) hold in the weak sense, meaning that for any test functions

$$\begin{aligned}&\boldsymbol{v} \in \mathcal{C}_c^1([0,T);V), \quad \psi \in \mathcal{C}_c^1([0,T);[H_0^1(\mathcal{O})]^3),\\ &\psi_1 + \nabla\phi \in \mathcal{C}_c^1([0,T);V_{\text{div}}^1 \oplus \mathcal{H}),\end{aligned}$$

we have

$$\begin{aligned}
&-\int_{Q_T} \boldsymbol{u}^\gamma \cdot \partial_t \boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} (\boldsymbol{u}^\gamma\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{u}^\gamma\,\mathrm{d}x\,\mathrm{d}t + \nu\int_{Q_T}\nabla\boldsymbol{u}^\gamma : \nabla\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\mu_0\int_{Q_T}[(\boldsymbol{M}^\gamma+\boldsymbol{H}^\gamma)\cdot\nabla]\boldsymbol{v}\cdot\boldsymbol{H}^\gamma\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_{\mathcal{O}}\boldsymbol{u}_0\cdot\boldsymbol{v}(0)\,\mathrm{d}x - \alpha\int_{Q_T}(\operatorname{curl}\boldsymbol{u}^\gamma - 2\mathbf{w}^\gamma)\cdot\operatorname{curl}\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t,\\
&-\int_{Q_T}\mathbf{w}^\gamma\cdot\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T}(\boldsymbol{u}^\gamma\cdot\nabla)\psi\cdot\mathbf{w}^\gamma\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+(\lambda_1+\lambda_2)\int_{Q_T}\operatorname{div}\mathbf{w}^\gamma\operatorname{div}\psi\,\mathrm{d}x\,\mathrm{d}t + \lambda_1\int_{Q_T}\nabla\mathbf{w}^\gamma:\nabla\psi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_{\mathcal{O}}\mathbf{w}_0\cdot\psi(0)\,\mathrm{d}x + 2\alpha\int_{Q_T}\boldsymbol{u}^\gamma\cdot\operatorname{curl}\psi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-4\alpha\int_{Q_T}\mathbf{w}^\gamma\cdot\psi\,\mathrm{d}x\,\mathrm{d}t + \mu_0\int_{Q_T}(\boldsymbol{M}^\gamma\times\boldsymbol{H}^\gamma)\cdot\psi\,\mathrm{d}x\,\mathrm{d}t,\\
&-\int_{Q_T}\boldsymbol{M}^\gamma\cdot\partial_t\psi_1\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T}(\boldsymbol{u}^\gamma\cdot\nabla)\psi_1\cdot\boldsymbol{M}^\gamma\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\gamma\int_{Q_T}\operatorname{curl}\boldsymbol{M}^\gamma\cdot\operatorname{curl}\psi_1\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_{\mathcal{O}}\boldsymbol{M}_0\cdot\psi_1(0)\,\mathrm{d}x + \int_{Q_T}(\mathbf{w}^\gamma\times\boldsymbol{M}^\gamma)\cdot\psi_1\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-\frac{1}{\tau}\int_{Q_T}(\boldsymbol{M}^\gamma-\chi_0\boldsymbol{H}^\gamma)\cdot\psi_1\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.4}$$

and

$$-\int_{Q_T}\boldsymbol{M}^\gamma\cdot\partial_t(\nabla\phi)\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T}(\boldsymbol{u}^\gamma\cdot\nabla)\nabla\phi\cdot\boldsymbol{M}^\gamma\,\mathrm{d}x\,\mathrm{d}t$$

$$-\gamma \int_{Q_T} \operatorname{div} \boldsymbol{M}^\gamma \Delta\phi \, \mathrm{d}x \, \mathrm{d}t$$
$$= \int_{\mathcal{O}} \boldsymbol{M}_0 \cdot \nabla\phi(0) \, \mathrm{d}x + \int_{Q_T} (\mathbf{w}^\gamma \times \boldsymbol{M}^\gamma) \cdot \nabla\phi \, \mathrm{d}x \, \mathrm{d}t$$
$$-\frac{1}{\tau} \int_{Q_T} (\boldsymbol{M}^\gamma - \chi_0 \boldsymbol{H}^\gamma) \cdot \nabla\phi \, \mathrm{d}x \, \mathrm{d}t,$$
$$-\mu_0 \int_{Q_T} (\boldsymbol{M}^\gamma + \boldsymbol{H}^\gamma) \cdot \partial_t \psi_1 \, \mathrm{d}x \, \mathrm{d}t + \frac{1}{\sigma} \int_{Q_T} \operatorname{curl} \boldsymbol{H}^\gamma \cdot \operatorname{curl} \psi_1 \, \mathrm{d}x \, \mathrm{d}t$$
$$= \mu_0 \int_{\mathcal{O}} (\boldsymbol{M}_0 + \boldsymbol{H}_0) \cdot \psi_1 \, \mathrm{d}x + \mu_0 \int_{Q_T} (\boldsymbol{u}^\gamma \times (\boldsymbol{M}^\gamma + \boldsymbol{H}^\gamma)) \cdot \operatorname{curl} \psi_1 \, \mathrm{d}x \, \mathrm{d}t.$$

(iii) We also require that $(\boldsymbol{u}^\gamma, \mathbf{w}^\gamma, \boldsymbol{M}^\gamma, \boldsymbol{H}^\gamma)$ is right-continuous at $t = 0$, i.e., as $t \to 0^+$

$$\begin{aligned}
\boldsymbol{u}^\gamma(t) \to \boldsymbol{u}^\gamma(0) = \boldsymbol{u}_0 \qquad & \text{in } \mathbb{L}^2(\mathcal{O}),\\
\boldsymbol{M}^\gamma(t) \to \boldsymbol{M}^\gamma(0) = \boldsymbol{M}_0 \qquad & \text{in } \mathbb{L}^2(\mathcal{O}),\\
\mathbf{w}^\gamma(t) \to \mathbf{w}^\gamma(0) = \mathbf{w}_0 \qquad & \text{in } \mathbb{L}^2(\mathcal{O}),\\
\boldsymbol{H}^\gamma(t) \to \boldsymbol{H}^\gamma(0) = \boldsymbol{H}_0 \qquad & \text{in } \mathbb{L}^2(\mathcal{O}).
\end{aligned}$$

(iv) The energy inequality holds

$$\begin{aligned}
&\mathcal{E}_{\text{tot}}(\boldsymbol{u}^\gamma(t), \mathbf{w}^\gamma(t), \boldsymbol{M}^\gamma(t), \boldsymbol{H}^\gamma(t)) + \int_0^t [\|\nabla \boldsymbol{u}^\gamma\|^2 + \|\operatorname{div} \mathbf{w}^\gamma\|^2 + \|\nabla \mathbf{w}^\gamma\|^2] \mathrm{d}s\\
&+ \int_0^t [\|\nabla \times \boldsymbol{M}^\gamma\|^2 + \|\nabla \times \boldsymbol{u}^\gamma - 2\mathbf{w}^\gamma\|^2] \mathrm{d}s + \int_0^t \left[\|\operatorname{div} \boldsymbol{M}^\gamma\|^2 + \|\boldsymbol{M}^\gamma\|^2\right] \mathrm{d}s\\
&+ \int_0^t \left[\|\operatorname{div} \boldsymbol{H}^\gamma\|^2 + \|\operatorname{div} \boldsymbol{H}^\gamma\|^2 + \|\nabla \times \boldsymbol{H}^\gamma\|^2 + \|\boldsymbol{H}^\gamma\|^2\right] \mathrm{d}s\\
&\leqslant C(\nu, \lambda_1, \lambda_2, \alpha, \gamma, \tau, \mu_0, \chi_0, \sigma, T) \mathcal{E}_{\text{tot}}(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{M}_0, \boldsymbol{H}_0)
\end{aligned} \tag{3.5}$$

for all $t \in [0, T]$, where

$$\mathcal{E}_{\text{tot}}(\boldsymbol{u}^\gamma, \mathbf{w}^\gamma, \boldsymbol{M}^\gamma, \boldsymbol{H}^\gamma) = \|\boldsymbol{u}^\gamma\|^2 + \|\mathbf{w}^\gamma\|^2 + \|\boldsymbol{M}^\gamma\|^2 + \mu_0 \|\boldsymbol{H}^\gamma\|^2.$$

We now recall the following result, which follows from [13, Theorem 1].

Proposition 3.2 *Assume that the initial data $(\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{M}_0, \boldsymbol{H}_0)$ satisfies (2.2). Then, for any $\gamma > 0$ there exists a global weak solution $(\boldsymbol{u}^\gamma, \boldsymbol{w}^\gamma, \boldsymbol{M}^\gamma, \boldsymbol{H}^\gamma) \in L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V})$ of the BT-ECREs.*

In the next subsection we will derive an energy inequality, which is similar to (3.5) but uniform in the parameter γ. Before doing so, we state several important remarks and results.

Remark 3.3 Notice that

$$\int_{Q_T} (\operatorname{curl} \boldsymbol{M}^\gamma) \cdot \operatorname{curl}(\nabla\phi) \, \mathrm{d}x \, \mathrm{d}t = 0,$$
$$\int_{Q_T} (\operatorname{curl} \boldsymbol{H}^\gamma) \cdot \operatorname{curl}(\nabla\phi) \, \mathrm{d}x \, \mathrm{d}t = 0,$$
$$\int_{Q_T} (\boldsymbol{u}^\gamma \times (\boldsymbol{M}^\gamma + \boldsymbol{H}^\gamma)) \cdot \operatorname{curl}(\nabla\phi) \, \mathrm{d}x \, \mathrm{d}t = 0,$$

for all $\nabla\phi \in \mathcal{C}_c^1([0,T);\mathcal{H})$ so that

$$\frac{1}{\sigma}\int_{Q_T} \operatorname{curl} \boldsymbol{H}^\gamma \cdot \operatorname{curl} \boldsymbol{\psi}_1 \, \mathrm{d}x \, \mathrm{d}t - \mu_0 \int_{Q_T} (\boldsymbol{u}^\gamma \times (\boldsymbol{M}^\gamma + \boldsymbol{H}^\gamma)) \cdot \operatorname{curl} \boldsymbol{\psi}_1 \, \mathrm{d}x \, \mathrm{d}t$$
$$= \frac{1}{\sigma}\int_{Q_T} \operatorname{curl} \boldsymbol{H}^\gamma \cdot \operatorname{curl} \psi_1 \, \mathrm{d}x \, \mathrm{d}t - \mu_0 \int_{Q_T} (\boldsymbol{u}^\gamma \times (\boldsymbol{M}^\gamma + \boldsymbol{H}^\gamma)) \cdot \operatorname{curl} \psi_1 \, \mathrm{d}x \, \mathrm{d}t$$

for all $\boldsymbol{\psi}_1 := \psi_1 + \nabla\phi \in \mathcal{C}_c^1([0,T); V_{\mathrm{div}}^1 \oplus \mathcal{H})$.

When the parameter $\gamma > 0$ is sufficiently small, $\boldsymbol{M}^\gamma$ and $\boldsymbol{H}^\gamma$ will eventually have spatial regularity no more than $\mathbb{L}^2(\mathcal{O})$. Thus, in order to control the Kelvin force $\mu_0(\boldsymbol{M}^\gamma \cdot \nabla)\boldsymbol{H}^\gamma$ (cf. (3.44)) and properly define the weak solutions in Definition 3.1, we shall use the following identity.

Lemma 3.4 *Let $\boldsymbol{v} \in V$. Then, $\boldsymbol{M}^\gamma$ and $\boldsymbol{H}^\gamma$ satisfy*

$$\begin{aligned}\int_{\mathcal{O}} (\boldsymbol{M}^\gamma \cdot \nabla)\boldsymbol{H}^\gamma \cdot \boldsymbol{v} \, \mathrm{d}x &= -\int_{\mathcal{O}} [(\boldsymbol{M}^\gamma + \boldsymbol{H}^\gamma) \cdot \nabla]\boldsymbol{v} \cdot \boldsymbol{H}^\gamma \, \mathrm{d}x \\ &\quad - \int_{\mathcal{O}} \operatorname{curl} \boldsymbol{H}^\gamma \cdot (\boldsymbol{H}^\gamma \times \boldsymbol{v}) \, \mathrm{d}x. \end{aligned} \tag{3.6}$$

Proof Let $\boldsymbol{v} \in V$ be arbitrary. Using (3.1e), the fact that $\boldsymbol{v}|_{\partial\mathcal{O}} = 0$, $\operatorname{div} \boldsymbol{v} = 0$ along with an integration by parts, we obtain the chain of equations

$$\begin{aligned}
&\int_{\mathcal{O}} (\boldsymbol{M}^\gamma \cdot \nabla)\boldsymbol{H}^\gamma \cdot \boldsymbol{v} \, \mathrm{d}x = \int_{\mathcal{O}} \boldsymbol{M}_i^\gamma (\partial x_i \boldsymbol{H}_j^\gamma)\boldsymbol{v}_j \, \mathrm{d}x \\
&= \int_{\mathcal{O}} [\boldsymbol{M}_i^\gamma + \boldsymbol{H}_i^\gamma](\partial x_i \boldsymbol{H}_j^\gamma)\boldsymbol{v}_j \, \mathrm{d}x - \int_{\mathcal{O}} \boldsymbol{H}_i^\gamma (\partial x_i \boldsymbol{H}_j^\gamma - \partial x_j \boldsymbol{H}_i^\gamma)\boldsymbol{v}_j \, \mathrm{d}x \\
&\quad - \int_{\mathcal{O}} \boldsymbol{H}_i^\gamma (\partial x_j \boldsymbol{H}_i^\gamma)\boldsymbol{v}_j \, \mathrm{d}x \\
&= -\int_{\mathcal{O}} [\boldsymbol{M}_i^\gamma + \boldsymbol{H}_i^\gamma](\partial x_i \boldsymbol{v}_j)\boldsymbol{H}_j^\gamma \, \mathrm{d}x - \int_{\mathcal{O}} [\partial x_i (\boldsymbol{M}_i^\gamma + \boldsymbol{H}_i^\gamma)]\boldsymbol{H}_j^\gamma \boldsymbol{v}_j \, \mathrm{d}x \\
&\quad - \int_{\mathcal{O}} \boldsymbol{H}_i^\gamma (\partial x_i \boldsymbol{H}_j^\gamma - \partial x_j \boldsymbol{H}_i^\gamma)\boldsymbol{v}_j \, \mathrm{d}x - \frac{1}{2}\int_{\mathcal{O}} \frac{\partial}{\partial x_j} |\boldsymbol{H}_i^\gamma|^2 \boldsymbol{v}_j \, \mathrm{d}x \\
&= -\int_{\mathcal{O}} [(\boldsymbol{M}^\gamma + \boldsymbol{H}^\gamma) \cdot \nabla]\boldsymbol{v} \cdot \boldsymbol{H}^\gamma \, \mathrm{d}x - \int_{\mathcal{O}} \operatorname{curl} \boldsymbol{H}^\gamma \cdot (\boldsymbol{H}^\gamma \times \boldsymbol{v}) \, \mathrm{d}x.
\end{aligned}$$

Here we have also used the summation convention on repeated indices. The last line of the chain of equations completes the proof of the lemma. □

3.2 Energy Inequality of the Approximating Sequences

Let us set

$$\mathcal{E}_{\text{tot}}(u, w, m, h) = \|u\|^2 + \|w\|^2 + \|m\|^2 + \mu_0\|h\|^2,$$

and

$$\begin{aligned}\mathcal{F}(u, w, m, h) &= \nu\|\nabla u\|^2 + (\lambda_1 + \lambda_2)\|\operatorname{div} w\|^2 + \lambda_1\|\nabla w\|^2 + \gamma\|\operatorname{div} m\|^2 \\ &\quad+\alpha\,\|\operatorname{curl} u - 2w\|^2 + \frac{\gamma}{2}\|\operatorname{curl} m\|^2 + \frac{1}{2\tau}\|m\|^2 \\ &\quad+\gamma\|\operatorname{div} h\|^2 + \frac{1}{2\sigma}\|\nabla \times h\|^2,\end{aligned}$$

where (u, w, m, h) are taken in a subspace of $[\mathbb{H}^1(\mathcal{O})]^4$.

Let

$$I_\gamma = \begin{cases}(0, \frac{2}{\sigma\mu_0^2}) & \text{if} \quad \sigma\mu_0^2 > 2 \\ (0, 1) & \text{if} \quad \sigma = \mu_0 = 1.\end{cases} \tag{3.7}$$

Hereafter, we will derive an energy inequality similar to (3.5) but independent of the parameter γ. More precisely, we will prove the following result.

Proposition 3.5 *Assume that the hypotheses of Theorem 2.4 hold. Let $(\boldsymbol{u}^\gamma, \mathbf{w}^\gamma, \boldsymbol{M}^\gamma, \boldsymbol{H}^\gamma)$ be the global weak solution to the BT-ECREs given by Proposition 3.2. Then, there exists a positive constant $C > 0$ such that for all $\gamma \in I_\gamma$, we have*

$$\begin{aligned}&\mathcal{E}_{\text{tot}}(\boldsymbol{u}^\gamma(t), \mathbf{w}^\gamma(t), \boldsymbol{M}^\gamma(t), \boldsymbol{H}^\gamma(t)) + 2\int_0^t \mathcal{F}(\boldsymbol{u}^\gamma(s), \mathbf{w}^\gamma(s), \boldsymbol{M}^\gamma(s), \boldsymbol{H}^\gamma(s))\,\mathrm{d}s \\ &\quad\leqslant C[1 + \mathcal{E}_{\text{tot}}(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{M}_0, \boldsymbol{H}_0)].\end{aligned} \tag{3.8}$$

As mentioned in the proposition statement, the existence of a global weak solution $(\boldsymbol{u}^\gamma, \mathbf{w}^\gamma, \boldsymbol{M}^\gamma, \boldsymbol{H}^\gamma) \in L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V})$ of the BT-ECREs satisfying the energy inequality (3.5) is insured by Proposition 3.2 which was proved in [13, Theorem 1]. However, since the uniform energy estimates (3.8) is a refinement of (3.5), we will repeat some of the arguments in [13] in the proof of Proposition 3.5 below.

Proof of Proposition 3.5 In this proof, we will restrict ourselves to the case where σ and μ_0 satisfy

$$2 < \sigma\mu_0^2. \tag{3.9}$$

Note that the analysis below still holds for $\sigma = \mu_0 = 1$. With this in mind, we fix an arbitrary $\gamma \in (0, \frac{1}{\sigma\mu_0^2})$. Observe that due to (3.9), we have $\gamma \in (0, \frac{1}{\sigma\mu_0^2}) \subset (0, 1)$.

The proof of Proposition 3.5 will now be done in several steps. □

Step 1: Galerkin approximation scheme

Let A be the Stokes operator, see, for instance [11, Appendix B] for its definition and some of its properties. We consider the family of eigenfunctions $\{\upsilon_j\}_{j=1}^{\infty}$ of A as a Hilbert basis in V, the family[1] $\{\Lambda_j\}_{j=1}^{\infty}$ as a Hilbert basis in $[H_0^1(\mathcal{O})]^3$, and the family $\{\bar{\psi}_j\}_{j=1}^{\infty} \cup \{\nabla\bar{\phi}_j\}_{j=1}^{\infty}$ given by [13, Lemma 5] as a Hilbert basis in V_1. For any integer $m \geqslant 1$, we define the finite-dimensional subspaces of V, $[H_0^1(\mathcal{O})]^3$ and V_1, respectively, by

$$\begin{aligned} V_m &= \text{span}\{\upsilon_1, \dots, \upsilon_m\}, \\ H_{0m}^1 &= \text{span}\{\Lambda_1, \dots, \Lambda_m\}, \\ V_{1m} &= \text{span}\{\bar{\psi}_1, \dots, \bar{\psi}_m\} \cup \text{span}\{\nabla\bar{\phi}_1, \dots, \nabla\bar{\phi}_m\}. \end{aligned}$$

We denote by $\mathcal{P}_m^1$, $\mathcal{P}_m^2$ and $\mathcal{P}_m^3$ the orthogonal projections on V_m, H_{0m}^1 and V_{1m} with respect to the inner product in H, $\mathbb{L}^2(\mathcal{O})$ and in $\mathbb{L}^2(\mathcal{O})$, respectively.
For any $m \in \mathbb{N}$, we consider the quadriplet $(\boldsymbol{u}_{0m}^{\gamma}, \mathbf{w}_{0m}^{\gamma}, \boldsymbol{M}_{0m}^{\gamma}, \boldsymbol{H}_{0m}^{\gamma})$, where

$$\boldsymbol{u}_{0m}^{\gamma} = \mathcal{P}_m^1\boldsymbol{u}_0, \quad \mathbf{w}_{0m}^{\gamma} = \mathcal{P}_m^2\mathbf{w}_0, \quad \boldsymbol{M}_{0m}^{\gamma} = \mathcal{P}_m^3\boldsymbol{M}_0, \quad \text{and } \boldsymbol{H}_{0m}^{\gamma} = \mathcal{P}_m^3\boldsymbol{H}_0.$$

As $m \to \infty$ we have

$$\begin{aligned} \boldsymbol{u}_{0m}^{\gamma} &\to \boldsymbol{u}_0 \text{ strongly in } \mathbb{L}^2(\mathcal{O}), \\ \mathbf{w}_{0m}^{\gamma} &\to \mathbf{w}_0 \text{ strongly in } \mathbb{L}^2(\mathcal{O}), \\ \boldsymbol{M}_{0m}^{\gamma} &\to \boldsymbol{M}_0 \text{ strongly in } \mathbb{L}^2(\mathcal{O}), \\ \boldsymbol{H}_{0m}^{\gamma} &\to \boldsymbol{H}_0 \text{ strongly in } \mathbb{L}^2(\mathcal{O}). \end{aligned} \tag{3.10}$$

Thus, from (3.10), we see that

$$\|\boldsymbol{u}_{0m}^{\gamma}\| \leqslant \|\boldsymbol{u}_0\|, \; \|\mathbf{w}_{0m}^{\gamma}\| \leqslant \|\mathbf{w}_0\|, \; \|\boldsymbol{M}_{0m}^{\gamma}\| \leqslant \|\boldsymbol{M}_0\|, \; \|\boldsymbol{H}_{0m}^{\gamma}\| \leqslant \|\boldsymbol{H}_0\|.$$

We now look for approximate solutions $(\boldsymbol{u}_m^{\gamma}, \mathbf{w}_m^{\gamma}, \boldsymbol{M}_m^{\gamma}, \boldsymbol{H}_m^{\gamma})$ of the BT-ECREs (3.1)–(3.3) of the form

$$\begin{aligned} \boldsymbol{u}_m^{\gamma}(t) &= \sum_{k=1}^{m} a_k^{\gamma,m}(t)\upsilon_k, \\ \mathbf{w}_m^{\gamma}(t) &= \sum_{k=1}^{m} b_k^{\gamma,m}(t)\Lambda_k, \end{aligned}$$

[1] $\{\Lambda_j\}_{j=1}^{\infty}$ is a Hilbert basis in $[H_0^1(\mathcal{O})]^3$ ($[H_0^1(\mathcal{O})]^3$ is endowed with the natural inner product), orthogonal in $[H_0^1(\mathcal{O})]^3$, orthonormal in $\mathbb{L}^2(\mathcal{O})$ and satisfying the spectral problem $-\Delta\Lambda_j = \iota_j\Lambda_j$ in $\mathcal{O}$.

$$\boldsymbol{M}_m^\gamma(t) = \boldsymbol{M}_{m,1}^\gamma(t) + \boldsymbol{M}_{m,2}^\gamma(t) = \sum_{k=1}^m c_k^{\gamma,m}(t)\bar{\psi}_k + \sum_{k=1}^m d_k^{\gamma,m}(t)\nabla\bar{\phi}_k,$$
$$\boldsymbol{H}_m^\gamma(t) = \boldsymbol{H}_{m,1}^\gamma(t) + \boldsymbol{H}_{m,2}^\gamma(t) = \sum_{k=1}^m e_k^{\gamma,m}(t)\bar{\psi}_k - \sum_{k=1}^m d_k^{\gamma,m}(t)\nabla\bar{\phi}_k. \quad (3.11)$$

The decompositions of $\boldsymbol{M}_m^\gamma$ and $\boldsymbol{H}_m^\gamma$ were chosen in such a way that $\boldsymbol{B}_m^\gamma = \mu_0(\boldsymbol{M}_m^\gamma + \boldsymbol{H}_m^\gamma)$ satisfies

$$\operatorname{div} \boldsymbol{B}_m^\gamma = 0 \quad \text{in} \quad Q_T.$$

We observe that $\boldsymbol{B}_m^\gamma(t) = \sum_{k=1}^m (c_k^{\gamma,m}(t) + e_k^{\gamma,m}(t))\bar{\psi}_k$.
Let $k = 1, \ldots, m$. Then, arguing as in [13], there exists a short interval $[0, T_m) \subset [0, T]$ and sequences $a_k^{\gamma,m}, b_k^{\gamma,m}, c_k^{\gamma,m}, d_k^{\gamma,m}, e_k^{\gamma,m} \in \mathcal{C}([0, T_m]) \cap \mathcal{C}^1(]0, T_m[)$ such that the sequence $(\boldsymbol{u}_m^\gamma, \mathbf{w}_m^\gamma, \boldsymbol{M}_m^\gamma, \boldsymbol{H}_m^\gamma, \boldsymbol{B}_m^\gamma)$ defined in (3.11) solve the following approximating problem

$$(\partial_t \boldsymbol{u}_m^\gamma, \upsilon_k) + ((\boldsymbol{u}_m^\gamma \cdot \nabla)\boldsymbol{u}_m^\gamma, \upsilon_k) + \nu(\nabla \boldsymbol{u}_m^\gamma, \nabla \upsilon_k) - \mu_0((\boldsymbol{M}_m^\gamma \cdot \nabla)\boldsymbol{H}_m^\gamma, \upsilon_k) \quad (3.12a)$$
$$= \mu_0((\nabla \times \boldsymbol{H}_m^\gamma) \times \boldsymbol{H}_m^\gamma, \upsilon_k) - \alpha(\nabla \times (\nabla \times \boldsymbol{u}_m^\gamma - 2\mathbf{w}_m^\gamma), \upsilon_k),$$
$$(\partial_t \mathbf{w}_m^\gamma, \Lambda_k) + ((\boldsymbol{u}_m^\gamma \cdot \nabla)\mathbf{w}_m^\gamma, \Lambda_k) + (\lambda_1 + \lambda_2)(\operatorname{div} \mathbf{w}_m^\gamma, \operatorname{div} \Lambda_k) + \lambda_1(\nabla \mathbf{w}_m^\gamma, \nabla \Lambda_k) \quad (3.12b)$$
$$= 2\alpha((\nabla \times \boldsymbol{u}_m^\gamma - 2\mathbf{w}_m^\gamma), \Lambda_k) + \mu_0(\boldsymbol{M}_m^\gamma \times \boldsymbol{H}_m^\gamma, \Lambda_k),$$
$$(\partial_t \boldsymbol{M}_m^\gamma, \bar{\psi}_k) + ((\boldsymbol{u}_m^\gamma \cdot \nabla)\boldsymbol{M}_m^\gamma, \bar{\psi}_k) + \gamma(\nabla \times \boldsymbol{M}_m^\gamma, \nabla \times \bar{\psi}_k) - \gamma(\nabla \operatorname{div} \boldsymbol{M}_m^\gamma, \bar{\psi}_k) \quad (3.12c)$$
$$= (\mathbf{w}_m^\gamma \times \boldsymbol{M}_m^\gamma, \bar{\psi}_k) - \frac{1}{\tau}((\boldsymbol{M}_m^\gamma - \chi_0 \boldsymbol{H}_m^\gamma), \bar{\psi}_k) \text{ and}$$
$$(\partial_t \boldsymbol{M}_m^\gamma, \nabla\bar{\phi}_k) + ((\boldsymbol{u}_m^\gamma \cdot \nabla)\boldsymbol{M}_m^\gamma, \nabla\bar{\phi}_k) - \gamma(\nabla \operatorname{div} \boldsymbol{M}_m^\gamma, \nabla\bar{\phi}_k) \quad (3.12d)$$
$$= (\mathbf{w}_m^\gamma \times \boldsymbol{M}_m^\gamma, \nabla\bar{\phi}_k) - \frac{1}{\tau}((\boldsymbol{M}_m^\gamma - \chi_0 \boldsymbol{H}_m^\gamma), \nabla\bar{\phi}_k),$$
$$(\partial_t \boldsymbol{B}_m^\gamma, \bar{\psi}_k) + \frac{1}{\sigma}(\nabla \times (\nabla \times \boldsymbol{H}_m^\gamma), \bar{\psi}_k) = (\nabla \times (\boldsymbol{u}_m^\gamma \times \boldsymbol{B}_m^\gamma), \bar{\psi}_k) \text{ and} \quad (3.12e)$$
$$(\partial_t \boldsymbol{B}_m^\gamma, \nabla\bar{\phi}_k) = 0. \quad (3.12f)$$

It will follow from the a priori estimates below that $(\boldsymbol{u}_m^\gamma, \mathbf{w}_m^\gamma, \boldsymbol{M}_m^\gamma, \boldsymbol{H}_m^\gamma)$ may be extended to the whole interval $[0, T]$.

Step 2: Uniform estimates for the approximating sequences

Multiplying (3.12a) by $a_k^{\gamma,m}$ and summing over k, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_m^\gamma\|^2 + \nu\|\nabla \boldsymbol{u}_m^\gamma\|^2 - \mu_0 \int_{\mathcal{O}} (\boldsymbol{M}_m^\gamma \cdot \nabla)\boldsymbol{H}_m^\gamma \cdot \boldsymbol{u}_m^\gamma \,\mathrm{d}x$$
$$= \mu_0 \int_{\mathcal{O}} [(\nabla \times \boldsymbol{H}_m^\gamma) \times \boldsymbol{H}_m^\gamma] \cdot \boldsymbol{u}_m \,\mathrm{d}x - \alpha \int_{\mathcal{O}} [\nabla \times (\nabla \times \boldsymbol{u}_m^\gamma - 2\mathbf{w}_m^\gamma)] \cdot \boldsymbol{u}^\gamma \,\mathrm{d}x. \quad (3.13)$$

As in [13], we rewrite the third term on the left-hand side of (3.13) as follows

$$\begin{aligned}\int_{\mathcal{O}} (\boldsymbol{M}_m^\gamma \cdot \nabla)\boldsymbol{H}_m^\gamma \cdot \boldsymbol{u}_m^\gamma \,\mathrm{d}x &= \int_{\mathcal{O}} \operatorname{curl} \boldsymbol{H}_m^\gamma \cdot (\boldsymbol{M}_m^\gamma \times \boldsymbol{u}_m^\gamma)\,\mathrm{d}x \\ &\quad - \int_{\mathcal{O}} (\boldsymbol{u}_m^\gamma \cdot \nabla)\boldsymbol{M}_m^\gamma \cdot \boldsymbol{H}_m^\gamma \,\mathrm{d}x \\ &= \int_{\mathcal{O}} (\operatorname{curl} \boldsymbol{H}_m^\gamma \times \boldsymbol{M}_m^\gamma) \cdot \boldsymbol{u}_m^\gamma \,\mathrm{d}x \\ &\quad - \int_{\mathcal{O}} (\boldsymbol{u}_m^\gamma \cdot \nabla)\boldsymbol{M}_m^\gamma \cdot \boldsymbol{H}_m^\gamma \,\mathrm{d}x. \end{aligned} \tag{3.14}$$

Now, plugging (3.14) into (3.13) yields:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_m^\gamma\|^2 + \nu\|\nabla \boldsymbol{u}_m^\gamma\|^2 - \int_{\mathcal{O}} [(\nabla \times \boldsymbol{H}_m^\gamma) \times \boldsymbol{B}_m^\gamma] \cdot \boldsymbol{u}_m^\gamma \,\mathrm{d}x \\ &\quad + \mu_0 \int_{\mathcal{O}} (\boldsymbol{u}_m^\gamma \cdot \nabla)\boldsymbol{M}_m^\gamma \cdot \boldsymbol{H}_m^\gamma \,\mathrm{d}x \\ &\quad = -\alpha \int_{\mathcal{O}} [\nabla \times (\nabla \times \boldsymbol{u}_m^\gamma)] \cdot \boldsymbol{u}_m^\gamma \,\mathrm{d}x + 2\alpha \int_{\mathcal{O}} [\nabla \times \mathbf{w}_m^\gamma] \cdot \boldsymbol{u}_m^\gamma \,\mathrm{d}x. \end{aligned} \tag{3.15}$$

Multiplying (3.12b) by $b_k^{\gamma,m}$ and summing over k the resulting equation, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}_m^\gamma\|^2 + (\lambda_1 + \lambda_2)\|\operatorname{div} \mathbf{w}_m^\gamma\|^2 + \lambda_1\|\nabla \mathbf{w}_m^\gamma\|^2 \\ &\quad = \mu_0 \int_{\mathcal{O}} (\boldsymbol{M}_m^\gamma \times \boldsymbol{H}_m^\gamma) \cdot \mathbf{w}_m^\gamma \,\mathrm{d}x + 2\alpha \int_{\mathcal{O}} (\nabla \times \boldsymbol{u}_m^\gamma - 2\mathbf{w}_m^\gamma) \cdot \mathbf{w}_m^\gamma \,\mathrm{d}x. \end{aligned}$$

Moreover, in light of the relations

$$\begin{aligned}&2\alpha \int_{\mathcal{O}} (\nabla \times \boldsymbol{u}_m^\gamma - 2\mathbf{w}_m^\gamma) \cdot \mathbf{w}_m^\gamma \,\mathrm{d}x = 4\alpha \int_{\mathcal{O}} \left(\frac{1}{2}\nabla \times \boldsymbol{u}_m^\gamma - \mathbf{w}_m^\gamma\right) \cdot \mathbf{w}_m^\gamma \,\mathrm{d}x \\ &= -4\alpha \left\|\frac{1}{2}\nabla \times \boldsymbol{u}_m^\gamma - \mathbf{w}_m^\gamma\right\|^2 + \alpha \int_{\mathcal{O}} (\nabla \times \boldsymbol{u}_m^\gamma) \cdot (\nabla \times \boldsymbol{u}_m^\gamma)\,\mathrm{d}x \\ &\quad -2\alpha \int_{\mathcal{O}} [\nabla \times \boldsymbol{u}_m^\gamma] \cdot \mathbf{w}_m^\gamma \,\mathrm{d}x \\ &= -4\alpha \left\|\frac{1}{2}\nabla \times \boldsymbol{u}_m^\gamma - \mathbf{w}_m^\gamma\right\|^2 + \alpha \int_{\mathcal{O}} [\nabla \times (\nabla \times \boldsymbol{u}_m^\gamma)] \cdot \boldsymbol{u}_m^\gamma dx \\ &\quad -2\alpha \int_{\mathcal{O}} [\nabla \times \mathbf{w}_m^\gamma] \cdot \boldsymbol{u}_m^\gamma \,\mathrm{d}x, \end{aligned}$$

we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}_m^\gamma\|^2 + (\lambda_1 + \lambda_2)\|\operatorname{div} \mathbf{w}_m^\gamma\|^2 + \lambda_1\|\nabla \mathbf{w}_m^\gamma\|^2 + 4\alpha \left\|\frac{1}{2}\nabla \times \boldsymbol{u}_m^\gamma - \mathbf{w}_m^\gamma\right\|^2$$

$$
= \mu_0 \int_{\mathcal{O}} (\boldsymbol{M}_m^\gamma \times \boldsymbol{H}_m^\gamma) \cdot \mathbf{w}_m^\gamma \, \mathrm{d}x + \alpha \int_{\mathcal{O}} [\nabla \times (\nabla \times \boldsymbol{u}_m^\gamma)] \cdot \boldsymbol{u}_m^\gamma \, \mathrm{d}x
- 2\alpha \int_{\mathcal{O}} [\nabla \times \mathbf{w}_m^\gamma] \cdot \boldsymbol{u}_m^\gamma \, \mathrm{d}x. \tag{3.16}
$$

Adding up (3.15) and (3.16) side by side, we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{u}_m^\gamma\|^2 + \|\mathbf{w}_m^\gamma\|^2) + \nu\|\nabla \boldsymbol{u}_m^\gamma\|^2 + (\lambda_1 + \lambda_2)\|\operatorname{div} \mathbf{w}_m^\gamma\|^2 + \lambda_1\|\nabla \mathbf{w}_m^\gamma\|^2 \\
&\quad + 4\alpha \left\| \frac{1}{2}\nabla \times \boldsymbol{u}_m^\gamma - \mathbf{w}_m^\gamma \right\|^2 \\
&= \int_{\mathcal{O}} [(\nabla \times \boldsymbol{H}_m^\gamma) \times \boldsymbol{B}_m^\gamma] \cdot \boldsymbol{u}_m^\gamma \, \mathrm{d}x - \mu_0 \int_{\mathcal{O}} (\boldsymbol{u}_m^\gamma \cdot \nabla)\boldsymbol{M}_m^\gamma \cdot \boldsymbol{H}_m^\gamma \, \mathrm{d}x \\
&\quad + \mu_0 \int_{\mathcal{O}} (\boldsymbol{M}_m^\gamma \times \boldsymbol{H}_m^\gamma) \cdot \mathbf{w}_m^\gamma \, \mathrm{d}x.
\end{aligned} \tag{3.17}
$$

Multiplying (3.12c) and (3.12d) by $c_k^{\gamma,m}$ and by $d_k^{\gamma,m}$, respectively, summing over k and adding up the resulting equalities side by side, we obtain

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{M}_m^\gamma\|^2 + \gamma\|\nabla \times \boldsymbol{M}_m^\gamma\|^2 + \gamma\|\operatorname{div} \boldsymbol{M}_m^\gamma\|^2 + \frac{1}{\tau}\|\boldsymbol{M}_m^\gamma\|^2 = \frac{\chi_0}{\tau}\int_{\mathcal{O}} \boldsymbol{H}_m^\gamma \cdot \boldsymbol{M}_m^\gamma \, \mathrm{d}x. \tag{3.18}
$$

We note that in (3.18), we used the fact that

$$
\begin{aligned}
\gamma \sum_{k=1}^{m} (\nabla \times \boldsymbol{M}_m^\gamma, \nabla \times b_k^{\gamma,m}\bar{\psi}_k) &= \gamma \sum_{k=1}^{m} (\nabla \times \boldsymbol{M}_m^\gamma, \nabla \times b_k^{\gamma,m}\bar{\psi}_k + \nabla \times d_k^{\gamma,m}(\nabla\bar{\phi}_k)) \\
&= \gamma \int_{\mathcal{O}} \nabla \times \boldsymbol{M}_m^\gamma \cdot \nabla \times \boldsymbol{M}_m^\gamma \, \mathrm{d}x = \gamma\|\nabla \times \boldsymbol{M}_m^\gamma\|^2.
\end{aligned}
$$

We multiply (3.12c) by $e_k^{\gamma,m}$ and (3.12d) by $-d_k^{\gamma,m}$, add up the corresponding equalities and sum from $k = 1$ to $k = m$. Then we obtain

$$
\begin{aligned}
&\int_{\mathcal{O}} \partial_t \boldsymbol{M}_m^\gamma \cdot \boldsymbol{H}_m^\gamma \, \mathrm{d}x + \int_{\mathcal{O}} (\boldsymbol{u}_m^\gamma \cdot \nabla)\boldsymbol{M}_m^\gamma \cdot \boldsymbol{H}_m^\gamma \, \mathrm{d}x - \gamma \int_{\mathcal{O}} \nabla \operatorname{div} \boldsymbol{M}_m^\gamma \cdot \boldsymbol{H}_m^\gamma \, \mathrm{d}x \\
&\quad + \gamma \int_{\mathcal{O}} (\nabla \times \boldsymbol{M}_m^\gamma) \cdot (\nabla \times \boldsymbol{H}_m^\gamma) \, \mathrm{d}x \\
&\quad = \int_{\mathcal{O}} (\mathbf{w}_m^\gamma \times \boldsymbol{M}_m^\gamma) \cdot \boldsymbol{H}_m^\gamma \, \mathrm{d}x - \frac{1}{\tau}\int_{\mathcal{O}} \boldsymbol{M}_m^\gamma \cdot \boldsymbol{H}_m^\gamma \, \mathrm{d}x + \frac{\chi_0}{\tau}\|\boldsymbol{H}_m^\gamma\|^2.
\end{aligned}
$$

Besides, since $\operatorname{div}(\boldsymbol{H}_m^\gamma + \boldsymbol{M}_m^\gamma) = 0$ and $(\mathbf{w}_m^\gamma \times \boldsymbol{M}_m^\gamma) \cdot \boldsymbol{H}_m^\gamma = (\boldsymbol{M}_m^\gamma \times \boldsymbol{H}_m^\gamma) \cdot \mathbf{w}_m^\gamma$, we further obtain

$$
\begin{aligned}
&-\mu_0 \int_{\mathcal{O}} \partial_t \boldsymbol{M}_m^\gamma \cdot \boldsymbol{H}_m^\gamma \,\mathrm{d}x - \gamma \mu_0 \int_{\mathcal{O}} (\nabla \times \boldsymbol{M}_m^\gamma) \cdot (\nabla \times \boldsymbol{H}_m^\gamma) \,\mathrm{d}x \\
&\quad + \gamma \mu_0 \| \operatorname{div} \boldsymbol{H}_m^\gamma \|^2 + \frac{\mu_0 \chi_0}{\tau} \| \boldsymbol{H}_m^\gamma \|^2 \\
&= \mu_0 \int_{\mathcal{O}} (\boldsymbol{u}_m^\gamma \cdot \nabla) \boldsymbol{M}_m^\gamma \cdot \boldsymbol{H}_m^\gamma \,\mathrm{d}x - \mu_0 \int_{\mathcal{O}} [\boldsymbol{M}_m^\gamma \times \boldsymbol{H}_m^\gamma] \cdot \mathbf{w}_m^\gamma \,\mathrm{d}x \\
&\quad + \frac{\mu_0}{\tau} \int_{\mathcal{O}} \boldsymbol{M}_m^\gamma \cdot \boldsymbol{H}_m^\gamma \,\mathrm{d}x.
\end{aligned} \tag{3.19}
$$

Multiplying (3.12e) and (3.12f) by $e_k^{\gamma,m}$ and by $-d_k^{\gamma,m}$, respectively, and summing over k, we find

$$
\begin{aligned}
&\frac{\mu_0}{2} \frac{\mathrm{d}}{\mathrm{d}t} \| \boldsymbol{H}_m^\gamma \|^2 + \mu_0 \int_{\mathcal{O}} \partial_t \boldsymbol{M}_m^\gamma \cdot \boldsymbol{H}_m^\gamma \,\mathrm{d}x + \frac{1}{\sigma} \| \nabla \times \boldsymbol{H}_m^\gamma \|^2 \\
&\quad = \int_{\mathcal{O}} [\nabla \times (\boldsymbol{u}_m^\gamma \times \boldsymbol{B}_m^\gamma)] \cdot \boldsymbol{H}_m^\gamma \,\mathrm{d}x.
\end{aligned} \tag{3.20}
$$

Adding (3.19) and (3.20) side by side, we get

$$
\begin{aligned}
&\frac{\mu_0}{2} \frac{\mathrm{d}}{\mathrm{d}t} \| \boldsymbol{H}_m^\gamma \|^2 + \gamma \mu_0 \| \operatorname{div} \boldsymbol{H}_m^\gamma \|^2 + \frac{\mu_0 \chi_0}{\tau} \| \boldsymbol{H}_m^\gamma \|^2 + \frac{1}{\sigma} \| \nabla \times \boldsymbol{H}_m^\gamma \|^2 \\
&= \gamma \mu_0 \int_{\mathcal{O}} (\nabla \times \boldsymbol{M}_m^\gamma) \cdot (\nabla \times \boldsymbol{H}_m^\gamma) \,\mathrm{d}x + \int_{\mathcal{O}} [\nabla \times (\boldsymbol{u}_m^\gamma \times \boldsymbol{B}_m^\gamma)] \cdot \boldsymbol{H}_m^\gamma \,\mathrm{d}x \\
&\quad + \mu_0 \int_{\mathcal{O}} (\boldsymbol{u}_m^\gamma \cdot \nabla) \boldsymbol{M}_m^\gamma \cdot \boldsymbol{H}_m^\gamma \,\mathrm{d}x - \mu_0 \int_{\mathcal{O}} [\boldsymbol{M}_m^\gamma \times \boldsymbol{H}_m^\gamma] \cdot \mathbf{w}_m^\gamma \,\mathrm{d}x \\
&\quad + \frac{\mu_0}{\tau} \int_{\mathcal{O}} \boldsymbol{M}_m^\gamma \cdot \boldsymbol{H}_m^\gamma \,\mathrm{d}x.
\end{aligned} \tag{3.21}
$$

Notice that

$$
\int_{\mathcal{O}} [(\nabla \times \boldsymbol{H}_m^\gamma) \times \boldsymbol{B}_m^\gamma] \cdot \boldsymbol{u}_m^\gamma \,\mathrm{d}x = - \int_{\mathcal{O}} [\nabla \times (\boldsymbol{u}_m^\gamma \times \boldsymbol{B}_m^\gamma)] \cdot \boldsymbol{H}_m^\gamma \,\mathrm{d}x.
$$

Summing (3.17), (3.18) and (3.21), integrating the corresponding equality over $[0, t]$, $t \in [0, T_m]$, we arrive at

$$
\begin{aligned}
&12\mathcal{E}_{\mathrm{tot}}(\boldsymbol{u}_m^\gamma(t), \mathbf{w}_m^\gamma(t), \boldsymbol{M}_m^\gamma(t), \boldsymbol{H}_m^\gamma(t)) + \int_0^t [\nu \| \nabla \boldsymbol{u}_m^\gamma \|^2 + (\lambda_1 + \lambda_2) \| \operatorname{div} \mathbf{w}_m^\gamma \|^2] \,\mathrm{d}s \\
&\quad + \int_0^t [\lambda_1 \| \nabla \mathbf{w}_m^\gamma \|^2 + \alpha \left\| \nabla \times \boldsymbol{u}_m^\gamma - 2\mathbf{w}_m^\gamma \right\|^2 + \gamma \| \nabla \times \boldsymbol{M}_m^\gamma \|^2 + \gamma \| \operatorname{div} \boldsymbol{M}_m^\gamma \|^2] \,\mathrm{d}s \\
&\quad + \int_0^t \left[\frac{1}{\tau} \| \boldsymbol{M}_m^\gamma \|^2 + \gamma \mu_0 \| \operatorname{div} \boldsymbol{H}_m^\gamma \|^2 + \frac{\mu_0 \chi_0}{\tau} \| \boldsymbol{H}_m^\gamma \|^2 + \frac{1}{\sigma} \| \nabla \times \boldsymbol{H}_m^\gamma \|^2 \right] \mathrm{d}s
\end{aligned}
$$

$$= \frac{1}{2}\mathcal{E}_{\text{tot}}(\boldsymbol{u}_{0m}, \mathbf{w}_{0m}, \boldsymbol{M}_{0m}, \boldsymbol{H}_{0m}) + \frac{\mu_0 + \chi_0}{\tau} \int_{Q_t} \boldsymbol{H}_m^\gamma \cdot \boldsymbol{M}_m^\gamma \, \mathrm{d}x \, \mathrm{d}s$$
$$+\gamma \mu_0 \int_{Q_t} (\nabla \times \boldsymbol{M}_m^\gamma) \cdot (\nabla \times \boldsymbol{H}_m^\gamma) \, \mathrm{d}x \, \mathrm{d}s \quad \forall t \in [0, T_m], \tag{3.22}$$

having set

$$\mathcal{E}_{\text{tot}}(\boldsymbol{u}_m^\gamma, \mathbf{w}_m^\gamma, \boldsymbol{M}_m^\gamma, \boldsymbol{H}_m^\gamma) = \|\boldsymbol{u}_m^\gamma\|^2 + \|\mathbf{w}_m^\gamma\|^2 + \|\boldsymbol{M}_m^\gamma\|^2 + \mu_0 \|\boldsymbol{H}_m^\gamma\|^2.$$

By Young's inequality, we have

$$\begin{aligned} &\frac{\mu_0 + \chi_0}{\tau} \int_{Q_t} \boldsymbol{H}_m^\gamma \cdot \boldsymbol{M}_m^\gamma \, \mathrm{d}x \, \mathrm{d}s \\ &\leqslant \frac{\mu_0 + \chi_0}{\tau} \int_0^t \|\boldsymbol{H}_m^\gamma\| \|\boldsymbol{M}_m^\gamma\| \mathrm{d}s \\ &\leqslant \frac{1}{2\tau} \int_0^t \|\boldsymbol{M}_m^\gamma\|^2 \mathrm{d}s + \frac{(\mu_0 + \chi_0)^2}{2\tau} \int_0^t \|\boldsymbol{H}_m^\gamma\|^2 \, \mathrm{d}s, \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} &\gamma \mu_0 \int_{Q_t} (\nabla \times \boldsymbol{M}_m^\gamma) \cdot (\nabla \times \boldsymbol{H}_m^\gamma) \, \mathrm{d}x \, \mathrm{d}s \leqslant \gamma \mu_0 \int_{Q_t} \|\nabla \times \boldsymbol{M}_m^\gamma\| \|\nabla \times \boldsymbol{H}_m^\gamma\| \, \mathrm{d}x \, \mathrm{d}s \\ &\leqslant \int_0^t \left[\frac{1}{2\sigma} \|\nabla \times \boldsymbol{H}_m^\gamma\|^2 + \frac{\sigma \gamma^2 \mu_0^2}{2} \|\nabla \times \boldsymbol{M}_m^\gamma\|^2 \right] \mathrm{d}s. \end{aligned} \tag{3.24}$$

Plugging (3.23) and (3.24) into (3.22), for m sufficiently large, we obtain

$$\begin{aligned} &\frac{1}{2}\mathcal{E}_{\text{tot}}(\boldsymbol{u}_m^\gamma(t), \mathbf{w}_m^\gamma(t), \boldsymbol{M}_m^\gamma(t), \boldsymbol{H}_m^\gamma(t)) + \int_0^t [\nu \|\nabla \boldsymbol{u}_m^\gamma\|^2 + (\lambda_1 + \lambda_2) \|\operatorname{div} \mathbf{w}_m^\gamma\|^2] \, \mathrm{d}s \\ &+ \int_0^t [\lambda_1 \|\nabla \mathbf{w}_m^\gamma\|^2 + \alpha \left\|\nabla \times \boldsymbol{u}_m^\gamma - 2\mathbf{w}_m^\gamma\right\|^2 + \gamma \|\operatorname{div} \boldsymbol{M}_m^\gamma\|^2] \, \mathrm{d}s \\ &+ \left(\gamma - \frac{\sigma \gamma^2 \mu_0^2}{2}\right) \int_0^t \|\nabla \times \boldsymbol{M}_m^\gamma\|^2 \, \mathrm{d}s \\ &+ \int_0^t \left[\frac{1}{2\tau} \|\boldsymbol{M}_m^\gamma\|^2 + \gamma \mu_0 \|\operatorname{div} \boldsymbol{H}_m^\gamma\|^2 + \frac{\mu_0 \chi_0}{\tau} \|\boldsymbol{H}_m^\gamma\|^2 + \frac{1}{2\sigma} \|\nabla \times \boldsymbol{H}_m^\gamma\|^2 \right] \mathrm{d}s \\ &\leqslant \frac{1}{2}\mathcal{E}_{\text{tot}}(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{M}_0, \boldsymbol{H}_0) + \frac{(\mu_0 + \chi_0)^2}{2\mu_0 \tau} \int_0^t \mathcal{E}_{\text{tot}}(\boldsymbol{u}_m^\gamma(s), \mathbf{w}_m^\gamma(s), \boldsymbol{M}_m^\gamma(s), \boldsymbol{H}_m^\gamma(s)) \, \mathrm{d}s, \end{aligned} \tag{3.25}$$

where we used the inequality

$$\mathcal{E}_{\text{tot}}(\boldsymbol{u}_m^\gamma, \mathbf{w}_m^\gamma, \boldsymbol{M}_m^\gamma, \boldsymbol{H}_m^\gamma) \leq \mathcal{E}_{\text{tot}}(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{M}_0, \boldsymbol{H}_0).$$

Thus, applying Grönwall's inequality to (3.25) for the function

$$y(t)=\frac{1}{2}\mathcal{E}_{\text{tot}}(\boldsymbol{u}_m^\gamma(t),\mathbf{w}_m^\gamma(t),\boldsymbol{M}_m^\gamma(t),\boldsymbol{H}_m^\gamma(t)),$$

we find

$$\begin{aligned}
&\frac{1}{2}\mathcal{E}_{\text{tot}}(\boldsymbol{u}_m^\gamma(t),\mathbf{w}_m^\gamma(t),\boldsymbol{M}_m^\gamma(t),\boldsymbol{H}_m^\gamma(t))+\int_0^t[\nu\|\nabla\boldsymbol{u}_m^\gamma\|^2+(\lambda_1+\lambda_2)\|\operatorname{div}\mathbf{w}_m^\gamma\|^2]\,\mathrm{d}s\\
&\quad+\int_0^t[\lambda_1\|\nabla\mathbf{w}_m^\gamma\|^2+\alpha\left\|\nabla\times\boldsymbol{u}_m^\gamma-2\mathbf{w}_m^\gamma\right\|^2+\gamma\|\operatorname{div}\boldsymbol{M}_m^\gamma\|^2]\,\mathrm{d}s\\
&\quad+\left(\gamma-\frac{\sigma\gamma^2\mu_0^2}{2}\right)\int_0^t\|\nabla\times\boldsymbol{M}_m^\gamma\|^2\,\mathrm{d}s\\
&\quad+\int_0^t\left[\frac{1}{2\tau}\|\boldsymbol{M}_m^\gamma\|^2+\gamma\mu_0\|\operatorname{div}\boldsymbol{H}_m^\gamma\|^2+\frac{\mu_0\chi_0}{\tau}\|\boldsymbol{H}_m^\gamma\|^2+\frac{1}{2\sigma}\|\nabla\times\boldsymbol{H}_m^\gamma\|^2\right]\mathrm{d}s\\
&\leqslant C(\mathcal{E}_{\text{tot}}(\boldsymbol{u}_0,\mathbf{w}_0,\boldsymbol{M}_0,\boldsymbol{H}_0)+1),
\end{aligned}\tag{3.26}$$

where C only depends on χ_0, μ_0, and τ. With (3.26) we can conclude that the solution $(\boldsymbol{u}_m^\gamma,\mathbf{w}_m^\gamma,\boldsymbol{M}_m^\gamma,\boldsymbol{H}_m^\gamma)$ can be extended on the whole interval $[0,T]$. Moreover, owing to (3.26), we have the following uniform bounds with respect to m and γ:

$$\begin{aligned}
&\|\boldsymbol{u}_m^\gamma\|_{L^\infty(0,T;H)}\leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\quad \|\boldsymbol{u}_m^\gamma\|_{L^2(0,T;V)}\leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
&\|\mathbf{w}_m^\gamma\|_{L^\infty(0,T;\mathbb{L}^2(\mathcal{O}))}\leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\ \|\mathbf{w}_m^\gamma\|_{L^2(0,T;[H_0^1(\mathcal{O})]^3)}\leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
&\|\boldsymbol{M}_m^\gamma\|_{L^\infty(0,T;H_{\boldsymbol{n}})}\leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\ \ \|\boldsymbol{H}_m^\gamma\|_{L^\infty(0,T;H_{\boldsymbol{n}})}\leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}).
\end{aligned}\tag{3.27}$$

Additionally,

$$\begin{aligned}
\|\nabla\times\boldsymbol{H}_m^\gamma\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))}&\leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
\|\sqrt{\gamma}\operatorname{div}\boldsymbol{M}_m^\gamma\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))}&\leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
\|\sqrt{\gamma}\operatorname{div}\boldsymbol{H}_m^\gamma\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))}&\leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
\|\sqrt{\gamma}(\nabla\times\boldsymbol{M}_m^\gamma)\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))}&\leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
\left\|\nabla\times\boldsymbol{u}_m^\gamma-2\mathbf{w}_m^\gamma\right\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))}&\leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}).
\end{aligned}$$

Here we used the notation $\mathcal{E}_{\text{tot}}=\mathcal{E}_{\text{tot}}(\boldsymbol{u}_0,\mathbf{w}_0,\boldsymbol{M}_0,\boldsymbol{H}_0)$.

Step 3: Passage to the limit

Thanks to (3.27), we infer that up to a subsequence

$$(\boldsymbol{u}_m^\gamma,\mathbf{w}_m^\gamma,\boldsymbol{M}_m^\gamma,\boldsymbol{H}_m^\gamma)\ \overset{*}{\rightharpoonup}\ (\boldsymbol{u}^\gamma,\mathbf{w}^\gamma,\boldsymbol{M}^\gamma,\boldsymbol{H}^\gamma)\text{ weakly-star in }\ L^\infty(0,T;\mathbb{H}),$$

$$(\boldsymbol{u}_m^\gamma, \mathbf{w}_m^\gamma, \boldsymbol{M}_m^\gamma, \boldsymbol{H}_m^\gamma) \rightharpoonup (\boldsymbol{u}^\gamma, \mathbf{w}^\gamma, \boldsymbol{M}^\gamma, \boldsymbol{H}^\gamma) \text{ weakly in } L^2(0,T;\mathbb{V}). \tag{3.28}$$

Moreover, from uniform bounds (3.27) and similar reasoning as in [13], we infer that up to a subsequence

$$\begin{aligned}(\boldsymbol{u}_m^\gamma, \mathbf{w}_m^\gamma) &\to (\boldsymbol{u}^\gamma, \mathbf{w}^\gamma) \text{ strongly in } L^2(0,T;\mathbb{L}^2(\mathcal{O}) \times \mathbb{L}^2(\mathcal{O})),\\ (\boldsymbol{M}_m^\gamma, \boldsymbol{H}_m^\gamma) &\to (\boldsymbol{M}^\gamma, \boldsymbol{H}^\gamma) \text{ strongly in } L^2(0,T;\mathbb{L}^2(\mathcal{O}) \times \mathbb{L}^2(\mathcal{O})).\end{aligned} \tag{3.29}$$

Next we multiply (3.22) by $\upsilon(t)$, where $\upsilon \in \mathcal{C}_c^\infty(0,T)$ such that $\upsilon(t) \geqslant 0$. After integrating in time over the interval $[0,T]$, we pass to the limit by exploiting (3.10), (3.28) and (3.29). After passage to the limit we find that

$$\begin{aligned}&\int_0^T \left[\frac{1}{2}\mathcal{E}_{\text{tot}}(\boldsymbol{u}^\gamma(t), \mathbf{w}^\gamma(t), \boldsymbol{M}^\gamma(t), \boldsymbol{H}^\gamma(t))\right]\upsilon(t)\,\mathrm{d}t\\ &\quad+\int_0^T\int_0^t [\nu\|\nabla \boldsymbol{u}^\gamma(s)\|^2+(\lambda_1+\lambda_2)\|\operatorname{div}\mathbf{w}^\gamma(s)\|^2]\,\mathrm{d}s\,\upsilon(t)\,\mathrm{d}t\\ &\quad+\int_0^T\int_0^t [\lambda_1\|\nabla \mathbf{w}^\gamma(s)\|^2+\alpha\left\|\nabla\times \boldsymbol{u}^\gamma(s)-2\mathbf{w}^\gamma(s)\right\|^2]\,\mathrm{d}s\,\upsilon(t)\,\mathrm{d}t\\ &\quad+\int_0^T\int_0^t [\gamma\|\nabla\times \boldsymbol{M}^\gamma(s)\|^2+\gamma\|\operatorname{div}\boldsymbol{M}^\gamma(s)\|^2]\,\mathrm{d}s\,\upsilon(t)\,\mathrm{d}t\\ &\quad+\int_0^T\int_0^t \left[\frac{\mu_0\chi_0}{\tau}\|\boldsymbol{H}^\gamma(s)\|^2+\frac{1}{\sigma}\|\nabla\times \boldsymbol{H}^\gamma(s)\|^2\right]\mathrm{d}s\,\upsilon(t)\,\mathrm{d}t\\ &\quad+\int_0^T\int_0^t \left[\frac{1}{\tau}\|\boldsymbol{M}^\gamma(s)\|^2+\gamma\mu_0\|\operatorname{div}\boldsymbol{H}^\gamma(s)\|^2\right]\mathrm{d}s\,\upsilon(t)\,\mathrm{d}t\\ &\leqslant \int_0^T \frac{1}{2}\mathcal{E}_{\text{tot}}(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{M}_0, \boldsymbol{H}_0)\upsilon(t)\,\mathrm{d}t\\ &\quad+\frac{\mu_0+\chi_0}{\tau}\int_0^T\left[\int_{Q_t}\boldsymbol{H}^\gamma(s)\cdot \boldsymbol{M}^\gamma(s)\,\mathrm{d}x\,\mathrm{d}s\right]\upsilon(t)\mathrm{d}t\\ &\quad+\gamma\mu_0\int_0^T\left[\int_{Q_t}(\nabla\times \boldsymbol{M}^\gamma(s))\cdot(\nabla\times \boldsymbol{H}^\gamma(s))\,\mathrm{d}x\,\mathrm{d}s\right]\upsilon(t)\,\mathrm{d}t,\end{aligned} \tag{3.30}$$

which in turn implies that

$$\begin{aligned}&\frac{1}{2}\mathcal{E}_{\text{tot}}(\boldsymbol{u}^\gamma(t), \mathbf{w}^\gamma(t), \boldsymbol{M}^\gamma(t), \boldsymbol{H}^\gamma(t))+\int_0^t [\nu\|\nabla \boldsymbol{u}^\gamma(s)\|^2+(\lambda_1+\lambda_2)\|\operatorname{div}\mathbf{w}^\gamma(s)\|^2]\,\mathrm{d}s\\ &\quad+\int_0^t [\lambda_1\|\nabla \mathbf{w}^\gamma(s)\|^2+\alpha\left\|\nabla\times \boldsymbol{u}^\gamma(s)-2\mathbf{w}^\gamma(s)\right\|^2+\gamma\|\nabla\times \boldsymbol{M}^\gamma(s)\|^2]\,\mathrm{d}s\\ &\quad+\int_0^t [\gamma\|\operatorname{div}\boldsymbol{M}^\gamma(s)\|^2+\frac{1}{\tau}\|\boldsymbol{M}^\gamma(s)\|^2+\gamma\mu_0\|\operatorname{div}\boldsymbol{H}^\gamma(s)\|^2]\,\mathrm{d}s\\ &\quad+\int_0^t \left[\frac{\mu_0\chi_0}{\tau}\|\boldsymbol{H}^\gamma(s)\|^2+\frac{1}{\sigma}\|\nabla\times \boldsymbol{H}^\gamma(s)\|^2\right]\mathrm{d}s\\ &\leqslant \frac{1}{2}\mathcal{E}_{\text{tot}}(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{M}_0, \boldsymbol{H}_0)+\frac{\mu_0+\chi_0}{\tau}\int_{Q_t}\boldsymbol{H}^\gamma(s)\cdot \boldsymbol{M}^\gamma(s)\,\mathrm{d}x\,\mathrm{d}s\end{aligned}$$

$$+\gamma\mu_0 \int_{Q_t} (\nabla \times \boldsymbol{M}^\gamma(s)) \cdot (\nabla \times \boldsymbol{H}^\gamma(s))\, \mathrm{d}x\, \mathrm{d}s, \quad \forall t \in [0, T]. \tag{3.31}$$

Arguing as in [13], we can show that $(\boldsymbol{u}^\gamma, \mathbf{w}^\gamma, \boldsymbol{M}^\gamma, \boldsymbol{H}^\gamma)$ is a weak solution to the problem (3.1a)–(3.1f) in the sense of Definition 3.1.

In order to obtain the energy inequality (3.5), we multiply (3.26) by $\upsilon(t)$, with $\upsilon \in \mathcal{C}_c^\infty(0, T)$ such that $\upsilon(t) \geqslant 0$. Then, after integrating in time, we can pass to the limit by using (3.10), (3.28), and (3.29), and the weak lower semicontinuity of the norms.

Let us now derive an energy inequality similar to (3.5) but which is independent of the parameter γ.
By Young's inequality, we have

$$\frac{\mu_0+\chi_0}{\tau} \int_{Q_t} |\boldsymbol{H}^\gamma \cdot \boldsymbol{M}^\gamma|\, \mathrm{d}x\, \mathrm{d}s \leqslant \frac{1}{2\tau} \int_0^t \|\boldsymbol{M}^\gamma(s)\|^2\, \mathrm{d}s + \frac{(\mu_0+\chi_0)^2}{2\tau} \int_0^t \|\boldsymbol{H}^\gamma(s)\|^2\, \mathrm{d}s.$$

Similarly,

$$\begin{aligned}
&\gamma\mu_0 \int_{Q_t} |(\nabla \times \boldsymbol{M}^\gamma) \cdot (\nabla \times \boldsymbol{H}^\gamma)|\, \mathrm{d}x\, \mathrm{d}s \\
&\quad \leqslant \frac{\sigma\gamma^2\mu_0^2}{2} \int_0^t \|\nabla \times \boldsymbol{M}^\gamma(s)\|^2\, \mathrm{d}s + \frac{1}{2\sigma} \int_0^t \|\nabla \times \boldsymbol{H}^\gamma(s)\|^2\, \mathrm{d}s.
\end{aligned}$$

Plugging the two previous estimates into (3.31) and using the assumption (3.9), we obtain

$$\begin{aligned}
&\frac{1}{2}\mathcal{E}_{\mathrm{tot}}(\boldsymbol{u}^\gamma(t), \mathbf{w}^\gamma(t), \boldsymbol{M}^\gamma(t), \boldsymbol{H}^\gamma(t)) + \int_0^t \mathcal{F}(\boldsymbol{u}^\gamma(s), \mathbf{w}^\gamma(s), \boldsymbol{M}^\gamma(s), \boldsymbol{H}^\gamma(s))\, \mathrm{d}s \\
&\quad \leqslant \frac{1}{2}\mathcal{E}_{\mathrm{tot}}(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{M}_0, \boldsymbol{H}_0) + \frac{\mu_0^2 + \chi_0^2}{2\tau} \int_0^t \|\boldsymbol{H}^\gamma(s)\|^2\, \mathrm{d}s.
\end{aligned} \tag{3.32}$$

We note that in (3.32) we used the fact that $\frac{\gamma}{2} < \frac{\gamma(2-\sigma\gamma\mu_0^2)}{2}$, since $\gamma \in (0, \frac{1}{\sigma\mu_0^2})$.
Applying Grönwall's inequality to (3.32) for the function

$$\begin{aligned}
X(t) &:= \mathcal{E}_{\mathrm{tot}}(\boldsymbol{u}^\gamma(t), \mathbf{w}^\gamma(t), \boldsymbol{M}^\gamma(t), \boldsymbol{H}^\gamma(t)) \\
&= \|\boldsymbol{u}^\gamma(t)\|^2 + \|\mathbf{w}^\gamma(t)\|^2 + \|\boldsymbol{M}^\gamma(t)\|^2 + \mu_0\|\boldsymbol{H}^\gamma(t)\|^2,
\end{aligned}$$

we find

$$\begin{aligned}
&\mathcal{E}_{\mathrm{tot}}(\boldsymbol{u}^\gamma(t), \mathbf{w}^\gamma(t), \boldsymbol{M}^\gamma(t), \boldsymbol{H}^\gamma(t)) + 2\int_0^t \mathcal{F}(\boldsymbol{u}^\gamma(s), \mathbf{w}^\gamma(s), \boldsymbol{M}^\gamma(s), \boldsymbol{H}^\gamma(s))\, \mathrm{d}s \\
&\quad \leqslant C[1 + \mathcal{E}_{\mathrm{tot}}(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{M}_0, \boldsymbol{H}_0)],
\end{aligned} \tag{3.33}$$

for some positive constant C which may depend on μ_0, χ_0, τ, and T, but is independent of $\gamma \in (0, 1)$. This completes the proof of the Proposition 3.5.

3.3 Completion of the Proof of Theorem 2.4: Passage to the Limit

We will now prove the existence of weak solutions to Problem (1.1)–(1.3). Before we do that, we shall state and prove a property of $\boldsymbol{H}^\gamma$ which will play a crucial role in the subsequent analysis.

Proposition 3.6 *Let the assumptions of Proposition 3.2 hold. Let $(\boldsymbol{u}^\gamma, \mathbf{w}^\gamma, \boldsymbol{M}^\gamma, \boldsymbol{H}^\gamma)$ be a weak solution of (3.1)–(3.3). Then, there exist two functions $\boldsymbol{H}_a^\gamma$ and $\boldsymbol{H}_d^\gamma$ such that $\boldsymbol{H}^\gamma = \boldsymbol{H}_a^\gamma + \boldsymbol{H}_d^\gamma$ and*

$$\begin{aligned}
&\boldsymbol{H}_a^\gamma \in L^\infty(0, T; H_{\boldsymbol{n}}) \cap L^2(0, T; V),\\
&\boldsymbol{H}_d^\gamma \in L^\infty(0, T; \mathbb{L}^2(\mathcal{O})) \cap L^2(0, T; \mathbb{H}^1(\mathcal{O})),\\
&\operatorname{div} \boldsymbol{H}_a^\gamma = 0 \quad \textit{and} \quad \operatorname{curl} \boldsymbol{H}_a^\gamma = \operatorname{curl} \boldsymbol{H}^\gamma \ \textit{in} \ \ Q_T,\\
&\boldsymbol{H}_d^\gamma = \nabla \varphi_d^\gamma, \quad \operatorname{curl} \boldsymbol{H}_d^\gamma = 0, \ \ \operatorname{div} \boldsymbol{H}_d^\gamma = -\operatorname{div} \boldsymbol{M}^\gamma \ \textit{in} \ \ Q_T,
\end{aligned} \tag{3.34}$$

where for almost every $t \in [0, T]$, $\nabla \varphi_d^\gamma(t) \in E_2(\mathcal{O})$ and the potential φ_d^γ solves the problem

$$\begin{cases} -\Delta \varphi_d^\gamma = \operatorname{div} \boldsymbol{M}^\gamma & \textit{in} \ \ Q_T, \\ \frac{\partial \varphi_d^\gamma}{\partial \boldsymbol{n}} = -\boldsymbol{M}^\gamma \cdot \boldsymbol{n} & \textit{on} \ \ \Sigma, \end{cases} \tag{3.35}$$

in the distributional sense.

Proof Let $(\boldsymbol{u}^\gamma, \mathbf{w}^\gamma, \boldsymbol{M}^\gamma, \boldsymbol{H}^\gamma)$, $\gamma > 0$, be a weak solution to the BT-ECREs. Then, by Definition 3.1 $(\boldsymbol{u}^\gamma, \mathbf{w}^\gamma, \boldsymbol{M}^\gamma, \boldsymbol{H}^\gamma) \in L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V})$. Furthermore, since a.e. $t \in [0, T]$, $\boldsymbol{H}^\gamma \in \mathbb{L}^2(\mathcal{O}) \cap E(\mathcal{O})$, we infer from [23, Corollary 5.5] that for a.e. $t \in [0, T]$, there exists a unique $\boldsymbol{H}_a^\gamma \in E(\mathcal{O})$ with $\operatorname{div} \boldsymbol{H}_a^\gamma = 0$ in $\mathcal{O}$, and $\boldsymbol{H}_a^\gamma \cdot \boldsymbol{n} = 0$ on $\partial\mathcal{O}$ and a unique $\nabla \varphi_d^\gamma \in E_2(\mathcal{O}) \cap E_3(\mathcal{O})$ such that $\boldsymbol{H}^\gamma = \boldsymbol{H}_a^\gamma + \nabla \varphi_d^\gamma$ and

$$\begin{cases} \Delta \varphi_d^\gamma = \operatorname{div} \boldsymbol{H}^\gamma = -\operatorname{div} \boldsymbol{M}^\gamma & \text{in} \ \ \mathcal{O}, \\ \frac{\partial \varphi_d^\gamma}{\partial \boldsymbol{n}} = \nabla \varphi_d^\gamma \cdot \boldsymbol{n} = \boldsymbol{H}^\gamma \cdot \boldsymbol{n} & \text{on} \ \ \partial\mathcal{O}. \end{cases}$$

The spaces $E_2(\mathcal{O})$ and $E_3(\mathcal{O})$ have been defined in Sect. 2. We put $\boldsymbol{H}_d^\gamma = \nabla \varphi_d^\gamma$. It is clear that $\operatorname{curl} \boldsymbol{H}_d^\gamma = 0$ and $\boldsymbol{H}^\gamma \cdot \boldsymbol{n} = \boldsymbol{H}_d^\gamma \cdot \boldsymbol{n} = \frac{\partial \varphi_d^\gamma}{\partial \boldsymbol{n}} = -\boldsymbol{M}^\gamma \cdot \boldsymbol{n}$. Now, since by the energy inequality (3.33), $\boldsymbol{H}^\gamma \in L^\infty(0, T; H_{\boldsymbol{n}})$, thus we infer from [23, Theorem 1.4] that

$$\boldsymbol{H}_a^\gamma \in L^\infty(0, T; H) \ \text{ and } \ \boldsymbol{H}_d^\gamma = \nabla \varphi_d^\gamma \in L^\infty(0, T; \mathbb{L}^2(\mathcal{O})).$$

In addition, we have $\operatorname{curl} \boldsymbol{H}^\gamma \in L^2(0, T; \mathbb{L}^2(\mathcal{O}))$. Thus, we obtain $\boldsymbol{H}_a^\gamma \in L^2(0, T; V)$ because $\operatorname{curl} \boldsymbol{H}_a^\gamma = \operatorname{curl} \boldsymbol{H}^\gamma$ and $\operatorname{div} \boldsymbol{H}_a^\gamma = 0$. Finally, since $\boldsymbol{H}^\gamma \in L^2(0, T; V_1)$,

$\boldsymbol{H}^\gamma \in L^\infty(0,T;H_{\boldsymbol{n}})$ and curl $\boldsymbol{H}^\gamma \in L^2(0,T;\mathbb{L}^2(\mathcal{O}))$, we easily see that $\nabla\varphi_d^\gamma = \boldsymbol{H}_d^\gamma = \boldsymbol{H}^\gamma - \boldsymbol{H}_a^\gamma \in L^\infty(0,T;\mathbb{L}^2(\mathcal{O})) \cap L^2(0,T;\mathbb{H}^1(\mathcal{O}))$. □

We now state and prove the following proposition, which is the main step of the proof of Theorem 2.4.

Proposition 3.7 *Let the assumptions of Theorem 2.4 hold. Let I_γ be the set defined in (3.7). Let $\{\gamma_n\}_{n\in\mathbb{N}} \subset I_\gamma$ be a real sequence converging to 0 as $n \to \infty$, and let $(\boldsymbol{u}^n, \mathbf{w}^n, \boldsymbol{M}^n, \boldsymbol{H}^n)$ be the solution of the BT-ECREs (3.1)–(3.3) with γ replaced by γ_n. Then, as $n \to \infty$ a subsequence of $\{(\boldsymbol{u}^n, \mathbf{w}^n, \boldsymbol{M}^n, \boldsymbol{H}^n)\}_{n\in\mathbb{N}}$ converges to a weak solution of (1.1)–(1.3) satisfying the results of Theorem 2.4.*

Proof From (3.33), we infer that there exists a positive constant C depending possibly on $\mu_0, \sigma, \nu, \lambda_1, \lambda_2, \chi_0, \tau, \alpha$, and T, such that

$$
\begin{aligned}
&\|\boldsymbol{u}^n\|_{L^\infty(0,T;H)} \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
&\|\boldsymbol{u}^n\|_{L^2(0,T;V)} \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
&\|\mathbf{w}^n\|_{L^\infty(0,T;\mathbb{L}^2(\mathcal{O}))} \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
&\|\mathbf{w}^n\|_{L^2(0,T;[H_0^1(\mathcal{O})]^3)} \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
&\|\boldsymbol{M}^n\|_{L^\infty(0,T;H_{\boldsymbol{n}})} \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
&\|\boldsymbol{H}^n\|_{L^\infty(0,T;H_{\boldsymbol{n}})} \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
&\|\nabla\times\boldsymbol{H}^n\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))} \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
&\|\sqrt{\gamma_n}\,\mathrm{div}\,\boldsymbol{M}^n\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))} \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
&\|\sqrt{\gamma_n}\,\mathrm{div}\,\boldsymbol{H}^n\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))} \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
&\|\sqrt{\gamma_n}(\nabla\times\boldsymbol{M}^n)\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))} \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
&\left\|\nabla\times\boldsymbol{u}^n - 2\mathbf{w}^n\right\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))} \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}).
\end{aligned}
\tag{3.36}
$$

Here we used the notation $\mathcal{E}_{\text{tot}} = \mathcal{E}_{\text{tot}}(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{M}_0, \boldsymbol{H}_0)$, and the positive constant C doesn't depend on n.
It follows from (3.36) and the Banach–Alaoglu theorem that (up to a subsequence)

$$
\begin{aligned}
\boldsymbol{u}^n &\overset{*}{\rightharpoonup} \boldsymbol{u} \quad \text{weakly-star in } L^\infty(0,T;H),\\
\boldsymbol{u}^n &\rightharpoonup \boldsymbol{u} \quad \text{weakly in } L^2(0,T;V),\\
\mathbf{w}^n &\overset{*}{\rightharpoonup} \mathbf{w} \quad \text{weakly-star in } L^\infty(0,T;\mathbb{L}^2(\mathcal{O})),\\
\mathbf{w}^n &\rightharpoonup \mathbf{w} \quad \text{weakly in } L^2(0,T;[H_0^1(\mathcal{O})]^3),\\
\boldsymbol{M}^n &\overset{*}{\rightharpoonup} \boldsymbol{M} \quad \text{weakly-star in } L^\infty(0,T;\mathbb{L}^2(\mathcal{O})),\\
\boldsymbol{H}^n &\overset{*}{\rightharpoonup} \boldsymbol{H} \quad \text{weakly-star in } L^\infty(0,T;\mathbb{L}^2(\mathcal{O})).
\end{aligned}
\tag{3.37}
$$

Let $\boldsymbol{B} = \mu_0(\boldsymbol{M} + \boldsymbol{H})$. Since $\operatorname{div}(\boldsymbol{M}^n + \boldsymbol{H}^n) = 0$ in Q_T, then

$$\operatorname{div}(\boldsymbol{M} + \boldsymbol{H}) = 0 \text{ in } \quad Q_T.$$

Estimates of the Time Derivatives

We now proceed to the estimates of the time derivatives of $\boldsymbol{u}^n$ and $\mathbf{w}^n$, which will enable us to apply the Aubin–Lions compactness lemma (see [24]).
By using (3.1b), $(3.36)_1$, $(3.36)_4$–$(3.36)_6$ and $(3.36)_{11}$, we deduce that

$$\int_0^T \|\partial_t \mathbf{w}^n(s)\|^2_{\mathbb{H}_0^{-2}(\mathcal{O})} \, \mathrm{d}s \leqslant C(1 + \mathcal{E}_{\text{tot}}^{\frac{1}{2}})^4, \tag{3.38}$$

with C independent of n. Hence, the sequence $\{\partial_t \mathbf{w}^n\}_n$ is uniformly bounded in $L^2(0, T; \mathbb{H}_0^{-2}(\mathcal{O}))$.

Let $\mathbb{L}^2_{(0)}(\mathcal{O})$ be the space of $\mathbb{L}^2(\mathcal{O})$-functions with vanishing trace. By the embeddings $[H_0^1(\mathcal{O})]^3 \overset{c}{\hookrightarrow} \mathbb{L}^2_{(0)}(\mathcal{O}) \hookrightarrow \mathbb{H}_0^{-2}(\mathcal{O})$, where the first embedding is compact and the second one continuous, we can apply Aubin–Lions compactness lemma and obtain (up to a subsequence) that as $n \to \infty$

$$\mathbf{w}^n \to \mathbf{w} \text{ in } L^2(0, T; \mathbb{L}^2_{(0)}(\mathcal{O})). \tag{3.39}$$

Using an interpolation argument, the Hölder inequality and $(3.36)_4$, we infer that

$$\begin{aligned}
&\int_0^T \|\mathbf{w}^n(t) - \mathbf{w}(t)\|^2_{\mathbb{L}^p(\mathcal{O})} \, \mathrm{d}t \\
&\leqslant C(\mathcal{O}) \int_0^T \|\mathbf{w}^n - \mathbf{w}\|^{\frac{6-p}{p}} \|\nabla(\mathbf{w}^n - \mathbf{w})\|^{\frac{3(p-2)}{p}} \, \mathrm{d}t \\
&\leqslant C(\mathcal{O}) \left(\int_0^T \|\mathbf{w}^n - \mathbf{w}\|^2 \, \mathrm{d}t \right)^{\frac{6-p}{2p}} \left(\int_0^T \|\nabla(\mathbf{w}^n - \mathbf{w})\|^2 \mathrm{d}t \right)^{\frac{3(p-2)}{2p}} \\
&\leqslant C(\mathcal{O}, p)(1 + \mathcal{E}_{\text{tot}}^{\frac{1}{2}})^{\frac{3(p-2)}{p}} \left(\int_0^T \|\mathbf{w}^n - \mathbf{w}\|^2 \, \mathrm{d}t \right)^{\frac{6-p}{2p}},
\end{aligned}$$

from which along with the strong convergence (3.39) of $\mathbf{w}^n$ in $L^2(0, T; \mathbb{L}^2_{(0)}(\mathcal{O}))$, we get

$$\mathbf{w}^n \to \mathbf{w} \quad \text{in} \quad L^2(0, T; \mathbb{L}^p(\mathcal{O})), \quad \forall p \in [2, 6). \tag{3.40}$$

From the variational formulation (3.4), we have

$$\langle \partial_t \boldsymbol{u}^n, \boldsymbol{v} \rangle \; = \int_{\mathcal{O}} (\boldsymbol{u}^n \cdot \nabla) \boldsymbol{v} \cdot \boldsymbol{u}^n \, \mathrm{d}x - \nu \int_{\mathcal{O}} \nabla \boldsymbol{u}^n \cdot \nabla \boldsymbol{v} \, \mathrm{d}x$$

$$
\begin{aligned}
&+\mu_0 \int_{\mathcal{O}} [(\boldsymbol{M}^n + \boldsymbol{H}^n) \cdot \nabla] \boldsymbol{v} \cdot \boldsymbol{H}^n \,\mathrm{d}x \\
&-\alpha \int_{\mathcal{O}} (\operatorname{curl} \boldsymbol{u}^n - 2\mathbf{w}^n) \cdot \operatorname{curl} \boldsymbol{v} \,\mathrm{d}x,
\end{aligned} \tag{3.41}
$$

for all $\boldsymbol{v} \in V$, in particular for any $\boldsymbol{v} \in \mathbb{H}^3_{\text{div}}(\mathcal{O}) = V \cap \mathbb{H}^3(\mathcal{O}) \subset V$.
Using $(3.36)_1$, we deduce that

$$
\begin{aligned}
&\int_0^T \left| \int_{\mathcal{O}} (\boldsymbol{u}^n \cdot \nabla) \boldsymbol{v} \cdot \boldsymbol{u}^n \mathrm{d}x \right|^2 \mathrm{d}s \leqslant \int_{Q_T} |(\boldsymbol{u}^n \cdot \nabla) \boldsymbol{v} \cdot \boldsymbol{u}^n|^2 \,\mathrm{d}x \,\mathrm{d}s \\
&\leqslant C \|\boldsymbol{v}\|^2_{\mathbb{H}^3(\mathcal{O})} \int_0^T \|\boldsymbol{u}_n\|^4 \,\mathrm{d}s \leqslant CT \|\boldsymbol{v}\|^2_{\mathbb{H}^3(\mathcal{O})} (1 + \mathcal{E}_{\text{tot}}^{\frac{1}{2}})^4.
\end{aligned} \tag{3.42}
$$

By $(3.36)_2$, we obtain

$$
\begin{aligned}
\int_0^T \left| \nu \int_{\mathcal{O}} \nabla \boldsymbol{u}^n \cdot \nabla \boldsymbol{v} \,\mathrm{d}x \right|^2 \mathrm{d}s &\leqslant \nu^2 \|\boldsymbol{v}\|_V^2 \int_0^T \|\nabla \boldsymbol{u}^n(t)\|^2 \,\mathrm{d}t \\
&\leqslant C\nu^2 \|\boldsymbol{v}\|_V^2 (1 + \mathcal{E}_{\text{tot}}^{\frac{1}{2}})^2.
\end{aligned} \tag{3.43}
$$

Using the Hölder inequality together with the embedding of H^2 in L^∞, we derive that

$$
\begin{aligned}
\int_{\mathcal{O}} \mu_0 |[(\boldsymbol{M}^n + \boldsymbol{H}^n) \cdot \nabla] \boldsymbol{v} \cdot \boldsymbol{H}^n| \mathrm{d}x &\leqslant \mu_0 \|\nabla \boldsymbol{v}\|_{\mathbb{L}^\infty(\mathcal{O})} \|\boldsymbol{M}^n + \boldsymbol{H}^n\| \|\boldsymbol{H}^n\| \\
&\leqslant C\mu_0 \|\boldsymbol{v}\|_{\mathbb{H}^3(\mathcal{O})} (\|\boldsymbol{M}^n\| + \|\boldsymbol{H}^n\|) \|\boldsymbol{H}^n\|.
\end{aligned}
$$

This, together with $(3.36)_5$ and $(3.36)_6$, implies

$$
\begin{aligned}
&\int_0^T \left| \int_{\mathcal{O}} \mu_0 [(\boldsymbol{M}^n + \boldsymbol{H}^n) \cdot \nabla] \boldsymbol{v} \cdot \boldsymbol{H}^n \mathrm{d}x \right|^2 \mathrm{d}s \\
&\leqslant C\mu_0^2 \|\boldsymbol{v}\|^2_{\mathbb{H}^3(\mathcal{O})} \sup_{t \in [0,T]} \|\boldsymbol{M}^n(t)\|^2 \sup_{t \in [0,T]} \|\boldsymbol{H}^n(t)\|^2 \\
&\quad + C\mu_0^2 \|\boldsymbol{v}\|^2_{\mathbb{H}^3(\mathcal{O})} \sup_{t \in [0,T]} \|\boldsymbol{H}^n(t)\|^4 \leqslant C\mu_0^2 \|\boldsymbol{v}\|^2_{\mathbb{H}^3(\mathcal{O})} (1 + \mathcal{E}_{\text{tot}}^{\frac{1}{2}})^4.
\end{aligned} \tag{3.44}
$$

Notice that

$$
\begin{aligned}
\alpha \int_{\mathcal{O}} |(\operatorname{curl} \boldsymbol{u}^n - 2\mathbf{w}^n) \cdot \operatorname{curl} \boldsymbol{v}| \mathrm{d}x &\leqslant \alpha \|\operatorname{curl} \boldsymbol{u}^n - 2\mathbf{w}^n\| \|\operatorname{curl} \boldsymbol{v}\| \\
&\leqslant \alpha \|\operatorname{curl} \boldsymbol{u}^n - 2\mathbf{w}^n\| \|\boldsymbol{v}\|_V,
\end{aligned}
$$

which along with $(3.36)_{11}$ leads us to

$$
\int_0^T \left| \alpha \int_{\mathcal{O}} (\operatorname{curl} \boldsymbol{u}^n - 2\mathbf{w}^n) \cdot \operatorname{curl} \boldsymbol{v} \,\mathrm{d}x \right|^2 \mathrm{d}s \leqslant C\alpha^2 \|\boldsymbol{v}\|_V^2 (1 + \mathcal{E}_{\text{tot}}^{\frac{1}{2}})^2. \tag{3.45}
$$

Here $C > 0$ is a positive constant that does not depend on n.
Collecting now the estimates (3.42)–(3.45), we may deduce from (3.41) that

$$\int_0^T \|\partial_t \boldsymbol{u}^n(s)\|^2_{\mathbb{H}^{-3}_{\text{div}}(\mathcal{O})} \, ds \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}})^2 + C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}})^4, \tag{3.46}$$

where C is a positive large constant which is independent of n. This proves that the sequence $\{\partial_t \boldsymbol{u}^n\}_n$ is uniformly bounded in $L^2(0,T;\mathbb{H}^{-3}_{\text{div}}(\mathcal{O}))$. Furthermore as $\boldsymbol{u}^n \in L^2(0,T;V)$ (cf. (3.36)$_2$), and since $V \overset{c}{\hookrightarrow} H \hookrightarrow \mathbb{H}^{-3}_{\text{div}}(\mathcal{O})$, where the first embedding is compact and the second one continuous, we can apply again the Aubin–Lions compactness lemma (see [24]) to have

$$\boldsymbol{u}^n \to \boldsymbol{u} \ \text{ in } \ L^2(0,T;H), \tag{3.47}$$

as $n \to \infty$ (up to a subsequence). Hence, by a similar reasoning as in (3.40), we get

$$\boldsymbol{u}^n \to \boldsymbol{u} \ \text{ in } \ L^2(0,T;\mathbb{L}^p(\mathcal{O})), \quad \forall p \in [2,6). \tag{3.48}$$

By integration by parts, using the Hölder inequality, and the embedding of $\mathbb{H}^1(\mathcal{O})$ in $\mathbb{L}^4(\mathcal{O})$, we obtain

$$\begin{aligned}\int_0^T \|(\boldsymbol{u}^n(t)\cdot\nabla)\boldsymbol{M}^n(t)\|^2_{\mathbb{H}^{-2}(\mathcal{O})} \, dt &\leqslant C \sup_{t\in[0,T]} \|\boldsymbol{M}^n(t)\|^2 \int_0^T \|\nabla \boldsymbol{u}^n(t)\|^2 \, dt \\ &\leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}})^4,\end{aligned}$$

where we have also used (3.36)$_2$ and (3.36)$_5$.
From (3.36)$_4$, (3.36)$_5$, and the embedding of $\mathbb{H}^1(\mathcal{O})$ in $\mathbb{L}^4(\mathcal{O})$, we deduce that

$$\begin{aligned}\int_0^T \|\mathbf{w}^n(t)\times \boldsymbol{M}^n(t)\|^2_{(\mathbb{L}^4(\mathcal{O}))'} \, dt &\leqslant C \sup_{t\in[0,T]} \|\boldsymbol{M}^n(t)\|^2 \int_0^T \|\mathbf{w}^n(t)\|^2_{[H_0^1(\mathcal{O})]^3} \, dt \\ &\leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}})^4.\end{aligned}$$

Using the Hölder inequality, the uniform bounds (3.36)$_{10}$ and (3.36)$_8$ together with the fact that $0 < \gamma_n < 1$, we obtain

$$\begin{aligned}\int_0^T \|\gamma_n \nabla\times(\nabla\times \boldsymbol{M}^n)\|^2_{\mathbb{H}^{-1}(\mathcal{O})} \, dt &\leqslant \gamma_n \int_0^T \gamma_n \|\nabla\times \boldsymbol{M}^n\|^2 \, dt \\ &\leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}})^2 \gamma_n \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}})^2,\end{aligned}$$

and

$$\int_0^T \| -\gamma_n \nabla \operatorname{div} \boldsymbol{M}^n(t)\|^2_{\mathbb{H}^{-1}(\mathcal{O})} \,\mathrm{d}t \leqslant \gamma_n \int_0^T \gamma_n \| \operatorname{div} \boldsymbol{M}^n(t)\|^2 \,\mathrm{d}t$$
$$\leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}})^2.$$

Therefore, we are led to

$$\int_0^T \|\partial_t \boldsymbol{M}^n(s)\|^2_{\mathbb{H}^{-2}(\mathcal{O})} \,\mathrm{d}s \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}})^4 + C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}})^2, \tag{3.49}$$

where C is a positive constant which is independent of n. This proves that $\{\partial_t \boldsymbol{M}^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;\mathbb{H}^{-2}(\mathcal{O}))$.
Notice that the following equality holds

$$\partial_t \boldsymbol{H}^n = -\frac{1}{\sigma}\nabla\times(\nabla\times\boldsymbol{H}^n) + \nabla\times(\boldsymbol{u}^n\times\boldsymbol{B}^n) - \partial_t\boldsymbol{M}^n$$

in the weak sense, i.e.,

$$\langle \partial_t \boldsymbol{H}^n, \psi\rangle_{\boldsymbol{X}',\boldsymbol{X}} = -\frac{1}{\sigma}\int_{\mathcal{O}} \operatorname{curl} \boldsymbol{H}^n \cdot \operatorname{curl}\psi \,\mathrm{d}x$$
$$+ \int_{\mathcal{O}} (\boldsymbol{u}^n\times\boldsymbol{B}^n)\cdot \operatorname{curl}\psi \,\mathrm{d}x - \langle \partial_t \boldsymbol{M}^n, \psi\rangle_{\boldsymbol{X}',\boldsymbol{X}} \tag{3.50}$$

for all $\psi \in \boldsymbol{X} := V_1 \cap \mathbb{H}^2(\mathcal{O}) \subset V_1$. This will enable us to obtain a uniform estimate for $\partial_t \boldsymbol{H}^n$. Indeed, let $\psi \in \boldsymbol{X}$ with $\|\psi\|_{\boldsymbol{X}} \leqslant 1$. We denote by $\boldsymbol{X}' = V_1' + \mathbb{H}^{-2}(\mathcal{O})$ its dual space.
Thanks to the Hölder inequality in conjunction with (3.36)$_7$, we obtain

$$\int_0^T \left| \int_{\mathcal{O}} \frac{1}{\sigma} \operatorname{curl} \boldsymbol{H}^n(s)\cdot \operatorname{curl}\psi \,\mathrm{d}x\right|^2 \mathrm{d}s \leqslant \frac{1}{\sigma^2}\int_0^T \|\operatorname{curl} \boldsymbol{H}^n(s)\|^2 \|\operatorname{curl}\psi\|^2 \mathrm{d}s$$
$$\leqslant \frac{1}{\sigma^2}\|\psi\|^2_{\boldsymbol{X}} \|\operatorname{curl} \boldsymbol{H}^n\|^2_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))}$$
$$\leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}})^2. \tag{3.51}$$

Next we claim that the second term on the right-hand side of (3.50) can be split as follows:

$$\int_{\mathcal{O}} (\boldsymbol{u}^n\times\boldsymbol{B}^n)\cdot\operatorname{curl}\psi \,\mathrm{d}x = -\int_{\mathcal{O}} (\boldsymbol{B}^n\cdot\nabla)\psi\cdot\boldsymbol{u}^n \,\mathrm{d}x$$
$$+ \int_{\mathcal{O}} (\boldsymbol{u}^n\cdot\nabla)\psi\cdot\boldsymbol{B}^n \,\mathrm{d}x \quad \text{a.e. in } (0,T). \tag{3.52}$$

For this, we consider the sequence $(\boldsymbol{u}_m^\gamma, \boldsymbol{M}_m^\gamma, \boldsymbol{H}_m^\gamma, \boldsymbol{B}_m^\gamma)$ introduced in **Step 1** and which satisfies (3.27)–(3.29).
By Green's formula (see [12, Eq. (2.18), p. 21]) and since $\psi \times \boldsymbol{n}|_{\partial\mathcal{O}} = 0$, we have

$$\begin{aligned}\int_{\mathcal{O}} (\boldsymbol{u}_m^\gamma \times \boldsymbol{B}_m^\gamma) \cdot \operatorname{curl} \psi \, \mathrm{d}x &= \int_{\mathcal{O}} \operatorname{curl}(\boldsymbol{u}_m^\gamma \times \boldsymbol{B}_m^\gamma) \cdot \psi \, \mathrm{d}x \\ &= \int_{\mathcal{O}} [(\boldsymbol{B}_m^\gamma \cdot \nabla)\boldsymbol{u}_m^\gamma - (\boldsymbol{u}_m^\gamma \cdot \nabla)\boldsymbol{B}_m^\gamma] \cdot \psi \, \mathrm{d}x.\end{aligned}$$

Besides, by integration by parts and the fact that $\operatorname{div} \boldsymbol{B}_m^\gamma = 0$ and $\operatorname{div} \boldsymbol{u}_m^\gamma = 0$ in $\mathcal{O}$, along with the boundary condition for u_m^γ, we obtain

$$\begin{aligned}\int_{\mathcal{O}} [(\boldsymbol{B}_m^\gamma \cdot \nabla)\boldsymbol{u}_m^\gamma - (\boldsymbol{u}_m^\gamma \cdot \nabla)\boldsymbol{B}_m^\gamma] \cdot \psi \, \mathrm{d}x = &- \int_{\mathcal{O}} (\boldsymbol{B}_m^\gamma \cdot \nabla)\psi \cdot \boldsymbol{u}_m^\gamma \, \mathrm{d}x \\ &+ \int_{\mathcal{O}} (\boldsymbol{u}_m^\gamma \cdot \nabla)\psi \cdot \boldsymbol{B}^n \, \mathrm{d}x.\end{aligned}$$

Consequently,

$$\begin{aligned}\int_{\mathcal{O}} (\boldsymbol{u}_m^\gamma(t) \times \boldsymbol{B}_m^\gamma(t)) \cdot \operatorname{curl} \psi \, \mathrm{d}x = &- \int_{\mathcal{O}} (\boldsymbol{B}_m^\gamma(t) \cdot \nabla)\psi \cdot \boldsymbol{u}_m^\gamma(t) \, \mathrm{d}x \\ &+ \int_{\mathcal{O}} (\boldsymbol{u}_m^\gamma(t) \cdot \nabla)\psi \cdot \boldsymbol{B}_m^\gamma(t) \, \mathrm{d}x\end{aligned}$$

for a.e. $t \in [0, T]$. Passing to the limit as $m \to \infty$ and using (3.29), we deduce that

$$\begin{aligned}\int_{\mathcal{O}} (\boldsymbol{u}^\gamma \times \boldsymbol{B}^\gamma) \cdot \operatorname{curl} \psi \, \mathrm{d}x = &- \int_{\mathcal{O}} (\boldsymbol{B}^\gamma \cdot \nabla)\psi \cdot \boldsymbol{u}^\gamma \, \mathrm{d}x \\ &+ \int_{\mathcal{O}} (\boldsymbol{u}^\gamma \cdot \nabla)\psi \cdot \boldsymbol{B}^\gamma \, \mathrm{d}x \quad \text{a.e. in } (0, T).\end{aligned} \tag{3.53}$$

Hence, we reach the desired conclusion, i.e., the claim (3.52) holds.
In light of (3.52), $(3.36)_2$, $(3.36)_5$ and $(3.36)_6$, we deduce that

$$\begin{aligned}\int_0^T \left| \int_{\mathcal{O}} (\boldsymbol{u}^\gamma \times \boldsymbol{B}^\gamma) \cdot \operatorname{curl} \psi \, \mathrm{d}x \right|^2 \mathrm{d}s &\leqslant C \sup_{s \in [0,T]} \|\boldsymbol{B}^n(s)\|^2 \int_0^T \|\boldsymbol{u}^n(s)\|^2_{\mathbb{L}^4(\mathcal{O})} \, \mathrm{d}s \\ &\leqslant C \sup_{s \in [0,T]} \|\boldsymbol{B}^n(s)\|^2 \int_0^T \|\nabla \boldsymbol{u}^n(s)\|^2 \, \mathrm{d}s \\ &\leqslant C(1 + \mathcal{E}_{\text{tot}}^{\frac{1}{2}})^4.\end{aligned} \tag{3.54}$$

Finally, invoking (3.49), (3.51), (3.54), and making use of (3.50), we infer that

$$\{\partial_t \boldsymbol{H}^n\}_n \quad \text{is uniformly bounded in} \quad L^2(0, T; \boldsymbol{X}') \quad \text{with respect to} \quad n. \tag{3.55}$$

Next we will estimate $\partial_t \boldsymbol{H}_d^n$.
Let $\boldsymbol{Y} = E_3(\mathcal{O}) \cap E(\mathcal{O}) \cap \mathbb{H}^2(\mathcal{O})$ and $\boldsymbol{Y}'$ its dual space. Let $\psi = \nabla\zeta \in \boldsymbol{Y}$. Then, By integrations by parts and the fact that $\boldsymbol{H}^n \cdot \boldsymbol{n} = \frac{\partial \varphi_d^n}{\partial \boldsymbol{n}} = \nabla\varphi_d^n \cdot \boldsymbol{n}$, we have

$$\begin{aligned}
\int_{\mathcal{O}} \partial_t \boldsymbol{H}_d^n \cdot \psi \,\mathrm{d}x = \int_{\mathcal{O}} \partial_t \nabla\varphi_d^n \cdot \nabla\zeta \,\mathrm{d}x &= \int_{\partial\mathcal{O}} \zeta\, \partial_t \nabla\varphi_d^n \cdot \boldsymbol{n} \,\mathrm{d}S - \int_{\mathcal{O}} \zeta\, \partial_t \Delta\varphi_d^n \,\mathrm{d}x \\
&= \int_{\partial\mathcal{O}} \zeta\, \partial_t \nabla\varphi_d^n \cdot \boldsymbol{n} \,\mathrm{d}S + \int_{\mathcal{O}} \zeta\, \partial_t \operatorname{div} \boldsymbol{M}^n \,\mathrm{d}x \\
&= \int_{\partial\mathcal{O}} \zeta\, \partial_t \nabla\varphi_d^n \cdot \boldsymbol{n} \,\mathrm{d}S - \int_{\mathcal{O}} \zeta \operatorname{div} \partial_t \boldsymbol{H}^n \,\mathrm{d}x \\
&= \int_{\partial\mathcal{O}} \zeta\, \partial_t \nabla\varphi_d^n \cdot \boldsymbol{n}\mathrm{d}S \\
&\quad - \left(\int_{\partial\mathcal{O}} \zeta\, \partial_t \boldsymbol{H}^n \cdot \boldsymbol{n}\mathrm{d}S - \int_{\mathcal{O}} \nabla\zeta \cdot \partial_t \boldsymbol{H}^n \,\mathrm{d}x \right) \\
&= \int_{\mathcal{O}} \nabla\zeta \cdot \partial_t \boldsymbol{H}^n \mathrm{d}x = \int_{\mathcal{O}} \partial_t \boldsymbol{H}^n \cdot \psi \,\mathrm{d}x.
\end{aligned} \tag{3.56}$$

In (3.56), we have also used the fact $\operatorname{div} \boldsymbol{H}^n = -\operatorname{div} \boldsymbol{M}^n$ in $\mathcal{O}$, and

$$\partial_t \Delta\varphi_d^n = \operatorname{div} \partial_t \boldsymbol{H}_d^n = -\operatorname{div} \partial_t \boldsymbol{M}^n = \operatorname{div} \partial_t \boldsymbol{H}^n \quad \text{in} \quad Q_T.$$

Hence, from (3.56), we infer that

$$\begin{aligned}
\sup_{\|\psi\|_{\boldsymbol{Y}} \leqslant 1} |\langle \partial_t \boldsymbol{H}_d^n, \psi \rangle| &\leqslant \sup_{\|\psi\|_{\boldsymbol{Y}} \leqslant 1} \|\partial_t \boldsymbol{H}^n\|_{\mathbb{H}^{-2}(\mathcal{O})} \|\psi\|_{\mathbb{H}^2(\mathcal{O})} \\
&\leqslant \sup_{\|\psi\|_{\boldsymbol{Y}} \leqslant 1} \|\partial_t \boldsymbol{H}^n\|_{\boldsymbol{X}'} \|\psi\|_{\boldsymbol{Y}} = \|\partial_t \boldsymbol{H}^n\|_{\boldsymbol{X}'},
\end{aligned}$$

from which and (3.55), we derive that $\{\partial_t \boldsymbol{H}_d^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(0, T; \boldsymbol{Y}')$ with respect to n. Now, since $\partial_t \boldsymbol{H}_a^n = \partial_t \boldsymbol{H}^n - \partial_t \boldsymbol{H}_d^n$, we will see that $\partial_t \boldsymbol{H}_a^n$ is uniformly bounded in $L^2(0, T; \boldsymbol{X}' + \boldsymbol{Y}')$.
In fact, let $\boldsymbol{Y}_1 = H \cap \mathbb{H}^2(\mathcal{O})$ and $\boldsymbol{Y}_1'$ its dual space. Let $\psi \in \boldsymbol{Y}_1$. Then, we have

$$\begin{aligned}
\int_{\mathcal{O}} \partial_t \boldsymbol{H}_a^n \cdot \psi \,\mathrm{d}x &= \int_{\mathcal{O}} \partial_t \boldsymbol{H}^n \cdot \psi \,\mathrm{d}x - \int_{\mathcal{O}} \partial_t \boldsymbol{H}_d^n \cdot \psi \,\mathrm{d}x \\
&= \int_{\mathcal{O}} \partial_t \boldsymbol{H}^n \cdot \psi \, dx - \int_{\mathcal{O}} \partial_t \nabla\varphi_d^n \cdot \psi \,\mathrm{d}x \\
&= \int_{\mathcal{O}} \partial_t \boldsymbol{H}^n \cdot \psi \,\mathrm{d}x - \int_{\mathcal{O}} \partial_t \varphi_d^n \operatorname{div} \psi \,\mathrm{d}x \\
&= \int_{\mathcal{O}} \partial_t \boldsymbol{H}^n \cdot \psi \,\mathrm{d}x,
\end{aligned}$$

where we used an integration by parts and the fact that div $\psi = 0$ in $\mathcal{O}$. Hence,

$$\sup_{\|\psi\|_{Y_1} \leqslant 1} |\langle \partial_t \boldsymbol{H}_a^n, \psi \rangle| \leqslant \|\partial_t \boldsymbol{H}^n\|_{\mathbb{H}^{-2}(\mathcal{O})} \leqslant \|\partial_t \boldsymbol{H}^n\|_{\boldsymbol{X}'},$$

from which and (3.55), we also derive that

$$\{\partial_t \boldsymbol{H}_a^n\}_{n\in\mathbb{N}} \text{ is uniformly bounded in } L^2(0, T; \boldsymbol{Y}_1') \text{ with respect to } n. \quad (3.57)$$

Thus, as claimed, $\{\partial_t \boldsymbol{H}_a^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(0, T; \boldsymbol{Y}_1') \cap L^2(0, T; \boldsymbol{X}' + \boldsymbol{Y}')$.

Due to $(3.36)_6$–$(3.36)_7$ and Proposition 3.6, we infer that $\boldsymbol{H}_a^n$ is uniformly bounded in $L^\infty(0, T; \mathbb{L}^2(\mathcal{O}))$, $L^2(0, T; \mathbb{L}^2(\mathcal{O}))$ and $L^2(0, T; \mathbb{H}^1(\mathcal{O}))$, respectively.
Now, since $\{\partial_t \boldsymbol{H}_a^n\}_n$ and $\{\boldsymbol{H}_a^n\}_{n\in\mathbb{N}}$ are uniformly bounded in $L^2(0, T; \boldsymbol{Y}_1')$ and $L^2(0, T; V)$, respectively, and since $\boldsymbol{Y}_1 \overset{c}{\hookrightarrow} H \hookrightarrow \boldsymbol{Y}_1'$, where the first embedding is compact and the second one continuous, we can apply the Aubin–Lions compactness lemma and obtain that up to a subsequence

$$\boldsymbol{H}_a^n \to \boldsymbol{H}_a \text{ in } L^2(0, T; H), \quad (3.58)$$

as $n \to \infty$. We also note that the limit function $\boldsymbol{H}_a$ satisfies

$$\boldsymbol{H}_a \in L^\infty(0, T; H) \cap L^2(0, T; V).$$

We now claim that up to a subsequence as $n \to \infty$

$$\boldsymbol{M}^n \to \boldsymbol{M} \text{ strongly in } L^2(0, T; \mathbb{L}^2(\mathcal{O})), \boldsymbol{H}^n \to \boldsymbol{H} \text{ strongly in } L^2(0, T; \mathbb{L}^2(\mathcal{O})). \quad (3.59)$$

This is the most difficult part of the proof of Theorem 2.4 and we will need to use new technique in order to prove these convergence results. The main idea is to use the notion of renormalisation and the decomposition given in Proposition 3.6. We start with noticing that similar to (3.18), we have

$$\begin{aligned} &\frac{1}{2}\|\boldsymbol{M}^n(t)\|^2 + \gamma_n \int_0^t (\|\operatorname{curl} \boldsymbol{M}^n(s)\|^2 + \|\operatorname{div} \boldsymbol{M}^n(s)\|^2)\,\mathrm{d}s \\ &\quad = \frac{1}{2}\|\boldsymbol{M}_0\|^2 - \frac{1}{\tau}\int_{Q_t} (|\boldsymbol{M}^n(s)|^2 - \chi_0 \boldsymbol{H}^n(s)\cdot \boldsymbol{M}^n(s))\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \quad (3.60)$$

By dropping the second (positive) term on the left-hand side of (3.60), we obtain

$$\begin{aligned} \frac{1}{2}\int_{\mathcal{O}} |\boldsymbol{M}^n(t,x)|^2\,\mathrm{d}x \leqslant \frac{1}{2}\|\boldsymbol{M}_0\|^2 - \frac{1}{\tau}\int_0^t \int_{\mathcal{O}} &\times (|\boldsymbol{M}^n(s)|^2 \\ &- \chi_0 \boldsymbol{H}^n(s)\cdot \boldsymbol{M}^n(s))\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \quad (3.61)$$

Let $\psi_1 \in \mathcal{C}_c^1([0,T); V^1_{\text{div}} \cap \mathbb{H}^2(\mathcal{O}))$ be arbitrary but fixed. Next multiplying (3.1c) by ψ_1, integrating over Q_T the resulting equation and using an integration by parts, we infer that

$$\begin{aligned}&-\int_{Q_T} \boldsymbol{M}^n \cdot \partial_t \psi_1 \,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} (\boldsymbol{u}^n \cdot \nabla)\psi_1 \cdot \boldsymbol{M}^n \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \gamma_n \int_{Q_T} (\nabla \times \boldsymbol{M}^n) \cdot (\nabla \times \psi_1)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\mathcal{O}} \boldsymbol{M}_0 \cdot \psi_1(0)\,\mathrm{d}x + \int_{Q_T} (\mathbf{w}^n \times \boldsymbol{M}^n) \cdot \psi_1 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \frac{1}{\tau}\int_{Q_T} (\boldsymbol{M}^n - \chi_0 \boldsymbol{H}^n)\cdot \psi_1 \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.62}$$

We rewrite the second term on the right-hand side of (3.62) as follows

$$\begin{aligned}&\int_{Q_T} (\mathbf{w}^n \times \boldsymbol{M}^n) \cdot \psi_1 \,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} (\mathbf{w} \times \boldsymbol{M}) \cdot \psi_1 \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_{Q_T} \boldsymbol{M}^n \cdot ((\mathbf{w}^n - \mathbf{w}) \times \psi_1)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_{Q_T} (\boldsymbol{M}^n - \boldsymbol{M}) \cdot (\mathbf{w} \times \psi_1)\,\mathrm{d}x\,\mathrm{d}t \equiv I^2_{1n} + I^2_{2n}. \end{aligned} \tag{3.63}$$

By the Hölder inequality and $(3.36)_5$, we see that

$$\begin{aligned}|I^2_{1n}| &\leqslant \int_0^T \|\boldsymbol{M}^n(t)\| \|\mathbf{w}^n(t) - \mathbf{w}(t)\|_{\mathbb{L}^4(\mathcal{O})} \|\psi_1(t)\|_{\mathbb{L}^4(\mathcal{O})}\,\mathrm{d}t \\ &\leqslant T^{\frac{1}{2}} \sup_{t\in[0,T]} \|\boldsymbol{M}^n(t)\| \sup_{t\in[0,T)} \|\psi_1(t)\|_{\mathbb{L}^4(\mathcal{O})} \|\|\mathbf{w}^n - \mathbf{w}\|_{L^2(0,T;\mathbb{L}^4(\mathcal{O}))} \\ &\leqslant C \sup_{t\in[0,T)} \|\psi_1(t)\|_{\mathbb{L}^4(\mathcal{O})} (1 + \mathcal{E}_{\text{tot}}^{\frac{1}{2}}) \|\mathbf{w}^n - \mathbf{w}\|_{L^2(0,T;\mathbb{L}^4(\mathcal{O}))}, \end{aligned}$$

where the constant $C > 0$ is independent of n. Hence, from this inequality and (3.48), we infer that $I^2_{1n} \to 0$ as $n \to \infty$. Furthermore, thanks to $(3.37)_5$, one has

$$I^2_{2n} = -\int_{Q_T} (\boldsymbol{M}^n - \boldsymbol{M}) \cdot (\mathbf{w} \times \psi_1)\,\mathrm{d}x\,\mathrm{d}t \to 0 \text{ as } n \to \infty.$$

Hence, we deduce that

$$\int_{Q_T} (\mathbf{w}^n \times \boldsymbol{M}^n) \cdot \psi_1 \,\mathrm{d}x\,\mathrm{d}t \to \int_{Q_T} (\mathbf{w} \times \boldsymbol{M}) \cdot \psi_1 \,\mathrm{d}x\,\mathrm{d}t \quad \text{as} \quad n \to \infty. \tag{3.64}$$

For the first nonlinear term on the left-hand side of (3.62), we have

$$\int_{Q_T} (\boldsymbol{u}^n \cdot \nabla)\psi_1 \cdot \boldsymbol{M}^n \,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} (\boldsymbol{u} \cdot \nabla)\psi_1 \cdot \boldsymbol{M} \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{Q_T} [(\boldsymbol{u}^n - \boldsymbol{u}) \cdot \nabla]\psi_1 \cdot \boldsymbol{M}^n \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} (\boldsymbol{u} \cdot \nabla)\psi_1 \cdot (\boldsymbol{M}^n - \boldsymbol{M}) \,\mathrm{d}x\,\mathrm{d}t.$$

Using the Hölder inequality, the estimate $(3.36)_5$ and the strong convergence (3.48), we obtain

$$\left| \int_{Q_T} [(\boldsymbol{u}^n - \boldsymbol{u}) \cdot \nabla]\psi_1 \cdot \boldsymbol{M}^n \,\mathrm{d}x\,\mathrm{d}t \right|$$
$$\leqslant \int_0^T \|\nabla\psi_1(t)\|_{\mathbb{L}^4(\mathcal{O})} \|\boldsymbol{u}^n(t) - \boldsymbol{u}(t)\|_{\mathbb{L}^4(\mathcal{O})} \|\boldsymbol{M}^n(t)\| \,\mathrm{d}t$$
$$\leqslant C \sup_{t\in[0,T)} \|\psi_1(t)\|_{\mathbb{H}^2(\mathcal{O})} \sup_{t\in[0,T]} \|\boldsymbol{M}^n(t)\| \|\mathbf{w}^n - \mathbf{w}\|_{L^2(0,T;\mathbb{L}^4(\mathcal{O}))}$$
$$\leqslant C \sup_{t\in[0,T)} \|\psi_1(t)\|_{\mathbb{H}^2(\mathcal{O})} (1 + \mathcal{E}_{\text{tot}}^{\frac{1}{2}}) \|\mathbf{w}^n - \mathbf{w}\|_{L^2(0,T;\mathbb{L}^4(\mathcal{O}))} \to 0 \quad \text{as} \quad n \to \infty.$$

Since $(\boldsymbol{u} \cdot \nabla)\psi_1 \in L^1(0, T; \mathbb{L}^2(\mathcal{O}))$, we deduce from $(3.37)_6$ that

$$\int_{Q_T} (\boldsymbol{u} \cdot \nabla)\psi_1 \cdot (\boldsymbol{M}^n - \boldsymbol{M}) \,\mathrm{d}x\,\mathrm{d}t \to 0 \text{ as } n \to \infty.$$

Using the two previous convergences, we see that

$$\int_{Q_T} (\boldsymbol{u}^n \cdot \nabla)\psi_1 \cdot \boldsymbol{M}^n \,\mathrm{d}x\,\mathrm{d}t \to \int_{Q_T} (\boldsymbol{u} \cdot \nabla)\psi_1 \cdot \boldsymbol{M} \,\mathrm{d}x\,\mathrm{d}t.$$

By $(3.36)_{10}$, we obtain

$$\gamma_n \int_{Q_T} (\nabla \times \boldsymbol{M}^n) \cdot (\nabla \times \psi_1) \,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \sqrt{\gamma_n} \|\sqrt{\gamma_n}(\nabla \times \boldsymbol{M}^n)\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))} \|\nabla \times \psi_1\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))}$$
$$\leqslant C(1 + \mathcal{E}_{\text{tot}}^{\frac{1}{2}}) \|\nabla \times \psi_1\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))} \sqrt{\gamma_n} \to 0,$$

since the constant $C > 0$ is independent of n and $\gamma_n \to 0$ as $n \to \infty$. Therefore, we proved that

$$\gamma_n \int_{Q_T} (\nabla \times \boldsymbol{M}^n) \cdot (\nabla \times \psi_1) \,\mathrm{d}x\,\mathrm{d}t \to 0 \quad \text{as} \quad n \to \infty. \tag{3.65}$$

Combining (3.64)–(3.65) along with $(3.37)_5$ and $(3.37)_6$, after passing to the limit in (3.62), we are led to

$$
\begin{aligned}
&-\int_{Q_T} \boldsymbol{M}\cdot\partial_t\psi_1\,\mathrm{d}x\,\mathrm{d}t-\int_{Q_T}(\boldsymbol{u}\cdot\nabla)\psi_1\cdot\boldsymbol{M}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_{\mathcal{O}}\boldsymbol{M}_0\cdot\psi_1(0)\,\mathrm{d}x+\int_{Q_T}(\mathbf{w}\times\boldsymbol{M})\cdot\psi_1\,\mathrm{d}x\,\mathrm{d}t-\frac{1}{\tau}\int_{Q_T}(\boldsymbol{M}-\chi_0\boldsymbol{H})\cdot\psi_1\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{3.66}
$$

Let $\nabla\phi := \zeta_1\nabla\zeta_2 \in \mathcal{C}^1_c([0,T];\mathcal{H})$ be arbitrary but fixed, with $\zeta_1 \in \mathcal{C}([0,T])\cap\mathcal{C}^1(0,T)$, $\zeta_1(T)=0$, and $\nabla\zeta_2\in\mathcal{H}$. Then from (3.1c) and Remark 3.3, we infer that

$$
\begin{aligned}
&-\int_{Q_T}\boldsymbol{M}^n\cdot\partial_t(\nabla\phi)\,\mathrm{d}x\,\mathrm{d}t-\int_{Q_T}(\boldsymbol{u}^n\cdot\nabla)\nabla\phi\cdot\boldsymbol{M}^n\,\mathrm{d}x\,\mathrm{d}t-\gamma_n\int_{Q_T}\operatorname{div}\boldsymbol{M}^n\,\Delta\phi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_{\mathcal{O}}\boldsymbol{M}_0\cdot\nabla\phi(0)\,\mathrm{d}x+\int_{Q_T}(\mathbf{w}^n\times\boldsymbol{M}^n)\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t-\frac{1}{\tau}\int_{Q_T}(\boldsymbol{M}^n-\chi_0\boldsymbol{H}^n)\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Next arguing similarly as we did in (3.66), we can pass to the limit in the previous equality. Then we obtain

$$
\begin{aligned}
&-\int_{Q_T}\boldsymbol{M}\cdot\partial_t(\nabla\phi)\,\mathrm{d}x\,\mathrm{d}t-\int_{Q_T}(\boldsymbol{u}\cdot\nabla)\nabla\phi\cdot\boldsymbol{M}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_{\mathcal{O}}\boldsymbol{M}_0\cdot\nabla\phi(0)\,\mathrm{d}x+\int_{Q_T}(\mathbf{w}\times\boldsymbol{M})\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-\frac{1}{\tau}\int_{Q_T}(\boldsymbol{M}-\chi_0\boldsymbol{H})\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{3.67}
$$

From (3.66) and (3.67), we can write

$$
\partial_t\boldsymbol{M}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{M}=\mathbf{w}\times\boldsymbol{M}-\frac{1}{\tau}(\boldsymbol{M}-\chi_0\boldsymbol{H})
\tag{3.68}
$$

in the sense of distributions in Q_T. We are now in position to apply Lemma 2.6 since all its hypotheses are satisfied. Hence, by Lemma 2.6 the weak limit $\boldsymbol{M}$ given by (3.37) satisfies

$$
\frac{1}{2}\int_{\mathcal{O}}|\boldsymbol{M}(t,x)|^2\,\mathrm{d}x=\frac{1}{2}\|\boldsymbol{M}_0\|^2-\frac{1}{\tau}\int_{Q_t}(|\boldsymbol{M}|^2-\chi_0\boldsymbol{M}\cdot\boldsymbol{H})\mathrm{d}x\,\mathrm{d}s\quad\forall t\in[0,T].
\tag{3.69}
$$

It then follows from (3.61) and (3.69) that

$$
\begin{aligned}
\frac{1}{2}\int_{\mathcal{O}}(|\boldsymbol{M}^n(t,x)|^2-|\boldsymbol{M}(t,x)|^2)\mathrm{d}x\leqslant&-\frac{1}{\tau}\int_{Q_t}(|\boldsymbol{M}^n|^2-|\boldsymbol{M}|^2-\chi_0\\
&\times(\boldsymbol{H}^n\cdot\boldsymbol{M}^n-\boldsymbol{H}\cdot\boldsymbol{M}))\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
\tag{3.70}
$$

We recall that $\boldsymbol{H}_d^n = \nabla\varphi_d^n$. Now, combining $(3.36)_5$ with Proposition 3.6, we infer that $\boldsymbol{H}_d^n$ is uniformly bounded in $L^\infty(0, T; \mathbb{L}^2(\mathcal{O}))$. Hence, from the Banach–Alaoglu theorem, we obtain that $\boldsymbol{H}_d^n$ has a weak-star convergent subsequence, which is still denoted by $\boldsymbol{H}_d^n$ for convenience, such that as $n \to \infty$

$$\boldsymbol{H}_d^n \overset{*}{\rightharpoonup} \boldsymbol{H}_d \text{ weakly-star in } \quad L^\infty(0, T; \mathbb{L}^2(\mathcal{O})), \tag{3.71}$$

where the limit satisfies

$$\boldsymbol{H}_d \in L^\infty(0, T; \mathbb{L}^2(\mathcal{O})).$$

Besides, recalling that $\boldsymbol{H}^n = \boldsymbol{H}_a^n + \boldsymbol{H}_d^n$, we have

$$\boldsymbol{H}_a + \boldsymbol{H}_d = \boldsymbol{H} \in L^\infty(0, T; \mathbb{L}^2(\mathcal{O})),$$

due to $(3.37)_6$, (3.58), $(3.71)_1$ and the uniqueness of the limit. Furthermore, we have $\operatorname{div} \boldsymbol{H}_a = 0$ a.e. in Q_T, $\boldsymbol{H}_a \cdot \boldsymbol{n} = 0$, $\boldsymbol{H} \cdot \boldsymbol{n} = \boldsymbol{H}_d \cdot \boldsymbol{n}$ a.e. in Σ.
Now, since $\boldsymbol{H} \in \mathbb{L}^2(\mathcal{O})$ for a.e. $t \in [0, T]$, by the Helmholtz–Leray decomposition (see [23, Corollary 5.5]), there exists a unique $\tilde{\boldsymbol{H}}_a \in E(\mathcal{O})$ with $\operatorname{div} \tilde{\boldsymbol{H}}_a = 0$, $\tilde{\boldsymbol{H}}_a \cdot \boldsymbol{n} = 0$ and a unique $\nabla\varphi_d \in E_2(\mathcal{O})$ such that $\boldsymbol{H} = \tilde{\boldsymbol{H}}_a + \nabla\varphi_d$, $\operatorname{div}(\boldsymbol{H} - \nabla\varphi_d) = 0$ in $\mathcal{O}$ and $(\boldsymbol{H} - \nabla\varphi_d)|_{\partial\mathcal{O}} = 0$. Because of uniqueness of the Helmholtz decomposition, it then follows that $\tilde{\boldsymbol{H}}_a = \boldsymbol{H}_a$ and $\boldsymbol{H}_d = \nabla\varphi_d$ a.e. in $\mathcal{O}$.
Next we claim that

$$\begin{aligned}\int_{\mathcal{O}} |\boldsymbol{H}^n|^2 \mathrm{d}x &= -\int_{\mathcal{O}} \boldsymbol{M}^n \cdot \boldsymbol{H}^n \mathrm{d}x + \int_{\mathcal{O}} (\boldsymbol{M}^n + \boldsymbol{H}^n) \cdot \boldsymbol{H}_a^n \,\mathrm{d}x \\ &= -\int_{\mathcal{O}} \boldsymbol{M}^n \cdot \boldsymbol{H}^n \mathrm{d}x \\ &\quad + \int_{\mathcal{O}} (\boldsymbol{M}^n + \boldsymbol{H}^n) \cdot (\boldsymbol{H}_a^n - \boldsymbol{H}_a) \mathrm{d}x + \int_{\mathcal{O}} (\boldsymbol{M}^n + \boldsymbol{H}^n) \cdot \boldsymbol{H}_a \mathrm{d}x.\end{aligned} \tag{3.72}$$

In fact, by integration by parts along with the fact that $\operatorname{div} \boldsymbol{H}^n = -\operatorname{div} \boldsymbol{M}^n$, $\boldsymbol{H}_d^n = \nabla\varphi_d^n$, $\boldsymbol{H}^n = \boldsymbol{H}_d^n + \boldsymbol{H}_a^n$ in $\mathcal{O}$, and $\boldsymbol{H}^n \cdot \boldsymbol{n} = -\boldsymbol{M}^n \cdot \boldsymbol{n}$ on $\partial\mathcal{O}$, we obtain

$$\begin{aligned}\int_{\mathcal{O}} |\boldsymbol{H}^n|^2 \,\mathrm{d}x &= \int_{\mathcal{O}} \boldsymbol{H}^n \cdot \boldsymbol{H}_d^n \,\mathrm{d}x + \int_{\mathcal{O}} \boldsymbol{H}^n \cdot \boldsymbol{H}_a^n \,\mathrm{d}x \\ &= \int_{\mathcal{O}} \boldsymbol{H}^n \cdot \boldsymbol{H}_d^n \,\mathrm{d}x - \int_{\mathcal{O}} \boldsymbol{M}^n \cdot \boldsymbol{H}_a^n \,\mathrm{d}x + \int_{\mathcal{O}} (\boldsymbol{H}^n + \boldsymbol{M}^n) \cdot \boldsymbol{H}_a^n \,\mathrm{d}x \\ &= \int_{\mathcal{O}} \boldsymbol{H}^n \cdot \nabla\varphi_d^n \,\mathrm{d}x - \int_{\mathcal{O}} \boldsymbol{M}^n \cdot \boldsymbol{H}_a^n \,\mathrm{d}x + \int_{\mathcal{O}} (\boldsymbol{H}^n + \boldsymbol{M}^n) \cdot \boldsymbol{H}_a^n \,\mathrm{d}x \\ &= \int_{\partial\mathcal{O}} \varphi_d^n \boldsymbol{H}^n \cdot \boldsymbol{n} \,\mathrm{d}S - \int_{\mathcal{O}} \varphi_d^n \operatorname{div} \boldsymbol{H}^n \,\mathrm{d}x - \int_{\mathcal{O}} \boldsymbol{M}^n \cdot \boldsymbol{H}_a^n \,\mathrm{d}x \\ &\quad + \int_{\mathcal{O}} (\boldsymbol{H}^n + \boldsymbol{M}^n) \cdot \boldsymbol{H}_a^n \,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \int_{\partial\mathcal{O}} \varphi_d^n \boldsymbol{H}^n \cdot \boldsymbol{n} \, \mathrm{d}S + \int_{\mathcal{O}} \varphi_d^n \operatorname{div} \boldsymbol{M}^n \, \mathrm{d}x - \int_{\mathcal{O}} \boldsymbol{M}^n \cdot \boldsymbol{H}_a^n \, \mathrm{d}x \\
&\quad + \int_{\mathcal{O}} (\boldsymbol{H}^n + \boldsymbol{M}^n) \cdot \boldsymbol{H}_a^n \, \mathrm{d}x \\
&= \int_{\partial\mathcal{O}} \varphi_d^n (\boldsymbol{H}^n + \boldsymbol{M}^n) \cdot \boldsymbol{n} \, \mathrm{d}S \\
&\quad + \int_{\mathcal{O}} [-\boldsymbol{M}^n \cdot \nabla\varphi_d^n - \boldsymbol{M}^n \cdot \boldsymbol{H}_a^n + (\boldsymbol{H}^n + \boldsymbol{M}^n) \cdot \boldsymbol{H}_a^n] \mathrm{d}x \\
&= -\int_{\mathcal{O}} \boldsymbol{M}^n \cdot \boldsymbol{H}_d^n \, \mathrm{d}x - \int_{\mathcal{O}} \boldsymbol{M}^n \cdot \boldsymbol{H}_a^n \mathrm{d}x + \int_{\mathcal{O}} (\boldsymbol{H}^n + \boldsymbol{M}^n) \cdot \boldsymbol{H}_a^n \, \mathrm{d}x \\
&= -\int_{\mathcal{O}} \boldsymbol{M}^n \cdot (\boldsymbol{H}_d^n + \boldsymbol{H}_a^n) \, \mathrm{d}x + \int_{\mathcal{O}} (\boldsymbol{H}^n + \boldsymbol{M}^n) \cdot \boldsymbol{H}_a^n \, \mathrm{d}x \\
&= -\int_{\mathcal{O}} \boldsymbol{M}^n \cdot \boldsymbol{H}^n \, \mathrm{d}x + \int_{\mathcal{O}} (\boldsymbol{H}^n + \boldsymbol{M}^n) \cdot \boldsymbol{H}_a^n \, \mathrm{d}x.
\end{aligned}
$$

Passing to the limit as $n \to \infty$ in (3.72), using $(3.37)_5$, $(3.37)_6$ and (3.58), we obtain

$$
\begin{aligned}
\int_{\mathcal{O}} \overline{|\boldsymbol{H}|^2} \, \mathrm{d}x &= -\int_{\mathcal{O}} \overline{\boldsymbol{M} \cdot \boldsymbol{H}} \, \mathrm{d}x + \int_{\mathcal{O}} (\boldsymbol{M} + \boldsymbol{H}) \cdot \boldsymbol{H}_a \, \mathrm{d}x \\
&= -\int_{\mathcal{O}} \overline{\boldsymbol{M} \cdot \boldsymbol{H}} \, \mathrm{d}x + \int_{\mathcal{O}} (\boldsymbol{M} + \boldsymbol{H}) \cdot \boldsymbol{H} \, \mathrm{d}x \\
&\quad - \int_{\mathcal{O}} (\boldsymbol{M} + \boldsymbol{H}) \cdot \boldsymbol{H}_d \, \mathrm{d}x, \ \text{a.e. } t,
\end{aligned} \tag{3.73}
$$

where $\overline{|\boldsymbol{H}|^2}$ and $\overline{\boldsymbol{M} \cdot \boldsymbol{H}}$ denote the weak limits of the sequences $|\boldsymbol{H}^n|^2$ and $\boldsymbol{M}^n \cdot \boldsymbol{H}^n$, respectively.
Using an integration by parts together with the fact that $\boldsymbol{H}_d = \nabla\varphi_d$ in $\mathcal{O}$, we get

$$
\begin{aligned}
\int_{\mathcal{O}} (\boldsymbol{M} + \boldsymbol{H}) \cdot \boldsymbol{H}_d \, \mathrm{d}x &= \int_{\mathcal{O}} (\boldsymbol{M} + \boldsymbol{H}) \cdot \nabla\varphi_d \, \mathrm{d}x \\
&= \int_{\partial\mathcal{O}} \varphi_d (\boldsymbol{H} + \boldsymbol{M}) \cdot \boldsymbol{n} \, \mathrm{d}S \\
&\quad - \int_{\mathcal{O}} \operatorname{div}(\boldsymbol{M} + \boldsymbol{H}) \varphi_d \, \mathrm{d}x = 0,
\end{aligned} \tag{3.74}
$$

for a.e. $t \in [0, T]$. Here we used the fact that $\operatorname{div}(\boldsymbol{M}+\boldsymbol{H}) = 0$ in $\mathcal{O}$ and $\boldsymbol{H} \cdot \boldsymbol{n} = -\boldsymbol{M} \cdot \boldsymbol{n}$ on $\partial\mathcal{O}$. Finally, by inserting (3.74) into the right-hand side of (3.73), we further obtain

$$
\int_{\mathcal{O}} \overline{|\boldsymbol{H}|^2} \, \mathrm{d}x - \int_{\mathcal{O}} |\boldsymbol{H}|^2 \, \mathrm{d}x = -\int_{\mathcal{O}} \overline{\boldsymbol{M} \cdot \boldsymbol{H}} \, \mathrm{d}x + \int_{\mathcal{O}} \boldsymbol{M} \cdot \boldsymbol{H} \, \mathrm{d}x, \ \text{a.e. } t \in [0, T]. \tag{3.75}
$$

Letting $n \to \infty$ in (3.70), we find

$$\frac{1}{2}\int_{\mathcal{O}}\left(\overline{|\boldsymbol{M}(t,x)|^2} - |\boldsymbol{M}(t,x)|^2\right)\mathrm{d}x \leqslant -\frac{1}{\tau}\int_{Q_t}\left(\overline{|\boldsymbol{M}|^2} - |\boldsymbol{M}|^2 - \chi_0(\overline{\boldsymbol{H}\cdot\boldsymbol{M}} - \boldsymbol{H}\cdot\boldsymbol{M})\right)\mathrm{d}x\,\mathrm{d}s, \tag{3.76}$$

where we denote the weak limit of $|\boldsymbol{M}^n|^2$ by $\overline{|\boldsymbol{M}|^2}$.
From (3.75), we can replace $-\overline{\boldsymbol{H}\cdot\boldsymbol{M}} + \boldsymbol{H}\cdot\boldsymbol{M}$ by $\overline{|\boldsymbol{H}|^2} - |\boldsymbol{H}|^2$ in (3.76), and we obtain

$$\frac{1}{2}\int_{\mathcal{O}}\left(\overline{|\boldsymbol{M}(t,x)|^2} - |\boldsymbol{M}(t,x)|^2\right)\mathrm{d}x \leqslant -\frac{1}{\tau}\int_{Q_t}\left[\overline{|\boldsymbol{M}|^2} - |\boldsymbol{M}|^2 + \chi_0(\overline{|\boldsymbol{H}|^2} - |\boldsymbol{H}|^2)\right]\mathrm{d}x\,\mathrm{d}s. \tag{3.77}$$

We recall that in view of (see [18, Corollary 3.33])

$$\int_D |\boldsymbol{v}|^2\,\mathrm{d}x \leqslant \liminf_{n\to\infty}\int_D |\boldsymbol{v}_n|^2\,\mathrm{d}x \leqslant \int_D \overline{|\boldsymbol{v}|^2}\,\mathrm{d}x \ \text{ and } \ |\boldsymbol{v}|^2 \leqslant \overline{|\boldsymbol{v}|^2} \ \text{ a.e. on } \ D, \tag{3.78}$$

with $\boldsymbol{v} \in \{\boldsymbol{H}, \boldsymbol{M}\}$, $D \in \{\mathcal{O}, [0,T]\times\mathcal{O}\}$.
Due to (3.78) and since $\tau > 0$, $\chi_0 > 0$, the left-hand side of (3.77) is nonnegative while the right-hand side is nonpositive, respectively. Hence, we deduce that

$$\frac{1}{2}\int_{\mathcal{O}}\overline{|\boldsymbol{M}(t,x)|^2}\,\mathrm{d}x - \frac{1}{2}\int_{\mathcal{O}}|\boldsymbol{M}(t,x)|^2\,\mathrm{d}x = 0. \tag{3.79}$$

Thus, by (3.78), we have $\overline{|\boldsymbol{M}(t,x)|^2} = |\boldsymbol{M}(t,x)|^2$ for a.e. $(t,x) \in Q_T$. By applying now [9, Theorem 1.1.1, item (iii)], we are led to

$$\boldsymbol{M}^n \to \boldsymbol{M} \ \text{ in } \ L^2(0,T;\mathbb{L}^2(\mathcal{O})). \tag{3.80}$$

Exploiting (3.75) and (3.80), we obtain

$$\int_{Q_T}(\overline{|\boldsymbol{H}(t,x)|^2} - |\boldsymbol{H}(t,x)|^2)\mathrm{d}x\,\mathrm{d}t = 0.$$

Hence, $\boldsymbol{H}^n \to \boldsymbol{H}$ strongly in $L^2(0,T;\mathbb{L}^2(\mathcal{O}))$. This completes the proof of our claim (3.59). Let $\Psi \in \mathcal{C}^1_c([0,T); V^1_{\mathrm{div}} \cap \mathbb{H}^2(\mathcal{O}))$ be arbitrary but fixed. From the weak formulation of (3.1d) and under the assumption (3.9), we have

$$-\int_{Q_T}\boldsymbol{B}^n\cdot\partial_t\Psi\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{O}}\boldsymbol{B}_0\cdot\Psi(0)\,\mathrm{d}x - \frac{1}{\sigma}\int_{Q_T}\operatorname{curl}\boldsymbol{H}^n\cdot\operatorname{curl}\Psi\,\mathrm{d}x\,\mathrm{d}t$$

$$+ \int_{Q_T} [(\boldsymbol{u}^n \times \boldsymbol{B}^n)] \cdot \operatorname{curl} \Psi \, \mathrm{d}x \, \mathrm{d}t. \tag{3.81}$$

We claim that

$$\begin{aligned} &\int_{Q_T} \operatorname{curl} \boldsymbol{H}^n \cdot \operatorname{curl} \Psi \, \mathrm{d}x \, \mathrm{d}t \to \int_{Q_T} \operatorname{curl} \boldsymbol{H} \cdot \operatorname{curl} \Psi \, \mathrm{d}x \, \mathrm{d}t, \\ &\int_{Q_T} (\boldsymbol{u}^n \times \boldsymbol{B}^n) \cdot \operatorname{curl} \Psi \, \mathrm{d}x \, \mathrm{d}t \to - \int_{Q_T} (\boldsymbol{B} \cdot \nabla)\Psi \cdot \boldsymbol{u} \, \mathrm{d}x \, \mathrm{d}t + \int_{Q_T} (\boldsymbol{u} \cdot \nabla)\Psi \cdot \boldsymbol{B} \, \mathrm{d}x \, \mathrm{d}t, \end{aligned} \tag{3.82}$$

as $n \to \infty$ (up to a subsequence), with $\Psi \in \mathcal{C}_c^1([0, T); V_{\mathrm{div}}^1 \cap \mathbb{H}^2(\mathcal{O}))$.

Let us point out that, since $\{\nabla \times \boldsymbol{H}^n\}_n$ is uniformly bounded in $L^2(0, T; \mathbb{L}^2(\mathcal{O}))$ (cf. $(3.36)_7$) and due to (3.59), we have up to a subsequence

$$\operatorname{curl} \boldsymbol{H}^n = \operatorname{curl} \boldsymbol{H}_a^n \rightharpoonup \operatorname{curl} \boldsymbol{H} = \operatorname{curl} \boldsymbol{H}_a \quad \text{weakly in } \; L^2(0, T; \mathbb{L}^2(\mathcal{O})). \tag{3.83}$$

Hence,

$$\int_{Q_T} \operatorname{curl} \boldsymbol{H}^n \cdot \operatorname{curl} \Psi \, \mathrm{d}x \, \mathrm{d}t \to \int_{Q_T} \operatorname{curl} \boldsymbol{H} \cdot \operatorname{curl} \Psi \, \mathrm{d}x \, \mathrm{d}t.$$

Let us now move to the proof of $(3.82)_2$. First, we recall that (cf. (3.52))

$$\begin{aligned} \int_{\mathcal{O}} (\boldsymbol{u}^n \times \boldsymbol{B}^n) \cdot \operatorname{curl} \Psi \, \mathrm{d}x = &- \int_{\mathcal{O}} (\boldsymbol{B}^n \cdot \nabla)\Psi \cdot \boldsymbol{u}^n \, \mathrm{d}x \\ &+ \int_{\mathcal{O}} (\boldsymbol{u}^n \cdot \nabla)\Psi \cdot \boldsymbol{B}^n \, \mathrm{d}x \quad \text{a.e. in } (0, T). \end{aligned} \tag{3.84}$$

We now split the first term on the right-hand side of (3.84) as follows

$$\begin{aligned} - \int_{\mathcal{O}} (\boldsymbol{B}^n \cdot \nabla)\Psi \cdot \boldsymbol{u}^n \, \mathrm{d}x \; = &- \int_{\mathcal{O}} (\boldsymbol{B}^n \cdot \nabla)\Psi \cdot (\boldsymbol{u}^n - \boldsymbol{u}) \, \mathrm{d}x \\ &- \int_{\mathcal{O}} [(\boldsymbol{B}^n - \boldsymbol{B}) \cdot \nabla]\Psi \cdot \boldsymbol{u} \, \mathrm{d}x \\ &- \int_{\mathcal{O}} (\boldsymbol{B} \cdot \nabla)\Psi \cdot \boldsymbol{u} \, \mathrm{d}x, \end{aligned}$$

and its second term as

$$\begin{aligned} \int_{\mathcal{O}} (\boldsymbol{u}^n \cdot \nabla)\Psi \cdot \boldsymbol{B}^n \, \mathrm{d}x \; = &\int_{\mathcal{O}} ((\boldsymbol{u}^n - \boldsymbol{u}) \cdot \nabla)\Psi \cdot \boldsymbol{B}^n \, \mathrm{d}x + \int_{\mathcal{O}} (\boldsymbol{u} \cdot \nabla)\Psi \cdot (\boldsymbol{B}^n - \boldsymbol{B}) \, \mathrm{d}x \\ &+ \int_{\mathcal{O}} (\boldsymbol{u} \cdot \nabla)\Psi \cdot \boldsymbol{B} \, \mathrm{d}x. \end{aligned}$$

Adding the two previous equalities, we further obtain

$$\begin{aligned}
&\int_{Q_T} (\boldsymbol{u}^n \times \boldsymbol{B}^n) \cdot \operatorname{curl} \Psi \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{Q_T} (\boldsymbol{B}^n \cdot \nabla)\Psi \cdot (\boldsymbol{u}^n - \boldsymbol{u}) \,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} [(\boldsymbol{B}^n - \boldsymbol{B}) \cdot \nabla]\Psi \cdot \boldsymbol{u} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{Q_T} ((\boldsymbol{u}^n - \boldsymbol{u}) \cdot \nabla)\Psi \cdot \boldsymbol{B}^n \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} (\boldsymbol{u} \cdot \nabla)\Psi \cdot (\boldsymbol{B}^n - \boldsymbol{B}) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_{Q_T} (\boldsymbol{B} \cdot \nabla)\Psi \cdot \boldsymbol{u} \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} (\boldsymbol{u} \cdot \nabla)\Psi \cdot \boldsymbol{B} \,\mathrm{d}x\,\mathrm{d}t. \qquad (3.85)
\end{aligned}$$

In light of (3.36)$_5$, (3.36)$_6$ and (3.47), we obtain

$$\int_{Q_T} (\boldsymbol{B}^n \cdot \nabla)\Psi \cdot (\boldsymbol{u}^n - \boldsymbol{u}) \,\mathrm{d}x\,\mathrm{d}t \to 0, \quad \int_{Q_T} (\boldsymbol{u} \cdot \nabla)\Psi \cdot (\boldsymbol{B}^n - \boldsymbol{B}) \,\mathrm{d}x\,\mathrm{d}t \to 0, \tag{3.86}$$

as $n \to \infty$.
From (3.37)$_5$ and (3.37)$_6$, we deduce that

$$\int_{Q_T} [(\boldsymbol{B}^n - \boldsymbol{B}) \cdot \nabla]\Psi \cdot \boldsymbol{u} \,\mathrm{d}x\,\mathrm{d}t \to 0 \quad \text{and} \quad \int_{Q_T} (\boldsymbol{u} \cdot \nabla)\Psi \cdot (\boldsymbol{B}^n - \boldsymbol{B}) \,\mathrm{d}x\,\mathrm{d}t \to 0. \tag{3.87}$$

Passing to the limit in (3.85) and using (3.86) and (3.87), we find

$$\begin{aligned}
\int_{\mathcal{O}} (\boldsymbol{u}^n \times \boldsymbol{B}^n) \cdot \operatorname{curl} \Psi \,\mathrm{d}x\,\mathrm{d}t \to & -\int_{Q_T} (\boldsymbol{B} \cdot \nabla)\Psi \cdot \boldsymbol{u} \,\mathrm{d}x\,\mathrm{d}t \\
& + \int_{Q_T} (\boldsymbol{u} \cdot \nabla)\Psi \cdot \boldsymbol{B} \,\mathrm{d}x\,\mathrm{d}t. \qquad (3.88)
\end{aligned}$$

This completes the proof of (3.82).
Exploiting now (3.37)$_5$, (3.37)$_6$ and (3.82), from (3.81), we infer that

$$\begin{aligned}
-\int_{Q_T} \boldsymbol{B} \cdot \partial_t \Psi \,\mathrm{d}x\,\mathrm{d}t &= \int_{\mathcal{O}} \boldsymbol{B}_0 \cdot \Psi(0) \,\mathrm{d}x - \frac{1}{\sigma} \int_{Q_T} \operatorname{curl} \boldsymbol{H} \cdot \operatorname{curl} \Psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_{Q_T} (\boldsymbol{B} \cdot \nabla)\Psi \cdot \boldsymbol{u} \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} (\boldsymbol{u} \cdot \nabla)\Psi \cdot \boldsymbol{B} \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.89}$$

We are now in position to pass to the limit in the weak formulations of (3.1a)–(3.1c). For this, let

$$\boldsymbol{v} \in \mathcal{C}_c^1([0,T); V \cap \mathbb{H}^3(\mathcal{O})), \quad \psi \in \mathcal{C}_c^1([0,T); [H_0^1(\mathcal{O})]^3 \cap \mathbb{H}^2(\mathcal{O}))$$

be arbitrary but fixed. Recalling the assumption (3.9), these weak formulations are respectively given by

$$
\begin{aligned}
&-\int_{Q_T} \boldsymbol{u}^n \cdot \partial_t \boldsymbol{v} \,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{v} \cdot \boldsymbol{u}^n \,\mathrm{d}x\,\mathrm{d}t + \nu \int_{Q_T} \nabla \boldsymbol{u}^n : \nabla \boldsymbol{v} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{Q_T} \mu_0[(\boldsymbol{M}^n + \boldsymbol{H}^n)\cdot\nabla]\boldsymbol{v} \cdot \boldsymbol{H}^n \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\mathcal{O}} \boldsymbol{u}_0 \cdot \boldsymbol{v}(0)\,\mathrm{d}x - \alpha \int_{Q_T} (\nabla \times \boldsymbol{u}^n - 2\mathbf{w}^n)\cdot \operatorname{curl} \boldsymbol{v} \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{3.90}
$$

and

$$
\begin{aligned}
&-\int_{Q_T} \mathbf{w}^n \cdot \partial_t \psi \,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} (\boldsymbol{u}^n \cdot \nabla)\psi \cdot \mathbf{w}^n \,\mathrm{d}x\,\mathrm{d}t + (\lambda_1+\lambda_2)\int_{Q_T} \operatorname{div} \mathbf{w}^n \operatorname{div} \psi \,\mathrm{d}x\,\mathrm{d}t \\
&+ \lambda_1 \int_{Q_T} \nabla \mathbf{w}^n : \nabla \psi \,\mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{O}} \mathbf{w}_0 \cdot \psi(0)\,\mathrm{d}x + 2\alpha \int_{Q_T} \boldsymbol{u}^n \cdot \operatorname{curl} \psi \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - 4\alpha \int_{Q_T} \mathbf{w}^n \cdot \psi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} \mu_0(\boldsymbol{M}^n \times \boldsymbol{H}^n)\cdot \psi \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.91}
$$

Using the convergence (3.37), (3.40), (3.48), (3.59), and the estimates (3.36), we can pass to the limit in (3.90) and (3.91). Hence, under the validity of (3.9), we deduce that (by passing to the limit in (3.90) and (3.91), respectively)

$$
\begin{aligned}
&-\int_{Q_T} \boldsymbol{u} \cdot \partial_t \boldsymbol{v} \,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} \cdot \boldsymbol{u} \,\mathrm{d}x\,\mathrm{d}t + \nu \int_{Q_T} \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{Q_T} \mu_0[(\boldsymbol{M} + \boldsymbol{H})\cdot\nabla]\boldsymbol{v} \cdot \boldsymbol{H} \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\mathcal{O}} \boldsymbol{u}_0 \cdot \boldsymbol{v}(0)\,\mathrm{d}x - \alpha \int_{Q_T} (\nabla \times \boldsymbol{u} - 2\mathbf{w})\cdot \operatorname{curl} \boldsymbol{v} \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{3.92}
$$

$$
\begin{aligned}
&-\int_{Q_T} \mathbf{w} \cdot \partial_t \psi \,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} (\boldsymbol{u} \cdot \nabla)\psi \cdot \mathbf{w} \,\mathrm{d}x\,\mathrm{d}t + (\lambda_1+\lambda_2)\int_{Q_T} \operatorname{div} \mathbf{w} \operatorname{div} \psi \,\mathrm{d}x\,\mathrm{d}t \\
&+\lambda_1 \int_{Q_T} \nabla \mathbf{w} : \nabla \psi \,\mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{O}} \mathbf{w}_0 \cdot \psi(0)\,\mathrm{d}x + 2\alpha \int_{Q_T} \boldsymbol{u} \cdot \operatorname{curl} \psi \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - 4\alpha \int_{Q_T} \mathbf{w} \cdot \psi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} \mu_0(\boldsymbol{M} \times \boldsymbol{H})\cdot \psi \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.93}
$$

The identities (3.66), (3.67), (3.89), (3.92), and (3.93) allow us to conclude that $\boldsymbol{u}$, $\mathbf{w}$, $\boldsymbol{H}$, and $\boldsymbol{M}$ satisfy exactly the same weak formulations as in Definition 2.2 (cf. item (ii)).

The Weak Time Continuity

Because of $(3.37)_2$, $(3.37)_5$ and the embedding of $\mathbb{H}^1(\mathcal{O})$ in $\mathbb{L}^4(\mathcal{O})$, we have

$$\int_0^T \|(\boldsymbol{u}(t)\cdot\nabla)\boldsymbol{M}(t)\|^2_{(\mathbb{H}^2(\mathcal{O}))'}\,\mathrm{d}t \leqslant C \sup_{t\in[0,T]} \|\boldsymbol{M}(t)\|^2 \int_0^T \|\nabla \boldsymbol{u}(t)\|^2_{\mathbb{L}^2(\mathcal{O})}\,\mathrm{d}t < \infty.$$

Thanks to $(3.37)_4$, $(3.37)_5$ and the embedding of $\mathbb{H}^1(\mathcal{O})$ in $\mathbb{L}^4(\mathcal{O})$, we infer that

$$\int_0^T \|\mathbf{w}(t)\times\boldsymbol{M}(t)\|^2_{(\mathbb{L}^4(\mathcal{O}))'}\,\mathrm{d}t \leqslant C \sup_{t\in[0,T]} \|\boldsymbol{M}(t)\|^2 \int_0^T \|\mathbf{w}(t)\|^2_{[H_0^1(\mathcal{O})]^3}\,\mathrm{d}t < \infty.$$

Recalling that (3.68) holds almost everywhere and using the above estimates, we infer that $\partial_t \boldsymbol{M} \in L^2(0,T;(\mathbb{H}^2(\mathcal{O}))')$. Furthermore, since $\boldsymbol{M} \in L^2(0,T;(\mathbb{H}^2(\mathcal{O}))')$ due to $(3.37)_5$, we can apply [24, Lemma 4] to deduce that $\boldsymbol{M} \in \mathcal{C}([0,T];(\mathbb{H}^2(\mathcal{O}))')$. Finally, by applying Lemma 2.1, we get

$$\boldsymbol{M} \in \mathcal{C}_w([0,T];\mathbb{L}^2(\mathcal{O})). \tag{3.94}$$

Due to (3.38) and $(3.37)_5$, (3.46) and $(3.37)_1$, we can argue as in the proof of (3.94) so as to get

$$\mathbf{w} \in \mathcal{C}_w([0,T];\mathbb{L}^2(\mathcal{O})) \quad \text{and} \quad \boldsymbol{u} \in \mathcal{C}_w([0,T];H).$$

Let

$$\boldsymbol{X}^3_{\mathrm{div}} = V^1_{\mathrm{div}} \cap \mathbb{H}^2(\mathcal{O}) \quad \text{and} \quad \boldsymbol{X}^{-3}_{\mathrm{div}} \quad \text{its dual space.}$$

Also, let

$$\boldsymbol{L}^2_{\mathrm{div}}(\mathcal{O}) = \{\boldsymbol{B} \in \mathbb{L}^2(\mathcal{O});\ \operatorname{div}\boldsymbol{B} = 0 \text{ in } \mathcal{O},\ \boldsymbol{B}\times\boldsymbol{n} = 0 \text{ on } \partial\mathcal{O}\}.$$

We observe that

$$\begin{aligned}\langle \partial_t \boldsymbol{B}, \psi\rangle = &-\frac{1}{\sigma}\int_{\mathcal{O}} \operatorname{curl}\boldsymbol{H}\cdot\operatorname{curl}\psi\,\mathrm{d}x \\ &-\int_{\mathcal{O}} (\boldsymbol{B}\cdot\nabla)\psi\cdot\boldsymbol{u}\,\mathrm{d}x + \int_{\mathcal{O}} (\boldsymbol{u}\cdot\nabla)\psi\cdot\boldsymbol{B}\,\mathrm{d}x,\end{aligned} \tag{3.95}$$

for all $\psi \in \boldsymbol{X}^3_{\mathrm{div}}$, where $\langle\cdot,\cdot\rangle$ stands for the duality product between $\boldsymbol{X}^3_{\mathrm{div}}$ and $\boldsymbol{X}^{-3}_{\mathrm{div}}$. Next by Hölder's inequality together with the fact that $\operatorname{curl}\boldsymbol{H} \in \mathbb{L}^2(\mathcal{O})$ a.e. in $(0,T)$, we find

$$\left|\int_{\mathcal{O}} \frac{1}{\sigma}\operatorname{curl}\boldsymbol{H}\cdot\operatorname{curl}\psi\,\mathrm{d}x\right|^2 \leqslant \frac{1}{\sigma^2}\|\operatorname{curl}\boldsymbol{H}\|^2\|\operatorname{curl}\psi\|^2$$

$$\leqslant \frac{1}{\sigma^2}\|\psi\|^2_{\boldsymbol{X}^3_{\rm div}}\|\operatorname{curl}\boldsymbol{H}\|^2. \tag{3.96}$$

We control the second term on the right-hand side of (3.95) as follows:

$$\begin{aligned}\left|\int_{\mathcal{O}}(\boldsymbol{B}\cdot\nabla)\psi\cdot\boldsymbol{u}\,\mathrm{d}x\right| &\leqslant \|\boldsymbol{B}\|\|\nabla\psi\|_{\mathbb{L}^4(\mathcal{O})}\|\boldsymbol{u}\|_{\mathbb{L}^4(\mathcal{O})}\\ &\leqslant C(\mathcal{O})\|\boldsymbol{B}\|\|\psi\|_{\boldsymbol{X}^3_{\rm div}}\|\nabla\boldsymbol{u}\|,\end{aligned}$$

where we used the embedding of $\mathbb{H}^1(\mathcal{O})$ in $\mathbb{L}^4(\mathcal{O})$. Here $C(\mathcal{O})$ is a positive constant depending only on $\mathcal{O}$.
Hence,

$$\left|\int_{\mathcal{O}}(\boldsymbol{B}\cdot\nabla)\psi\cdot\boldsymbol{u}\,\mathrm{d}x\right|^2 \leqslant C(\mathcal{O})\|\boldsymbol{B}\|^2\|\psi\|^2_{\boldsymbol{X}^3_{\rm div}}\|\nabla\boldsymbol{u}\|^2 \quad \forall\psi\in\boldsymbol{X}^3_{\rm div}. \tag{3.97}$$

In a similar way, we have

$$\left|\int_{\mathcal{O}}(\boldsymbol{u}\cdot\nabla)\psi\cdot\boldsymbol{B}\,\mathrm{d}x\right|^2 \leqslant C(\mathcal{O})\|\psi\|^2_{\boldsymbol{X}^3_{\rm div}}\|\boldsymbol{B}\|^2\|\nabla\boldsymbol{u}\|^2 \quad \forall\psi\in\boldsymbol{X}^3_{\rm div}. \tag{3.98}$$

Notice that

$$\begin{aligned}\int_0^T\|\partial_t\boldsymbol{B}(t)\|^2_{\boldsymbol{X}^{-3}_{\rm div}}\,\mathrm{d}t &= \int_0^T\sup_{\|\psi\|_{\boldsymbol{X}^3_{\rm div}}\leqslant 1}|\langle\partial_t\boldsymbol{B}(t),\psi\rangle|^2\mathrm{d}t\\ &\leqslant 3\int_0^T\sup_{\|\psi\|_{\boldsymbol{X}^3_{\rm div}}\leqslant 1}\left|\frac{1}{\sigma}\int_{\mathcal{O}}\operatorname{curl}\boldsymbol{H}(t)\cdot\operatorname{curl}\psi\,\mathrm{d}x\right|^2\mathrm{d}t\\ &\quad+3\int_0^T\sup_{\|\psi\|_{\boldsymbol{X}^3_{\rm div}}\leqslant 1}\left|\int_{\mathcal{O}}(\boldsymbol{B}(t)\cdot\nabla)\psi\cdot\boldsymbol{u}(t)\,\mathrm{d}x\right|^2\\ &\quad+3\int_0^T\sup_{\|\psi\|_{\boldsymbol{X}^3_{\rm div}}\leqslant 1}\left|\int_{\mathcal{O}}(\boldsymbol{u}(t)\cdot\nabla)\psi\cdot\boldsymbol{B}(t)\,\mathrm{d}x\right|^2\mathrm{d}t,\end{aligned}$$

from which and (3.96)–(3.98), we infer that

$$\begin{aligned}&\int_0^T\|\partial_t\boldsymbol{B}(t)\|^2_{\boldsymbol{X}^{-3}_{\rm div}}\,\mathrm{d}t \leqslant C(\mathcal{O})\\ &\times\left(\sup_{t\in[0,T]}\|\boldsymbol{B}(t)\|^2\int_0^T\|\nabla\boldsymbol{u}(t)\|^2\mathrm{d}t+\frac{1}{\sigma^2}\int_0^T\|\operatorname{curl}\boldsymbol{H}(t)\|^2\mathrm{d}t\right)<\infty,\end{aligned} \tag{3.99}$$

since $\boldsymbol{u}\in L^2(0,T;V)$ and $\operatorname{curl}\boldsymbol{H}\in L^2(0,T;\mathbb{L}^2(\mathcal{O}))$.
In light of (3.99), we find that $\partial_t\boldsymbol{B}\in L^2(0,T;\boldsymbol{X}^{-3}_{\rm div})$. Now, since $\boldsymbol{B}\in L^2(0,T;\mathbb{L}^2(\mathcal{O}))$

due to $(3.37)_5$ and $(3.37)_6$; div $\boldsymbol{B} = \operatorname{div}(\boldsymbol{M}+\boldsymbol{H}) = 0$ in Q_T and $\boldsymbol{B}\times\boldsymbol{n} = 0$ on $\partial\mathcal{O}$, we can say that $\boldsymbol{B} \in L^2(0, T; \boldsymbol{X}^{-3}_{\mathrm{div}})$. So, by [24, Lemma 4] one has $\boldsymbol{B} \in \mathcal{C}([0, T]; \boldsymbol{X}^{-3}_{\mathrm{div}})$. It then follows from Lemma 2.1 that $\boldsymbol{B} \in \mathcal{C}_w([0, T]; \boldsymbol{L}^2_{\mathrm{div}}(\mathcal{O}))$. In particular, we have $\boldsymbol{B} \in \mathcal{C}_w([0, T]; \mathbb{L}^2(\mathcal{O}))$. Finally, since $\boldsymbol{M} \in \mathcal{C}_w([0, T]; \mathbb{L}^2(\mathcal{O}))$ and $\boldsymbol{B} \in \mathcal{C}_w([0, T]; \mathbb{L}^2(\mathcal{O}))$, with $\boldsymbol{B} = \boldsymbol{M}+\boldsymbol{H}$, we conclude that $\boldsymbol{H} \in \mathcal{C}_w([0, T]; \mathbb{L}^2(\mathcal{O}))$. This completes the proof of item (i) in Definition 2.2.
Moreover, using the above estimates (cf. (3.36)) and convergence results (3.39), (3.47), (3.59), $(3.37)_2$ and $(3.37)_4$, we can pass to the limit in (3.31), in order to obtain (2.5).

The Initial Data

Recalling that $\boldsymbol{u} \in \mathcal{C}_w([0, T]; \mathbb{L}^2(\mathcal{O}))$, $\mathbf{w} \in \mathcal{C}_w([0, T]; \mathbb{L}^2(\mathcal{O}))$, $\boldsymbol{M} \in \mathcal{C}_w([0, T]; \mathbb{L}^2(\mathcal{O}))$ and $\boldsymbol{H} \in \mathcal{C}_w([0, T]; \mathbb{L}^2(\mathcal{O}))$, we have

$$\begin{aligned}\int_{\mathcal{O}} \frac{1}{2}|\boldsymbol{u}_0|^2\,\mathrm{d}x &\leqslant \liminf_{t\to 0^+} \int_{\mathcal{O}} \frac{1}{2}|\boldsymbol{u}(t)|^2\,\mathrm{d}x, \quad \int_{\mathcal{O}} \frac{1}{2}|\mathbf{w}_0|^2 dx\\ &\leqslant \liminf_{t\to 0^+} \int_{\mathcal{O}} \frac{1}{2}|\mathbf{w}(t)|^2\,\mathrm{d}x, \int_{\mathcal{O}} \frac{1}{2}|\boldsymbol{M}_0|^2\,\mathrm{d}x\\ &\leqslant \liminf_{t\to 0^+} \int_{\mathcal{O}} \frac{1}{2}|\boldsymbol{M}(t)|^2\,\mathrm{d}x, \quad \int_{\mathcal{O}} \frac{1}{2}|\boldsymbol{H}_0|^2\,\mathrm{d}x\\ &\leqslant \liminf_{t\to 0^+} \int_{\mathcal{O}} \frac{1}{2}|\boldsymbol{H}(t)|^2\,\mathrm{d}x. \end{aligned} \tag{3.100}$$

Taking the upper limit in (2.5) as $t \to 0^+$, we have

$$\begin{aligned}&\limsup_{t\to 0^+}\Big[\int_{\mathcal{O}} \frac{1}{2}|\boldsymbol{u}(t)|^2\,\mathrm{d}x + \int_{\mathcal{O}} \frac{1}{2}|\mathbf{w}(t)|^2\,\mathrm{d}x + \int_{\mathcal{O}} \frac{1}{2}|\boldsymbol{M}(t)|^2\,\mathrm{d}x\\ &\quad + \int_{\mathcal{O}} \frac{1}{2}|\boldsymbol{H}(t)|^2\,\mathrm{d}x\Big]\\ &\leqslant \int_{\mathcal{O}} \frac{1}{2}|\boldsymbol{u}_0|^2\,\mathrm{d}x + \int_{\mathcal{O}} \frac{1}{2}|\mathbf{w}_0|^2\,\mathrm{d}x + \int_{\mathcal{O}} \frac{1}{2}|\boldsymbol{M}_0|^2\,\mathrm{d}x + \int_{\mathcal{O}} \frac{1}{2}|\boldsymbol{H}_0|^2\,\mathrm{d}x.\end{aligned} \tag{3.101}$$

In light of (3.100) and (3.101), we find

$$\begin{aligned}&\lim_{t\to 0^+}\Big[\int_{\mathcal{O}} \frac{1}{2}|\boldsymbol{u}(t)|^2\,\mathrm{d}x + \int_{\mathcal{O}} \frac{1}{2}|\mathbf{w}(t)|^2\,\mathrm{d}x + \int_{\mathcal{O}} \frac{1}{2}|\boldsymbol{M}(t)|^2\,\mathrm{d}x\\ &\quad + \int_{\mathcal{O}} \frac{1}{2}|\boldsymbol{H}(t)|^2\,\mathrm{d}x\Big]\\ &= \int_{\mathcal{O}} \frac{1}{2}|\boldsymbol{u}_0|^2\,\mathrm{d}x + \int_{\mathcal{O}} \frac{1}{2}|\mathbf{w}_0|^2\,\mathrm{d}x + \int_{\mathcal{O}} \frac{1}{2}|\boldsymbol{M}_0|^2\,\mathrm{d}x + \int_{\mathcal{O}} \frac{1}{2}|\boldsymbol{H}_0|^2\,\mathrm{d}x.\end{aligned} \tag{3.102}$$

Exploiting (3.102) in conjunction with the fact that $\boldsymbol{u} \in \mathcal{C}_w([0,T];\mathbb{L}^2(\mathcal{O}))$, $\mathbf{w} \in \mathcal{C}_w([0,T];\mathbb{L}^2(\mathcal{O}))$, $\boldsymbol{M} \in \mathcal{C}_w([0,T];\mathbb{L}^2(\mathcal{O}))$, and $\boldsymbol{H} \in \mathcal{C}_w([0,T];\mathbb{L}^2(\mathcal{O}))$, we deduce that

$$\lim_{t\to 0^+}\left[\int_{\mathcal{O}}|\boldsymbol{u}(t)-\boldsymbol{u}_0|^2\,\mathrm{d}x+\int_{\mathcal{O}}|\mathbf{w}(t)-\mathbf{w}_0|^2\,\mathrm{d}x+\int_{\mathcal{O}}|\boldsymbol{M}(t)-\boldsymbol{M}_0|^2\,\mathrm{d}x\right.$$
$$\left.+\int_{\mathcal{O}}|\boldsymbol{H}(t)-\boldsymbol{H}_0|^2\,\mathrm{d}x\right]=0.$$

This concludes the proof of Theorem 2.4. □

4 Zero Limit of the Relaxation Time for the System (1.1)–(1.3)

In this section, we aim to study the asymptotic behaviour of the solutions to the problem (1.1)–(1.3) when $\tau \to 0^+$. More precisely, we prove that as $\tau \to 0^+$, the solutions $(\boldsymbol{u}_\tau, \mathbf{w}_\tau, \boldsymbol{M}_\tau, \boldsymbol{H}_\tau)$ of (1.1)–(1.3) converge to a solution $(\boldsymbol{U}, \boldsymbol{W}, \boldsymbol{m}, \boldsymbol{h})$ of the following system

$$\partial_t\boldsymbol{U}+(\boldsymbol{U}\cdot\nabla)\boldsymbol{U}+\nabla p-\nu\Delta\boldsymbol{U}-\mu_0(\boldsymbol{m}\cdot\nabla)\boldsymbol{h}$$
$$=\mu_0(\nabla\times\boldsymbol{h})\times\boldsymbol{h}-\alpha\nabla\times(\nabla\times\boldsymbol{U}-2\boldsymbol{W}), \tag{4.1a}$$
$$\partial_t\boldsymbol{W}+(\boldsymbol{U}\cdot\nabla)\boldsymbol{W}-(\lambda_1+\lambda_2)\nabla\operatorname{div}\boldsymbol{W}-\lambda_1\Delta\boldsymbol{W}=2\alpha(\nabla\times\boldsymbol{U}-2\boldsymbol{W}), \tag{4.1b}$$
$$\boldsymbol{m}=\chi_0\boldsymbol{h}, \tag{4.1c}$$
$$\partial_t\boldsymbol{b}+\frac{1}{\sigma}\nabla\times(\nabla\times\boldsymbol{h})=\nabla\times(\boldsymbol{U}\times\boldsymbol{b}), \tag{4.1d}$$
$$\boldsymbol{b}=\mu_0(1+\chi_0)\boldsymbol{h},\quad \operatorname{div}\boldsymbol{b}=0, \tag{4.1e}$$
$$\operatorname{div}\boldsymbol{U}=0\quad\text{in } Q_T, \tag{4.1f}$$
$$\boldsymbol{U}=0,\quad \boldsymbol{W}=0,\quad \boldsymbol{m}\times\boldsymbol{n}=0,\quad \boldsymbol{h}\times\boldsymbol{n}=0\qquad \text{on }\ \Sigma, \tag{4.1g}$$
$$\begin{cases}(\boldsymbol{U},\boldsymbol{W},\boldsymbol{m},\boldsymbol{h})(0)=(\boldsymbol{U}_0,\boldsymbol{W}_0,\boldsymbol{m}_0,\boldsymbol{h}_0) & \text{in }\ \mathcal{O},\\ \operatorname{div}\boldsymbol{U}_0=0,\quad \operatorname{div}\boldsymbol{m}_0=0, & \text{in }\ \mathcal{O}.\end{cases} \tag{4.1h}$$

Before stating and proving the main result of this section, we define what we mean by a weak solution to the problem (4.1a)–(4.1h).

Definition 4.1 A quadriplet of functions $(\boldsymbol{U}, \boldsymbol{W}, \boldsymbol{m}, \boldsymbol{h})$ is a weak solution to the problem (4.1a)–(4.1h) if:

(i) The functions $\boldsymbol{U}, \boldsymbol{W}, \boldsymbol{m}, \boldsymbol{h}$ satisfy

$$\boldsymbol{U},\boldsymbol{W},\boldsymbol{m},\boldsymbol{h})\in L^\infty(0,T;\mathbb{H})\cap L^2(0,T;V\times[H_0^1(\mathcal{O})]^3\times V^1_{\mathrm{div}}\times V^1_{\mathrm{div}}),$$
$$(\boldsymbol{U},\boldsymbol{W},\boldsymbol{m},\boldsymbol{h})\in\mathcal{C}_w([0,T];H\times\mathbb{L}^2(\mathcal{O})\times\mathbb{L}^2(\mathcal{O})\times\mathbb{L}^2(\mathcal{O})),$$
$$\boldsymbol{m}=\chi_0\boldsymbol{h},\quad \operatorname{div}\boldsymbol{h}=0,\quad \text{a.e. in }\ Q_T.$$

(ii) For any $\boldsymbol{v} \in \mathcal{C}_c^1([0,T); V \cap \mathbb{H}^2(\mathcal{O}))$, $\psi \in \mathcal{C}_c^1([0,T); [H_0^1(\mathcal{O})]^3 \cap \mathbb{H}^2(\mathcal{O}))$ and $\psi_1 + \nabla\phi \in \mathcal{C}_c^1([0,T); [V_{\mathrm{div}}^1 \cap \mathbb{H}^2(\mathcal{O})] \oplus \mathcal{H})$, the following equations hold

$$
\begin{aligned}
&-\int_{Q_T} \boldsymbol{U} \cdot \partial_t \boldsymbol{v} \,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} (\boldsymbol{U}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{U}\,\mathrm{d}x\,\mathrm{d}t + \nu\int_{Q_T} \nabla\boldsymbol{U} : \nabla\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \mu_0(1+\chi_0)\int_{Q_T} [\boldsymbol{h}\cdot\nabla]\boldsymbol{v}\cdot\boldsymbol{h}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad = \int_{\mathcal{O}} \boldsymbol{U}_0\cdot\boldsymbol{v}(0)\,\mathrm{d}x - \alpha\int_{Q_T} (\nabla\times\boldsymbol{U} - 2\boldsymbol{W})\cdot\nabla\times\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t,\\
&\quad -\int_{Q_T} \boldsymbol{W}\cdot\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} (\boldsymbol{U}\cdot\nabla)\psi\cdot\boldsymbol{W}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + (\lambda_1+\lambda_2)\int_{Q_T} \mathrm{div}\,\boldsymbol{W}\,\mathrm{div}\,\psi\,\mathrm{d}x\,\mathrm{d}t + \lambda_1\int_{Q_T} \nabla\boldsymbol{W} : \nabla\psi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad = \int_{\mathcal{O}} \boldsymbol{W}_0\cdot\psi(0)\,\mathrm{d}x + 2\alpha\int_{Q_T} \boldsymbol{U}\cdot\nabla\times\psi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - 4\alpha\int_{Q_T} \boldsymbol{W}\cdot\psi\,\mathrm{d}x\,\mathrm{d}t,\\
&\quad - \mu_0(1+\chi_0)\int_{Q_T} \boldsymbol{h}\cdot\partial_t\psi_1\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\sigma}\int_{Q_T} \mathrm{curl}\,\boldsymbol{h}\cdot\mathrm{curl}\,\psi_1\,\mathrm{d}x\,\mathrm{d}t\\
&\quad = \mu_0(1+\chi_0)\int_{\mathcal{O}} \boldsymbol{h}_0\cdot\psi_1\,\mathrm{d}x + \mu_0(1+\chi_0)\int_{Q_T} (\boldsymbol{u}\times\boldsymbol{h})\cdot\mathrm{curl}\,\psi_1\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

(iii) The initial data is assumed in the following sense: $\boldsymbol{U}(t) \to \boldsymbol{U}(0) = \boldsymbol{U}_0$ in $\mathbb{L}^2(\mathcal{O})$, $\boldsymbol{m}(t) \to \boldsymbol{m}(0) = \boldsymbol{m}_0$ in $\mathbb{L}^2(\mathcal{O})$, $\boldsymbol{W}(t) \to \boldsymbol{W}(0) = \boldsymbol{W}_0$ in $\mathbb{L}^2(\mathcal{O})$, $\boldsymbol{h}(t) \to \boldsymbol{h}(0) = \boldsymbol{h}_0$ in $\mathbb{L}^2(\mathcal{O})$, as $t \to 0^+$.

With this definition in mind, we now state in precise manner the main result of this section.

Theorem 4.2 *Let the assumptions of Theorem 2.4 be satisfied. In addition, we assume that the physical parameters μ_0 and χ_0 satisfied*

$$\mu_0 = \chi_0. \tag{4.2}$$

Let $\{(\boldsymbol{u}_\tau, \boldsymbol{w}_\tau, \boldsymbol{M}_\tau, \boldsymbol{H}_\tau)\}_{\tau>0}$ be the sequence of global weak solutions to the ECREs (1.1)–(1.3) given by Theorem 2.4, where $\tau > 0$ is a small relaxation time. Then, when $\tau \to 0^+$, there exists a subsequence of $\{(\boldsymbol{u}_\tau, \boldsymbol{w}_\tau, \boldsymbol{M}_\tau, \boldsymbol{H}_\tau)\}_{\tau>0}$ convergent in $L^2(0,T;\mathbb{L}^2(\mathcal{O}))$ to a weak solution $(\boldsymbol{U}, \boldsymbol{W}, \boldsymbol{m}, \boldsymbol{h})$ of problem (4.1a)–(4.1h).

Before proving the theorem, we state and prove the following auxiliary result.

Lemma 4.3 *Let $T > 0$ be a fixed positive time. Let $(\boldsymbol{u}_\tau, \boldsymbol{w}_\tau, \boldsymbol{M}_\tau, \boldsymbol{H}_\tau)$ be the solution of problem (1.1)–(1.3) provided by Theorem 2.4, with $\tau > 0$ be given. We have*

$$\frac{\mu_0+\chi_0}{\tau}\int_{Q_t} \boldsymbol{H}_\tau(s)\cdot\boldsymbol{M}_\tau(s)\,\mathrm{d}x\,\mathrm{d}s - \frac{1}{\tau}\int_0^t \|M_\tau(s)\|^2\,\mathrm{d}s - \frac{\mu_0\chi_0}{\tau}\int_0^t \|\boldsymbol{H}_\tau(s)\|^2\,\mathrm{d}s$$

$$\leqslant -\frac{1}{2\tau}\int_0^t \|\boldsymbol{M}_\tau(s)-\chi_0\boldsymbol{H}_\tau(s)\|^2\,\mathrm{d}s+\frac{(\mu_0-\chi_0)^2}{2\tau}\int_0^t \|\boldsymbol{H}_\tau(s)\|^2\,\mathrm{d}s,\quad \forall t\in[0,T]. \tag{4.3}$$

Proof of Lemma 4.3 Let $t \in [0, T]$ be fixed. Notice that

$$\begin{aligned}
&\frac{\mu_0+\chi_0}{\tau}\int_{Q_t}\boldsymbol{H}_\tau(s)\cdot\boldsymbol{M}_\tau(s)\,\mathrm{d}x\,\mathrm{d}s-\int_0^t\frac{1}{\tau}\|M_\tau(s)\|^2\mathrm{d}s-\frac{\mu_0\chi_0}{\tau}\int_0^t\|\boldsymbol{H}_\tau(s)\|^2\,\mathrm{d}s\\
&=\frac{\mu_0}{\tau}\int_{Q_t}\boldsymbol{H}_\tau(s)\cdot\boldsymbol{M}_\tau(s)\mathrm{d}x\,\mathrm{d}s-\frac{1}{2\tau}\int_0^t\|\boldsymbol{M}_\tau(s)\|^2\mathrm{d}s\\
&\quad+\frac{\chi_0(\chi_0-2\mu_0)}{2\tau}\int_0^t\|\boldsymbol{H}_\tau(s)\|^2\mathrm{d}s-\frac{1}{2\tau}\int_0^t\|\boldsymbol{M}_\tau(s)-\chi_0\boldsymbol{H}_\tau(s)\|^2\mathrm{d}s.
\end{aligned} \tag{4.4}$$

By Hölder's and Young's inequalities, we have

$$\begin{aligned}
\frac{\mu_0}{\tau}\int_{Q_t}\boldsymbol{H}_\tau\cdot\boldsymbol{M}_\tau\,\mathrm{d}x\,\mathrm{d}s &\leqslant \frac{\mu_0}{\tau}\int_0^t\|\boldsymbol{M}_\tau(s)\|\|\boldsymbol{H}_\tau(s)\|\,\mathrm{d}s\\
&\leqslant\frac{1}{2\tau}\int_0^t\|\boldsymbol{M}_\tau(s)\|^2\,\mathrm{d}s+\frac{\mu_0^2}{2\tau}\int_0^t\|\boldsymbol{H}_\tau(s)\|^2\,\mathrm{d}s.
\end{aligned}$$

Hence, by inserting this inequality into (4.4), we easily derive (4.3). □

Remark 4.4 The estimate (4.3) in Lemma 4.3 and the assumption (4.2) are necessary and sufficient for the proof of Theorem 4.2. In fact,

(1) if for instance $\mu_0 \neq \chi_0$, the term on the left-hand side of (2.5) can't be bounded independently of τ by using (4.3) in (2.5).
(2) If $\mu_0 \neq \chi_0$, we can derive another estimate different from (4.3) and then bound all the terms on the left-hand side of (2.5) independently of τ. The problem is that with another bound different from (4.3), the left-hand side of (2.5) should contain the terms $\frac{1}{\iota\tau}\int_0^t\|\boldsymbol{M}_\tau\|^2ds$ or $\frac{c}{\iota\tau}\int_0^t\|\boldsymbol{H}_\tau\|^2ds$, with $\iota > 0$ sufficiently small and c being a positive constant which may possibly depend on μ_0 and χ_0. This may lead to an issue such as $\|\boldsymbol{M}_\tau\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))} \leqslant C\sqrt{\tau} \to 0$ if $\tau \to 0^+$. This does not make sense since $\boldsymbol{M}_\tau$ can't be equal to zero a.e. in $[0, T] \times \mathcal{O}$.

Following this important observation, we now provide the proof of Theorem 4.2.

Proof of Theorem 4.2 From (4.3), (4.2), and (2.5), we obtain

$$\begin{aligned}
&\frac{1}{2}\mathcal{E}_{\text{tot}}(\boldsymbol{u}_\tau(t),\mathbf{w}_\tau(t),\boldsymbol{M}_\tau(t),\boldsymbol{H}_\tau(t))+\int_0^t[\nu\|\nabla\boldsymbol{u}_\tau(s)\|^2+(\lambda_1+\lambda_2)\|\operatorname{div}\mathbf{w}_\tau(s)\|^2]\mathrm{d}s\\
&\quad+\int_0^t[\lambda_1\|\nabla\mathbf{w}_\tau(s)\|^2+\alpha\|\operatorname{curl}\boldsymbol{u}_\tau(s)-2\mathbf{w}_\tau(s)\|^2]\mathrm{d}s+\frac{1}{\sigma}\int_0^t\|\operatorname{curl}\boldsymbol{H}_\tau(s)\|^2\mathrm{d}s\\
&\quad+\frac{1}{2\tau}\int_0^t\|\boldsymbol{M}_\tau(s)-\chi_0\boldsymbol{H}_\tau(s)\|^2\mathrm{d}s\leqslant\frac{1}{2}\mathcal{E}_{\text{tot}}(\boldsymbol{u}_0,\mathbf{w}_0,\boldsymbol{M}_0,\boldsymbol{H}_0).
\end{aligned} \tag{4.5}$$

In view of (4.5), we see that $(\boldsymbol{u}_\tau, \mathbf{w}_\tau, \boldsymbol{M}_\tau, \boldsymbol{H}_\tau)$ satisfies the following bounds

$$\begin{aligned}
&|\boldsymbol{u}_\tau\|_{L^\infty(0,T;H)} + \|\boldsymbol{u}_\tau\|_{L^2(0,T;V)} \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
&\|\mathbf{w}_\tau\|_{L^\infty(0,T;\mathbb{L}^2(\mathcal{O}))} + \|\mathbf{w}_\tau\|_{L^2(0,T;[H_0^1(\mathcal{O})]^3)} \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
&\|\boldsymbol{M}_\tau\|_{L^\infty(0,T;H_{\boldsymbol{n}})} \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
&\|\boldsymbol{H}_\tau\|_{L^\infty(0,T;H_{\boldsymbol{n}})} \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
&\|\operatorname{curl}\boldsymbol{u}_\tau - 2\mathbf{w}_\tau\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))} \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
&\|\operatorname{curl}\boldsymbol{H}_\tau\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))} \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),\\
&\tau^{-1/2}\|\boldsymbol{M}_\tau - \chi_0\boldsymbol{H}_\tau\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))} \leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}}),
\end{aligned} \tag{4.6}$$

for a given $\tau > 0$. Here $\mathcal{E}_{\text{tot}} = \mathcal{E}_{\text{tot}}(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{M}_0, \boldsymbol{H}_0)$, and C is a large positive constant depending only on σ, ν, λ_1, λ_2, α, and χ_0.
Arguing similarly as in (3.38) and (3.46), using (4.5) in place of (3.33), we can prove that

$$\begin{aligned}
&\{\partial_t\mathbf{w}_\tau\}_{\tau>0} \text{ is uniformly bounded in } L^2(0,T;\mathbb{H}_0^{-2}(\mathcal{O})) \text{ with respect to } \tau,\\
&\{\partial_t\boldsymbol{u}_\tau\}_{\tau>0} \text{ is uniformly bounded in } L^2(0,T;\mathbb{H}_{\text{div}}^{-3}(\mathcal{O})) \text{ with respect to } \tau.
\end{aligned} \tag{4.7}$$

By the Banach–Alaoglu theorem and up to some subsequences, the estimates $(4.6)_1$, $(4.6)_2$, and (4.7) imply that

$$\begin{aligned}
&\boldsymbol{u}_\tau \overset{*}{\rightharpoonup} \boldsymbol{U} \text{ weakly-star in } L^\infty(0,T;H),\\
&\boldsymbol{u}_\tau \rightharpoonup \boldsymbol{U} \text{ weakly in } L^2(0,T;V),\\
&\mathbf{w}_\tau \overset{*}{\rightharpoonup} \boldsymbol{W} \text{ weakly-star in } L^\infty(0,T;\mathbb{L}^2(\mathcal{O})),\\
&\mathbf{w}_\tau \rightharpoonup \boldsymbol{W} \text{ weakly in } L^2(0,T;[H_0^1(\mathcal{O})]^3),\\
&\partial_t\boldsymbol{u}_\tau \rightharpoonup \partial_t\boldsymbol{U} \text{ weakly in } L^2(0,T;\mathbb{H}_0^{-2}(\mathcal{O})),\\
&\partial_t\mathbf{w}_\tau \rightharpoonup \partial_t\boldsymbol{W} \text{ weakly in } L^2(0,T;\mathbb{H}_{\text{div}}^{-3}(\mathcal{O})), \text{ as } \tau \to 0^+.
\end{aligned}$$

Besides, by $(4.6)_1$, $(4.6)_2$, (4.7) and $[H_0^1(\mathcal{O})]^3 \overset{c}{\hookrightarrow} \mathbb{L}^2(\mathcal{O}) \hookrightarrow \mathbb{H}_0^{-2}(\mathcal{O})$, $V \overset{c}{\hookrightarrow} H \hookrightarrow \mathbb{H}_{\text{div}}^{-3}(\mathcal{O})$, where every first embedding is compact and every second one continuous, we can use the Aubin–Lions compactness lemma so that

$$\boldsymbol{u}_\tau \to \boldsymbol{U}, \quad \mathbf{w}_\tau \to \boldsymbol{W} \quad \text{strongly in } L^2(0,T;\mathbb{L}^p(\mathcal{O})), \quad p \in [2,6), \tag{4.8}$$

up to a subsequence, for some limiting functions $\boldsymbol{U} \in L^\infty(0,T;H) \cap L^2(0,T;V)$ and $\boldsymbol{W} \in L^\infty(0,T;\mathbb{L}^2(\mathcal{O})) \cap L^2(0,T;[H_0^1(\mathcal{O})]^3)$.

By (4.6)$_3$ and (4.6)$_4$ in conjunction with the Banach–Alaoglu theorem, we infer that

$$\begin{aligned} &\boldsymbol{M}_\tau \overset{*}{\rightharpoonup} \boldsymbol{m} \text{ weakly-star in } L^\infty(0,T;\mathbb{L}^2(\mathcal{O})),\\ &\boldsymbol{H}_\tau \overset{*}{\rightharpoonup} \boldsymbol{h} \text{ weakly-star in } L^\infty(0,T;\mathbb{L}^2(\mathcal{O})), \end{aligned} \tag{4.9}$$

for limiting functions $\boldsymbol{m}, \boldsymbol{h} \in L^\infty(0,T;\mathbb{L}^2(\mathcal{O}))$. Moreover, we have $\boldsymbol{m} = \chi_0 \boldsymbol{h}$ a.e. in Q_T. In fact, for any test function $\psi \in L^2(0,T;\mathbb{L}^2(\mathcal{O}))$, using the Hölder inequality along with the uniform bounds (4.6)$_7$, we have

$$\begin{aligned} \left|\int_{Q_T} (\boldsymbol{m} - \chi_0 \boldsymbol{h})\cdot\psi \,\mathrm{d}x\,\mathrm{d}s\right| &= \left|\lim_{\tau\to 0^+}\int_{Q_T} (\boldsymbol{M}_\tau - \chi_0 \boldsymbol{H}_\tau)\cdot\psi \,\mathrm{d}x\,\mathrm{d}s\right|\\ &\leqslant \lim_{\tau\to 0^+}(\|\boldsymbol{M}_\tau - \chi_0\boldsymbol{H}_\tau\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))})\|\psi\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))}\\ &\leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}})\|\psi\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))}\lim_{\tau\to 0^+}(\sqrt{\tau}) = 0. \end{aligned}$$

This proves that $\boldsymbol{m} = \chi_0 \boldsymbol{h}$ in $L^2(0,T;\mathbb{L}^2(\mathcal{O}))$.
We recall that the following equalities hold true (since for fix $\tau > 0$, $\{\boldsymbol{u}_\tau, \mathbf{w}_\tau, \boldsymbol{M}_\tau, \boldsymbol{H}_\tau\}$ is a weak solutions of problem (1.1)–(1.3) provided by Theorem 2.4):

$$\begin{aligned} \boldsymbol{H}_\tau &= \boldsymbol{H}_a^\tau + \boldsymbol{H}_d^\tau,\\ \operatorname{div}\boldsymbol{H}_a^\tau &= 0, \quad \operatorname{curl}\boldsymbol{H}_a^\tau = \operatorname{curl}\boldsymbol{H}_\tau \quad \text{in} \quad Q_T,\\ \operatorname{curl}\boldsymbol{H}_d^\tau &= 0, \quad \boldsymbol{H}_d^\tau = \nabla\varphi_d^\tau, \quad \operatorname{div}\boldsymbol{H}_d^\tau = -\operatorname{div}\boldsymbol{M}_\tau \quad \text{in} \quad Q_T, \end{aligned}$$

$$\begin{aligned} \int_{\mathcal{O}} \nabla\varphi_d^\tau\cdot\nabla\psi\,\mathrm{d}x &= \int_{\mathcal{O}}(\boldsymbol{H}_\tau - \boldsymbol{H}_a^\tau)\cdot\nabla\psi\,\mathrm{d}x = \int_{\mathcal{O}}\boldsymbol{H}_\tau\cdot\nabla\psi\mathrm{d}x\\ &= -\int_{\mathcal{O}}\boldsymbol{M}_\tau\cdot\nabla\psi\,\mathrm{d}x, \quad \forall\psi\in H^1(\mathcal{O}), \end{aligned}$$

and

$$\boldsymbol{H}_d^\tau\cdot\boldsymbol{n} = \frac{\partial\varphi_d^\tau}{\partial\boldsymbol{n}} = \nabla\varphi_d^\tau\cdot\boldsymbol{n} = -\boldsymbol{M}_\tau\cdot\boldsymbol{n} \quad \text{on} \quad \Sigma.$$

In view of (4.6)$_4$ and [23, Theorem 1.4], we deduce that $\boldsymbol{H}_d^\tau = \nabla\varphi_d^\tau$ is uniformly bounded in $L^\infty(0,T;\mathbb{L}^2(\mathcal{O}))$ with respect to τ. Hence, by the Banach–Alaoglu theorem, we derive that

$$\boldsymbol{H}_d^\tau \overset{*}{\rightharpoonup} \boldsymbol{H}_d^* \text{ weakly-star in } L^\infty(0,T;\mathbb{L}^2(\mathcal{O})),$$

for a limiting function $\boldsymbol{H}_d^* \in L^\infty(0,T;\mathbb{L}^2(\mathcal{O}))$.
We recall that $\boldsymbol{H}_a^\tau = \boldsymbol{H}_\tau - \boldsymbol{H}_d^\tau$ in Q_T. Since $\boldsymbol{H}_d^\tau$ and $\boldsymbol{H}_\tau$ are uniformly bounded in $L^\infty(0,T;\mathbb{L}^2(\mathcal{O}))$, respectively, we find that $\boldsymbol{H}_a^\tau$ is also uniformly bounded in

$L^\infty(0,T;H)$. Moreover, since curl $\boldsymbol{H}_a^\tau = $ curl $\boldsymbol{H}_\tau$, div $\boldsymbol{H}_a^\tau = 0$, and curl $\boldsymbol{H}_\tau$ is uniformly bounded in $L^2(0,T;\mathbb{L}^2(\mathcal{O}))$ (due to (4.6)$_6$), we deduce that $\boldsymbol{H}_a^\tau$ is uniformly bounded in $L^\infty(0,T;H)\cap L^2(0,T;V)$. By applying again the Banach–Alaoglu theorem, we deduce that

$$\begin{aligned}&\boldsymbol{H}_a^\tau \overset{*}{\rightharpoonup} \boldsymbol{H}_a^* \text{ weakly-star in } L^\infty(0,T;H),\\&\boldsymbol{H}_a^\tau \rightharpoonup \boldsymbol{H}_a^* \text{ weakly in } L^2(0,T;V),\end{aligned}$$

for a limiting function $\boldsymbol{H}_a^* \in L^\infty(0,T;H)\cap L^2(0,T;V)$.
Passing to the limit as $\tau \to 0^+$ in $\boldsymbol{H}_\tau = \boldsymbol{H}_a^\tau + \boldsymbol{H}_d^\tau$, $\operatorname{div}(\boldsymbol{M}_\tau + \boldsymbol{H}_\tau) = 0$ and $\operatorname{div} \boldsymbol{H}_d^\tau = -\operatorname{div} \boldsymbol{M}_\tau$, we obtain

$$\begin{aligned}&\boldsymbol{h} = \boldsymbol{H}_a^* + \boldsymbol{H}_d^*, \quad 0 = \operatorname{div}(\boldsymbol{m}+\boldsymbol{h}) = (1+\chi_0)\operatorname{div}\boldsymbol{h},\\&\operatorname{div}\boldsymbol{H}_d^* = -\operatorname{div}\boldsymbol{m} = \operatorname{div}\boldsymbol{h} = 0 \text{ in } \mathcal{O}.\end{aligned} \tag{4.10}$$

On the other hand, owing to [23, Corollary 5.5] and the uniqueness of the Helmholtz–Leray decomposition, we find that there exists $\nabla\varphi_d^* \in E_2(\mathcal{O})$ such that $\boldsymbol{h} := \boldsymbol{H}_a^* + \boldsymbol{H}_d^* = \boldsymbol{H}_a^* + \nabla\varphi_d^*$, $\Delta\varphi_d^* = 0$ in Q_T and $\nabla\varphi_d^*\cdot\boldsymbol{n} = \boldsymbol{h}\cdot\boldsymbol{n}$ on Σ. Furthermore, we observe that

$$\nabla\varphi_d^* = \boldsymbol{H}_d^* \in L^\infty(0,T;\mathbb{L}^2(\mathcal{O}))\cap L^2(0,T;\mathbb{H}^1(\mathcal{O})).$$

We will now give an estimate for $\partial_t \boldsymbol{M}_\tau$.
Exploiting the Hölder, (4.6)$_1$ and (4.6)$_3$, we have

$$\begin{aligned}\int_0^T \|(\boldsymbol{u}_\tau(t)\cdot\nabla)\boldsymbol{M}_\tau(t)\|^2_{\mathbb{H}^{-2}(\mathcal{O})}\,\mathrm{d}t &\leqslant C \sup_{t\in[0,T]}\|\boldsymbol{M}_\tau(t)\|^2 \int_0^T \|\nabla\boldsymbol{u}_\tau(t)\|^2\,\mathrm{d}t\\&\leqslant C(1+\mathcal{E}_{\mathrm{tot}}^{\frac12})^4,\end{aligned}$$

where the constant C is independent of τ.
Due to (4.6)$_1$, (4.6)$_2$ and the embedding of $\mathbb{H}^1(\mathcal{O})$ in $\mathbb{L}^4(\mathcal{O})$, we see that

$$\begin{aligned}\int_0^T \|\mathbf{w}_\tau(t)\times\boldsymbol{M}_\tau(t)\|^2_{(\mathbb{L}^4(\mathcal{O}))'}\,\mathrm{d}t &\leqslant C \sup_{t\in[0,T]}\|\boldsymbol{M}_\tau(t)\|^2 \int_0^T \|\mathbf{w}_\tau(t)\|^2_{[H_0^1(\mathcal{O})]^3}\,\mathrm{d}t\\&\leqslant C(1+\mathcal{E}_{\mathrm{tot}}^{\frac12})^4.\end{aligned}$$

Here the positive constant C is independent of τ. Hence, by recalling also the uniform bounds (4.6)$_3$ and (4.6)$_4$ for $\boldsymbol{M}_\tau$ and $\boldsymbol{H}_\tau$, respectively, we easily see that $\{\partial_t\boldsymbol{M}_\tau\}_{\tau>0}$ is uniformly bounded in $L^2(0,T;\mathbb{H}^{-2}(\mathcal{O}))$.
Let $\boldsymbol{X} := V_1 \cap \mathbb{H}^2(\mathcal{O})$. Then arguing as in (3.54), we can prove that

$$\int_0^T \|\nabla\times(\boldsymbol{u}_\tau(t)\times\boldsymbol{B}_\tau(t))\|^2_{\boldsymbol{X}'}\,\mathrm{d}t \leqslant C \sup_{t\in[0,T]}\|\boldsymbol{B}_\tau(t)\|^2 \int_0^T \|\nabla\boldsymbol{u}_\tau(t)\|^2\,\mathrm{d}t$$

$$\leqslant C(1+\mathcal{E}_{\text{tot}}^{\frac{1}{2}})^4, \tag{4.11}$$

and

$$\int_0^T \frac{1}{\sigma^2}\|\nabla\times(\nabla\times \boldsymbol{H}_\tau(t))\|_{\boldsymbol{X}'}^2\,\mathrm{d}t \leqslant \frac{1}{\sigma^2}\int_0^T \|\operatorname{curl}\boldsymbol{H}_a^\tau(t)\|^2\,\mathrm{d}t. \tag{4.12}$$

We note that in (4.11) we have also used the uniform bounds $(4.6)_1$, $(4.6)_3$ and $(4.6)_4$. Furthermore, since $\boldsymbol{H}_a^\tau$ is uniformly bounded in $L^2(0,T;\mathbb{H}^1(\mathcal{O}))$, more precisely in $L^2(0,T;V)$, we see that the left-hand side of (4.12) is uniformly bounded with respect to τ. Finally, since (1.1d) holds almost everywhere, we derive that $\{\partial_t\boldsymbol{H}_\tau\}_{\tau>0}$ is uniformly bounded in $L^2(0,T;\boldsymbol{X}')$ with respect to τ.
As in (3.57), we can prove that $\{\partial_t\boldsymbol{H}_a^\tau\}_\tau$ is uniformly bounded in $L^2(0,T;\boldsymbol{Y}_1') := L^2(0,T;(H\cap\mathbb{H}^2(\mathcal{O}))')$, and consequently (cf. (3.58))

$$\boldsymbol{H}_a^\tau \to \boldsymbol{H}_a^* \quad \text{in} \quad L^2(0,T;H), \tag{4.13}$$

as $\tau\to 0^+$, for a limiting function $\boldsymbol{H}_a^*\in L^\infty(0,T;H)\cap L^2(0,T;V)$.
Next using and integration by parts and the fact that $\operatorname{div}\boldsymbol{h}=0$ in $\mathcal{O}$, we find

$$(1+\chi_0)\int_{\mathcal{O}}\boldsymbol{h}\cdot\boldsymbol{H}_d^*\,\mathrm{d}x = (1+\chi_0)\int_{\mathcal{O}}\boldsymbol{h}\cdot\nabla\varphi_d^*\,\mathrm{d}x = -(1+\chi_0)\int_{\mathcal{O}}\varphi_d^*\operatorname{div}\boldsymbol{h}\,\mathrm{d}x = 0.$$

Notice that $\int_{\mathcal{O}}\boldsymbol{h}\cdot\boldsymbol{H}_d^*\,\mathrm{d}x=0$ and $\int_{\mathcal{O}}\boldsymbol{m}\cdot\boldsymbol{H}_d^*\,\mathrm{d}x=0$, since $\operatorname{div}\boldsymbol{h}=\operatorname{div}\boldsymbol{m}=0$ and $\boldsymbol{H}_d^*=\nabla\varphi_d^*$. Hence, as $\boldsymbol{h}=\boldsymbol{H}_d^*+\boldsymbol{H}_a^*$ (cf. (4.10)), we obtain

$$\begin{aligned}\int_{\mathcal{O}}|\boldsymbol{h}|^2\,\mathrm{d}x &= \int_{\mathcal{O}}\boldsymbol{h}\cdot\boldsymbol{H}_a^*\,\mathrm{d}x + \int_{\mathcal{O}}\boldsymbol{h}\cdot\boldsymbol{H}_d^*\,\mathrm{d}x = \int_{\mathcal{O}}\boldsymbol{h}\cdot\boldsymbol{H}_a^*\,\mathrm{d}x\\ &= \int_{\mathcal{O}}\boldsymbol{h}\cdot\boldsymbol{H}_a^*\,\mathrm{d}x - \int_{\mathcal{O}}\boldsymbol{m}\cdot\boldsymbol{H}_d^*\,\mathrm{d}x\\ &= \int_{\mathcal{O}}\boldsymbol{h}\cdot\boldsymbol{H}_a^*\,\mathrm{d}x - \int_{\mathcal{O}}\boldsymbol{m}\cdot(\boldsymbol{h}-\boldsymbol{H}_a^*)\,\mathrm{d}x.\end{aligned}$$

Thus, we deduce that

$$\int_{\mathcal{O}}|\boldsymbol{h}|^2\,\mathrm{d}x = -\int_{\mathcal{O}}\boldsymbol{m}\cdot\boldsymbol{h}\,\mathrm{d}x + \int_{\mathcal{O}}(\boldsymbol{m}+\boldsymbol{h})\cdot\boldsymbol{H}_a^*\,\mathrm{d}x. \tag{4.14}$$

Owing to (4.14) and the fact that $\boldsymbol{m}=\chi_0\boldsymbol{h}$, we have

$$\int_{\mathcal{O}}(\boldsymbol{m}+\boldsymbol{h})\cdot\boldsymbol{H}_a^*\,\mathrm{d}x = (1+\chi_0)\int_{\mathcal{O}}|\boldsymbol{h}|^2\,\mathrm{d}x. \tag{4.15}$$

Next we observe that

$$\int_{\mathcal{O}}(\boldsymbol{H}_\tau+\boldsymbol{M}_\tau)\cdot\boldsymbol{H}_\tau\,\mathrm{d}x = \int_{\mathcal{O}}(\boldsymbol{H}_\tau+\boldsymbol{M}_\tau)\cdot(\boldsymbol{H}_a^\tau+\boldsymbol{H}_d^\tau)\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{\mathcal{O}} (\boldsymbol{H}_\tau + \boldsymbol{M}_\tau) \cdot (\boldsymbol{H}_a^\tau - \boldsymbol{H}_a^*)\, \mathrm{d}x \\
&\quad + \int_{\mathcal{O}} (\boldsymbol{H}_\tau + \boldsymbol{M}_\tau) \cdot \boldsymbol{H}_a^*\, \mathrm{d}x + \int_{\mathcal{O}} (\boldsymbol{H}_\tau + \boldsymbol{M}_\tau) \cdot \nabla \varphi_d^\tau\, \mathrm{d}x \\
&= \int_{\mathcal{O}} (\boldsymbol{H}_\tau + \boldsymbol{M}_\tau) \cdot (\boldsymbol{H}_a^\tau - \boldsymbol{H}_a^*) dx + \int_{\mathcal{O}} (\boldsymbol{H}_\tau + \boldsymbol{M}_\tau) \cdot \boldsymbol{H}_a^*\, \mathrm{d}x. \qquad (4.16)
\end{aligned}
$$

Owing to the uniform bounds $(4.6)_3$ and $(4.6)_4$ for $\boldsymbol{M}_\tau$ and $\boldsymbol{H}_\tau$, respectively, and the strong convergence (4.13), we obtain that the first term on the right-hand side of (4.16) converges to zero, as $\tau \to 0^+$. Hence,

$$
\int_{\mathcal{O}} \overline{(\boldsymbol{H} + \boldsymbol{M}) \cdot \boldsymbol{H}}\, \mathrm{d}x = 0 + \int_{\mathcal{O}} (\boldsymbol{h} + \boldsymbol{m}) \cdot \boldsymbol{H}_a^*\, \mathrm{d}x \quad \text{a.e.} \quad t \in [0, T]. \qquad (4.17)
$$

It then follows from (4.15) and (4.17) that

$$
\begin{aligned}
\int_{Q_T} \overline{(\boldsymbol{H} + \boldsymbol{M}) \cdot \boldsymbol{H}}\, \mathrm{d}x\, \mathrm{d}t &= \lim_{\tau \to 0^+} \int_{Q_T} (\boldsymbol{H}_\tau + \boldsymbol{M}_\tau) \cdot \boldsymbol{H}_\tau\, \mathrm{d}x\, \mathrm{d}t \\
&= \int_{Q_T} (\boldsymbol{h} + \boldsymbol{m}) \cdot \boldsymbol{H}_a^*\, \mathrm{d}x\, \mathrm{d}t = (1 + \chi_0) \int_{Q_T} |\boldsymbol{h}|^2\, \mathrm{d}x\, \mathrm{d}t.
\end{aligned}
\qquad (4.18)
$$

Besides, from $(4.6)_4$ and $(4.6)_7$, we obtain

$$
\begin{aligned}
&\left| (1 + \chi_0) \lim_{\tau \to 0^+} \int_{Q_T} |\boldsymbol{H}_\tau|^2\, \mathrm{d}x\, \mathrm{d}t - \int_{Q_T} \overline{(\boldsymbol{H} + \boldsymbol{M}) \cdot \boldsymbol{H}}\, \mathrm{d}x\, \mathrm{d}t \right| \\
&= \left| (1 + \chi_0) \lim_{\tau \to 0^+} \int_{Q_T} |\boldsymbol{H}_\tau|^2\, \mathrm{d}x\, \mathrm{d}t - \lim_{\tau \to 0^+} \int_{Q_T} (\boldsymbol{H}_\tau + \boldsymbol{M}_\tau) \cdot \boldsymbol{H}_\tau\, \mathrm{d}x\, \mathrm{d}t \right| \\
&= \left| \lim_{\tau \to 0^+} \int_{Q_T} \boldsymbol{H}_\tau \cdot (\chi_0 \boldsymbol{H}_\tau - \boldsymbol{M}_\tau)\, \mathrm{d}x\, \mathrm{d}t \right| \\
&\leqslant \limsup_{\tau \to 0^+} \left| \int_{Q_T} \boldsymbol{H}_\tau \cdot (\chi_0 \boldsymbol{H}_\tau - \boldsymbol{M}_\tau)\, \mathrm{d}x\, \mathrm{d}t \right| \\
&\leqslant \limsup_{\tau \to 0^+} \int_{Q_T} |\boldsymbol{H}_\tau| |(\chi_0 \boldsymbol{H}_\tau - \boldsymbol{M}_\tau)|\, \mathrm{d}x\, \mathrm{d}t \\
&\leqslant C (1 + \mathcal{E}_{\text{tot}}^{\frac{1}{2}})^2 \limsup_{\tau \to 0^+} \sqrt{\tau} \to 0, \qquad (4.19)
\end{aligned}
$$

By (4.19) and (4.18), we deduce that

$$
\begin{aligned}
(1 + \chi_0) \lim_{\tau \to 0^+} \int_{Q_T} |\boldsymbol{H}_\tau|^2\, \mathrm{d}x\, \mathrm{d}t &= \int_{Q_T} \overline{(\boldsymbol{H} + \boldsymbol{M}) \cdot \boldsymbol{H}}\, \mathrm{d}x\, \mathrm{d}t \\
&= (1 + \chi_0) \int_{Q_T} |\boldsymbol{h}|^2\, \mathrm{d}x\, \mathrm{d}t, \qquad (4.20)
\end{aligned}
$$

which along with (4.20) and (4.9) implies

$$\begin{aligned}\lim_{\tau\to 0^+}\int_{Q_T}|\boldsymbol{H}_\tau-\boldsymbol{h}|^2\,\mathrm{d}x\,\mathrm{d}t &= \lim_{\tau\to 0^+}\int_{Q_T}|\boldsymbol{H}_\tau|^2\,\mathrm{d}x\,\mathrm{d}t-2\lim_{\tau\to 0^+}\int_{Q_T}\boldsymbol{H}_\tau\cdot\boldsymbol{h}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_{Q_T}|\boldsymbol{h}|^2\,\mathrm{d}x\,\mathrm{d}t\\ &=2\int_{Q_T}|\boldsymbol{h}|^2\,\mathrm{d}x\,\mathrm{d}t-2\lim_{\tau\to 0^+}\int_{Q_T}\boldsymbol{H}_\tau\cdot\boldsymbol{h}\,\mathrm{d}x\,\mathrm{d}t=0.\end{aligned}$$

This proves that a subsequence of $\{\boldsymbol{H}_\tau\}_{\tau>0}$ converges strongly to $L^2(0,T;\mathbb{L}^2(\mathcal{O}))$. We observe that

$$\begin{aligned}\int_{Q_T}|\boldsymbol{M}_\tau-\chi_0\boldsymbol{h}|\,\mathrm{d}x\,\mathrm{d}t &\leqslant \int_{Q_T}|\boldsymbol{M}_\tau-\chi_0\boldsymbol{H}_\tau|\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_T}|\chi_0\boldsymbol{H}_\tau-\chi_0\boldsymbol{h}|\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant T^{\frac12}|\mathcal{O}|^{\frac12}\|\boldsymbol{M}_\tau-\chi_0\boldsymbol{H}_\tau\|_{L^2(0,T;\mathbb{L}^2(\mathcal{O}))}\\ &\quad+\int_{Q_T}|\chi_0\boldsymbol{H}_\tau-\chi_0\boldsymbol{h}|\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Using $(4.6)_7$ and the fact that $\boldsymbol{H}_\tau\to\boldsymbol{h}$ in $L^2(0,T;\mathbb{L}^2(\mathcal{O}))$, as $\tau\to 0^+$, we find

$$\begin{aligned}&\lim_{\tau\to 0^+}\int_{Q_T}|\boldsymbol{M}_\tau-\chi_0\boldsymbol{h}|\,\mathrm{d}x\,\mathrm{d}t\\ &\quad\leqslant C(1+\mathcal{E}_{\mathrm{tot}}^{\frac12})\lim_{\tau\to 0^+}\sqrt{\tau}+\lim_{\tau\to 0^+}\int_{Q_T}|\chi_0\boldsymbol{H}_\tau-\chi_0\boldsymbol{h}|\,\mathrm{d}x\,\mathrm{d}t=0,\end{aligned}$$

which along with the fact that $\boldsymbol{m}=\chi_0\boldsymbol{h}$ yields that $\boldsymbol{M}_\tau\to\boldsymbol{m}$ strongly in $L^2(0,T;\mathbb{L}^2(\mathcal{O}))$, as $\tau\to 0^+$.
Let $\Psi\in\mathcal{C}_c^1([0,T);V_{\mathrm{div}}^1\cap\mathbb{H}^2(\mathcal{O}))$ be arbitrary but fixed. From $(4.6)_6$, we deduce that (up to a subsequence)

$$\operatorname{curl}\boldsymbol{H}_\tau\rightharpoonup\operatorname{curl}\boldsymbol{h}\quad\text{weakly in}\ \ L^2(0,T;\mathbb{L}^2(\mathcal{O})).$$

Consequently,

$$\int_{Q_T}\operatorname{curl}\boldsymbol{H}_\tau\cdot\operatorname{curl}\Psi\,\mathrm{d}x\,\mathrm{d}t\to\int_{Q_T}\operatorname{curl}\boldsymbol{h}\cdot\operatorname{curl}\Psi\,\mathrm{d}x\,\mathrm{d}t\quad\text{as}\quad\tau\to 0^+.$$

Observe that by Green's formula (see [12, Eq. (2.18), p. 21]) and since $\Psi\times\boldsymbol{n}|_{\partial\mathcal{O}}=0$, we have

$$\begin{aligned}\int_{\mathcal{O}}(\boldsymbol{U}\times\boldsymbol{b})\cdot\operatorname{curl}\Psi\,\mathrm{d}x &= \int_{\mathcal{O}}\operatorname{curl}(\boldsymbol{U}\times\boldsymbol{b})\cdot\Psi\,\mathrm{d}x\\ &=\int_{\mathcal{O}}[(\boldsymbol{b}\cdot\nabla)\boldsymbol{U}-(\boldsymbol{U}\cdot\nabla)\boldsymbol{b}]\cdot\Psi\,\mathrm{d}x\end{aligned}$$

$$= -\int_{\mathcal{O}} (\boldsymbol{b} \cdot \nabla)\Psi \cdot \boldsymbol{U}\, \mathrm{d}x + \int_{\mathcal{O}} (\boldsymbol{U} \cdot \nabla)\Psi \cdot \boldsymbol{b}\, \mathrm{d}x.$$

Hence, from (4.8) and (4.9), and arguing as we did in (3.82), we obtain

$$\int_{Q_T} (\boldsymbol{u}_\tau \times \boldsymbol{B}_\tau) \cdot \operatorname{curl}\Psi\, \mathrm{d}x\, \mathrm{d}t \to -\int_{Q_T} (\boldsymbol{b} \cdot \nabla)\Psi \cdot \boldsymbol{U}\, \mathrm{d}x\, \mathrm{d}t + \int_{Q_T} (\boldsymbol{U} \cdot \nabla)\Psi \cdot \boldsymbol{b}\, \mathrm{d}x\, \mathrm{d}t$$
$$= \int_{Q_T} (\boldsymbol{U} \times \boldsymbol{b}) \cdot \operatorname{curl}\Psi\, \mathrm{d}x\, \mathrm{d}t,$$

as $\tau \to 0^+$ (up to a subsequence). So, we can pass to the limit in the weak formulation of (1.1d) and derive (4.1d) in the sense of distributions.
The next step is to pass to the limit, but before doing that, we multiply both sides of the weak formulation of (1.1d). With this in mind, the convergence results established in this section enable us to pass to the limit in the weak formulation of (1.1)–(1.3) (cf. (2.3)–(2.4)) and obtain that the limit $(\boldsymbol{U}, \boldsymbol{W}, \boldsymbol{m}, \boldsymbol{h})$ satisfies (4.1a)–(4.1f). This completes the proof of Theorem 4.2. □

5 Conclusions

In this work, we studied a nonlinear coupling system of partial differential equations proposed by Rosensweig for the dynamics of an incompressible viscous ferrofluid. This system of partial differential equations, which we called the electrically conductive Rosensweig equations (ECREs for short), has been studied in a connected and bounded open domain in $\mathbb{R}^3$, with no-slip boundary condition both for the velocity $\boldsymbol{u}$ and angular velocity $\mathbf{w}$, and electric boundary conditions for the magnetization field $\boldsymbol{M}$ and the magnetic field $\boldsymbol{H}$. First, we showed the existence of global weak solutions to the non-regularized electrically conductive Rosensweig equations if the electric conductivity coefficient σ is not too small. This enables us to send a message to ferrofluid users that ferrofluids are naturally very poor conductors of electric current or dielectric, and that one cannot develop non-conductive ferrofluid from conductive ferrofluid with very small electric conductivity. Second, motivated by the smallness of the ferrofluids' relaxation time, which is the average time needed by the ferrofluid to recover an equilibrium state once perturbed, we gave a rigorous analysis of the behavior of the global weak solutions in the relaxation time limit regime $\tau \to 0$ (cf. Theorem 4.2). We mainly proved that as $\tau \to 0$, a sequence of global weak solutions of the ECREs converges in appropriate topology towards the quasi-equilibrium, which is basically a solution to the Navier–Stokes–Maxwell system with internal rotation. We conclude this paper by mentioning some few open questions which will be the object of future investigations:

- **Well-posedness of strong solutions** An interesting issue is to prove the existence and uniqueness of the regular strong solutions to (4.1a)–(4.1h) whose existence of the global weak solutions is proved in Theorem 4.2. More precisely, with smooth initial data, is it possible to establish the existence and uniqueness of the limiting system (4.1a)–(4.1h), as discussed in [17, Appendix A] or [28], in the case where the magnetic field is described by the magnetostatic equations?

- **Stochastic case** An extension of this paper to the analysis of the stochastic version of the ECREs with a special type of noise is the subject of a work in progress. This is motivated by the fact that the randomness of the environment and the stochastic motions of the ferromagnetic particles in the ferrofluids may induce phenomena that might be absent in the deterministic system. Despite the randomness of the dynamics, previous mathematical analyses in the literature only considered deterministic models.
- **Boundary conditions** Further interesting questions concern the analysis of system (1.1a)–(1.1f) with Dirichlet boundary condition both for the velocity and angular velocity, and other types of conditions such the magnetic boundary conditions or the Robin-like boundary conditions (cf. [16, p. 5]) for the magnetization field and the magnetic field, respectively.

Author Contributions All authors contributed to the mathematical analysis of this manuscript. They also read and approved the final manuscript.

Funding Open access funding provided by North-West University. On behalf of all authors, the corresponding author declares that no funds, grants, or other support were received during the preparation of this manuscript.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. Amirat, Y., Hamdache, K.: Global weak solutions to a ferrofluid flow model. Math. Methods Appl. Sci. **31**(2), 123–151 (2008)
2. Amirat, Y., Hamdache, K.: Weak solutions to the equations of motion for compressible magnetic fluids. J. Math. Pures Appl. **91**(5), 433–467 (2009)
3. Amirat, Y., Hamdache, K.: Strong solutions to the equations of a ferrofluid flow model. J. Math. Anal. Appl. **353**(1), 271–294 (2009)
4. Amirat, Y., Hamdache, K.: Unique solvability of equations of motion for ferrofluids. Nonlinear Anal. **73**(2), 471–494 (2010)
5. Amirat, Y., Hamdache, K.: Strong solutions to the equations of electrically conductive magnetic fluids. J. Math. Anal. Appl. **421**(1), 75–104 (2015)
6. Amirat, Y., Hamdache, K., Murat, F.: Global weak solutions to equations of motion for magnetic fluids. J. Math. Fluid Mech. **10**(3), 326–351 (2008)

7. De Anna, F., Scrobogna, S.: A global well-posedness result for the Rosensweig system of ferrofluids. Rev. Mat. Iberoam. **36**(3), 895–938 (2020)
8. DiPerna, R.J., Lions, P.-L.: Ordinary differential equations, transport theory and Sobolev spaces. Invent. Math. **98**, 511–547 (1989). https://doi.org/10.1007/BF01393835
9. Evans, L.C.: Weak Convergence Methods for Nonlinear Partial Differential Equations. CBMS Regional Conference Series in Mathematics, vol. 74. American Mathematical Society, Providence (1990). https://doi.org/10.1090/cbms/074
10. Feireisl, E.: Dynamics of Viscous Compressible Fluids. Oxford Lecture Series in Mathematics and Its Applications, vol. 26. Oxford University Press, Oxford (2004)
11. Giorgini, A., Miranville, A., Temam, R.: Uniqueness and regularity for the Navier-Stokes-Cahn-Hilliard system. SIAM J. Math. Anal. **51**(3), 2535–2574 (2019)
12. Girault, V., Raviart, P.-A.: Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms. Springer Series in Computational Mathematics, vol. 5. Springer, Berlin (1986)
13. Hamdache, K., Hamroun, D.: Weak solutions to unsteady and steady models of conductive magnetic fluids. Appl. Math. Optim. **81**(2), 479–509 (2020)
14. Kole, M., Khandekar, S.: Engineering applications of ferrofluids: a review. J. Magn. Magn. Mater. **537**, Paper No. 168222 (2021)
15. Lions, P.-L.: Mathematical Topics in Fluid Mechanics—Volumes 1 & 2, Compressible Models. Oxford Lecture Series in Mathematics and Its Applications 10. Clarendon Press, Oxford University Press, New York (1998)
16. Nochetto, R.H., Salgado, A., Tomas, I.: The equations of ferrohydrodynamics: modeling and numerical methods. Math. Models Methods Appl. Sci. **26**(13), 2393–2449 (2016)
17. Nochetto, R.H., Trivisa, K., Weber, F.: On the dynamics of ferrofluids: global weak solutions to the Rosensweig system and rigorous convergence to equilibrium. SIAM J. Math. Anal. **51**(6), 4245–4286 (2019)
18. Novotný, A., Straskraba, I.: Introduction to the Mathematical Theory of Compressible Flow. Oxford Lecture Series in Mathematics and Its Applications, vol. 27. Oxford University Press, Oxford (2004)
19. Rosensweig, R.E.: Ferrohydrodynamics. Cambridge University Press, Cambridge; 344 pp. J. Fluid Mech. **200**, 597–599 (1985). https://doi.org/10.1017/S0022112089220773
20. Scherer, C., Figueiredo Neto, A.M.: Ferrofluids: properties and applications. Braz. J. Phys. **35**(3A), 718–727 (2005)
21. Scrobogna, S.: Zero limit of entropic relaxation time for the Shliomis model of ferrofluids. J. Math. Anal. Appl. **501**(2), 125213 (2018)
22. Scrobogna, S.: On the global well-posedness of a class of 2D solutions for the Rosensweig system of ferrofluids. J. Differ. Equ. **266**(5), 2718–2761 (2019)
23. Simader, C.G., Sohr, H.: A new approach to the Helmholtz decomposition and the Neumann problem in L^q-spaces for bounded and exterior domains. In: Galdi, G.P. (ed.) Mathematical Problems Relating to the Navier-Stokes Equations, vol. 11 of Advances in Mathematics for Applied Sciences, pp. 1–35. World Scientific, London (1992)
24. Simon, J.: Compact sets in the space $L^p(0, T; B)$. Ann. Math. Pura Appl. **146**(4), 65–96 (1987)
25. Strauss, W.A.: On continuity of functions with values in various Banach spaces. Pac. J. Math. **19**, 543–551 (1996)
26. Temam, R.: Navier-Stokes Equations. Theory and Numerical Analysis. North-Holland Publishing, Amsterdam. Reedition in the AMS-Chelsea Series. AMS, Providence (1997)
27. Temam, R.: Navier-Stokes Equations. Theory and Numerical Analysis. Reprint of the 1984 edition. AMS Chelsea Publishing, Providence (2001)
28. Xie, C.: Global solvability of the Rosensweig system for ferrofluids in bounded domains. Nonlinear Anal. Real World Appl. **48**, 1–11 (2019)