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# Non-concave Expected Utility Optimization with Uncertain Time Horizon

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## Abstract

We consider an expected utility maximization problem where the utility function is not necessarily concave and the time horizon is uncertain. We establish a necessary and sufficient condition for the optimality for general non-concave utility function in a complete financial market. We show that the general concavification approach of the utility function to deal with non-concavity, while being still applicable when the time horizon is a stopping time with respect to the financial market filtration, leads to sub-optimality when the time horizon is independent of the financial risk, and hence can not be directly applied. For the latter case, we suggest a recursive procedure which is based on the dynamic programming principle. We illustrate our findings by carrying out a multi-period numerical analysis for optimal investment problem under a convex option compensation scheme with random time horizon. We observe that the distribution of the non-concave portfolio in both certain and uncertain random time horizon is right-skewed with a long right tail, indicating that the investor expects frequent small losses and a few large gains from the investment. While the (certain) average time horizon portfolio at a premature stopping date is unimodal, the random time horizon portfolio is multimodal distributed which provides the investor a certain flexibility of switching between the local maximizers, depending on the market performance. The multimodal structure with multiple peaks of different heights can be explained by the concavification procedure, whereas the distribution of the time horizon has significant impact on the amplitude between the modes.

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## 1 Introduction

A classical problem in optimal control theory and mathematical finance is to maximize the expected reward or utility over all admissible terminal positions (portfolios) starting with an initial investment in the time horizon $[0, T]$, where $T > 0$ is given upfront and the objective (utility) function is a concave. Such a utility maximization problem in a continuous-time setting dates back to Merton [21] with the underlying stochastic processes representing a financial market. Merton's pioneering work has been extended in several directions e.g. by assuming more general structures of preferences, by incorporating additional randomness to the underlying risk processes, or by including a risk constraint to the optimization problem, see among many others e.g. Biagini [5], Wong et al. [28], or Karatzas et al. [19] for a broad discussion.

In this work, we investigate an extension of the Merton problem to the case where the utility function is not necessarily concave and the time horizon is random. Let us briefly mention some of the most relevant literature. Most optimal control-type problems have a fixed known time horizon. However, in reality such a natural fixed maturity does not exist and instead exogenous or endogenous events determine the end of the optimal control/optimal investment problem. An early paper by Yaari [29] looks at the investment problem of an individual with an uncertain time of death in a simplified case with purely deterministic investment opportunities. Yaari's paper is extended to discrete-time settings with multiple risky assets. Optimal life-cycle consumption and investment is studied by Merton [21], where the time horizon uncertainty is reflected by the first jump of an independent Poisson process with constant intensity. Richard [25] solves in closed-form an optimal portfolio choice problem with an uncertain time of death and the presence of life insurance. In these works, the time horizon uncertainty can be treated as additional discount factor and closed-form solution can be provided by using dynamic programming principle for concave utility functions. A more complete setting for concave utility maximization with a continuous time horizon distribution in a complete financial market has been studied in Blanchet et al. [10]. Bouchard and Pham [11] investigate a concave utility maximization in an incomplete market with general uncertain time horizon structure. All the mentioned works leave the case where the objective utility is not necessarily concave e.g. [13] as an open problem. To the best of our knowledge, the non-concave utility maximization problem under random time horizon has not yet been investigated.

The literature of non-concave optimization with certain time horizon is vast, see for instance Aumann and Perles [2], Basak and Makarov [3], Bensoussan et al. [4], Bichuch and Sturm [8], Carassus and Pham [12], Carpenter [13], Chen et al. [14], Larsen [20], Reichlin [24], Rieger [26] and Ross [27]. For non-concave optimization with constraints see Nguyen and Stadje [22] or Dai et al. [15]. In these works in the finance and the OR literature, the non-concavity arises typically from non-linear, option-type managerial compensations. Such remuneration schemes have been seen in industry as one way to overcome potential principal-agent issues and are supposed to align the incentives of managers with the ones of owners.

Another important application of non-concave investment relates to participating insurance contracts which have been extensively used in European and non-European life insurance markets. Typically, to buy a participating insurance policy, the policyholder pays a lump sum premium upfront and the capital saved is invested in a self-financing way, subject to annual interest, where the insurance company offers a (minimal) guarantee. An example is given by so called "flexibility rider contract" which have gained popularity recently due to the current low interest rate development where the decision variable is the riskiness of the investment pool, see [14] and the references within. In positive economic developments, the policyholder receives a surplus, while in case of bad economic developments, the insurance company carries the loss. Hence, a participating insurance contract may be regarded as an option-type financial instrument, leading to a non-concave utility function. In such a context our work to the best of our knowledge is the first one to be able to include the randomness of the lifetime into the investment problem (instead of simply assuming a fixed pre-specified time-horizon). As we aim to obtain some explicit result in the illustration section, we extensively consider the option compensation problem in [13] where the utility function admits only one concavification interval but with a random time horizon which has a discrete distribution on the universal time interval $[0, T]$. We remark that our results can be extended to settings with a continuous distribution time horizon.

Our contribution is fourth-fold. First, we show that when the time horizon is a stopping time with respect to the financial market filtration, the general approach of concavificiation techniques as described in [26] to deal with non-concavity can be applied. This is an extension of the result in [13], complementing the result in [11] (Proposition 3.3) to random time horizons in complete markets. Second, when $\tau$ is independent of the financial risk and the market is therefore incomplete, we establish necessary and sufficient conditions for the optimality for general utility functions. Third, also for the case where $\tau$ is independent of the financial risk, we show that optimizing the concavified version of the utility function will lead to sub-optimality with a potentially significant expected utility loss and suggest a recursive procedure which is based on the dynamic programming principle to solve the optimization problem in this situation. Fourth, we illustrate our finding by carrying out a multiple period numerical analysis for the non-concave option compensation problem with random time horizon thoroughly exploring the effect of randomness on managerial compensation schemes and participing insurance contracts. This is computationally challenging because the optimal multiplier obtained by the concavified problem in one period is a random variable that depends on the market realizations at the end of the previous period.

We numerically show that under an uncertain time horizon which imposes a new randomness that cannot be fully hedged by only using the available financial instruments, the concavified problem strategy is sub-optimal and leads to an expected utility loss. In addition, due to concavification, the distribution of the wealth at exiting times of the non-concave optimization problems is right-skewed with a long right tail, indicating that the investor can expect frequent small losses and a few large gains from the investment. Intuitively, a positively skewed distribution of investment returns is generally desirable by the agent with option-liked compensation payoff because there is some probability to gain huge profits that can cover all the frequent small losses. Under the premature exiting risk, the wealth at an exiting time exhibits a bimodal distribution with peaks of different heights. The bimodal structure can be explained by the concavification procedure whereas the distribution of the exiting time $\tau$ has significant impact on the amplitude between the two modes. When the concavified utility at an exiting time is affine in many open intervals, the corresponding wealth is expected to be of multimodal distribution.

The remainder of the paper is organized as follows: First, we describe a specific complete financial market setting and introduce the uncertain investment time in Sect. 2. We present our necessary and sufficient condition for optimality for non-concave general utility functions in Sect. 3. We show that the concavification technique is not applicable in a non-concave setting with random time horizon which induces additional risk to the financial market, and derive a dynamic programming principle for such a non-concave optimization with uncertain time horizon in Sect. 4. In Sect. 5, we investigate the case of power utility and perform a numerical study for non-concave optimization with time horizon uncertainty. We study the case when the time horizon is a stopping time with respect to the financial market filtration in Sect. 6. Finally, Sect. 7 summarizes our main results. Some additional lemmas can be found in the Appendix.

## 2 Financial Market and the Optimal Investment Problem

Let $[0, T]$ with $0 < T < \infty$ be the maximal time span of the economy and $W$ is an $n$-dimensional Brownian motion in a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

### 2.1 The Financial Market

For the market setup, we assume that the prices of $n$ risky assets $S$ are modelled as a geometric Brownian motion, i.e.,

$$\frac{dS_t^i}{S_t^i} = \mu_t^i dt + \sum_{j=1}^{n} \sigma_t^{i,j} dW_t^j, \; i = 1, \dots, n,$$

where the superscript $i$ denotes the $i$-th entry of the corresponding vector or $(i, j)$ the entry in the $i$-th row and $j$-th column of a matrix and we use the subscript $t$ to denote the time index $t$. We use the notation $\mu = (\mu^i)_{1 \le i \le n}$ and $\sigma = (\sigma^{i,j})_{1 \le i,j \le n}$ for the corresponding vector or matrix, respectively. Additionally to these risky assets, we consider a risk-free asset (e.g. a *bond*) $B$, given by $dB_t = B_t r_t dt$, where $r$ denotes the (deterministic) interest rate. The information in the market is captured by the augmented filtration $\mathcal{F} = (\mathcal{F}_t)_{t \ge 0}$ generated by the Brownian motion, satisfying the usual conditions and $\mathcal{F}_0$ is trivial. We assume that the coefficients $\mu, r \ge 0$ are bounded and deterministic and the volatility $\sigma$ is bounded, deterministic, invertible with bounded inverse $\sigma^{-1}$.

In this arbitrage-free financial market, there exists a unique equivalent martingale measure $\mathbb{Q}$ with Radon-Nikodym density $M$ as the solution of $dM_t =$

$-M_t\theta_t dW_t$ with $M_0 = 1$, where $\theta_t := \sigma_t^{-1}(\mu_t - r_t\mathbf{1})$. Furthermore, we define $\xi_t := \exp\left(-\int_0^t r_s ds\right) M_t$. By Itô's formula, we have $d\xi_t = -\xi_t r_t dt - \xi_t\theta_t dW_t$, and

$$\xi_t = \exp\left(-\int_0^t (r_s + \frac{1}{2}\theta_s^T\theta_s)ds - \int_0^t \theta_s dW_s\right).$$

We consider the economy in the usual frictionless setting, where stocks and bonds are infinitely divisible and there are no market frictions, no transaction costs etc. Additional to the financial market setting, we consider a random time-horizon $\tau$, where $\tau$ is a positive discrete random variable independent of $\mathcal{F}$. In particular, $\tau$ is not an $\mathcal{F}$-stopping time. Let $\mathcal{F}^\tau = (\mathcal{F}_t^\tau)_{0\le t\le T}$ with $\mathcal{F}_t^\tau$ being the $\sigma$-algebra generated by $(\mathbf{1}_{\tau\le s})_{0\le s\le t}$. Define $\mathcal{G} = \mathcal{F} \vee \mathcal{F}^\tau$. The equivalent martingale measure $\mathbb{Q}$ can be extended to $\mathcal{G}_T$ by defining $\mathbb{Q}(A) := \mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}\mathbf{1}_A]$ for any $A \in \mathcal{G}_T$. We note that any $\mathcal{G}$-martingale is also an $\mathcal{F}$-martingale, see e.g. [1].

### 2.2 Utility Function

In the sequel we consider a general not necessarily concave utility function $U : [0,\infty) \to \mathbb{R}$ which is non-constant, increasing, continuous, has left- and right-hand side derivatives and satisfies the growth condition

$$\lim_{x\to\infty} \frac{U(x)}{x} = 0. \tag{2.1}$$

We set $U(x) = -\infty$ for $x < 0$ to avoid ambiguity and define $U(\infty) := \lim_{x\to\infty} U(x)$. We do *not* assume that $U$ is concave or strictly increasing. In a concave setting, Eq. (2.1) is equivalent to $U'(\infty) = 0$, which is part of the Inada condition. We note that Eq. (2.1) and the assumption $U(\infty) > 0$ imply that there exists a concave function $U^c : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ that dominates $U$, i.e., $U^c \ge U$. The following result explains why we can consider $\mathcal{F}$-predictable, instead of $\mathcal{G}$-predictable investment strategies. We call a $\mathcal{G}$-predictable or a $\mathcal{F}$-predictable process $(\pi_s)$ locally square integrable if $\int_0^T |\pi_s|^2\, ds < \infty$ a.s.

**Lemma 1** *Suppose that $(\pi_s)$ is $\mathcal{G}$-predictable and locally square integrable. Then there exists a strategy $(\tilde{\pi}_s)$ which is $\mathcal{F}$-predictable, locally square integrable and*

$$\int_0^{\tau\wedge T} \pi_s\sigma_s dW_s^{\mathbb{Q}} = \int_0^{\tau\wedge T} \widetilde{\pi}_s\sigma_s dW_s^{\mathbb{Q}}.$$

***Proof*** Denote by $Y_s = \pi_s\sigma_s$ for $0 \le s \le T$. Prop. 2.11 in [1] yields that the $\mathcal{G}$-predictable process $Y$ can be expressed as $Y = y\mathbf{1}_{[0,\tau]} + g(\tau)\mathbf{1}_{]\tau,T]}$ where $(y_s)_{0\le s\le T}$ is $\mathcal{F}$-predictable and $g_t(\omega, u)$; $t \ge u$ is a $\mathcal{P}\otimes\mathcal{B}([0,T])$ random function with $\mathcal{P}$ being the $\mathcal{F}$-predictable $\sigma$-algebra. Set $\tilde{\pi}_s = y_s\sigma_s^{-1}$. Then

$$\pi_s\sigma_s = Y_s = y_s = \tilde{\pi}_s\sigma_s \quad \text{on } 0 \le s < \tau\wedge T.$$

This entails that $\int_0^{\tau\wedge T} \pi_s \sigma_s dW_s^{\mathbb{Q}} = \int_0^{\tau\wedge T} \tilde{\pi}_s \sigma_s dW_s^{\mathbb{Q}}$ and the lemma follows. □

### 2.3 Admissible Strategy

We consider an investor putting the amount $\pi_s^i$ in the risky asset $i$ at time $0 \le s \le T$, $i = 1, \dots, n$. By considering a self-financing portfolio, the amount $P_s - \sum_{i=1}^n \pi_s^i$ is invested in the bond. We use the notation $(P_s^{t,\pi,x}, t \le s \le T)$ for the wealth process at time $s$, developed from an initial capital $P_t^\pi := P_t^{t,\pi,x} = x \ge 0$ at time $t$ under a self-financing strategy $\pi$, where $\pi^i$ denotes the amount invested in asset $i$. At time $t = 0$ we assume that the initial capital is strictly positive. Then $P_s^{t,\pi,x}$ evolves according to the stochastic differential equation

$$dP_s^{t,\pi,x} = P_s^{t,\pi,x} r_s ds + \pi_s \left[ (\mu_s - \mathbf{1} r_s) ds + \sigma_s dW_s \right]. \tag{2.2}$$

We call $(\pi_t,\ 0 \le t \le T)$ *admissible*, if $\pi$ is progressively measurable w.r.t. $\mathcal{F}$, locally square-integrable, i.e., $\sum_{i=1}^n \int_0^T (\pi_s^i)^2 ds < \infty$ a.s., and the associated wealth process is non-negative. A wealth process corresponding to an admissible strategy is called admissible wealth process. By Girsanov's theorem as long as $\pi$ is locally square-integrable $(\xi_s P_s^{t,\pi,x})_s$ is always a local martingale. For the set of admissible wealth processes with initial capital $x$ at time $t$, we use the notation

$$\Pi(t,x) := \left\{ \pi_s : t \le s \le T,\ \pi \text{ is admissible and } P_s^{t,\pi,x} \ge 0 \right\}. \tag{2.3}$$

We define $\tilde{P}_t := P_t \exp\left( - \int_0^t r_s ds \right)$ as the *discounted wealth process*.

## 3 Non-concave Optimization Problem with Random Time Horizon

Since $U$ is minus infinity for negative outcomes we may throughout this paper restrict ourselves to analyse non-negative wealth processes. Specifically, assume that the agent evaluates his/her investment performance at times $0 =: T_0 < T_1 < T_2 < \cdots < T_n := T$ with respect to the weights $p_i := \mathbb{P}(\tau = T_i)$, $i = 1, \cdots, n-1$, and $p_n = \mathbb{P}[\tau \ge T_n]$, with $\sum_{i=1}^n p_i = 1$. Let $\widetilde{\Pi}$ be the set of all portfolios $\pi$ that are progressively measurable with respect to $\mathcal{G}$, locally square-integrable with non-negative associated wealth process. In our complete financial market setup, we consider the problem

$$\begin{aligned} V_\tau(x,U) &= \sup_{\pi \in \widetilde{\Pi}(0,x)} \mathbb{E}\left[ U(P_{T\wedge\tau}) \right] \\ &= \sup_{\pi \in \Pi(0,x)} \mathbb{E}\left[ U(P_{T\wedge\tau}) \right] = \sup_{\pi \in \Pi(0,x)} \mathbb{E}\left[ \sum_{i=1}^n p_i U(P_{T_i}) \right], \end{aligned} \tag{3.1}$$

where the second equality holds by Lemma 1. Define

$$C_\tau(x) := \Big\{ P = (P_0, \cdots, P_n) : P_0 = x > 0, \quad P_i \geq 0 \ \mathcal{F}_{T_i}\text{-measurable with}$$
$$\tilde{P}_i =: e^{-\int_0^{T_i} r_s\, ds} P_i = \tilde{P}_{i-1} + \int_{T_{i-1}}^{T_i} \sigma_s \pi_{i,s} dW_s^{\mathbb{Q}} \text{ and } \pi_i \in \Pi(T_{i-1}, P_{i-1}),\ i = \{1, \ldots, n\} \Big\}. \tag{3.2}$$

Note that for $P = (P_0, \cdots, P_n) \in C_\tau(x)$, we have $\mathbb{E}\left[\sum_{i=1}^n p_i \xi_{T_i} P_i\right] \leq x$. Furthermore, it is clear that the strategy

$$\pi_t^{(n)}(P) := \sum_{i=1}^n \pi_{i,t} \mathbf{1}_{(T_{i-1}, T_i]}(t) \in \Pi(0, x)$$

is locally square integrable. Denote by $P^{x,\pi^{(n)}} := P^{x,\pi^{(n)}(P)}$ the corresponding admissible wealth process (which is equal to $P_i$ at time $T_i$ for $i = 0, \ldots, n$). Note that $\xi P^{x,\pi^{(n)}} = \xi P^{x,\pi^{(n)}(P)}$ is a non-negative local martingale (hence a supermartingale). We say that the supermartingale $\xi P^{x,\pi^{(n)}(P)}$ is generated by the $n+1$-tuple $P = (P_0, \cdots, P_n) \in C_\tau(x)$.

The optimization problem (3.1) can be restated in the following way

$$V_\tau(x, U) = \sup_{P \in C_\tau(x)} \mathbb{E}\left[\sum_{i=1}^n p_i U(P_i)\right]. \tag{3.3}$$

Define

$$I(a) := \inf \left\{ y \geq 0 | U(y) - ya = \sup_{p \geq 0} \{U(p) - pa\} \right\}. \tag{3.4}$$

By condition (2.1), $I$ is well defined. By continuity the supremum and infimum are attained so that $I(x)$ is the smallest arg max of the function $y \mapsto U(y) - yx$. Below, $U'$ denotes the right-hand side derivative of $U$. The following result provides a sufficient condition for optimality of the optimization problem (3.3).

**Theorem 1** *Let $x > 0$. Suppose that there is an adapted process $\nu \geq 0$ with $\nu_0 = U'(x)$ such that the process $\xi P^{x,\pi^{(n)}(P^*)}$ generated by the $n+1$-tuple $P^* := (x, I(\nu_{T_1}\xi_{T_1}), \cdots, I(\nu_{T_n}\xi_{T_n}))$ is a martingale and $\sum_{i=1}^n p_i \nu_{T_i}$ is a constant. Then, $P^*$ solves the optimization problem (3.3).*

***Proof*** Let $\sum_{i=1}^n p_i \nu_{T_i} = y$ which is a constant by assumption. Then for any $P = (x, P_1, \ldots, P_n) \in C_\tau(x)$ by construction, the process $\xi P^{x,\pi^{(n)}(P)}$ is a non-negative local martingale. To omit cumbersome notation denote $Y = P^{x,\pi^{(n)}(P)}$. Let $\tau_m$ be the corresponding localizing sequence. Then, for any $m \geq 1$,

$$
\begin{aligned}
\mathbb{E}[\sum_{i=1}^{n} p_i \nu_{T_i} \xi_{T_i\wedge\tau_m} Y_{T_i\wedge\tau_m}] &= \sum_{i=1}^{n} \mathbb{E}\Big[p_i \nu_{T_i} \mathbb{E}[\xi_{T\wedge\tau_m} Y_{T\wedge\tau_m} | \mathcal{F}_{T_i}]\Big] \\
&= \sum_{i=1}^{n} \mathbb{E}\Big[p_i \nu_{T_i} \xi_{T\wedge\tau_m} Y_{T\wedge\tau_m}\Big] = \mathbb{E}\Big[(\sum_{i=1}^{n} p_i \nu_{T_i}) \xi_{T\wedge\tau_m} Y_{T\wedge\tau_m}\Big] = xy.
\end{aligned}
$$

Passing to the limit as $m \to \infty$ and using Fatou's lemma yields

$$
\mathbb{E}[\sum_{i=1}^{n} p_i \nu_{T_i} \xi_{T_i} Y_{T_i}] \leq xy. \tag{3.5}
$$

The process $Z := x^{-1}\xi P^{x,\pi^{(n)}(P^*)}$ defines a density process of a probability measure $\mathbb{Q}^\nu << \mathbb{P}$ as it is a martingale with initial value equal to 1. Due to the construction of $P^{x,\pi^{(n)}(P^*)}$ it can be observed that $Z_{T_i} = x^{-1}\xi_{T_i} I(\nu_{T_i}\xi_{T_i})$. Therefore, we obtain

$$
\mathbb{E}[\sum_{i=1}^{n} p_i \nu_{T_i} \xi_{T_i} I(\nu_{T_i}\xi_{T_i})] = x\mathbb{E}^{\mathbb{Q}^\nu}[\sum_{i=1}^{n} p_i \nu_{T_i}] = xy.
$$

For any admissible $Y$ we have therefore

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n} p_i U(I(\nu_{T_i}\xi_{T_i}))\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n} p_i U(I(\nu_{T_i}\xi_{T_i}))\right] - x\mathbb{E}^{\mathbb{Q}^\nu}[\sum_{i=1}^{n} p_i \nu_{T_i}] + xy \\
&= \mathbb{E}\left[\sum_{i=1}^{n} p_i U(I(\nu_{T_i}\xi_{T_i}))\right] - \mathbb{E}[\sum_{i=1}^{n} p_i \nu_{T_i} \xi_{T_i} I(\nu_{T_i}\xi_{T_i})] + xy \\
&= \mathbb{E}\left[\sum_{i=1}^{n} p_i \sup_{X\geq 0}\Big(U(X) - \nu_{T_i}\xi_{T_i} X\Big)\right] + xy \\
&\geq \mathbb{E}\left[\sum_{i=1}^{n} p_i \Big(U(Y_{T_i}) - \nu_{T_i}\xi_{T_i} Y_{T_i}\Big)\right] + xy \geq \mathbb{E}\left[\sum_{i=1}^{n} p_i U(Y_{T_i})\right],
\end{aligned}
$$

where we have used (3.5) in the last step. This implies the optimality of $P^*$. □

We now seek for a necessary condition for optimality. The following is the main theorem of this section which generalizes the results by Blanchet et al. [10] to non-concave settings. Let $U'$ be the right-hand side derivative of $U$. We need the following assumption.

**Assumption 1** We assume that $P^* = (x, P^*_{T_1}, \cdots, P^*_{T_n})$ is an optimal solution to Problem (3.3) such that $\mathbb{E}\Big[\max_i |U(P^*_{T_i})|\Big] < \infty$, and that $\xi P^*$ is a square integrable martingale (instead of only a local martingale). Furthermore, $\mathbb{E}\left[\max_i \frac{h\left((1-\delta)P^*_{T_i}\right)}{\xi_{T_i}}\right] < \infty$ for some $\delta > 0$ with $h$ being a decreasing function with $\max(U'_-, U') = \max(U'_-, U'_+) \leq h$.

If $U = U^c$ (concave hull) for all $x > x_0$ for an $x_0$ and if $U'$ is bounded on $[0, x_0]$, we can choose $h(x) = a + U'(x)$. Next, we will construct a set $A \subset \Omega$ such that $U'_-(P^*_{T_i}) = U'_+(P^*_{T_i})$ a.s. on $A$ for $i = 1, \dots, n$. Since by assumption the right- and left-hand side derivatives of $U$ exist at every point, by Theorem 17.9 in [18], the set where $U$ is not differentiable is countable. Hence, the set

$$M := \Big\{ y \geq 0 \mid U \text{ is not differentiable in } y \text{ and there exists a } T_i \text{ for } i \in \{1, \dots, n\} \text{ with } \mathbb{P}[P^*_{T_i} = y] > 0 \Big\},$$

is countable as well, and therefore Borel-measurable.

Denote the set of all $\omega$ with $P^*_{T_i}(\omega) \in M$ for at least one $i$ by $A^c$, and let $A$ be its complement. In other words, the set $A$ contains at most scenarios with portfolio outcomes at a $T_i$ where the utility function is differentiable modulo a zero set. Since $A^c = \bigcup_{i=1}^n \{P^*_{T_i} \in M\}$, $A$ is measurable.

**Theorem 2** *Assume that $(x, P^*_{T_1}, \cdots, P^*_{T_n})$ is an optimal solution to Problem (3.3) which satisfies Assumption 1. Define $\nu_{T_i} := \xi^{-1}_{T_i} U'(P^*_{T_i})$ for $i = 1, \cdots, n$. Then, it holds that the random variable $\sum_{i=1}^n p_i \nu_{T_i}$ is constant (a.s.) on $A$.*

***Proof*** If $A$ is a zero-set the theorem is obvious. So assume $\mathbb{P}(A) > 0$. Consider an admissible (non-negative wealth process) $Y$ with terminal value of the form $Y_T = P^*_T \mathbf{1}_{A^c} + \zeta \mathbf{1}_A$ where $\zeta$ is non-negative and $\mathcal{F}_T$ measurable, such that

a) $\xi Y$ is a martingale,
b) there exists a constant $C > 0$ such that $\max(0, P^*_T - \frac{C}{\xi_T}) \leq Y_T \leq P^*_T + \frac{C}{\xi_T}$. In particular, $\xi_T(P^*_T - Y_T)$ is bounded.

Hence, we consider a portfolio $Y$, which at time $T$ agrees with $P^*_T$ on $A^c$. Furthermore, since $Y$ is an admissible wealth process and $\xi Y$ is a martingale

$$\mathbb{E}[\xi_T P^*_T \mathbf{1}_{A^c}] + \mathbb{E}[\xi_T Y_T \mathbf{1}_A] = \mathbb{E}[\xi_T Y_T] = \mathbb{E}[\xi_0 Y_0] = Y_0 = x = P^*_0 = \mathbb{E}[\xi_T P^*_T].$$

In particular,

$$\mathbb{E}[\xi_T Y_T \mathbf{1}_A] = \mathbb{E}[\xi_T \zeta \mathbf{1}_A] = \mathbb{E}[\xi_T P^*_T \mathbf{1}_A]. \tag{3.6}$$

We define for $0 \leq \varepsilon \leq 1$ the functions $\Phi$ and $\chi$ by $\Phi(\varepsilon) := \mathbb{E}[\chi(\varepsilon)]$ and

$$\chi(\varepsilon, \omega) := \left( \sum_{i=1}^n U\left(\varepsilon P^*_{T_i} + (1-\varepsilon) Y_{T_i}\right) p_i \right).$$

We denote the right-hand side derivative of a continuous function $f$ by $f'_+$ and the left-hand side derivative by $f'_-$. Of course, in points where the function is differentiable both limits coincide and the "+" and "−" may be omitted, respectively. Note that $\varepsilon P^*_T + (1-\varepsilon)Y_T$ is the terminal condition of an admissible wealth process implying that

$$-\infty < -|U(0)| \le -\sum_{n=1}^{n} \mathbb{E}\left[\left(U(\epsilon P^*_{T_i} + (1-\epsilon)Y_{T_i})\right)^-\right] p_i$$

$$\text{and } \sum_{n=1}^{n} \mathbb{E}[U(\epsilon P^*_{T_i} + (1-\epsilon)Y_{T_i})] p_i \le \mathbb{E}[U(P^*_{T\wedge\tau})] < \infty$$

by Assumption 1, where $\left(U(\epsilon P^*_{T_i}+(1-\epsilon)Y_{T_i})\right)^-$ is the negative part of $U(\epsilon P^*_{T_i}+(1-\epsilon)Y_{T_i})$. Hence, $\chi$ is integrable. For $\varepsilon > 1-\delta$ we have $(1-\delta)P^*_{T_i} < \varepsilon P^*_{T_i} + (1-\varepsilon)Y_{T_i}$. Calculating

$$|\chi'_-(\varepsilon,\omega)| \le \sum_{i=1}^{n} p_i \left|h\left((1-\delta)P^*_{T_i}\right)\right| \left|P^*_{T_i} - Y_{T_i}\right| \le \max_i \frac{C\left|h\left((1-\delta)P^*_{T_i}\right)\right|}{\xi_{T_i}},$$

gives an integrable dominating random variable. Denote $\operatorname{sign}(x) := +$ if $x \ge 0$ and $\operatorname{sign}(x) := -$ else. Under Assumption 1 we obtain for $\varepsilon$ close enough to 1

$$\Phi'_-(\varepsilon) = \mathbb{E}\left[\sum_{i=1}^{n} U'_{-\operatorname{sign}(P^*_{T_i}-Y_{T_i})}\left(\varepsilon P^*_{T_i} + (1-\varepsilon)Y_{T_i}\right)\left(P^*_{T_i} - Y_{T_i}\right) p_i\right].$$

We know that the function $\Phi$ attains its maximum at $\varepsilon = 1$, since $P^*$ is the optimal solution by assumption. Hence, $0 \le \Phi'_-(1)$. Thus,

$$0 \le \mathbb{E}\left[\sum_{i=1}^{n} U'_{-\operatorname{sign}(P^*_{T_i}-Y_{T_i})}\left(P^*_{T_i}\right)\xi^{-1}_{T_i} p_i \left(\xi_{T_i}P^*_{T_i} - \xi_{T_i}Y_{T_i}\right)\right].$$

Using that $(\xi_t(P^*_t - Y_t))_{0\le t\le T}$ is a bounded martingale yields

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n} U'_{-\operatorname{sign}(P^*_{T_i}-Y_{T_i})}\left(P^*_{T_i}\right)\xi^{-1}_{T_i} p_i \left(\xi_{T_i}P^*_{T_i} - \xi_{T_i}Y_{T_i}\right)\right] \\
&= \mathbb{E}\left[(P^*_T - Y_T)\xi_T \sum_{i=1}^{n} U'_{-\operatorname{sign}(P^*_{T_i}-Y_{T_i})}\left(P^*_{T_i}\right)\xi^{-1}_{T_i} p_i\right] \\
&= \mathbb{E}\left[(P^*_T - Y_T)\xi_T \sum_{i=1}^{n} U'\left(P^*_{T_i}\right)\xi^{-1}_{T_i} p_i \mathbf{1}_A\right],
\end{aligned}$$

since by definition $P^*_T = Y_T$ on $A^c$ and $U(P^*_{T_i})$ is differentiable a.s. for each $i$ so that $U'(P^*_{T_i}) = U'_+(P^*_{T_i}) = U'_-(P^*_{T_i})$ on $A$. Hence,

$$0 \leq \mathbb{E}\left[\xi_T \left(P^*_T - Y_T\right) \left(\sum_{i=1}^n U'\left(P^*_{T_i}\right) \xi_{T_i}^{-1} p_i\right) \mathbf{1}_A\right]. \tag{3.7}$$

We note that this equality holds for any admissible wealth process $Y$ such that a) - b) hold and $Y_T$ is equal to $P^*_T$ on $A^c$. Define $Z := \mathbf{1}_A \sum_{i=1}^n U'\left(P^*_{T_i}\right) \xi_{T_i}^{-1} p_i$. Thus, $0 \leq \mathbb{E}\left[\xi_T (P^*_T - Y_T) Z \mathbf{1}_A\right]$, which is equivalent to

$$0 \leq \mathbb{E}\left[\xi_T (P^*_T - Y_T) Z | A\right]. \tag{3.8}$$

The theorem would follow if we can show that $Z$ is constant on $A$. By (3.6) we have $\mathbb{E}\left[\xi_T (P^*_T - Y_T) | A\right] = \mathbb{E}\left[\xi_T (P^*_T - \zeta) | A\right] = 0$. Hence, (3.8) implies

$$0 \leq \mathbb{E}\left[\xi_T (P^*_T - Y_T) \tilde{Z} | A\right], \tag{3.9}$$

with $\tilde{Z} := (Z - \mathbb{E}[Z|A]) \mathbf{1}_A$. By Lemma 2 below this entails that

$$0 \leq \mathbb{E}\left[X \tilde{Z} | A\right] \tag{3.10}$$

for any bounded $X$ with $\mathbb{E}[X|A] = 0$ satisfying $X \leq 0$ on $P^*_T = 0$.

Now if $\tilde{Z} \geq 0$ or $\leq 0$ on $A$ we have that $\tilde{Z} = 0$ on $A$ and we are done (since $\tilde{Z} = (Z - \mathbb{E}[Z|A]) \mathbf{1}_A$ and therefore, $Z$ must then be constant on $A$). On the other hand, if $\mathbb{P}\left[\tilde{Z} < 0 | A\right] > 0$ then

$$\mathbb{P}\left[P^*_T > 0, \tilde{Z} < 0 | A\right] = \mathbb{P}\left[\tilde{Z} < 0 | A\right] > 0,$$

where the first equation holds as the wealth process $P^*_T$ is non-negative and $\mathbb{P}\left[P^*_T = 0 | A\right] = 0$ by the definition of $A$, since $U(y)$ is not differentiable at $y = 0$. For $a, b > 0$ we can define $X = \left\{-b \mathbf{1}_{\tilde{Z} > 0} + a \mathbf{1}_{P^*_T > 0, \tilde{Z} < 0}\right\} \mathbf{1}_A$. Then $X \leq 0$ on $P^*_T = 0$. Choose $b, a > 0$ such that $\mathbb{E}[X|A] = 0$. Then by (3.10)

$$0 \leq \mathbb{E}\left[X \tilde{Z} | A\right] = -b \mathbb{E}\left[\tilde{Z}^+ | A\right] - a \mathbb{E}\left[\tilde{Z}^- \mathbf{1}_{P^*_T > 0} | A\right].$$

Hence, $\mathbf{1}_A \tilde{Z}^+ = 0$ and thus $\mathbf{1}_A \tilde{Z} = 0$, since $\mathbb{E}[\tilde{Z}|A] = 0$. By the definition of $\tilde{Z}$ above this entails that $Z$ is constant. To obtain the representation of $Z$, we recall our definition $\nu_t = \xi_t^{-1} U'(P^*_t)$ (for $t \in \{T_1, \ldots, T_n\}$) from the very beginning of this proof. □

**Lemma 2** $0 \le \mathbb{E}\left[X\tilde{Z}|A\right]$ *for any* $\mathcal{F}_T$*-measurable bounded* $X$ *with* $\mathbb{E}\left[X|A\right] = 0$ *satisfying* $X \le 0$ *on* $P_T^* = 0$.

***Proof*** Setting $\bar{X} = (P_T^* - Y_T)\mathbf{1}_A$, (3.9) implies that

$$\mathbb{E}\left[\xi_T \bar{X}\tilde{Z}|A\right] \ge 0. \tag{3.11}$$

On the other hand, (3.9) implies that (3.11) holds for any $\mathcal{F}_T$-measurable $\bar{X} \le P_T^*$ such that $\mathbb{E}\left[\xi_T \bar{X}|A\right] = 0$ and $\xi_T \bar{X}$ is bounded. [1] In particular,

$$\mathbb{E}\left[\tilde{X}\tilde{Z}|A\right] \ge 0, \tag{3.12}$$

for any bounded $\tilde{X}$ being $\mathcal{F}_T$-measurable such that $\mathbb{E}\left[\tilde{X}|A\right] = 0$ and $\tilde{X} \le P_T^*\xi_T$.

By definition the wealth process, $P^*$, is non-negative. Therefore, we actually have that

$$\mathbb{E}\left[\tilde{X}\tilde{Z}|A\right] \ge 0, \tag{3.13}$$

for all bounded $\mathcal{F}_T$-measurable $\tilde{X}$ with $\mathbb{E}[\tilde{X}|A] = 0$, such that $\tilde{X} \le 0$ on $P_T^*\xi_T \le \delta$ for some $\delta > 0$, and $\tilde{X}$ bounded by $K$ else. This can be seen as follows. Suppose $\tilde{X}$ is bounded and $\mathcal{F}_T$-measurable with $\mathbb{E}[\tilde{X}|A] = 0$, such that $\tilde{X} \le 0$ on $P_T^*\xi_T \le \delta$ for a $\delta > 0$, and $\tilde{X}$ bounded by $K$ else. Then on $P_T^*\xi_T \ge \delta$ we have

$$\frac{\delta}{K}\tilde{X} \le \delta \le P_T^*\xi_T,$$

while on $P_T^*\xi_T \le \delta$ we have

$$\frac{\delta}{K}\tilde{X} \le 0 \le P_T^*\xi_T.$$

In particular, $\frac{\delta}{K}\tilde{X} \le P_T^*\xi_T$. Since obviously $\mathbb{E}[\frac{\delta}{K}\tilde{X}|A] = 0$, by (3.12) $\mathbb{E}\left[\left(\frac{\delta}{K}\tilde{X}\right)\tilde{Z}|A\right] \ge 0$ implying (3.13). Since $K$ was arbitrary, (3.13) holds actually for any bounded $\mathcal{F}_T$-measurable $\tilde{X}$ with $\mathbb{E}\left[\tilde{X}|A\right] = 0$ and $\tilde{X} \le 0$ on $P_T\xi_T \le \delta$ for some $\delta > 0$.

Now we take an $\mathcal{F}_T$-measurable bounded $X$ satisfying $X \le 0$ if $P_T^* = 0$, and $\mathbb{E}\left[X|A\right] = 0$. If $X = 0$ on $A$ then clearly $E[X\tilde{Z}|A] \ge 0$ and the lemma follows. If $X \ne 0$ on $A$ then $A \cap \{X > 0\}$ is a non-zero set (since $\mathbb{E}\left[X|A\right] = 0$) and therefore by assumption $A \cap \{X > 0\} \cap \{P_T^* > 0\}$ is a non-zero set as well (see the definition

[1] Since by the martingale representation theorem for such a $\bar{X}$ there exists a corresponding admissible $Y$ with $\xi_T Y_T = \mathbf{1}_A \xi_T (P_T^* - \bar{X}) + \mathbf{1}_{A^c} \xi_T P_T^*$ such that $\xi Y$ is a martingale and $\xi_T(P_T^* - Y_T)$ is bounded.

of $A$ noting that $U$ is not differentiable at zero). Define

$$X^{\delta} = \Big(X + \hat{\delta}(\delta)\Big)\mathbf{1}_{\xi_T P_T^* > \delta, X > 0} + \mathbf{1}_{X \le 0} X$$

with $0 \le \hat{\delta}(\delta)$ chosen such that $\mathbb{E}\left[X^{\delta}|A\right] = 0$. The existence of $\hat{\delta}(\delta) \in [0, \infty)$ if $\delta$ is small enough such that $\mathbb{P}[\xi_T P_T^* > \delta, X > 0|A] > 0$ follows from the intermediate value theorem as

$$f(0) := \mathbb{E}\left[X\mathbf{1}_{\xi_T P_T^* > \delta, X > 0} + \mathbf{1}_{X \le 0} X|A\right] \le \mathbb{E}\left[X|A\right] = 0$$

and

$$\infty = \lim_{\hat{\delta} \to \infty} f(\hat{\delta}) := \lim_{\hat{\delta} \to \infty} \mathbb{E}\left[\left(X + \hat{\delta}\right)\mathbf{1}_{\xi_T P_T^* > \delta, X > 0} + \mathbf{1}_{X \le 0} X\middle|A\right] > \mathbb{E}[X|A] = 0.$$

Now $\hat{\delta}(\delta) \downarrow 0$ as $\delta \downarrow 0$. Furthermore, $X^{\delta} \le 0$ on $\xi_T P_T^* \le \delta$ so that $X^{\delta}$ satisfies (3.13). Hence, $0 \le \mathbb{E}\left[X^{\delta}\tilde{Z}|A\right] \overset{\delta\downarrow 0}{\to} \mathbb{E}\left[X\tilde{Z}|A\right]$. □

## 4 Dynamic Programming Approach with Random Time Horizon

Concavification has been widely applied to solve non-concave optimization problems, see e.g. [8, 12–14, 20, 22, 24, 26, 27] in various settings where the time horizon is fixed and the market is complete. The concavification argument is based on the fact that the concavified hull $U^c$ strictly dominates the initial function $U$ only in a union of finite number of open intervals and $U^c$ is affine in this union. The key idea is that in order to gain more expected utility, it is possible for the agent to put all the expensive states to the left points of these intervals in the concavification region, keeping the budget constraint unchanged.

In this section, we show that the concavification technique may no longer be directly applicable in settings with a random time horizon. Furthermore, we derive a dynamic programming principle for such a non-concave optimization.

We will start with the following useful lemma, where with a slight abuse of notation we write $\tau$ instead of $\tau \wedge T$.

**Lemma 3** *Let $\tilde{\tau}$ have the same distribution as $\tau$ conditioned on $\tau > t$ and be independent of $W$. Then we have*

$$\begin{aligned}
&\mathbb{E}\left[U\left(x + \int_0^{\tau} \pi_s \sigma_s dW_s^{\mathbb{Q}}\right)\middle|\mathcal{G}_t\right] \\
&\quad = \mathbf{1}_{\tau \le t} U\left(x + \int_0^{\tau} \pi_s \sigma_s dW_s^{\mathbb{Q}}\right) + \mathbb{E}\left[U\left(x + \int_0^{\tilde{\tau}} \pi_s \sigma_s dW_s^{\mathbb{Q}}\right)\middle|\mathcal{F}_t\right]\mathbf{1}_{\tau > t}.
\end{aligned}$$

***Proof*** Let Y be bounded and $\mathcal{G}_t$-measurable. By Jeulin (2006), Lemma 4.4 $Y(\omega) = \mathbf{1}_{\tau>t}X_t(\omega) + \mathbf{1}_{\tau\le t}g_t(\omega,\tau)$ for some $\mathcal{F}_t$-measurable random variable $X_t$ and some family of $\mathcal{F}_t \otimes \mathcal{B}([0,T])$-measurable random variables $g_t(\cdot,u)$; $t \ge u$. Let $\tilde{\tau}$ have the same distribution as $\tau$ conditioned on $\tau > t$ and be independent of $W$. Then we have

$$
\begin{aligned}
&\mathbb{E}\Big[U\Big(x+\int_0^\tau \pi_s\sigma_s dW_s^{\mathbb{Q}}\Big)Y\Big]\\
&=\mathbb{E}\Big[\mathbf{1}_{\tau\le t}U\Big(x+\int_0^\tau \pi_s\sigma_s dW_s^{\mathbb{Q}}\Big)Y\Big]+\mathbb{E}\Big[\mathbf{1}_{\tau>t}U(x+\int_0^\tau \pi_s\sigma_s dW_s^{\mathbb{Q}})Y\Big]\\
&=\mathbb{E}\Big[\mathbf{1}_{\tau\le t}U\Big(x+\int_0^\tau \pi_s\sigma_s dW_s^{\mathbb{Q}}\Big)Y\Big]+\mathbb{E}\Big[\mathbf{1}_{\tau>t}U\Big(x+\int_0^\tau \pi_s\sigma_s dW_s^{\mathbb{Q}}\Big)X_t\Big]\\
&=\mathbb{E}\Big[\mathbf{1}_{\tau\le t}U(x+\int_0^\tau \pi_s\sigma_s dW_s^{\mathbb{Q}})Y\Big]+\mathbb{E}\Big[\mathbb{E}\Big[\mathbf{1}_{\tau>t}U(x+\int_0^\tau \pi_s\sigma_s dW_s^{\mathbb{Q}})|\mathcal{F}_t\Big]X_t\Big]\\
&=\mathbb{E}\Big[\mathbf{1}_{\tau\le t}U\Big(x+\int_0^\tau \pi_s\sigma_s dW_s^{\mathbb{Q}}\Big)Y\Big]+\mathbb{E}\Big[\frac{\mathbf{1}_{\tau>t}\mathbb{E}\Big[\mathbf{1}_{\tau>t}U\Big(x+\int_0^\tau \pi_s\sigma_s dW_s^{\mathbb{Q}}\Big)\Big|\mathcal{F}_t\Big]}{P(\tau>t)}X_t\Big]\\
&=\mathbb{E}\Big[\mathbf{1}_{\tau\le t}U\Big(x+\int_0^\tau \pi_s\sigma_s dW_s^{\mathbb{Q}}\Big)Y\Big]+\mathbb{E}\Big[\mathbf{1}_{\tau>t}\mathbb{E}[U\Big(x+\int_0^{\tilde{\tau}} \pi_s\sigma_s dW_s^{\mathbb{Q}}\Big)\Big|\mathcal{F}_t]X_t\Big]\\
&=\mathbb{E}\Big[\mathbf{1}_{\tau\le t}U(x+\int_0^\tau \pi_s\sigma_s dW_s^{\mathbb{Q}})Y\Big]+\mathbb{E}\Big[\mathbf{1}_{\tau>t}\mathbb{E}[U(x+\int_0^{\tilde{\tau}} \pi_s\sigma_s dW_s^{\mathbb{Q}})\Big|\mathcal{F}_t\Big]Y\Big]\\
&=\mathbb{E}\Big[\Big\{\mathbf{1}_{\tau\le t}U\Big(x+\int_0^\tau \pi_s\sigma_s dW_s^{\mathbb{Q}}\Big)\Big]+\mathbf{1}_{\tau>t}\mathbb{E}\Big[U\Big(x+\int_0^{\tilde{\tau}} \pi_s\sigma_s dW_s^{\mathbb{Q}}\Big)\Big|\mathcal{F}_t\Big]\Big\}Y\Big],
\end{aligned}
$$

from which the lemma follows by the definition of a conditional expectation. □

Let $\tilde{\tau}$ have the same distribution as $\tau \wedge T$ conditioned on $\tau \wedge T > t$ and be independent of $W$. Let us define

$$
\begin{aligned}
V(t,x) &:= \operatorname*{ess\,sup}_{(\pi_s)_{t\le s\le T\wedge\tau}} \mathbb{E}\Big[U(P^{t,\pi,x}_{\tau\wedge T})\mathbf{1}_{\tau>t}|\mathcal{G}_t\Big]\\
&= \operatorname*{ess\,sup}_{(\pi_s)_{t\le s\le T}} \mathbb{E}\Big[U(P^{t,\pi,x}_{\tau\wedge T})\mathbf{1}_{\tau>t}|\mathcal{G}_t\Big] = \operatorname*{ess\,sup}_{(\pi_s)_{t\le s\le T}} \mathbb{E}\Big[U\Big(x+\int_t^{\tilde{\tau}} \pi_s\sigma_s dW_s^{\mathbb{Q}}\Big)\Big|\mathcal{F}_t\Big]\mathbf{1}_{\tau>t}
\end{aligned}
$$

and $\tilde{V}(t,x) := \operatorname*{ess\,sup}_{(\pi_s)_{t\le s\le T}} \mathbb{E}[U(P^{\tau\wedge t,\pi,x}_{\tau\wedge T})|\mathcal{G}_t] = U(x)\mathbf{1}_{\tau\le t} + V(t,x)\mathbf{1}_{\tau>t}$. Note that $V$ and $\tilde{V}$ depend on $\omega$ which is suppresed in the notation for the ease of exposition. We want to find $\tilde{V}(0,x)$. Below we show that $\tilde{V}(t,x)$ follows the usual dynamic programming principle.

**Proposition 1** (Dynamic Programming) *For any* $0 \le t \le t' \le T$, *we have*

$$
\tilde{V}(t,x) = \operatorname*{\mathit{ess\,sup}}_{(\pi_s)_{t\le s\le t'}} \mathbb{E}\Big[\tilde{V}(t', P^{\tau\wedge t,\pi,x}_{t'\wedge\tau})|\mathcal{G}_t\Big].
$$

***Proof*** Denote by $\tilde{\pi}_s$ the fraction of wealth invested in to each asset, i.e., $\tilde{\pi} = \frac{\pi}{P^\pi}$ where we set $\tilde{\pi}_t = 0$ if $P^\pi = 0$. Writing with a slide abuse of notation $P_s^{t,\tilde{\pi},x}$ for the corresponding wealth process we have

$$dP_s^{t,\tilde{\pi},x} = P_s^{t,\tilde{\pi},x}\big(r_s ds + \tilde{\pi}_s\left[(\mu_s - \mathbf{1}r_s)ds + \sigma_s dW_s\right]\big). \tag{4.1}$$

Note that (4.1) entails that $P_s^{t,\tilde{\pi},x}$ is an exponential Doléans-Dade exponential and in particular is non-negative. Below, for a (fixed) admissible strategies $(\widetilde{\pi}_s)$ and $0 < t' < T$, we define the concatenation of $(\widetilde{\pi}_s)$ with $(\hat{\pi}_s)_{t' \le s \le T\wedge T}$ at $t'$ by $(\tilde{\pi} \sqcup \hat{\pi})_s = \begin{cases} \tilde{\pi}_s, & 0 \le s < t' \\ \hat{\pi}_s, & t' \le s \le T\wedge\tau \end{cases}$. Assuming without loss of generality that $r = 0$ we have

$$\begin{aligned}
\widetilde{V}(t,x) &= \operatorname*{ess\,sup}_{(\tilde{\pi}_s)_{t\le s<t'}} \operatorname*{ess\,sup}_{(\hat{\pi}_s)_{t'\le s\le T\wedge\tau}} \mathbb{E}\Big[U(x)\mathbf{1}_{\tau\le t} + U\Big(x + \int_t^{T\wedge\tau} P_s^{t,\tilde{\pi}\sqcup\hat{\pi},x}(\tilde{\pi}\sqcup\hat{\pi})_s\sigma_s dW_s^{\mathbb{Q}}\Big)\mathbf{1}_{\tau>t}|\mathcal{G}_t\Big] \\
&= \operatorname*{ess\,sup}_{(\tilde{\pi}_s)_{t\le s<t'}} \mathbb{E}\Big[U(x)\mathbf{1}_{\tau\le t} + U\Big(x + \int_t^{T\wedge\tau} P_s^{t,\tilde{\pi},x}\tilde{\pi}_s\sigma_s dW_s^{\mathbb{Q}}\Big)\mathbf{1}_{t<\tau\le t'} \qquad (4.2)\\
&\quad + \operatorname*{ess\,sup}_{(\hat{\pi}_s)_{t'\le s\le T\wedge\tau}} \mathbb{E}\Big[\mathbf{1}_{\tau>t'}U\Big(x + \int_t^{t'} P_s^{t,\tilde{\pi},x}\tilde{\pi}_s\sigma_s dW_s^{\mathbb{Q}} + \int_{t'}^{T\wedge\tau} P_s^{t,\tilde{\pi}\sqcup\hat{\pi},x}\hat{\pi}_s\sigma_s dW_s^{\mathbb{Q}}\Big)|\mathcal{G}_{t'}\Big]\Big|\mathcal{G}_t\Big] \\
&= \operatorname*{ess\,sup}_{(\tilde{\pi}_s)_{t\le s<t'}} \mathbb{E}\Big[\Big\{U(P_{\tau\wedge T}^{\tau\wedge t,\tilde{\pi},x})\mathbf{1}_{\tau\le t'} \\
&\quad + \operatorname*{ess\,sup}_{(\hat{\pi}_s)_{t'\le s\le\tau\wedge T}} \mathbf{1}_{\tau>t'}\mathbb{E}\Big[U\Big(P_{t'}^{\tau\wedge t,\tilde{\pi},x} + \int_{t'}^{T\wedge\tau} P_s^{\tau\wedge t,\tilde{\pi}\sqcup\hat{\pi},x}\hat{\pi}_s\sigma_s dW_s^{\mathbb{Q}}\Big)|\mathcal{G}_{t'}\Big]\Big\}\Big|\mathcal{G}_t\Big] \\
&= \operatorname*{ess\,sup}_{(\tilde{\pi}_s)_{t\le s<t'}} \mathbb{E}\Big[U(P_{t'\wedge\tau}^{\tau\wedge t,\tilde{\pi},x})\mathbf{1}_{\tau\le t'} + V(t', P_{t'}^{\tau\wedge t,\tilde{\pi},x})\mathbf{1}_{\tau>t'}\Big|\mathcal{G}_t\Big] = \operatorname*{ess\,sup}_{(\tilde{\pi}_s)_{t\le s<t'}} \mathbb{E}\Big[\widetilde{V}(t', P_{t'\wedge\tau}^{\tau\wedge t,\tilde{\pi},x})\Big|\mathcal{G}_t\Big],
\end{aligned}$$

where $P_s^{t,\tilde{\pi}\sqcup\hat{\pi},x}$ denotes the wealth process corresponding to the strategy $\tilde{\pi}$ until time $t'$ and to $\hat{\pi}$ from $t'$ on. The first equality holds as $\operatorname{ess\,sup}_{x,y} f(x,y) = \operatorname{ess\,sup}_x \operatorname{ess\,sup}_y f(x,y)$. To see the second equality note that clearly "$\le$" holds. To show "$\ge$" we will show that actually for each fixed $(\tilde{\pi}_s)_{t\le s<t'}$, it holds that

$$\begin{aligned}
&\operatorname*{ess\,sup}_{(\hat{\pi}_s)_{t'\le s\le T\wedge\tau}} \mathbb{E}\Big[U(x)\mathbf{1}_{\tau\le t} + U\Big(x + \int_t^{T\wedge\tau} P_s^{t,\tilde{\pi}\sqcup\hat{\pi},x}(\tilde{\pi}\sqcup\hat{\pi})_s\sigma_s dW_s^{\mathbb{Q}}\Big)\mathbf{1}_{\tau>t}|\mathcal{G}_t\Big] \\
&\ge \mathbb{E}\Big[U(x)\mathbf{1}_{\tau\le t} + U\Big(x + \int_t^{T\wedge\tau} P_s^{t,\tilde{\pi},x}\tilde{\pi}_s\sigma_s dW_s^{\mathbb{Q}}\Big)\mathbf{1}_{t<\tau\le t'} \\
&+ \operatorname*{ess\,sup}_{(\hat{\pi}_s)_{t'\le s\le T\wedge\tau}} \mathbb{E}\Big[\mathbf{1}_{\tau>t'}U\Big(x + \int_t^{t'} P_s^{t,\tilde{\pi},x}\tilde{\pi}_s\sigma_s dW_s^{\mathbb{Q}} + \int_{t'}^{T\wedge\tau} P_s^{t,\tilde{\pi}\sqcup\hat{\pi},x}\hat{\pi}_s\sigma_s dW_s^{\mathbb{Q}}\Big)|\mathcal{G}_{t'}\Big]\Big|\mathcal{G}_t\Big].
\end{aligned}$$

To see this inequality we will show that there exists an admissible sequence of strategies starting at time $t'$ for which the conditional inner expectations converge to the essential supremum. For this we will first argue that the set over which the essential supremum is taken is directed upward, see Appendix A.5 in [17] for a definition.

Identifying each strategy with the corresponding terminal wealth, we can equivalently write the essential supremum as being taken over a set $\Phi$ of random variables defined as

$$\Phi = \Bigg\{\mathbb{E}\Bigg[\mathbf{1}_{\tau>t'}U\Bigg(x+\int_t^{t'} P_s^{t,\tilde{\pi},x}\tilde{\pi}_s\sigma_s dW_s^{\mathbb{Q}} + \int_{t'}^{T\wedge\tau} P_s^{t,\tilde{\pi}\sqcup\hat{\pi},x}\hat{\pi}_s\sigma_s dW_s^{\mathbb{Q}}\Bigg)\Bigg|\mathcal{G}_{t'}\Bigg], \\ (\hat{\pi}_s)_{t'\le s\le T\wedge\tau}\Bigg\}.$$

Next let us show that the set $\Phi$ is indeed directed upward. For $(\hat{\pi}_s^1)_{t'\le s\le T\wedge\tau}$ and $(\hat{\pi}_s^2)_{t'\le s\le T\wedge\tau}$, we define $\bar{\pi}_s := \mathbf{1}_{B_{t'}}\hat{\pi}_s^1\mathbf{1}_{[t',T]}(s) + \mathbf{1}_{B_{t'}^c}\hat{\pi}_s^2\mathbf{1}_{[t',T]}(s)$ for $t'\le s\le T\wedge\tau$, with

$$B_{t'} = \Bigg\{\omega\in\Omega\Bigg|\mathbb{E}\Bigg[\mathbf{1}_{\tau>t'}U\Bigg(x+\int_t^{t'} P_s^{t,\tilde{\pi},x}\tilde{\pi}_s\sigma_s dW_s^{\mathbb{Q}} + \int_{t'}^{T\wedge\tau} P_s^{t,\tilde{\pi}\sqcup\hat{\pi}^1,x}\hat{\pi}_s^1\sigma_s dW_s^{\mathbb{Q}}\Bigg)|\mathcal{G}_{t'}\Bigg](\omega) \\ > \mathbb{E}\Bigg[\mathbf{1}_{\tau>t'}U\Bigg(x+\int_t^{t'} P_s^{t,\tilde{\pi},x}\tilde{\pi}_s\sigma_s dW_s^{\mathbb{Q}} + \int_{t'}^{T\wedge\tau} P_s^{t,\tilde{\pi}\sqcup\hat{\pi}^2,x}\hat{\pi}_s^2\sigma_s dW_s^{\mathbb{Q}}\Bigg)|\mathcal{G}_{t'}\Bigg](\omega)\Bigg\},$$

where both conditional expectations are identified with particular a.s. versions. Then by definition we have for the concatenated strategy $\tilde{\pi}\sqcup\bar{\pi}$ that

$$\begin{aligned}
&\mathbb{E}\Bigg[\mathbf{1}_{\tau>t'}U\Bigg(x+\int_t^{t'} P_s^{t,\tilde{\pi},x}\tilde{\pi}_s\sigma_s dW_s^{\mathbb{Q}} + \int_{t'}^{T\wedge\tau} P_s^{t,\tilde{\pi}\sqcup\bar{\pi},x}\bar{\pi}_s\sigma_s dW_s^{\mathbb{Q}}\Bigg)\Bigg|\mathcal{G}_{t'}\Bigg] \\
&\quad= \mathbb{E}\Bigg[\mathbf{1}_{\tau>t'}\mathbf{1}_{B_{t'}}U\Bigg(x+\int_t^{t'} P_s^{t,\tilde{\pi},x}\tilde{\pi}_s\sigma_s dW_s^{\mathbb{Q}} + \int_{t'}^{T\wedge\tau} P_s^{t,\tilde{\pi}\sqcup\hat{\pi}^1,x}\hat{\pi}_s^1\sigma_s dW_s^{\mathbb{Q}}\Bigg)\Bigg|\mathcal{G}_{t'}\Bigg] \\
&\qquad+ \mathbb{E}\Bigg[\mathbf{1}_{\tau>t'}\mathbf{1}_{B_{t'}^c}U\Bigg(x+\int_t^{t'} P_s^{t,\tilde{\pi},x}\tilde{\pi}_s\sigma_s dW_s^{\mathbb{Q}} + \int_{t'}^{T\wedge\tau} P_s^{t,\tilde{\pi}\sqcup\hat{\pi}^2,x}\hat{\pi}_s^2\sigma_s dW_s^{\mathbb{Q}}\Bigg)\Bigg|\mathcal{G}_{t'}\Bigg] \\
&\quad= \mathbf{1}_{B_{t'}}\mathbb{E}\Bigg[\mathbf{1}_{\tau>t'}U\Bigg(x+\int_t^{t'} P_s^{t,\tilde{\pi},x}\tilde{\pi}_s\sigma_s dW_s^{\mathbb{Q}} + \int_{t'}^{T\wedge\tau} P_s^{t,\tilde{\pi}\sqcup\hat{\pi}^1,x}\hat{\pi}_s^1\sigma_s dW_s^{\mathbb{Q}}\Bigg)\Bigg|\mathcal{G}_{t'}\Bigg] \\
&\qquad+ \mathbf{1}_{B_{t'}^c}\mathbb{E}\Bigg[\mathbf{1}_{\tau>t'}U\Bigg(x+\int_t^{t'} P_s^{t,\tilde{\pi},x}\tilde{\pi}_s\sigma_s dW_s^{\mathbb{Q}} + \int_{t'}^{T\wedge\tau} P_s^{t,\tilde{\pi}\sqcup\hat{\pi}^2,x}\hat{\pi}_s^2\sigma_s dW_s^{\mathbb{Q}}\Bigg)\Bigg|\mathcal{G}_{t'}\Bigg] \\
&\quad= \mathbb{E}\Bigg[\mathbf{1}_{\tau>t'}U\Bigg(x+\int_t^{t'} P_s^{t,\tilde{\pi},x}\tilde{\pi}_s\sigma_s dW_s^{\mathbb{Q}} + \int_{t'}^{T\wedge\tau} P_s^{t,\tilde{\pi}\sqcup\hat{\pi}^1,x}\hat{\pi}_s^1\sigma_s dW_s^{\mathbb{Q}}\Bigg)\Bigg|\mathcal{G}_{t'}\Bigg] \\
&\qquad\vee \mathbb{E}\Bigg[\mathbf{1}_{\tau>t'}U\Bigg(x+\int_t^{t'} P_s^{t,\tilde{\pi},x}\tilde{\pi}_s\sigma_s dW_s^{\mathbb{Q}} + \int_{t'}^{T\wedge\tau} P_s^{t,\tilde{\pi}\sqcup\hat{\pi}^2,x}\hat{\pi}_s^2\sigma_s dW_s^{\mathbb{Q}}\Bigg)\Bigg|\mathcal{G}_{t'}\Bigg] \quad a.s,
\end{aligned}$$

since $B_{t'}$ and $B_{t'}^{\mathsf{c}}$ are $\mathcal{G}_{t'}$ measurable. The last equality holds by the definition of $B_{t'}$ as $\max(a,b) = a1_{a>b} + b1_{a\le b}$. Hence, the set $\Phi$ is directed upward.

By Theorem A.37(b) in [17] there exists then a sequence $\hat{\pi}^n$ such that the corresponding inner conditional expectations are increasing in $n$ and converging to the inner essential supremum. Therefore, by the monotone convergence theorem

$$\mathbb{E}\Bigg[U(x)\mathbf{1}_{\tau\le t}+U\Bigg(x+\int_t^{\tau}P_s^{t,\tilde\pi,x}\tilde\pi_s\sigma_s dW_s^{\mathbb{Q}}\Bigg)\mathbf{1}_{t<\tau\le t'}$$
$$+\operatorname*{ess\,sup}_{(\hat\pi_s)_{t'\le s\le T\wedge\tau}}\mathbb{E}\Bigg[\mathbf{1}_{\tau>t'}U\Bigg(x+\int_t^{t'}P_s^{t,\tilde\pi,x}\tilde\pi_s\sigma_s dW_s^{\mathbb{Q}}+\int_{t'}^{T\wedge\tau}P_s^{t,\tilde\pi\sqcup\hat\pi,x}\hat\pi_s\sigma_s dW_s^{\mathbb{Q}}\Bigg)\Bigg|\mathcal{G}_{t'}\Bigg]\Bigg|\mathcal{G}_t\Bigg]$$
$$=\lim_{n\to\infty}\mathbb{E}\Bigg[U(x)\mathbf{1}_{\tau\le t}+U\Bigg(x+\int_t^{\tau}P_s^{t,\tilde\pi,x}\tilde\pi_s\sigma_s dW_s^{\mathbb{Q}}\Bigg)\mathbf{1}_{t<\tau\le t'}$$
$$+\mathbb{E}\Bigg[\mathbf{1}_{\tau>t'}U\Bigg(x+\int_t^{t'}P_s^{t,\tilde\pi,x}\tilde\pi_s\sigma_s dW_s^{\mathbb{Q}}+\int_{t'}^{T\wedge\tau}P_s^{t,\tilde\pi\sqcup\hat\pi^n,x}\hat\pi_s^n\sigma_s dW_s^{\mathbb{Q}}\Bigg)\Bigg|\mathcal{G}_{t'}\Bigg]\Bigg|\mathcal{G}_t\Bigg]$$
$$\le\operatorname*{ess\,sup}_{(\hat\pi_s)_{t'\le s\le T\wedge\tau}}\mathbb{E}\Bigg[U(x)\mathbf{1}_{\tau\le t}+U\Bigg(x+\int_t^{T\wedge\tau}P_s^{t,\tilde\pi\sqcup\hat\pi,x}(\tilde\pi\sqcup\hat\pi)_s\sigma_s dW_s^{\mathbb{Q}}\Bigg)\mathbf{1}_{\tau>t}\Bigg|\mathcal{G}_t\Bigg].$$

□

We are now in a position to show that contrary to the case with a certain time horizon, concavifying the utility function is not applicable when the investment horizon is random. In other words, replacing $U$ with $U^c$ in the optimization (3.1) leads to a sub-optimal strategy. To this end, we need to investigate smoothness and concavity of the value function of the *certain* time horizon optimization problem

$$\bar V(t,x):=\sup_{\pi\in\Pi(t,x)}\mathbb{E}[U(P_T)|P_t=x]. \tag{4.3}$$

Smoothness and concavity of the value function has been also studied in [6] by working on the dual control problem and the dual HJB equation under the following assumption which we in the sequel will make as well[2]:

**Assumption (H)**: $U(0)=U^c(0)=0$, $U^c(\infty)=\infty$ and $U^c$ is strictly increasing and $U^c(x)\le C(1+x^p)$ for some constant $C>0$ and $0<p<1$.

**Proposition 2** *Under Assumption (H), the value function $\bar V(t,x)$ of Problem (4.3) is strictly concave and strictly increasing and $C^{1,2}$ in $[0,T)\times[0,\infty)$. Furthermore, $\bar V(T,x)=U^c(x)$, $\bar V(t,0)=0$ and $\bar V(t,x)\le\widetilde C(1+x^p)$ for some positive constant $\widetilde C$ and $\bar V(t,x)$ satisfies the Inada's condition at zero and infinity.*

***Proof*** By Theorem 4.1 in [24] the concavification argument can be applied and $U$ can be replaced by its concave hull $U^c$. By assumption, $U^c$ is increasing and concave and it follows that $\bar V$ is strictly increasing, in $C^{1,2}$ and satisfies the growth condition by applying Theorem 3.8 in [6]. An inspection of the proof of Theorem 3.8 together with Lemma 3.6 [6] also confirms that $\bar V$ satisfies the Inada condition at 0 and infinity. □

**Proposition 3** *Assume that Assumption (H) holds and the concavification region $\{U<U^c\}$ contains an interval $(0,\eta)$ for some $\eta>0$*[3]*. Suppose that $\tau$ is not identical zero and the original problem (3.1) has a solution $\pi^*$. Define $\sup_\pi\mathbb{E}[U^c(P_{\tau\wedge T}^{\pi,x})]=A_0$ and $\sup_\pi\mathbb{E}[U(P_{\tau\wedge T}^{\pi,x})]=\mathbb{E}[U(P_{\tau\wedge T}^{\pi^*,x})]=B_0$. Then $A_0>B_0$.*

[2] In [7], the authors obtain similar results under a Hölder-continuity condition (Theorem 4.2) by using the comparison principle of PDEs for the dual control problem.

[3] This assumption is satisfied in the option compensation problem with power utility, see Sect. 5.2

***Proof*** Denote by $\tilde{\pi}_s$ the fraction of wealth invested in to each asset, i.e., $\tilde{\pi} = \frac{\pi}{P^\pi}$ where we set $\tilde{\pi}_t = 0$ if $P^\pi = 0$. By Lemma 1 we may restrict ourselves to $\mathcal{F}$- predictable strategies. Writing again with a slide abuse of notation $P_s^{t,\tilde{\pi},x}$ for the corresponding wealth process we have

$$dP_s^{t,\tilde{\pi},x} = P_s^{t,\tilde{\pi},x}\big(r_s ds + \tilde{\pi}_s\,[(\mu_s - \mathbf{1}r_s)ds + \sigma_s dW_s]\big).$$

In the sequel of the proof let us assume without loss of generality that $\{\tau > T_{n-1}\}$ is a non-zero set. (Otherwise redefine $T_n$.) Assume then by contradiction that $A_0 = B_0$. Arguing as in (4.2), one can show that $(\tilde{\pi}^*)_{T_{n-2}\wedge\tau\le s\le T_n\wedge\tau}$ is actually also a maximizer for ess $\sup_{(\tilde{\pi}_s)_{T_{n-2}\wedge\tau\le s\le T_n\wedge\tau}} \mathbb{E}[U(P^{\tilde{\pi}}_{\tau\wedge T})|\mathcal{G}_{T_{n-2}\wedge\tau}]$, i.e.,

$$\mathbb{E}[U(P^{\tilde{\pi}^*}_{\tau\wedge T})|\mathcal{G}_{T_{n-2}\wedge\tau}] = \text{ess sup}_{(\tilde{\pi}_s)_{T_{n-2}\wedge\tau\le s\le T_n\wedge\tau}} \mathbb{E}[U(P^{\tilde{\pi}}_{\tau\wedge T})|\mathcal{G}_{T_{n-2}\wedge\tau}],$$

where all strategies $\tilde{\pi}$ are assumed to agree with $\tilde{\pi}^*$ until time $T_{n-2}\wedge\tau$. By the dynamic programming principle it is sufficient to show that on a non-zero set

$$\begin{aligned} A_{T_{n-2}}^{(\tilde{\pi}^*_s)_{0\le s<T_{n-2}\wedge\tau}} &:= \underset{(\tilde{\pi}_s)_{T_{n-2}\wedge\tau\le s<T_n\wedge\tau}}{\text{ess sup}} \mathbb{E}[U^c(P^{\tilde{\pi}}_{\tau\wedge T})|\mathcal{G}_{T_{n-2}\wedge\tau}] \\ &> \underset{(\tilde{\pi}_s)_{T_{n-2}\wedge\tau\le s\le T_n\wedge\tau}}{\text{ess sup}} \mathbb{E}[U(P^{\tilde{\pi}}_{\tau\wedge T})|\mathcal{G}_{T_{n-2}\wedge\tau}] =: B_{T_{n-2}}^{(\tilde{\pi}^*_s)_{0\le s\le T_{n-2}\wedge\tau}}, \end{aligned}$$

since then it follows that

$$A_0 \ge \mathbb{E}[A_{T_{n-2}}^{(\tilde{\pi}^*_s)_{0\le s<T_{n-2}\wedge\tau}}] > \mathbb{E}[B_{T_{n-2}}^{(\tilde{\pi}^*_s)_{0\le s<T_{n-2}\wedge\tau}}] = \mathbb{E}\Big[\mathbb{E}[U(P^{\tilde{\pi}^*}_{\tau\wedge T})|\mathcal{G}_{T_{n-2}\wedge\tau}]\Big] = B_0,$$

deriving a contradiction.

Let us remark that in our complete market setting, the market price density $\xi$ is atomless and $U$ is continuous by assumption. In particular, by Proposition 2 and the concavification techniques in the certain maturity case (see Sect. 5 of [24]), the last period value function on $\tau_n > T_{n-1}$ is given by

$$\begin{aligned} V_{T_{n-1}}(x) &= \underset{(\tilde{\pi}_s)_{T_{n-1}\wedge\tau\le s\le T_n\wedge\tau}}{\text{ess sup}} \mathbb{E}[U^c(P^{\tilde{\pi}}_{\tau\wedge T})|\mathcal{G}_{T_{n-1}\wedge\tau},\, P^{\tilde{\pi}}_{T_{n-1}\wedge\tau} = x] \\ &= \underset{(\tilde{\pi}_s)_{T_{n-1}\wedge\tau\le s\le T_n\wedge\tau}}{\text{ess sup}} \mathbb{E}[U^c(P^{\tilde{\pi}}_{\tau\wedge T})|\sigma(\tau \le T_{n-1}),\, P^{\tilde{\pi}}_{T_{n-1}\wedge\tau} = x] \\ &= \underset{(\tilde{\pi}_s)_{T_{n-1}\wedge\tau\le s\le T_n\wedge\tau}}{\text{ess sup}} \mathbb{E}[U^c(P^{\tilde{\pi}}_{\tau\wedge T})|\tau > T_{n-1},\; P^{\tilde{\pi}}_{T_{n-1}\wedge\tau} = x] \\ &= \underset{(\tilde{\pi}_s)_{T_{n-1}\wedge\tau\le s\le T_n\wedge\tau}}{\text{ess sup}} \mathbb{E}[U(P^{\tilde{\pi}}_{\tau\wedge T})|\tau > T_{n-1},\; P^{\tilde{\pi}}_{T_{n-1}\wedge\tau} = x], \end{aligned}$$

which is strictly increasing and strictly concave and $V_{T_{n-1}}(0) = U(0) = 0$. Therefore,

$$p_{n-1}U + p_n V_{T_{n-1}} \leq (p_{n-1}U + p_n V_{T_{n-1}})^c \leq p_{n-1}U^c + p_n V_{T_{n-1}}.$$

On $\tau_n > T_{n-1}$ by the Inada condition of $V_{T_{n-1}}$ and Lemma 8.3 there exists $\tilde{\varepsilon} > 0$ such that $(p_{n-1}U + p_n V_{T_{n-1}})^c$ is strictly concave on $[0, \tilde{\varepsilon}]$ and

$$(0, \tilde{\varepsilon}) \subset \{p_{n-1}U + p_n V_{T_{n-1}} < \big(p_{n-1}U + p_n V_{T_{n-1}}\big)^c\} \subset \{p_{n-1}U + p_n V_{T_{n-1}} < p_{n-1}U^c + p_n V_{T_{n-1}}\} = \{U < U^c\}.$$

It then follows

$$\begin{aligned}
A_{T_{n-2}} &= \sum_{i \leq n-2} \mathbf{1}_{\tau = T_i} U^c(P^{\tilde{\pi}^*}_{T_i}) \\
&\quad + \mathbf{1}_{\tau > T_{n-2}} \operatorname*{ess\,sup}_{(\tilde{\pi}_s)_{T_{n-2}\wedge\tau \leq s < T_{n-1}\wedge\tau}} \mathbb{E}[p_{n-1}U^c(P^{\tilde{\pi}}_{T_{n-1}}) + p_n V_{T_{n-1}}(P^{\tilde{\pi}}_{T_{n-1}}) | \mathcal{G}_{T_{n-2}\wedge\tau}]/(p_{n-1} + p_n) \\
&\geq \sum_{i \leq n-2} \mathbf{1}_{\tau = T_i} U^c(P^{\tilde{\pi}^*}_{T_i}) \\
&\quad + \mathbf{1}_{\tau > T_{n-2}} \mathbb{E}[p_{n-1}U^c(P^{\tilde{\pi}^*}_{T_{n-1}}) + p_n V_{T_{n-1}}(P^{\tilde{\pi}^*}_{T_{n-1}}) | \mathcal{G}_{T_{n-2}\wedge\tau}]/(p_{n-1} + p_n) \\
&> \sum_{i \leq n-2} \mathbf{1}_{\tau = T_i} U(P^{\tilde{\pi}^*}_{T_i}) + \mathbf{1}_{\tau > T_{n-2}} \\
&\quad \mathbb{E}[\big(p_{n-1}U(P^{\tilde{\pi}^*}_{T_{n-1}}) + p_n V_{T_{n-1}}(P^{\tilde{\pi}^*}_{T_{n-1}})\big)^c | \mathcal{G}_{T_{n-2}\wedge\tau}]/(p_{n-1} + p_n) \\
&= \sum_{i \leq n-2} \mathbf{1}_{\tau = T_i} U(P^{\tilde{\pi}^*}_{T_i}) \\
&\quad + \mathbf{1}_{\tau > T_{n-2}} \operatorname*{ess\,sup}_{(\tilde{\pi}_s)_{T_{n-2}\wedge\tau \leq s < T_{n-1}\wedge\tau}} \mathbb{E}[p_{n-1}U(P^{\tilde{\pi}}_{T_{n-1}}) + p_n V_{T_{n-1}}(P^{\tilde{\pi}}_{T_{n-1}}) | \mathcal{G}_{T_{n-2}\wedge\tau}]/(p_{n-1} + p_n) \\
&= B_{T_{n-2}}
\end{aligned}$$

with $\tilde{\pi}^* = \operatorname*{argsup}_{(\tilde{\pi}_s)_{T_{n-2}\wedge\tau \leq s < T_{n-1}\wedge\tau}} \mathbb{E}[p_{n-1}U(P^{\tilde{\pi}}_{T_{n-1}}) + p_n V_{T_{n-1}}(P^{\tilde{\pi}}_{T_{n-1}}) | \mathcal{G}_{T_{n-2}\wedge\tau}]$. The strict inequality above holds because on a non-zero set $\big(p_{n-1}U + p_n V_{T_{n-1}}\big)^c$ is not affine on the set $\{U < U^c\}$ and is strictly concave on an interval $(0, \tilde{\varepsilon}) \subset \{U < U^c\}$ due to Inada's condition at zero of $V_{T_{n-1}}$ (see Lemma 8.3). Hence, by a Merton-Lagrange-type analysis, $P^{\tilde{\pi}^*}_{T_{n-1}}$ takes values with positive probability in a non-zero set where $\big(p_{n-1}U + p_n V_{T_{n-1}}\big)^c < p_{n-1}U^c + p_n V_{T_{n-1}}$. □

It follows from Proposition 3 that concavification techniques (as for instance in [24]) cannot be directly applied to $U$ when the time horizon is random. The non-concave optimization in this case can however still be solved by a recursive procedure which is established by Proposition 1. This will be explicitly illustrated in the next section.

## 5 Example for Power Utility Function

In this section, we illustrate our main results established in the previous sections. In particular, we consider for $0 \leq \tau \leq T$ a discrete random variable, i.e., there are times

$T_0 := 0 < T_1 < T_2 < \cdots < T_n = T$ and probabilities $0 < p_i < 1$ for $1 \leq i \leq n$ with $\sum_{i=1}^n p_i = 1$ such that $\mathbb{P}(\tau = T_i) = p_i$, for $1 \leq i \leq n$. For simplicity, we assume that $\theta$ and $r$ are constant and we choose a power (CRRA) utility, i.e.,

$$U(x) := \frac{x^{1-\gamma}}{1-\gamma}, \text{ for } 0 < \gamma < 1. \tag{5.1}$$

### 5.1 Concave Optimization with Power Utility

Note first that since $U$ is strictly concave we have that $I = (U')^{-1}$. By Theorem 1, we need to find an adapted process $\nu \geq 0$ with $\nu_0 = U'(x)$ such that the process $\xi P^{x,\pi^{(n)}(x,I(\nu_{T_1}\xi_{T_1}),\cdots,I(\nu_{T_n}\xi_{T_n}))}$ generated by the $n+1$-tuple $(x, I(\nu_{T_1}\xi_{T_1}), \cdots, I(\nu_{T_n}\xi_{T_n}))$ is a martingale and $\sum_{i=1}^n p_i \nu_{T_i}$ is a constant. As shown below, for such a CRRA utility function we can find a $\nu$ which is deterministic, in particular, $\sum_{i=1}^n p_i \nu_{T_i}$ is a constant. We will not provide a proof.

**Proposition 4** *For power utility $U$ defined in (5.1), the optimal solution $P^*$ generated by the $n+1$-tuple $(x, I(\nu_{T_1}\xi_{T_1}), \cdots, I(\nu_{T_n}\xi_{T_n}))$, where*

$$\nu_{T_j} = \left(\frac{x}{f\left(\frac{\gamma-1}{\gamma}, 0, T_j\right)}\right)^{-\gamma}, \quad 1 \leq j \leq n.$$

*and $f(q,t,T) := \exp\left(-q\int_t^T (r_s + \frac{1}{2}\theta_s^2)ds + q^2 \int_t^T \frac{\theta_s^2}{2} ds\right)$, with $\theta$ being the market price of risk from Sect. 2.1. Furthermore, the optimal investment strategy is the Merton strategy, i.e., the optimal fraction of wealth invested in the risky asset at time $t$ is given by $\frac{\mu_t - r_t}{\gamma\sigma_t^2}$, which is independent of the distribution of the stopping time.*

Hence, in the concave optimization problem the optimal portfolio selection is not affected by the presence of an uncertain time horizon, even though the value function is not identical to the one corresponding to the standard fixed-horizon case. This result can be considered as a confirmation of Merton [21] and Richard [25] and is aligned with the findings in [10, 11].

### 5.2 Non-concave Optimization: Recursive Solution

Throughout the rest of this section we focus on the case where the random maturity has a binary distribution. We consider the special choice of a non-concave objective function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ as in (5.2), i.e., for given $K > 0$ and $B > 0$:

$$U(x) = \begin{cases} u(K + \alpha(x-B)^+) & \text{for } x \geq 0, \\ -\infty & \text{else.} \end{cases} \tag{5.2}$$

where $u(x) = x^{1-\gamma}/(1-\gamma)$, with $0 < \gamma < 1$. We remark that although in almost all optimal control problems considered in the literature a fixed known time horizon is assumed, in reality a fixed maturity is typically not naturally given and the target date itself instead is often of random-type. Hence, the problem considered in this chapter fits to all option type managerial compensation problems, for which in the case of a non-random time horizon there is already a rich literature in the finance & OR literature going back to [13, 27].

Note that not only considered in the managerial option compensation, the option-like payoffs of the form (5.2) also arise naturally for instance in flexibility rider insurance products which at the end of the life time of the policy holder, pay out a guarantee plus a participation rate the latter depending on the returns in the stock market. In these products, the policyholder is allowed to influence the investment decision of the life insurance product. An example for such products are in France for instance the life insurance products AXA Twin Star, in Germany the Swiss Life Champion and in the US, for example, Allianz Index Advanta, see [14] and the references within. In these cases, $\alpha$, $K$, $B$ are the participation rate, the guarantee and the threshold for the participation, respectively.

By Proposition 3, a concavification procedure cannot be directly applied and we will solve the optimization by a recursive procedure. For comparison purpose, we introduce the concave envelope $U^c : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ given by

$$U^c(x) := \begin{cases} -\infty & \text{for } x < 0, \\ U(0) + U'(\hat{x}(B))x & \text{for } 0 \le x \le \hat{x}(B), \\ U(x) & \text{for } x > \hat{x}(B), \end{cases} \tag{5.3}$$

where $\hat{x}(B) := \min\{x > 0 : U(x) = U^c(x)\}$. As in [13, 22], $\hat{x}(B)$ is defined by the following concavification equation:

$$U(\hat{x}(B)) - U(0) = U'(\hat{x}(B))\hat{x}(B). \tag{5.4}$$

Note that $U^c$ dominates $U$ with equality for $x = 0$ and $x \ge \hat{x}(B)$. Consistently with (3.4), we are able to define the function $I : (0, \infty) \to [0, \infty)$ by

$$I(y) := \left[\frac{1}{\alpha}\left(i\left(\frac{y}{\alpha}\right) - K\right) + B\right]\mathbf{1}_{\{y < U'(\hat{x}(B))\}}, \tag{5.5}$$

where $i(x) = x^{-1/\gamma}$ is the inverse of $u'$. We note that $I$ is the generalized inverse function of $(\partial U^c)$ in the sense that $y \in (\partial U^c)(I(y))$ for all $y > 0$.

In our last period we already know $\tau$ so that the problem can be treated as a static non-concave EU maximization problem.

Given $P_{T_1} = x > 0$, the wealth level at time $T_1$, the optimal terminal wealth of the conditional static problem, $\sup_{P-admissible, P_{T_1}=x} \mathbb{E}[U(P_T)|P_{T_1} = x]$, is given by

$$P_T^* = I(\lambda_T \xi_T) = \left[\frac{1}{\alpha}\left(i\left(\frac{\lambda_T \xi_T}{\alpha}\right) - K\right) + B\right]\mathbf{1}_{\{\lambda_T \xi_T < U'(\hat{x}(B))\}}, \tag{5.6}$$

where $\lambda_T$ is $\mathcal{F}_{T_1}$-measurable and defined by the budget constraint $\mathbb{E}[\xi_T \xi_{T_1}^{-1} P_T^* | \mathcal{F}_{T_1}] = x$, see e.g. [13, 22, 27] for more detail discussions. The optimal wealth process is given by the following lemma:

**Lemma 4** *Given a realized wealth level at time $T_1$, the optimal wealth process on $(T_1, T]$ is given by $P_t^* = P_{t,T}^*(\lambda_T \xi_t)$, where*

$$
\begin{aligned}
P_{t,T}^*(y) &:= f(1,t,T)\left(B - \frac{K}{\alpha}\right)\Phi[d(1,t,T,y)] \\
&\quad + \left(\frac{1}{\alpha}\right)^{1-\frac{1}{\gamma}} (y)^{-1/\gamma} f\left(1-\frac{1}{\gamma}, t, T\right)\Phi[d(1-1/\gamma, t, T, y)],
\end{aligned}
\tag{5.7}
$$

*where $\lambda_T$ satisfies the budget constraint at time $T_1$, $\Phi$ denotes the cumulative distribution function of the standard normal distribution and*

$$
d(q,t,T,\xi) = \frac{\log\left(U'(\hat{x}(B))/\xi\right) + \left(r + \frac{1}{2}\theta^2\right)(T-t)}{\theta\sqrt{T-t}} - q\theta\sqrt{T-t}. \tag{5.8}
$$

***Proof*** The lemma follows directly from (5.6) and Lemma 8.2, in the Appendix. □

Note that the wealth process $P_{t,T}^*(\lambda_T \xi_t)$, expressed as a functional of the product $\lambda_T \xi_t$, depends on $P_{T_1}$, the wealth level at time $T_1$, as the multiplier $\lambda_T = \lambda_T(P_{T_1})$ is characterized by the budget equation at time $T_1$.

**Lemma 5** *The indirect value function $V_{T_1}(x) := \mathbb{E}[U(P_T^*)|P_{T_1} = x]$ is given by*

$$
\begin{aligned}
V_{T_1}(x) =& \frac{1}{1-\gamma} f(q, T_1, T)\left(\frac{1}{\alpha}\right)^{1-\frac{1}{\gamma}} \\
& * (\lambda_T(x)\xi_{T_1})^{1-1/\gamma} f\left(1-\frac{1}{\gamma}, T_1, T\right)\Phi[d(1-1/\gamma, T_1, T, \xi_{T_1}\lambda_T(x)) \\
& + \frac{1}{1-\gamma} K^{1-\gamma}(1 - \Phi[d(0, T_1, T, \xi_{T_1}\lambda_T(x))]).
\end{aligned}
\tag{5.9}
$$

***Proof*** From (5.6) we have

$$
\begin{aligned}
&V_{T_1}(x) = \mathbb{E}[U(P_T^*)|P_{T_1}^* = x] = \mathbb{E}\left[\frac{1}{1-\gamma}\left(\frac{\lambda_T(x)\xi_T}{\alpha}\right)^{(-1/\gamma)\times(1-\gamma)} \mathbf{1}_{\{\lambda_T\xi_T < U'(\hat{x}(B))\}}\right] \\
&+ \frac{K^{1-\gamma}}{1-\gamma}\mathbb{E}\left[\mathbf{1}_{\{\lambda_T\xi_T \geq U'(\hat{x}(B))\}}\right]
\end{aligned}
$$

and the explicit formula follows directly from Lemma 8.2 in the Appendix. □

**Proposition 5** *$V_{T_1}(x)$ is a globally strictly concave function and its first two derivatives are given by*

$$V'_{T_1}(x) = \lambda_T(x)\xi_{T_1} \quad \textit{and} \quad V''_{T_1}(x) = \lambda'_T(x)\xi_{T_1}. \tag{5.10}$$

*The inverse of marginal indirect value function $(V_{T_1})'$ is given by*

$$\begin{aligned} I_{T_1}(X) =& f(1, T_1, T)\left(B - \frac{K}{\alpha}\right)\Phi[d(1, T_1, T, X)] \\ &+ \left(\frac{1}{\alpha}\right)^{1-\frac{1}{\gamma}}(X)^{-1/\gamma} f\left(1 - \frac{1}{\gamma}, T_1, T\right)\Phi[d(1 - 1/\gamma, T_1, T, X)]. \end{aligned} \tag{5.11}$$

***Proof*** By differentiating the budget constraint

$$\begin{aligned} x =& \left(\frac{1}{\alpha}\right)^{1-\frac{1}{\gamma}}(\lambda_T\xi_{T_1})^{-1/\gamma} f\left(1 - \frac{1}{\gamma}, T_1, T\right)\Phi[d(1 - 1/\gamma, T_1, T, \lambda_T\xi_{T_1})] \\ &+ f(1, T_1, T)\left(B - \frac{K}{\alpha}\right)\Phi[d(1, T_1, T, \lambda_T\xi_{T_1})], \end{aligned}$$

we obtain $\frac{\mathrm{d}x}{\mathrm{d}\lambda_T} = A_1 + A_2 + A_3$, where

$$\begin{aligned} A_1 &= \frac{-1}{\gamma}\left(\frac{1}{\alpha}\right)^{1-\frac{1}{\gamma}}\xi_{T_1}(\lambda_T\xi_{T_1})^{-1/\gamma-1} f\left(1 - \frac{1}{\gamma}, T_1, T\right)\Phi[d(1 - 1/\gamma, T_1, T, \lambda_T\xi_{T_1})], \\ A_2 &= \left(\frac{1}{\alpha}\right)^{1-\frac{1}{\gamma}}(\lambda_T\xi_{T_1})^{-1/\gamma} f\left(1 - \frac{1}{\gamma}, T_1, T\right)\varphi[d(1 - 1/\gamma, T_1, T, \lambda_T\xi_{T_1})]\frac{-1}{\lambda_T\theta\sqrt{T - T_1}}, \\ A_3 &= f(1, T_1, T)\left(B - \frac{K}{\alpha}\right)\varphi[d(1, T_1, T, \lambda_T\xi_{T_1})]\frac{-1}{\lambda_T\theta\sqrt{T - T_1}}. \end{aligned}$$

Similarly, by differentiating (5.9) we obtain that

$$\frac{\mathrm{d}x}{\mathrm{d}\lambda_T}V'_{T_1}(x) = \lambda_T\xi_{T_1}A_1 + \frac{1}{1-\gamma}\lambda_T\xi_{T_1}A_2 - \frac{1}{1-\gamma}K^{1-\gamma}\varphi[d(0, T_1, T, \xi_{T_1}\lambda_T)]\frac{-1}{\lambda_T\theta\sqrt{T - T_1}}.$$

Note that from (5.8) we have for any $q \in \mathbb{R}$,

$$
\begin{aligned}
&f(q, T_1, T)\varphi[d(q, T_1, T, \lambda_T \xi_{T_1})] \\
&= f(q, T_1, T)\varphi[d(0, T_1, T, \lambda_T \xi_{T_1}) - q\theta\sqrt{T - T_1}] \\
&= f(q, T_1, T)\varphi[d(0, T_1, T, \lambda_T \xi_{T_1})]e^{-\frac{1}{2}q^2\theta^2(T-T_1)}e^{qd(0,T_1,T,\lambda_T\xi_{T_1})\theta\sqrt{T-T_1}} \\
&= f(q, T_1, T)\varphi[d(0, T_1, T, \lambda_T \xi_{T_1})]\frac{e^{q\log\left(\frac{U'(\hat{x}_B)}{\lambda_T\xi_{T_1}}\right)}}{f(q, T_1, T)} \\
&= \left(\frac{U'(\hat{x}_B)}{\lambda_T \xi_{T_1}}\right)^q \varphi[d(0, T_1, T, \lambda_T \xi_{T_1})]. \qquad (5.12)
\end{aligned}
$$

By direct calculation, we can represent $V'_{T_1}(x)$ as

$$
\begin{aligned}
V'_{T_1}(x) &= \frac{\mathrm{d}\lambda_T}{\mathrm{d}x}\left(\lambda_T\xi_{T_1}A_1 + \frac{1}{1-\gamma}\lambda_T\xi_{T_1}A_2 - \frac{1}{1-\gamma}K^{1-\gamma}\varphi[d(0, T_1, T, \xi_{T_1}\lambda_T)]\frac{-1}{\lambda_T\theta\sqrt{T-T_1}}\right) \\
&= \frac{\mathrm{d}\lambda_T}{\mathrm{d}x}\lambda_T\xi_{T_1}(A_1 + A_2 + A_3) \\
&\quad + \frac{\mathrm{d}\lambda_T}{\mathrm{d}x}\left((\frac{1}{1-\gamma} - 1)\lambda_T\xi_{T_1}A_2 - \frac{1}{1-\gamma}K^{1-\gamma}\varphi[d(0, T_1, T, \xi_{T_1}\lambda_T)]\frac{-1}{\lambda_T\theta\sqrt{T-T_1}} - \lambda_T\xi_{T_1}A_3\right) \\
&= \lambda_T\xi_{T_1} + \frac{\mathrm{d}\lambda_T}{\mathrm{d}x}\left(\frac{\gamma}{1-\gamma}\lambda_T\xi_{T_1}A_2 - \frac{K^{1-\gamma}}{1-\gamma}\varphi[d(0, T_1, T, \xi_{T_1}\lambda_T)]\frac{-1}{\lambda_T\theta\sqrt{T-T_1}} - \lambda_T\xi_{T_1}A_3\right). \qquad (5.13)
\end{aligned}
$$

By applying (5.12) with $q = 1$ and $q = 1 - \frac{1}{\gamma}$ we obtain

$$
\begin{aligned}
f(1, T_1, T)\varphi[d(1, T_1, T, \lambda_T\xi_{T_1})] &= \left(\frac{U'(\hat{x}_B)}{\lambda_T\xi_{T_1}}\right)\varphi[d(0, T_1, T, \lambda_T\xi_{T_1})], \\
f(1 - \frac{1}{\gamma}, T_1, T)\varphi[d(1 - \frac{1}{\gamma}, T_1, T, \lambda_T\xi_{T_1})] &= \left(\frac{U'(\hat{x}_B)}{\lambda_T\xi_{T_1}}\right)^{1-\frac{1}{\gamma}}\varphi[d(0, T_1, T, \lambda_T\xi_{T_1})].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(\lambda_T\xi_{T_1})A_2 &= (\lambda_T\xi_{T_1})\left(\frac{1}{\alpha}\right)^{1-\frac{1}{\gamma}}(\lambda_T\xi_{T_1})^{-1/\gamma}\left(\frac{U'(\hat{x}_B)}{\lambda_T\xi_{T_1}}\right)^{1-\frac{1}{\gamma}}\varphi[d(0, T_1, T, \lambda_T\xi_{T_1})]\frac{-1}{\lambda_T\theta\sqrt{T-T_1}} \\
&= \left(\frac{U'(\hat{x}_B)}{\alpha}\right)^{1-\frac{1}{\gamma}}\varphi[d(0, T_1, T, \lambda_T\xi_{T_1})]\frac{-1}{\lambda_T\theta\sqrt{T-T_1}} \qquad (5.14)
\end{aligned}
$$

and

$$
\begin{aligned}
(\lambda_T \xi_{T_1}) A_3 &= (\lambda_T \xi_{T_1}) \left(B - \frac{K}{\alpha}\right) \left(\frac{U'(\hat{x}_B)}{\lambda_T \xi_{T_1}}\right) \varphi[d(0, T_1, T, \lambda_T \xi_{T_1})] \frac{-1}{\lambda_T \theta \sqrt{T - T_1}} \\
&= \left(B - \frac{K}{\alpha}\right) U'(\hat{x}_B) \varphi[d(0, T_1, T, \lambda_T \xi_{T_1})] \frac{-1}{\lambda_T \theta \sqrt{T - T_1}}. \qquad (5.15)
\end{aligned}
$$

The bracket in (5.13) can be expressed as

$$
\Bigg(\underbrace{\frac{\gamma}{1-\gamma}\left(\frac{1}{\alpha}\right)^{1-\frac{1}{\gamma}} (U'(\hat{x}_B))^{1-1/\gamma} - \frac{1}{1-\gamma} K^{1-\gamma} - (B - \frac{K}{\alpha}) U'(\hat{x}_B)}_{A_4}\Bigg) \frac{-\varphi[d(0, T_1, T, \lambda_T \xi_{T_1})]}{\lambda_T \theta \sqrt{T - T_1}}.
$$

From (5.2) we have

$$
U(0) = \frac{1}{1-\gamma} K^{1-\gamma}, \quad \frac{\gamma}{1-\gamma} (\frac{1}{\alpha})^{1-\frac{1}{\gamma}} (U'(\hat{x}_B))^{1-1/\gamma} = \gamma U(\hat{x}_B),
$$

and

$$
\begin{aligned}
&(1-\gamma) U(\hat{x}_B) - \hat{x}_B U'(\hat{x}_B) \\
&= (1-\gamma) \frac{(\alpha \hat{x}_B - \alpha B + K)^{1-\gamma}}{1-\gamma} - \hat{x}_B \alpha (\alpha \hat{x}_B - \alpha B + K)^{-\gamma} \\
&= (B - \frac{K}{\alpha}) U'(\hat{x}_B).
\end{aligned}
$$

This implies that $A_4 = U(\hat{x}_B) - U(0) - \hat{x}_B U'(\hat{x}_B) = 0$ due to the concavification equation (5.4). Hence, $V'_{T_1}(x) = \lambda_T \xi_{T_1}$. The above derivation also shows that (5.11) defines the inverse of $V_{T_1}$. □

For a power utility function, it is straightforward to compute the optimal investment strategy in the period $[T_1, T)$ given the wealth level at time $T_1$, see e.g. [13, 22]. Having determined the indirect utility function at time $T_1$, we now represent the optimization problem as

$$
\sup_{(\pi_t)_{t \in [0, T_1]}} \mathbb{E}\Big[ p U(P_{T_1}) + (1-p) V_{T_1}(P_{T_1}) \Big], \qquad (5.16)
$$

where $\mathbb{P}(\tau = T_1) = p$ and $\mathbb{P}(\tau = T) = 1 - p$. Note that (5.16) is expressed as a non-concave optimization problem in a complete market. To solve it we look at its static version

$$\begin{aligned} &\sup_{P\geq 0,\ \mathcal{F}_{T_1}-measurable} \mathbb{E}\Big[pU(P) + (1-p)V_{T_1}(P)\Big] \\ &= \sup_{P\geq 0,\ \mathcal{F}_{T_1}-measurable} \mathbb{E}\Big[U_1(P)\mathbf{1}_{P\leq B} + U_2(P)\mathbf{1}_{P>B}\Big], \end{aligned} \tag{5.17}$$

subject to the usual budget constraint $\mathbb{E}[\xi_{T_1} P] \leq x$, where $U_i, i = 1, 2$ are concave functions defined by

$$U_1(x) := pU(0) + (1-p)V_{T_1}(x), \quad U_2(x) := pU(x) + (1-p)V_{T_1}(x). \tag{5.18}$$

Since in the first period $[0, T_1]$, the problem becomes static, the solution of the non-concave optimization (5.17) is given by maximizing the concavified target function. Let $I_i, i = 1, 2$ be the corresponding inverse marginal utilities of $U_1$ and $U_2$ respectively. The optimal wealth at $T_1$ is given by the following expression.

**Proposition 6** *The optimal portfolio of Problem (5.17) is given by*

$$P^*_{T_1} = I_2(\lambda\xi_{T_1})\mathbf{1}_{\xi_{T_1}<\widehat{\xi}} + I_1(\lambda\xi_{T_1})\mathbf{1}_{\xi_{T_1}\geq\widehat{\xi}},$$

*where $\widehat{\xi}$ is defined by*

$$U_2(I_2(\lambda\widehat{\xi})) - U_1(I_1(\lambda\widehat{\xi})) = \lambda\widehat{\xi}\Big(I_2(\lambda\widehat{\xi}) - I_1(\lambda\widehat{\xi})\Big), \tag{5.19}$$

*and $\lambda$ is determined such that the budget constraint $\mathbb{E}[\xi_{T_1} P^*_{T_1}] = x$ is satisfied.*

Before presenting the proof let us remark that (5.19) defines the linear line that is jointly tangent to the curves of $U_1$ and $U_2$.

***Proof*** For $\widehat{\lambda} > 0$ and $\xi > 0$, consider the following Lagrangian

$$\Psi(x) := U_1(x)\mathbf{1}_{x<B} + U_2(x)\mathbf{1}_{x\geq B} - \lambda\xi x.$$

Note first that $\Psi$ is continuous and $U_i$ attains maximum at $I_i(\lambda\xi)$, $i = 1, 2$. Furthermore, it follows from (5.18) that $I_2(\lambda\xi) > I_1(\lambda\xi)$ for all $\lambda > 0$ and $\xi > 0$. Let $\xi_{B,1} := \frac{U_1'(B)}{\lambda}$. If $\xi \leq \xi_{B,1}$, then $I_1(\lambda\xi) > B$. Hence $\Psi$ is increasing in $[0, I_2(\lambda\xi))$ and decreasing in $[I_2(\lambda\xi), \infty)$. So $I_2(\lambda\xi)$ is the maximizer when $\xi < \xi_{B,1}$. Similarly, for $\xi \geq \xi_{B,2} := \frac{U_2'(B)}{\lambda} > \xi_{B,1}$ we observe that $\Psi$ is increasing in $[0, I_1(\lambda\xi))$ and decreasing in $[I_1(\lambda\xi), \infty)$. So $I_1(\lambda\xi)$ is the maximizer for $\xi \geq \xi_{B,2}$. It remains to consider the case $\xi_{B,1} \leq \xi \leq \xi_{B,2}$. The global optimality of $\Psi$ results from the comparison of $\Psi(I_2(\lambda\xi))$ and $\Psi(I_1(\lambda\xi))$. To this end, consider the continuous function

$$f(\xi) := \Psi(I_2(\lambda\xi)) - \Psi(I_1(\lambda\xi)) = U_2(I_2(\lambda\xi)) - U_1(I_1(\lambda\xi)) - \lambda\xi(I_2(\lambda\xi) - I_1(\lambda\xi)).$$

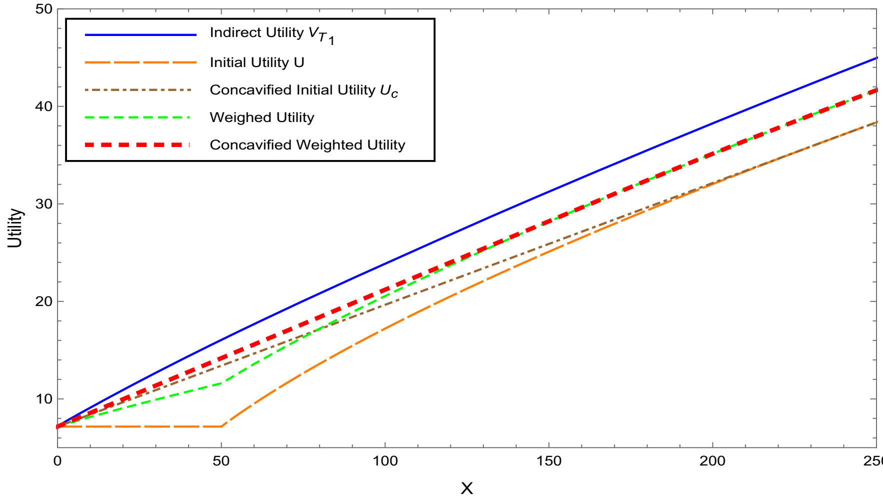



**Fig. 1** Weighted utility at time $T_1$ with $p = 1/2$

Obviously $f'(\xi) = -\lambda(I_2(\lambda\xi) - I_1(\lambda\xi)) < 0$, which implies that $f$ is decreasing in $\xi \in (0, \infty)$. Furthermore, noting that $U_1(B) = U_2(B)$ we obtain

$$f(\xi_{B,1}) = U_2(I_2(\lambda\xi_{B,1})) - U_2(B) - U_2'(I_2(\lambda\xi_{B,1}))(I_2(\lambda\xi_{B,1}) - B) > 0,$$

and

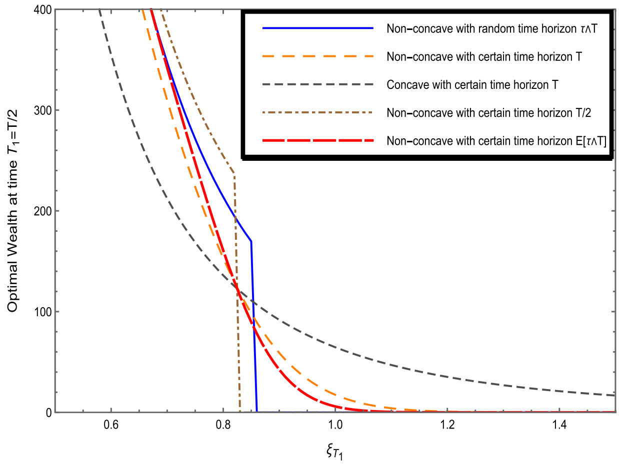
$$f(\xi_{B,2}) = U_1(B) - U_1(I_1(\lambda\xi_{B,2})) - U_1'(I_1(\lambda\xi_{B,2}))(B - I_1(\lambda\xi_{B,1}) < 0,$$

because $U_1$ and $U_2$ are strictly concave. Therefore, there exists $\widehat{\xi} \in [\xi_{B,1}, \xi_{B,2}]$ such that $f(\widehat{\xi}) = 0$ which gives the concavification equation (5.19). Note that $f$ is strictly positive in $[\xi_{B,1}, \widehat{\xi})$ and strictly negative in $(\widehat{\xi}, \xi_{B,2}]$. The global maximizer of $\Psi$ is then given by $I_2(\lambda\xi)$ if $\xi < \widehat{\xi}$ or by $I_1(\lambda\xi)$ if $\xi \geq \widehat{\xi}$. The existence of $\lambda$ is not difficult to see. □

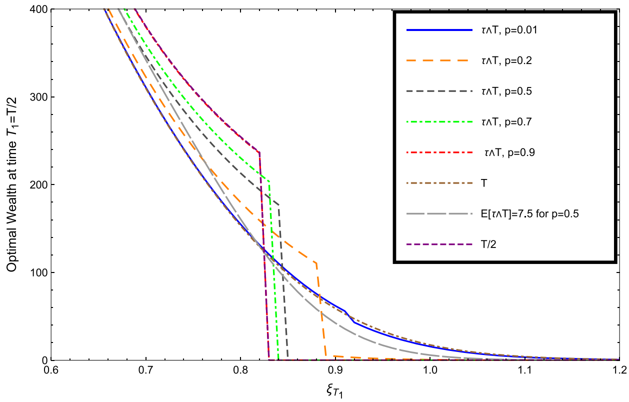
### 5.3 Numerical Illustration

We consider a classical Black-Scholes market with a risky asset $S$ and a bond $B$ and the $0 < T_1 = 5 < T = 10$ such that $\mathbb{P}(\tau = T_1) = \mathbb{P}(\tau = T) = 1/2$. We assume that $\mu = 0.08$ $r = 0.03$, $\sigma = 0.2$, $x_0 = 100$, $K = 10$, $B = 50$, $\gamma = 0.3$, $\alpha = 0.5$. We carry out a recursive procedure to determine the optimal solution for the non-concave problem with random time horizon $T \wedge \tau$.

Our numerical illustration relies on a Monte-Carlo simulation with 50000 paths of the market price density $\xi_{T_1}$ to determine the optimal multiplier $\lambda$ in the first period. This recursive procedure is computationally rather challenging. First, although the indirect value function of the last period can be computed in closed form in (5.9),




(a) $p = 0.5$




(b) Impact of $p$

**Fig. 2** Optimal wealth at $T_1 = T/2$

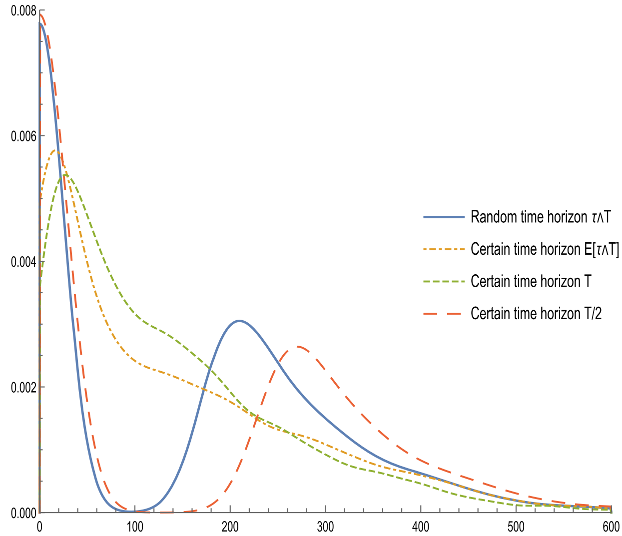
it implicitly depends on the price density $\xi_{T_1}$. Second, computation of the marginal utility functions $I_1$, $I_2$ of the corresponding concavified utilities is computationally intensive as concavification requires a root search step for each value of the market price density $\xi_{T_1}$. This is done using Brent's method with a careful choice of the starting values. We remark that $\xi_t(\omega)$ are the Arrow-Debreux prices of the economy with $\xi_t(\omega)$ corresponding to the value of \$1 per probability unit in the state of $\omega$ paid out at time $t$. Since this value is high during a depression and low in prosperous times, $\xi_{T_1}(\omega)$ may be interpreted as reflecting the overall state of the economy or the stock market. In particular, it is higher in bad market scenarious but lower in good market states than the corresponding wealth of the certain time horizon problem $T$ (resp. $T/2$). Below, we numerically test and confirm the theoretical result established in Sect. 5.2

In order to test the concavity of the weighted utility at time $T_1$, we plot in Fig. 1 the indirect valued function at time $T_1$ defined in (5.9). The graph numerically confirms the result in Proposition 5 that $V_{T_1}$ is strictly concave and dominates the initial utility $U$. In addition, the weighted utility defined by (5.16) is indeed non-concave and its concave hull is dominated by the indirect value function $V_{T_1}$. This implies that having a premature stopping time before $T$ leads to lower expected utility than the solution with certain time horizon $T$. In other words, this numerical example also confirms the result in Proposition 3 that optimizing the concavified version of the utility function will lead to sub-optimality.

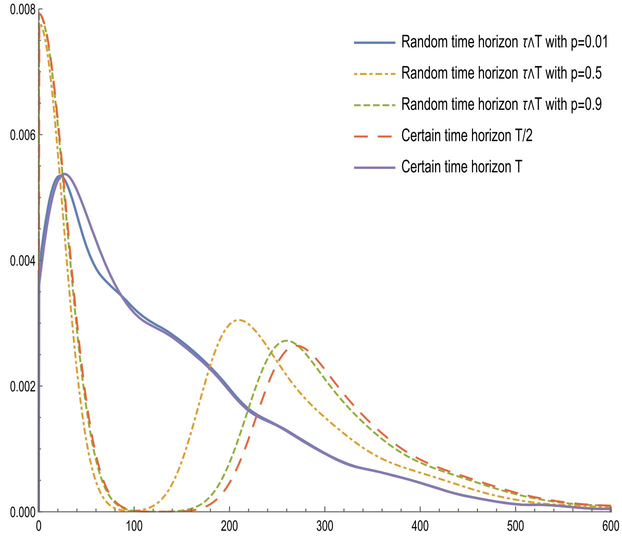
The optimal wealth at time $t$ is plotted in Fig. 2a which exhibits an intermediate investment behavior between the non-concave problems with certain time horizon $T = \max\{T \wedge \tau(\omega)\}$ and $T/2 = \min\{T \wedge \tau(\omega)\}$. In addition, there are ranges of intermediate market states in which the uncertain time wealth can be higher and lower than that of the non-concave problem with (certain) average time horizon $\mathbb{E}[\tau \wedge T] = pT_1+(1-p)T = 7.5$. As confirmed in Fig. 2b, the larger (resp. smaller) the probability of exiting at the smallest time horizon value $T/2$, the riskier (resp. less risky) the investment behavior at time $T/2$. Furthermore, the random horizon problem converges to the extreme cases with certain horizon $T$ and $T/2$ when $p$ approaches to 0 and 1 respectively.

To further understand this effect, we plot in Fig. 3 the estimated density of the optimal wealth at time $T_1$ from 5000 simulations of the market price density $\xi_{T_1}$. It is interesting to observe that the distribution of the wealth at time $T_1$ of the non-concave optimization problems is right-skewed with a long right tail, indicating that the investor expects frequent small losses and a few large gains from the investment. A positively skewed distribution of investment returns is generally desirable by the agent with option-liked compensation payoff. In addition, the premature (before time $T$) exiting risk forces the investor to follow a portfolio that is of right-skewed and bimodal distribution with peaks of different heights. The bimodal structure can be explained by the concavification procedure at $T_1$, whereas the binomial distribution of the exiting time $\tau$ has significant impact on the amplitude between the two modes. The higher the probability $p$, the larger the amplitude. While the (certain) average time horizon portfolio is right-skewed and unimodal, the random time horizon portfolio, due to the option-liked compensation payoff at time $T_1$, is bimodal distributed which provides the investor flexibility of switching between the two local maximizers $I_1$ and $I_2$, depending on the market performance. If the concavified utility at time $T_1$ is affine in many open intervals, the corresponding wealth is intuitively expected to be of multimodal distribution. Again, when $p$ approaches to 0 or 1, the wealth distribution of the random horizon problem converges to the extreme cases with certain horizon $T$ and $T/2$.

Figure 4 presents the estimated density of the optimal stopped wealth $P^*_{T\wedge\tau} = P^*_{T_1}\mathbf{1}_{\tau=T_1} + P^*_T\mathbf{1}_{\tau=T}$ simulated from 10000 scenarios of the price density and the binary random variable $\tau$. Aligned with what is observed in Fig. 3, the random time horizon payoff exhibits a right-skewed and multimodal distribution, confirming again that the optimizing investor under the premature (before time T) exiting risk is forced to accept frequent small losses and a few large gains from the investment. Compared to the non-concave problem with (certain) average horizon $\mathbb{E}[\tau \wedge T] = pT_1+(1-p)T$,




(a) Effect of randomization with $p = 0.5$.




(b) Effect of $p$.

**Fig. 3** Estimated density of the optimal wealth at time $T_1 = T/2 = 5$

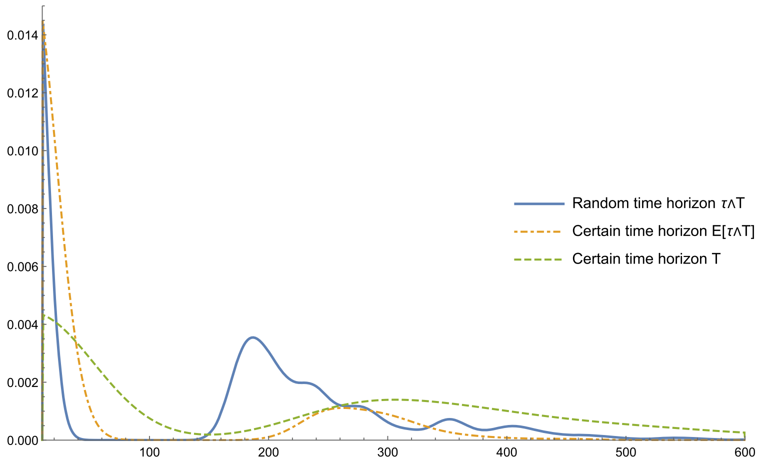


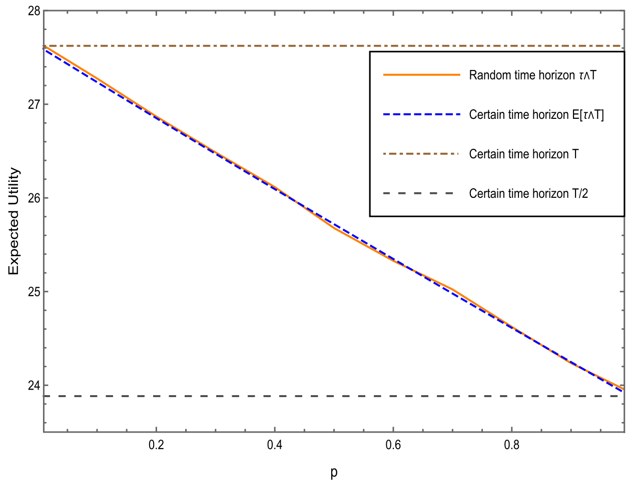

**Fig. 4** Estimated density of $X_{T\wedge\tau}$, the optimal wealth stopped at $\tau$

while less protected in states with large losses (i.e., when the wealth is smaller than 100), the random horizon payoff has a higher potential in intermediate and extreme gain scenarios. However, the early exiting risk makes the random horizon payoff not only less attractive in extreme gain scenarios but also riskier in large loss states than the optimal payoff with certain horizon $T$. Due to the budget constraint, the agent with the random time horizon on the other hand enjoys higher potentials in intermediate scenarios.

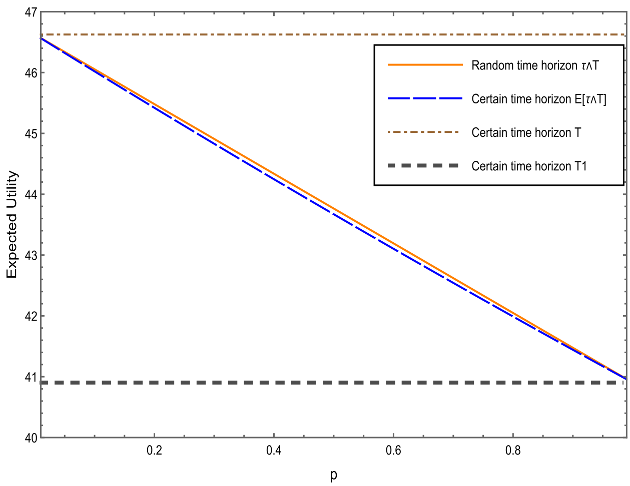
We now turn our attention to the impact of the time horizon uncertainty on the total expected utility. We first remark that in certain time horizon settings, it can numerically be shown that the value function of the concave and the non-concave problems is a convex function in the time horizon variable. Figure 5 reports the impact of exiting probability $p = \mathbb{P}(\tau = T/2)$ on the expected utility of the random time horizon $\tau \wedge T$ and the certain time horizon $\mathbb{E}[\tau \wedge T]$. As shown in the right panel, the expected utility of the random horizon concave problem is always higher than that of the certain horizon problem, which is due to the convexity in time horizon of the value function and the fact that investment strategies for both cases with certain and uncertain horizon time horizon are identical and given by the Merton fraction (see Proposition 4).

The left panel of Fig. 5 reports the expected utility of the non-concave optimization setting. We observe a similar expected utility dominance of the uncertain time horizon problem over the certain time horizon problem when $p$ is close to 0 and 1. However, this effect is hard to see for intermediate values of $p$ for the given parameters. Unlike concave problems, the optimal investment strategy of the non-concave optimization problem significantly depends on the time horizon.

Lastly, let $\nu_{T_i} := \xi_{T_i}^{-1} U'(P^*_{T_i})$, $i = 1, 2$ where $U'$ is the right-hand derivative of $U$. We now want to numerically verify that as shown in Theorem 2, the weighted multiplier $p_1 \nu_{T_1} + p_2 \nu_{T_2}$ is constant (a.s.) on the set $A = \{\omega : \ P^*_{T_i}(\omega) > 0\} = \{\omega : \ P^*_T(\omega) > 0\}$. We remark that $\{P^*_T = 0\}$ is a non-zero set and $U$ is not differentiable at 0. For the given parameters and for 50000 paths on the market price of risk, we obtain that




(a) Non-concave problem




(b) Concave problem

**Fig. 5** Impact of $p = \mathbb{P}(\tau = T_1 = T/2)$ on the expected utility

$p\nu_{T_1} + (1-p)\nu_{T_2} = 0.165183$ is constant on the set $A = \{P_T^* > 0\}$, confirming the result established in Theorem 2. Note that this weighted multiplier coincides with the multiplier of the first period. The result is consistent when different values of $p$ are considered, see Table 1.

**Table 1** Weighted multiplier on the set $A = \{P_T^* > 0\}$

| $p$ | $\lambda$ | $p\nu_{T_1} + (1-p)\nu_{T_2}$ |
|---|---|---|
| 0.1 | 0.17661847317422547 | 0.1766184731742255 |
| 0.2 | 0.17365388191218573 | 0.1736538819121857 |
| 0.3 | 0.170919 | 0.170919 |
| 0.4 | 0.16871475448916323 | 0.16871475448916323 |
| 0.5 | 0.165183 | 0.165183 |
| 0.6 | 0.1627175838160374 | 0.1627175838160374 |
| 0.7 | 0.1598441738970805 | 0.1598441738970805 |
| 0.8 | 0.15719801306953618 | 0.15719801306953618 |
| 0.9 | 0.15428390949982979 | 0.15428390949982979 |

## 6 Optimal Investment with an $\mathcal{F}$-Stopping Time

In this section we study the case where $\bar{\tau}$ is an $\mathcal{F}$- stopping time taking values at $0 < T_1 < \cdots < T_n = T$ or being greater than $T$. In particular the independence assumption on the random maturity is dropped. For simplicity, we consider again in this section the non-concave utility function $U$ defined in (5.2). We remark that the result obtained in this section can be extended to more general utilities. The optimization problem (3.1) becomes

$$V_{\bar{\tau}}(x, U) = \sup_{\pi \in \Pi(0,x)} \mathbb{E}\left[U(P_{T\wedge\bar{\tau}})\right]$$
$$= \sup_{\pi \in \Pi(0,x)} \mathbb{E}\left[\left(\sum_{i=1}^{n-1} U(P_{T_i})\mathbf{1}_{\bar{\tau}=T_i}\right) + U(P_T)\mathbf{1}_{\bar{\tau}\geq T}\right]. \tag{6.1}$$

Recall the generalized inverse marginal utility $I$ defined by (3.4). The function $U^c$ is not differentiable everywhere but the superdifferential $\partial U^c$ may be identified with the set-valued function

$$\partial U^c(x) := \begin{cases} [U'(\hat{x}(B)), \infty) & \text{for } x = 0, \\ \{U'(\hat{x}(B))\} & \text{for } 0 < x \leq \hat{x}(B), \\ \{U'(x)\} & \text{for } x > \hat{x}(B). \end{cases} \tag{6.2}$$

We denote by $\mathcal{X}(x)$ the set all admissible wealth processes $(P_t^{0,x,\pi})_{t\in[0,T]}$ which solve the SDE (2.2) for some $\pi \in \Pi(0, x)$. Note that for any $Y \in \mathcal{X}(x)$ and stopping time $\bar{\tau}$, the stopped supermartingale property implies that $\mathbb{E}[\xi_{\bar{\tau}\wedge T} Y_{\bar{\tau}\wedge T}] \leq x$.

**Proposition 7** *Assume that $\bar{\tau}$ is an $\mathcal{F}$-stopping time taking values at $0 < T_1 < \cdots < T_n = T$ or being larger than $T$. Suppose furthermore that there is an adapted process $\nu \geq 0$ with $\nu_0 = U'(x) > 0$ such that $(\xi_{\bar{\tau}\wedge t} I(\nu_{\bar{\tau}\wedge t}\xi_{\bar{\tau}\wedge t}))_{0\leq t\leq T}$ is a martingale and $\nu_{\bar{\tau}\wedge T} = \left(\sum_{i=1}^{n-1} \nu_{T_i}\mathbf{1}_{\bar{\tau}=T_i}\right) + \nu_T \mathbf{1}_{\bar{\tau}\geq T}$ is a constant. Then, $P_t^* := I(\nu_{\bar{\tau}\wedge t}\xi_{\bar{\tau}\wedge t})$ solves the optimization problem (6.1).*

***Proof*** Let $\left(\sum_{i=1}^{n-1}\nu_{T_i}\mathbf{1}_{\bar{\tau}=T_i}\right)+\nu_T\mathbf{1}_{\bar{\tau}\geq T}=y$ which is a constant by assumption. We first observe that $(\xi_{\bar{\tau}\wedge t}I(\nu_{\bar{\tau}\wedge t}\xi_{\bar{\tau}\wedge t}))_{0\leq t\leq T}$ is a martingale starting with initial value $P_0^*=I(\nu_0)=x$. Using the martingale representation theorem and Itô's formula, we can derive that $P^*$ satisfies the SDE (2.2) for some admissible strategy $\pi$ and hence, $P^*\in\mathcal{X}(x)$. Furthermore, clearly $P^*_{\bar{\tau}\wedge t}=P^*_t$. Now, for any admissible $Y\in\mathcal{X}(x)$ we have

$$\mathbb{E}[\nu_{\bar{\tau}\wedge T}\xi_{\bar{\tau}\wedge T}Y_{\bar{\tau}\wedge T}]=\mathbb{E}\left[\left(\sum_{i=1}^{n-1}\nu_{T_i}\xi_{T_i}Y_{T_i}\mathbb{1}_{\bar{\tau}=T_i}\right)+\nu_T\xi_TY_T\mathbb{1}_{\bar{\tau}\geq T}\right]\leq xy. \quad (6.3)$$

Note that the process $Z_t:=x^{-1}\xi_{\bar{\tau}\wedge t}I(\nu_{\bar{\tau}\wedge t}\xi_{\bar{\tau}\wedge t}),\ 0\leq t\leq T$ defines a density process of a probability measure $\mathbb{Q}^\nu<<\mathbb{P}$ as it is a martingale with initial value equal to 1. Therefore, by Bayes formula and the assumption $\nu_{\bar{\tau}\wedge T}=y$ we obtain

$$\begin{aligned}\mathbb{E}[\nu_{\bar{\tau}\wedge T}\xi_{\bar{\tau}\wedge T}P^*_{\bar{\tau}\wedge T}]&=\mathbb{E}\left[\left(\sum_{i=1}^{n-1}\mathbf{1}_{\bar{\tau}=T_i}\nu_{T_i}\xi_{T_i}I(\nu_{T_i}\xi_{T_i})\right)+\mathbf{1}_{\bar{\tau}\geq T}\nu_T\xi_TI(\nu_T\xi_T)\right]\\&=x\mathbb{E}^{\mathbb{Q}^\nu}\left[\left(\sum_{i=1}^{n-1}\mathbf{1}_{\bar{\tau}=T_i}\nu_{T_i}\right)+\mathbf{1}_{\bar{\tau}\geq T}\nu_T\right]=xy. \qquad (6.4)\end{aligned}$$

Now, for any admissible $Y$ we have by (6.4) that

$$\begin{aligned}&\mathbb{E}\left[\left(\sum_{i=1}^{n-1}\mathbf{1}_{\bar{\tau}=T_i}U(I(\nu_{T_i}\xi_{T_i}))\right)+\mathbf{1}_{\bar{\tau}\geq T}U(I(\nu_T\xi_T))\right]\\&=\mathbb{E}\left[\left(\sum_{i=1}^{n-1}\mathbf{1}_{\bar{\tau}=T_i}U(I(\nu_{T_i}\xi_{T_i}))\right)+\mathbf{1}_{\bar{\tau}\geq T}U(I(\nu_T\xi_T))\right]-x\mathbb{E}^{\mathbb{Q}^\nu}[(\sum_{i=1}^{n-1}\mathbf{1}_{\bar{\tau}=T_i}\nu_{T_i})+\mathbf{1}_{\bar{\tau}\geq T}\nu_T]+xy\\&=\mathbb{E}\left[\left\{\sum_{i=1}^{n-1}\mathbf{1}_{\bar{\tau}=T_i}\left(U(I(\nu_{T_i}\xi_{T_i}))-\nu_{T_i}\xi_{T_i}I(\nu_{T_i}\xi_{T_i})\right)\right\}+\mathbf{1}_{\bar{\tau}\geq T}(U(I(\nu_T\xi_T))-\nu_T\xi_TI(\nu_T\xi_T))\right]+xy\\&=\mathbb{E}\left[\left(\sum_{i=1}^{n-1}\mathbf{1}_{\bar{\tau}=T_i}\sup_{X\geq 0}\left(U(X)-\nu_{T_i}\xi_{T_i}X\right)\right)+\mathbf{1}_{\bar{\tau}\geq T}\sup_{X\geq 0}\left(U(X)-\nu_T\xi_TX\right)\right]+xy\\&\geq\mathbb{E}\left[\left(\sum_{i=1}^{n-1}\mathbf{1}_{\bar{\tau}=T_i}\left(U(Y_{T_i})-\nu_{T_i}\xi_{T_i}Y_{T_i}\right)\right)+\mathbf{1}_{\bar{\tau}\geq T}\left(U(Y_T)-\nu_T\xi_TY_T\right)\right]+xy\\&\geq\mathbb{E}\left[\left(\sum_{i=1}^{n-1}\mathbf{1}_{\bar{\tau}=T_i}U(Y_{T_i})\right)+\mathbf{1}_{\bar{\tau}\geq T}U(Y_T)\right],\end{aligned}$$

where we have used (6.3) in the last step. This implies the optimality of the process $P^*$. □

We now aim to solve the non-concave optimization problem when $\bar{\tau}$ is an $\mathcal{F}$-stopping time, namely

$$\sup_{\pi\in\Pi(0,x)} \mathbb{E}\left[U(P_{T\wedge\bar{\tau}})\right] = \sup_{\pi\in\Pi(0,x)} \mathbb{E}\left[\left(\sum_{i=1}^{n-1} U(P_{T_i})\mathbf{1}_{\bar{\tau}=T_i}\right) + U(P_T)\mathbf{1}_{\bar{\tau}\geq T}\right], \quad (6.5)$$

where $U$ is the non-concave utility function defined by (5.2). In particular, by applying Proposition 7 we prove below that Problem (6.5) can be solved by concavification arguments and the optimal wealth process can be characterized by the process $I(\nu_{\bar{\tau}\wedge t}\xi_{\bar{\tau}\wedge t})$, where $I$ is the generalized inverse marginal utility function defined by (5.5) and $\nu$ is an adapted process. We need the following integrability condition.

**Condition (C)**: For any $y > 0$, $\mathbb{E}\left[\xi_{\bar{\tau}\wedge T} I(y\xi_{\bar{\tau}\wedge T})\right] < \infty$.

Below we show that under condition (C) and the assumption that the stopping time is adapted to the financial market filtration, it is possible to construct an adapted process $\nu$ such that the process process $(\xi_{\bar{\tau}\wedge t} I(\nu_{\bar{\tau}\wedge t}\xi_{\bar{\tau}\wedge t}))_{0\leq t\leq T}$ is a martingale and $\nu_{\bar{\tau}\wedge T} = \left(\sum_{i=1}^{n-1} \mathbf{1}_{\bar{\tau}=T_i}\nu_{T_i}\right) + \mathbf{1}_{\bar{\tau}\geq T}\nu_T$ is a constant. The result is summarized in the following proposition.

**Proposition 8** (Non-concave problem with a stopping time horizon) *Assume that $\bar{\tau}$ is an $\mathcal{F}$-stopping time taking values at $0 < T_1 < \cdots < T_n = T$ or being larger than $T$, and Condition (C) holds. Then, there exists an $\mathcal{F}$-adapted process $\nu$ such that the optimal wealth of Problem (6.5) is given by $P_t^* := I(\nu_{\bar{\tau}\wedge t}\xi_{\bar{\tau}\wedge t})$, $0 \leq t \leq T$ and $\left(\sum_{i=1}^{n-1} \upsilon_T \mathbf{1}_{\bar{\tau}=T_i}\right) + \upsilon_T \mathbf{1}_{\bar{\tau}\geq T} = y^*$ is a constant satisfying $\mathbb{E}\left[\xi_{\bar{\tau}\wedge T} I(y^*\xi_{\bar{\tau}\wedge T})\right] = x$.*

***Proof*** Consider the mapping $y \longmapsto \mathbb{E}\left[\xi_{\bar{\tau}\wedge T} I(y\xi_{\bar{\tau}\wedge T})\right] = f(y)$ defined for $y \in (0,\infty)$ by Condition (C). Since the market price density $\xi$ is atomless, $f$ is continuous on $(0,\infty)$. Moreover, by Fatou's lemma, (5.5) and Inada's condition of the power utility function $U$ we obtain $\lim_{y\to 0} f(y) = \infty$ and $\lim_{y\to\infty} f(y) = 0$. Therefore, there exists $y^* \in (0,\infty)$ such that $\mathbb{E}\left[\xi_{\bar{\tau}\wedge T} I(y^*\xi_{\bar{\tau}\wedge T})\right] = f(y^*) = x$. Define for $0 \leq t \leq T$, $\zeta_t := I(y^*\xi_t)$ and

$$\nu_t \in \frac{1}{\xi_t}\partial U^c\left(\mathbb{E}\left[\xi_{\bar{\tau}\wedge t}^{-1}\xi_{\bar{\tau}\wedge T}\zeta_{\bar{\tau}\wedge T}|\mathcal{F}_{\bar{\tau}\wedge t}\right]\right),$$

where the superdifferential is defined by (6.2). Note that since the conditional expectation process $\mathbb{E}\left[\xi_{\bar{\tau}\wedge t}^{-1}\xi_{\bar{\tau}\wedge T}\zeta_{\bar{\tau}\wedge T}|\mathcal{F}_{\bar{\tau}\wedge t}\right] = \mathbb{E}\left[\xi_{\bar{\tau}\wedge t}^{-1}\xi_{\bar{\tau}\wedge T}\zeta_{\bar{\tau}\wedge T}|\xi_{\bar{\tau}\wedge t}\right] > 0$ is a non-negative martingale with initial value $x > 0$, almost surely $\partial U^c$ above corresponds to $U'$ and is invertible. Thus, by construction the process $\xi_{\bar{\tau}\wedge t} I(\nu_{\bar{\tau}\wedge t}\xi_{\bar{\tau}\wedge t}) = \mathbb{E}\left[\xi_{\bar{\tau}\wedge T} I(\nu_{\bar{\tau}\wedge T}\xi_{\bar{\tau}\wedge T})|\mathcal{F}_{\bar{\tau}\wedge t}\right]$ is a martingale with $\nu_0 = \partial^c U(x) = U'(x) > 0$ (see (6.2)) and $y^* = \nu_{\bar{\tau}\wedge T} = \left(\sum_{i=1}^{n-1} \upsilon_{T_i}\mathbf{1}_{\bar{\tau}=T_i}\right) + \upsilon_T \mathbf{1}_{\bar{\tau}\geq T}$ is a constant. Hence, by Proposition 7, $P_t^* = I(\nu_{\bar{\tau}\wedge t}\xi_{\bar{\tau}\wedge t})$ is an optimal solution to (6.5). □

The following is aligned with Proposition 3.3 in [11] when $\bar{\tau}$ is an $\mathcal{F}$-stopping time for strictly concave utility function $U$.

**Corollary 1** (Concave problem with a stopping time horizon) *Assume that $U$ is a strictly concave utility function for which Condition (C) holds and $\bar{\tau}$ is an $\mathcal{F}$-stopping time taking values at $0 < T_1 < \cdots < T_n = T$ or being larger than $T$. For any $x > 0$, there exists $y^* > 0$ such that $\mathbb{E}\left[\xi_{\bar{\tau}\wedge T} I(y^* \xi_{\bar{\tau}\wedge T})\right] = x$. Moreover, there exists an adapted process $\nu$ such that the optimal wealth of Problem (6.5) is given by $P_t^* := I(\nu_{\bar{\tau}\wedge t}\xi_{\bar{\tau}\wedge t}), \quad 0 \le t \le T$, and $\nu_{\bar{\tau}\wedge T} = \left(\sum_{i=1}^{n-1} \nu_{T_i} \mathbf{1}_{\bar{\tau}=T_i}\right) + \nu_T \mathbf{1}_{\bar{\tau}\ge T} = y^*$ is a constant.*

## 7 Conclusion

We studied a non-concave optimal investment with a random time horizon in a complete financial market setting. We established a necessary and sufficient condition for the optimality in this case for general utility functions with a random time horizon. When $\tau$ is independent of the financial risk, we showed that a direct concavification approach cannot be applied and suggest a recursive procedure based on the dynamic programming principle. We illustrated our finding by carrying out a multiple period numerical analysis for the non-concave option compensation problem with random time horizon. We numerically show that due to concavification, the distribution of the wealth at exiting times of the non-concave optimization problems is right-skewed with a long right tail, indicating that the investor can expect frequent small losses and a few large gains from the investment. Under the premature exiting risk, the wealth at an exiting time exhibits a bimodal distribution with peaks of different heights due to the concavification procedure and whereas the exiting time $\tau$ distribution has significant impact on the amplitude between the two modes.

Our work leaves several interesting directions for future work. For instance, it would be interesting to look at the case when the time horizon is correlated with the financial market information, or to investigate the problem in a general incomplete financial market as in [11]. Furthermore, our non-concave framework with random horizon might serve as an attempt to extend the results for contract design problems of term-life insurance or insurance contracts with surplus participation [14, 22] to an uncertain time horizon setting. We leave this for future work.

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## Declarations

**Conflict of interest** The authors have not disclosed any competing interests.

## Appendix

The following result can be shown directly using the lognormal distribution of $\xi$:

**Lemma 8.1** *Let $q \in \mathbb{R}$. With $f$ defined in Proposition 4 it holds for $0 \leq t \leq T$ that*

$$\mathbb{E}\left[\xi_T^q | \mathcal{F}_t\right] = \mathbb{E}\left[\left(\frac{\xi_T}{\xi_t}\right)^q \middle| \mathcal{F}_t\right] \xi_t^q = f(q, t, T)\, \xi_t^q. \tag{8.1}$$

The next result provides a generalization of Lemma 8.1 when the market parameters are constant.

**Lemma 8.2** *Let $q \in \mathbb{R}$, $0 \leq t < T$ and let $\lambda$ be a positive constant. With $\Phi$ the cdf of the standard normal distribution and $d$ defined in (5.8) it holds that*

$$\mathbb{E}\left[\xi_T^q \mathbf{1}_{\lambda \xi_T \leq U'(\hat{x}(B))} | \mathcal{F}_t\right] = \xi_t^q f(q, t, T) \Phi\left(d(q, t, T, \lambda \xi_t)\right). \tag{8.2}$$

**Lemma 8.3** *Let $U$, $V$ be continuous, increasing functions in $[0, \infty)$. Let $(a, b) \subset \{U < U^c\}$ be an open interval in the concavification region of $U$. Assume that there exists $x_0 \in [a, b)$ at which $U + V$ coincides with the affine line*

$$g(x) := U(a) + V(a) + \frac{(U(b) + V(b)) - (U(a) + V(a))}{b - a}(x - a)$$

*and the right derivative of the sum $U + V$ exists and*

$$U'(x_0^+) + V'(x_0^+) > \frac{(U(b) + V(b)) - (U(a) + V(b))}{b - a}. \tag{8.3}$$

*Then, the interval $(a, b)$ cannot be a concavification set of the sum $U + V$, i.e. there exists an open interval $(a', b') \subset (a, b)$ such that $U(x) + V(x) = (U + V)^c(x)$ for all $x \in (a', b')$.*

***Proof*** We have $U(x_0) + V(x_0) = g(x_0)$. By continuity and (8.3) it can be seen that $U + V > g(x)$ in a right-hand neighbourhood of $x_0$, which implies that the affine line $g$ is not the concave hull of $U + V$ on the whole interval $(a, b)$. □

## References


1. Aksamit, A., Jeanblanc, M.: Enlargement of Filtration with Finance in View. Springer, Switzerland (2017)
2. Aumann, R.J., Perles, M.: A variational problem arising in economics. J. Math. Anal. Appl. **11**, 488–503 (1965)
3. Basak, S., Makarov, D.: Strategic asset allocation in money management. J. Financ. **69**(1), 179–217 (2014)
4. Bensoussan, A.C., Koo, H.K.: Entrepreneurial decisions on effort and project with a nonconcave objective function. Math. Oper. Res. **40**(4), 902–914 (2015)
5. Biagini, S.: Expected utility maximization: duality methods. Encycl. Quant. Financ. (2010). https://doi.org/10.1002/9780470061602.eqf04009
6. Bian, B., Miao, S., Zheng, H.: Smooth value functions for a class of nonsmooth utility maximization problems. SIAM J. Financ. Math. **2**(1), 727–747 (2011)
7. Bian, B., Chen, X., Xu, Z.Q.: Utility maximization under trading constraints with discontinuous utility. SIAM J. Financ. Math. **10**(1), 243–260 (2019)
8. Bichuch, M., Sturm, S.: Portfolio optimization under convex incentive schemes. Financ. Stochast. **18**(4), 873–915 (2014)
9. Blanchet-Scalliet, C., El Karoui, N., Martellini, L.: Dynamic asset pricing theory with uncertain time-horizon. J. Econ. Dyn. Control **29**(10), 1737–1764 (2005)
10. Blanchet-Scalliet, C., El Karoui, N., Jeanblanc, M., Martellini, L.: Optimal investment decisions when time-horizon is uncertain. J. Math. Econ. **44**(11), 1100–1113 (2008)
11. Bouchard, B., Pham, H.: Wealth-path dependent utility maximization in incomplete markets. Financ. Stochast. **8**(4), 579–603 (2004)
12. Carassus, L., Pham, H.: Portfolio optimization for piecewise concave criteria functions. In: The 8th Workshop on Stochastic Numerics (2009)
13. Carpenter, J.N.: Does option compensation increase managerial risk appetite? J. Financ. **55**(5), 2311–2331 (2000)
14. Chen, A., Hieber, P., Nguyen, T.: Constrained non-concave utility maximization: an application to life insurance contracts with guarantees. Eur. J. Oper. Res. **273**(3), 1119–1135 (2019)
15. Dai, M., Kou, S., Qian, S., Wan, X.: Nonconcave utility maximization without the concavification principle (2019). https://ssrn.com/abstract=3422276
16. Dehm, C., Nguyen, N., Stadje, M.: Non-concave expected utility optimization with uncertain time horizon (2021). https://arxiv.org/pdf/2005.13831.pdf
17. Föllmer, H., Schied, A.: Stochastic finance. In: Stochastic Finance. de Gruyter, Berlin (2016)
18. Hewitt, E., Stromberg, K.: Real and Abstract Analysis: A Modern Treatment of the Theory of Functions of a Real Variable. Springer, New York (2013)
19. Karatzas, I., Lehoczky, J.P., Shreve, S.E., Xu, G.L.: Martingale and duality methods for utility maximization in an incomplete market. SIAM J. Control. Optim. **29**(3), 702–730 (1991)
20. Larsen, K.: Optimal portfolio delegation when parties have different coefficients of risk aversion. Quant. Financ. **5**(5), 503–512 (2005)
21. Merton, R.C.: Optimal consumption and portfolio rules in continuous time. J. Econ. Theory **3**, 373–413 (1971)
22. Nguyen, T., Stadje, M.: Nonconcave optimal investment with value-at-risk constraint: an application to life insurance contracts. SIAM J. Control. Optim. **58**(2), 895–936 (2020)
23. Pham, H.: Stochastic control under progressive enlargement of filtrations and applications to multiple defaults risk management. Stoch. Process. Appl. **120**(9), 1795–1820 (2010)
24. Reichlin, C.: Utility maximization with a given pricing measure when the utility is not necessarily concave. Math. Financ. Econ. **7**(4), 531–556 (2013)
25. Richard, S.F.: Optimal consumption, portfolio and life insurance rules for an uncertain lived individual in a continuous time model. J. Financ. Econ. **2**(2), 187–203 (1975)
26. Rieger, M.O.: Optimal financial investments for non-concave utility functions. Econ. Lett. **114**(3), 239–240 (2012)
27. Ross, S.A.: Compensation, incentives, and the duality of risk aversion and riskiness. J. Financ. **59**(1), 207–225 (2004)
28. Wong, K.C., Yam, S.C.P., Zheng, H.: Utility-deviation-risk portfolio selection. SIAM J. Control. Optim. **55**(3), 1819–1861 (2017)

29. Yaari, M.E.: Uncertain lifetime, life insurance, and the theory of the consumer. Rev. Econ. Stud. **32**(2), 137–150 (1965)