Applied Mathematics & Optimization (2023) 88:35
https://doi.org/10.1007/s00245-023-10010-8

Multidomain Optimal Control of Variational Subpotential Mixed Evolution Inclusions

Gonzalo Alduncin[1]

Accepted: 30 April 2023 / Published online: 31 May 2023

Abstract
Multidomain variational optimal control of evolution mixed subpotential inclusions, are formulated and analyzed, on the basis of a perturbation conjugate duality convex analysis theory developed by the author. For Lagrangian optimality mixed conditions, fixed point existence results are demonstrated with an strongly monotone qualifying condition. Governing multidomain state systems correspond to primal evolution macro-hybrid mixed subpotential problems, whose solvability is similarly achieved. Innovative multidomain optimization existence results of primal, dual, Lagrangian mixed, as well as coupled pair state-control problems are established. Applications to underground macro-hybrid mixed control transport flow processes, illustrate the theory.

Keywords Multidomain variational evolution optimal control · Perturbation conjugate duality method · Set-valued variational analysis · Evolution macro-hybrid mixed subpotential system · Transmission dual Lagrange multiplier · Underground control transport flow processes

Mathematics Subject Classification 35K90 · 49J40 · 58E30

1 Introduction

Multidomain optimal control of primal evolution mixed variational subpotential inclusions, in a macro-hybrid mixed evolution real functional framework of primal and dual product reflexive Banach spaces, $\boldsymbol{\mathcal{V}}_{MH} \equiv \prod_{e=1}^{E} V(\Omega_e)$ and $\boldsymbol{\mathcal{Y}}^*_{MH} \equiv \prod_{e=1}^{E} Y^*(\Omega_e)$, whith topological duals $\boldsymbol{\mathcal{V}}^*_{MH} \equiv \prod_{e=1}^{E} V^*(\Omega_e)$ and $\boldsymbol{\mathcal{Y}}_{MH} \equiv \prod_{e=1}^{E} Y(\Omega_e)$, are formulated and analyzed. All of this in relation with a spatial bounded domain $\Omega \subset \Re^d$,

✉ Gonzalo Alduncin
alduncin@geofisica.unam.mx

[1] Instituto de Geofísica, Universidad Nacional Autónoma de México, Ciudad de México, Coyoacán 04510, Mexico

$d \in \{1, 2, 3\}$, decomposed in terms of disjoint and connected subdomains $\{\Omega_e\}$, $\overline{\Omega} = \bigcup_{e=1}^{E} \overline{\Omega}_e$, with Lipschitz external boundaries $\Sigma_e = \partial\Omega_e \cap \partial\Omega$ and internal boundaries $\Gamma_e = \partial\Omega_e \cap \Omega$, $e = 1, 2, \ldots, E$, with interfaces $\Gamma_{ek} = \Gamma_e \cap \Gamma_k$, $1 \leq e < k \leq E$; and further with a corresponding reflexive Banach internal boundary trace space $\boldsymbol{B}_\Gamma \equiv \prod_{e=1}^{E} B(\Gamma_e)$ with topological dual $\boldsymbol{B}_\Gamma^* \equiv \prod_{e=1}^{E} B^*(\Gamma_e)$.

As a macro-hybrid mixed state system, in accordance with such a multidomain mixed functional frameworks, with respective primal and dual solution spaces, $\boldsymbol{\mathcal{W}}_{MH} = \{\boldsymbol{v} \in \boldsymbol{\mathcal{V}}_{MH} : d\boldsymbol{v}/dt \in \boldsymbol{\mathcal{V}}_{MH}^*\}$ and $\boldsymbol{\mathcal{Y}}_{MH}^* \times \boldsymbol{\mathcal{B}}_\Gamma^*$, we shall consider a primal evolution inclusion problem,

$$(\boldsymbol{\mathcal{MH}}_\kappa) \begin{cases} \text{Given } \boldsymbol{f}^* \in \boldsymbol{\mathcal{R}}\big(-\boldsymbol{\Lambda}^T\big) \subset \boldsymbol{\mathcal{V}}_{MH}^*,\ (\boldsymbol{g}, \boldsymbol{h}) \in \boldsymbol{\mathcal{R}}(\boldsymbol{\Lambda}) \times \boldsymbol{\mathcal{R}}\big(\boldsymbol{\pi}_\Gamma\big) \\ \subset \boldsymbol{\mathcal{Y}}_{MH} \times \boldsymbol{\mathcal{B}}_\Gamma,\ \text{and } \boldsymbol{v}_{0_\kappa} \in \boldsymbol{H}_{MH}\ , \\ \text{find } \boldsymbol{v}_\kappa \in \boldsymbol{\mathcal{W}}_{MH},\ (\boldsymbol{y}_\kappa^*, \boldsymbol{\lambda}_\kappa^*) \in \boldsymbol{\mathcal{Y}}_{MH}^* \times \boldsymbol{\mathcal{B}}_\Gamma^* : \\ -\boldsymbol{\Lambda}^T \boldsymbol{y}_\kappa^* - \boldsymbol{\pi}_\Gamma^T \boldsymbol{\lambda}_\kappa^* \in \dfrac{d\boldsymbol{v}_\kappa}{dt} + \boldsymbol{\partial} \widetilde{\boldsymbol{F}} \boldsymbol{v}_\kappa + \boldsymbol{B}^* \boldsymbol{\kappa} - \widetilde{\boldsymbol{f}^*}\ , \qquad \text{in } \boldsymbol{\mathcal{V}}_{MH}^*\ , \\ \boldsymbol{v}_\kappa(0) = \boldsymbol{v}_{0_\kappa}\ , \\ \big(\boldsymbol{\Lambda}, \boldsymbol{\pi}_\Gamma\big)(\boldsymbol{v}_\kappa, \boldsymbol{v}_\kappa) \in \big(\boldsymbol{\partial G}^*, \boldsymbol{\partial I}_{Q^*}\big)\big(\boldsymbol{y}_\kappa^*, \boldsymbol{\lambda}_\kappa^*\big) + (\boldsymbol{g}, \boldsymbol{h}),\ \text{in } \boldsymbol{\mathcal{Y}}_{MH} \times \boldsymbol{\mathcal{B}}_\Gamma\ , \end{cases}$$

governing the optimal control minimization problem of the theory, stated as follows:

$$(\boldsymbol{\mathcal{O}}_{MH}) \begin{cases} \text{Find } \boldsymbol{\kappa} \in \boldsymbol{\mathcal{C}}_{MH_{ad}} \subset \boldsymbol{\mathcal{C}}_{MH} : \\ J_{MH}\big(\boldsymbol{v}_\kappa, (\boldsymbol{y}_\kappa^*, \boldsymbol{\lambda}_\kappa^*), \boldsymbol{\kappa}\big) \\ \quad \leq J_{MH}\big(\boldsymbol{v}_{\boldsymbol{\kappa}}, (\boldsymbol{y}_\kappa^*, \boldsymbol{\lambda}_\kappa^*), \boldsymbol{\eta}\big), \quad \forall \boldsymbol{\eta} \in \boldsymbol{\mathcal{C}}_{\boldsymbol{MH}}, \end{cases}$$

related to $\boldsymbol{\mathcal{C}}_{MH} = \boldsymbol{L}^2\big(0, T; \boldsymbol{U}_{MH}\big)$, an evolution macro-hybrid Hilbert space of $\boldsymbol{U}_{MH} \equiv \prod_{e=1}^{E} U(\Omega_e)$-controls, and constrained in the sense of a given nonempty closed convex subset of admissible controls $\boldsymbol{\mathcal{C}}_{MH_{ad}}$.

Then for corresponding $\boldsymbol{\kappa}$-optimal macro-hybrid mixed states $\big(\boldsymbol{v}_\kappa, (\boldsymbol{y}_\kappa^*, \boldsymbol{\lambda}_\kappa^*)\big) \in \boldsymbol{\mathcal{W}}_{MH} \times \big(\boldsymbol{\mathcal{Y}}_{MH}^* \times \boldsymbol{\mathcal{B}}_\Gamma^*\big)$, the cost or objective functional $J_{MH} : \boldsymbol{\mathcal{W}}_{MH} \times \big(\boldsymbol{\mathcal{Y}}_{MH}^* \times \boldsymbol{\mathcal{B}}_{MH}^*\big) \times \boldsymbol{\mathcal{C}}_{MH} \to \Re \cup \{+\infty\}$, will be of a macro-hybrid mixed general form,

$$J_{MH}\left(\boldsymbol{w}_\kappa, \left(\boldsymbol{x}_\kappa^*, \boldsymbol{\chi}_\kappa^*\right), \boldsymbol{\eta}\right) = \int_0^T g_1(\boldsymbol{w}_\kappa)dt + \int_0^T (g_2, g_3)\left(\boldsymbol{x}_\kappa^*, \boldsymbol{\chi}_\kappa^*\right)dt + \int_0^T j(\boldsymbol{\eta})dt, \tag{1}$$

assumed to be a lower semicontinuous convex functional, whose integrand real functional components $g_1 : \boldsymbol{\mathcal{W}}_{MH} \to \Re$, $(g_2, g_3) : \big(\boldsymbol{\mathcal{Y}}_{MH}^* \times \boldsymbol{\mathcal{B}}_\Gamma^*\big) \to \Re$ and $j : \boldsymbol{\mathcal{C}}_{MH} \to \Re$, are dictated by proposed technological design-profiles, with appropriate multidomain variational properties.

Optimal control minimization problem $(\boldsymbol{\mathcal{O}}_{MH})$ governed by primal evolution mixed multisystem $(\boldsymbol{\mathcal{MH}}_\kappa)$, will be formulated and analyzed, on the basis of a perturbation conjugate duality convex analysis theory developed by the author [1], – in the spirit of Azé-Bolintinéanu's study on constrained convex parabolic control problems [2].

The governing state system $\left(\mathcal{MH}_\kappa\right)$ will correspond to a macro-hybrid primal evolution mixed subpotential problem, whose existence result will be demonstrated as a resolvent primal fixed point characterization consequence, for regular primal initial conditions, with a primal operator strong monotonicity qualifying property [3]. On the other hand, a Lagrangian optimality mixed condition will be established, analyzing its macro-hybrid mixed solvability via a resolvent fixed point characterization too, with an objective subdifferential strong monotonicity $\boldsymbol{\partial J}_{MH}$-qualifying condition.

As an innovative perturbation conjugate duality accomplishment of the present study, optimization multidomain macro-hybrid mixed existence results of primal, dual, Lagrangian mixed, as well as coupled pair state-control problems are established.

For some own applications of the original theory [1], to primal and dual variational evolution mechanical systems, we refer to [4–8]. Specifically, in paper [4] optimal control of nonlinear transport-flow mixed variational problems are studied; work [5] has to do with optimal control problems governed by primal and dual evolution macro-hybrid mixed variational state inclusions, in reflexive Banach spaces, with applications to nonlinear constrained mechanical problems; optimal control of quasistatic elasto-viscoplastic macro-hybrid mixed set-valued variational problems, are studied in [6], applying macro-hybrid variational formulations, as an strategy for multi-physics as well as parallel computing; paper [7] is concerned about optimality conditions of stationary macro-hybrid mixed variational inclusions, governing constrained optimal control problems, with applications to dual mechanical nonlinear distributed control steady diffusion processes, as well as to primal boundary static deformation systems. Lastly, macro-hybrid optimal control of dual evolution mixed variational transport flow processes, through elastoviscoplastic porous media, are studied in [8].

On the other hand, in contrast to our subpotential perturbation conjugate duality, operator variational approach, some optimal control studies based on diverse kinds of variational inequalities to be mentioned are the following [9–14]. In paper [9] existence and optimality conditions of optimal pairs are established for semilinear evolutionary variational inequality problems, with bilateral constraints; evolution implicit quasi-variational inequalities of optimal control problems are analyzed in work [10], for quasi-static processes of elastic contact problems with friction between a body and a rigid foundation; paper [11] is concerned with the optimal control of systems modeled by differential inclusions, with anti-periodic conditions in Banach spaces, analyzing evolutionary hemi-variational inequalities for trajectory-control pair solutions; a class of subdifferential evolution inclusions are considered in study [12], with history-dependent operators, establishing the existence of an optimal control for a dynamic frictional contact mechanical problem. Lastly, in paper [13] a result of optimal control to a minimization problem, applicable to problems of evolution differential inclusions is established, on the basis of quasi mixed equilibrium problems with a compact constraint set, bounded and unbounded; and in work [14] an optimal control problem for a differential quasivariational inequality is analyzed, applying the abstract results to a free boundary problem, for a viscoelastic body in a frictionless unilateral contact with a rigid foundation.

The present paper is organized as follows: In Sect. 2, the multidomain evolution mixed real functional framework, $\mathcal{W}_{MH} \times \mathcal{Y}^*_{MH} \times \mathcal{B}^*_\Gamma$, is introduced for the variational analysis; the primal evolution macro-hybrid mixed state system $\left(\mathcal{MH}_\kappa\right)$, governing optimal control problem $\left(\mathcal{O}_{MH}\right)$, is presented in Sect. 3, demonstrating its fixed point solvability result with a $\boldsymbol{\partial}\widetilde{\boldsymbol{F}}$-strong monotonicity qualifying condition; Sect. 4 has to do with the variational implementation of the multidomain transport flow mechanical underground processes, with intrinsic control constraints, that illustrate the theory of evolution macro-hybrid mixed state systems. Next Sect. 5 elaborates on the perturbation conjugate duality, multidomain evolution mixed optimal control theory, establishing the corresponding variational optimality mixed Lagrangian system, as a new optimization result; then in Sects. 5.1 and 5.2, the multidomain optimal control of the mechanical transport and flow state systems (of Sect. 4) are presented; in Sect. 6 the solvability analysis of the variational optimality macro-hybrid mixed Lagrangian system is proved, via a fixed point resolvent primal characterization; followed by some corresponding macro-hybrid mixed proximal penalty-duality algorithms. Then last Sect. 7 proceeds with the complementary innovating multidomain optimization analysis, to determine the primal, dual and Lagrangian mixed results, which provide the basis for concluding the central optimization result of the study: the solution existence of the own optimal control system $\left(\mathcal{O}_{MH}\right)$, as well as the corresponding variational solvability of the coupled macro-hybrid mixed pair state-control system $\left(\mathcal{MH}_{\kappa}\right)$-$\left(\mathcal{O}_{MH}\right)$. Section 8 states the conclusions of the paper at the end.

2 Multidomain Variational Evolution Functional Frameworks

For the multidomain evolution mixed functional setting of the analysis, we introduce a stationary mixed state real functional framework of primal and dual reflexive Banach spaces, $V(\Omega)$ and $Y^*(\Omega)$, with topological duals $V^*(\Omega)$ and $Y(\Omega)$, related to a spatial bounded domain $\Omega \subset \Re^d$, $d \in \{1, 2, 3\}$, with a Lipschitz boundary $\partial\Omega$. As corresponding Hilbert pivot spaces we shall have $H(\Omega)$ and $Z^*(\Omega)$, such that $V(\Omega) \subset H(\Omega) \subset V^*(\Omega)$ and $Y^*(\Omega) \subset Z^*(\Omega) \subset Y(\Omega)$, with continuous and dense embeddings. On the other hand, associated boundary primal and dual reflexive Banach trace spaces will be $B(\partial\Omega)$ and its topological dual $B^*(\partial\Omega)$.

For a corresponding multidomain macro-hybrid mixed functional framework, we shall consider that the spatial domain Ω is decomposed in terms of disjoint and connected subdomains $\{\Omega_e\}$, $\overline{\Omega} = \bigcup_{e=1}^{E} \overline{\Omega}_e$, with Lipschitz external and internal boundaries $\Sigma_e = \partial\Omega_e \cap \partial\Omega$ and $\Gamma_e = \partial\Omega_e \cap \Omega$, $e = 1, 2, \ldots, E$, and interfaces $\Gamma_{ek} = \Gamma_e \cap \Gamma_k$, $1 \le e < k \le E$.

Following our studies [15–17], we define primal and dual macro-hybrid local functional product spaces $\boldsymbol{V}_{MH} = \prod_{e=1}^{E} V(\Omega_e)$ and $\boldsymbol{Y}^*_{MH} = \prod_{e=1}^{E} Y^*(\Omega_e)$, with duals $\boldsymbol{V}^*_{MH} = \prod_{e=1}^{E} V^*(\Omega_e)$ and $\boldsymbol{Y}_{MH} = \prod_{e=1}^{E} Y(\Omega_e)$, as well as internal boundary functional product spaces $\boldsymbol{B}_{\Gamma} \equiv \prod_{e=1}^{E} B(\Gamma_e)$ and its dual $\boldsymbol{B}^*_{\Gamma} \equiv \prod_{e=1}^{E} B^*(\Gamma_e)$. Also, as proper trace operators, we consider the internal boundary primal and dual operators $\boldsymbol{\pi}_{\Gamma}$ and $\boldsymbol{\delta}^*_{\Gamma}$, continuous and linear, assumed to satisfy the compatibility conditions [18]

$$\begin{aligned} &\left(\boldsymbol{C}_{\pi_{\Gamma}}\right) \quad \boldsymbol{\pi}_{\Gamma} \in \mathcal{L}(\boldsymbol{V}_{MH}, \boldsymbol{B}_{\Gamma}) \text{ is surjective,} \\ &\left(\boldsymbol{C}_{\delta^*_{\Gamma}}\right) \quad \boldsymbol{\delta}^*_{\Gamma} \in \mathcal{L}(\boldsymbol{Y}^*_{MH}, \boldsymbol{B}^*_{\Gamma}) \text{ is surjective,} \end{aligned}$$

which are fundamental for macro-hybrid compositional duality results.

Then, for the multidomain transmission constraints, to which decomposed multi-systems are subjected in the context of mechanical applications, the primal Ω-stationary space $V(\Omega)$ is supposed to be decomposable in the sense

$$V(\Omega) = \left\{ \boldsymbol{w} \in \boldsymbol{V}_{MH} \equiv \prod_{e=1}^{E} V(\Omega_e) : \boldsymbol{\pi}_{\Gamma} \boldsymbol{w} \in \boldsymbol{Q} \right\}, \tag{2}$$

where $\boldsymbol{Q} \subset \boldsymbol{B}_{\Gamma}$ stands as the primal admissibility subspace of internal boundary weak interface continuity. On the other hand, the dual space is decomposable in the natural sense – without internal boundary constraints – $Y^*(\Omega) = \boldsymbol{Y}^*_{MH} \equiv \prod_{e=1}^{E} Y^*(\Omega_e)$, as well as the Hilbert pivot spaces such that, $H(\Omega) = \boldsymbol{H}_{MH} \equiv \prod_{e=1}^{E} H(\Omega_e)$ and $Z^*(\Omega) = \boldsymbol{Z}^*_{MH} \equiv \prod_{e=1}^{E} Z^*(\Omega_e)$.

Toward a macro-hybridization of the systems—a global local hybrid nonoverlapping decomposition, the following multidomain variational compositional dualization result is achieved via a convex dualization and its own compositional dualization [15].

Lemma 1 *Under the primal compatibility condition* $\left(\boldsymbol{C}_{\pi\Gamma}\right)$, *for* $\boldsymbol{w} \in \boldsymbol{V}_{MH}$ *and* $\boldsymbol{\chi}^* \in \boldsymbol{B}^*_{\Gamma}$, *macro-hybrid compositional dualization*

$$\boldsymbol{\pi}_{\Gamma} \boldsymbol{w} \in \boldsymbol{\partial} \boldsymbol{I}_{Q^*} \boldsymbol{\chi}^* \iff \boldsymbol{\pi}_{\Gamma}^T \boldsymbol{\chi}^* \in \boldsymbol{\partial}(I_Q \circ \boldsymbol{\pi}_{\Gamma}) \boldsymbol{w} \tag{3}$$

holds true, where I_{Q^*} *stands as the conjugate of indicator functional* I_Q.

Proof Indeed, by convex dualization $\boldsymbol{\pi}_{\Gamma} \boldsymbol{w} \in \boldsymbol{\partial} \boldsymbol{I}_{Q^*} \boldsymbol{\chi}^* \Leftrightarrow \boldsymbol{\chi}^* \in \boldsymbol{\partial} \boldsymbol{I}_Q(\boldsymbol{\pi}_{\Gamma} \boldsymbol{w})$, then the variational inequalities of primal inclusions $\boldsymbol{\chi}^* \in \boldsymbol{\partial} \boldsymbol{I}_Q(\boldsymbol{\pi}_{\Gamma} \boldsymbol{w})$ and $\boldsymbol{\pi}_{\Gamma}^T \boldsymbol{\chi}^* \in \boldsymbol{\partial}\big(I_Q \circ \boldsymbol{\pi}_{\Gamma}\big) \boldsymbol{w}$, turn out to be equivalent due to primal compatibility condition $(\boldsymbol{C}_{\pi\Gamma})$. □

Next, concerning the corresponding multidomain evolution macro-hybrid mixed variational functional framework of the theory, along a given fixed time interval $]0, T[$, well-known [19], primal and dual evolution reflexive Banach spaces are given by $\boldsymbol{\mathcal{V}}_{MH} = \boldsymbol{L}^p\big(]0, T[; \boldsymbol{V}_{MH}\big)$ and $\boldsymbol{\mathcal{Y}}^*_{MH} = \boldsymbol{L}^{q^*}\big(]0, T[; \boldsymbol{Y}^*_{MH}\big)$, for $2 \le p < \infty$ and $q^* = p/(p-1)$, with duals $\boldsymbol{\mathcal{V}}^*_{MH} = \boldsymbol{L}^{q^*}\big(]0, T[; \boldsymbol{V}^*_{MH}\big)$ and $\boldsymbol{\mathcal{Y}}_{MH} = \boldsymbol{L}^p\big(]0, T[; \boldsymbol{Y}_{MH}\big)$, respectively.

Lastly, the primal solution space, for the primal evolution macro-hybrid mixed governing state system, $\big(\boldsymbol{\mathcal{MH}}_{\kappa}\big)$, to be considered in the next section, is defined by

$$\boldsymbol{\mathcal{W}}_{MH} = \big\{ \boldsymbol{w} : \boldsymbol{w} \in \boldsymbol{\mathcal{V}}_{MH}, d\boldsymbol{w}/dt \in \boldsymbol{\mathcal{V}}^*_{MH} \big\}, \tag{4}$$

endowed with the norm $\|\boldsymbol{w}\|_{\mathcal{W}_{MH}} = \|\boldsymbol{w}\|_{\mathcal{V}_{MH}} + \|d\boldsymbol{w}/dt\|_{\mathcal{V}^*_{MH}}$, which turns out to be continuous and densely embedded in the space $\boldsymbol{C}\big(]0, T[; \boldsymbol{H}_{MH}\big)$ of $\boldsymbol{H}_{MH}$-continuos functions, with initial values set $\big\{\boldsymbol{w}(0) : \boldsymbol{w} \in \boldsymbol{\mathcal{W}}_{MH}\big\} = \boldsymbol{H}_{MH}$ [19]. Also, corresponding primal and dual evolution boundary trace spaces are defined by the reflexive Banach spaces $\boldsymbol{\mathcal{B}}_{MH} = \boldsymbol{L}^p\big(]0, T[; \boldsymbol{B}_{MH}\big)$ and its dual $\boldsymbol{\mathcal{B}}^*_{MH} = \boldsymbol{L}^{q^*}\big(]0, T[; \boldsymbol{B}^*_{MH}\big)$.

3 Primal Evolution Macro-hybrid Mixed State System

In this section, we present the multidomain variational primal evolution mixed governing κ-state multisystem, related to the macro-hybrid optimization problem $\left(\mathcal{O}_{MH}\right)$ of the Introduction.

In a general abstract sense [20], and in accordance with the evolution functional framework of previous Sect. 2, such a state multisystem has the variational form

$$\left(\mathcal{MH}_\kappa\right) \begin{cases} \text{Given } \boldsymbol{f}^* \in \mathcal{R}\left(-\boldsymbol{\Lambda}^T\right) \subset \boldsymbol{\mathcal{V}}^*_{MH}, \ (\boldsymbol{g}, \boldsymbol{h}) \in \mathcal{R}(\boldsymbol{\Lambda}) \times \mathcal{R}\left(\boldsymbol{\pi}_\Gamma\right) \\ \subset \boldsymbol{\mathcal{Y}}_{MH} \times \boldsymbol{\mathcal{B}}_\Gamma, \text{ and } \boldsymbol{v}_0 \in \boldsymbol{H}_{MH} , \\ \text{find } \boldsymbol{v} \in \boldsymbol{\mathcal{W}}_{MH}, \ (\boldsymbol{y}^*, \boldsymbol{\lambda}^*) \in \boldsymbol{\mathcal{Y}}^*_{MH} \times \boldsymbol{\mathcal{B}}^*_\Gamma : \\ -\boldsymbol{\Lambda}^T \boldsymbol{y}^* - \boldsymbol{\pi}_\Gamma^T \boldsymbol{\lambda}^* \in \dfrac{d\boldsymbol{v}}{dt} + \boldsymbol{\partial} \widetilde{\boldsymbol{F}} \boldsymbol{v} + \boldsymbol{B}^* \boldsymbol{\kappa} - \widetilde{\boldsymbol{f}^*} , \qquad \text{in } \boldsymbol{\mathcal{V}}^*_{MH} , \\ \boldsymbol{v}(0) = \boldsymbol{v}_0 , \\ \left(\boldsymbol{\Lambda}, \boldsymbol{\pi}_\Gamma\right)(\boldsymbol{v}, \boldsymbol{v}) \in \left(\boldsymbol{\partial} \boldsymbol{G}^*, \boldsymbol{\partial} \boldsymbol{I}_{\boldsymbol{Q}^*}\right)(\boldsymbol{y}^*, \boldsymbol{\lambda}^*) + (\boldsymbol{g}, \boldsymbol{h}), \ \text{in } \boldsymbol{\mathcal{Y}}_{MH} \times \boldsymbol{\mathcal{B}}_\Gamma , \end{cases}$$

where $\boldsymbol{B}^* \in \mathcal{L}\left(\mathcal{C}_{MH}, \boldsymbol{\mathcal{V}}^*_{MH}\right)$ is the primal macro-hybrid coupling optimal control operator, linear and continuous, and $\boldsymbol{\lambda}^* \in \boldsymbol{Q}^* \subset \boldsymbol{\mathcal{B}}^*_\Gamma$ is the dual macro-hybrid synchronizing internal boundary field implemented variationally by the conjugate indicator functional I_{Q^*}.

Here variational operator $\boldsymbol{\Lambda} \in \mathcal{L}\left(\boldsymbol{\mathcal{V}}_{MH}, \boldsymbol{\mathcal{Y}}_{MH}\right)$ corresponds to the primal linear continuous coupling operator with transpose $\boldsymbol{\Lambda}^T \in \mathcal{L}\left(\boldsymbol{\mathcal{Y}}^*_{MH}, \boldsymbol{\mathcal{V}}^*_{MH}\right)$. Also $\mathcal{R}\left(-\boldsymbol{\Lambda}^T\right)$ and $\mathcal{R}\left(\boldsymbol{\Lambda}\right)$, as well as $\mathcal{R}\left(\boldsymbol{\pi}_\Gamma\right)$, denote range subspaces of the respective functionals. Further, the macro-hybrid primal subdifferential and right hand side term of the problem are defined by

$$\begin{aligned} \boldsymbol{\partial} \widetilde{\boldsymbol{F}} &= \boldsymbol{\partial} \boldsymbol{F} + \boldsymbol{\partial}\left(I_{\widehat{v}} \circ \boldsymbol{\pi}_D\right) + \boldsymbol{\partial} \boldsymbol{\varphi} : \boldsymbol{\mathcal{V}}_{MH} \to \boldsymbol{2}^{\mathcal{V}^*_{MH}} , \\ -\widetilde{\boldsymbol{f}^*} &= \boldsymbol{\pi}_D^T \widehat{\boldsymbol{y}}^* - \boldsymbol{f}^* \in \boldsymbol{\mathcal{V}}^*_{MH} , \end{aligned} \tag{5}$$

where the subdifferential $\boldsymbol{\partial\varphi}$ that models the primal intrinsic controlling distributed constraint, imposed to the system, will be assumed to satisfy the qualifying condition

$$\left(\boldsymbol{C}_{\partial\varphi}\right) \quad \boldsymbol{\partial\varphi} : \boldsymbol{\mathcal{V}}_{MH} \to \boldsymbol{2}^{\mathcal{V}^*_{MH}}, \ \text{is a strongly monotone subpotential operator.}$$

On the other hand, for such a macro-hybrid mixed state system, in particular $\widehat{\boldsymbol{v}}$ and $\widehat{\boldsymbol{y}}^*$ stand as the primal Dirichlet (D-essential) and dual Neumman (N-natural) trace fields, prescribed on disjoint and complementary parts of the Ω-domain boundary, $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, of the system under control. Further, regarding the dual subdifferentials $\left(\boldsymbol{\partial G}^*, \boldsymbol{\partial I}_{Q^*}\right) : \boldsymbol{\mathcal{Y}}^*_{MH} \times \boldsymbol{\mathcal{B}}^*_\Gamma \to \boldsymbol{2}^{\mathcal{Y}_{MH} \times \mathcal{B}_\Gamma}$, they model variationally the dualized distributed primal constraint of the system, as well as the synchronization internal boundary transmission constraints due to the multidomain spatial decomposition. Recall that in this study all the subdifferential operators are assumed to be subpotential [21]; i.e, maximal monotone subdifferentials of proper convex semicontinuous functionals.

In order to determine the subpotential of primal subdifferential $\boldsymbol{\partial}\widetilde{\boldsymbol{F}}$, we incorporate the interior domain subpotential conditions, (cf. [21], Theorem 2.10),

$$\left(\boldsymbol{C}_{F,\left(I_{\widehat{\mathcal{V}}}\circ\pi_D\right),\varphi}\right)\begin{cases}\ \ \text{int}\,\boldsymbol{\mathcal{D}}(F)\ \cap\ \boldsymbol{\mathcal{D}}\big(I_{\widehat{\mathcal{V}}}\circ\boldsymbol{\pi}_D\big)\neq\emptyset\,,\\ \text{and}\\ \ \ \text{int}\,\boldsymbol{\mathcal{D}}\big(F+(I_{\widehat{\mathcal{V}}}\circ\boldsymbol{\pi}_D)\big)\ \cap\ \boldsymbol{\mathcal{D}}\big(\varphi\big)\neq\emptyset\,,\end{cases}$$

which guarantee the corresponding subdifferential subpotential sum relation

$$\boldsymbol{\partial}\widetilde{\boldsymbol{F}}=\boldsymbol{\partial}\big(F+I_{\widehat{\mathcal{V}}}\circ\boldsymbol{\pi}_D+\varphi\big). \tag{6}$$

Therefore, the desired primal subpotential characterization is concluded as follows.

Lemma 2 *Under functional compatibility condition* $\left(\boldsymbol{C}_{F,\left(I_{\widehat{\mathcal{V}}}\circ\pi_D\right),\varphi}\right)$*, the subpotential of primal subdifferential* $\boldsymbol{\partial}\widetilde{\boldsymbol{F}}:\boldsymbol{\mathcal{V}}\to\mathbf{2}^{\mathcal{V}^*}$, $(5)_1$, *is defined by*

$$\widetilde{F}=F+I_{\widehat{\mathcal{V}}}\circ\boldsymbol{\pi}_D+\varphi:\boldsymbol{\mathcal{V}}_{MH}\to\Re\cup\{+\infty\}, \tag{7}$$

and moreover, due to the intrinsic control condition $\left(\boldsymbol{C}_{\partial\varphi}\right)$ *state primal subdifferential* $\boldsymbol{\partial}\widetilde{\boldsymbol{F}}$ *turns out to be a strongly monotone maximal monotone operator.*

Example 1 A primal intrinsic controlling distributed constraint may be modeled by a subpotential strongly monotone sum subdifferential

$$\boldsymbol{\partial\varphi}=\boldsymbol{\partial}\widetilde{\boldsymbol{\varphi}}+\boldsymbol{\partial A}:\boldsymbol{\mathcal{V}}_{MH}\to\mathbf{2}^{\mathcal{V}^*_{MH}},$$

with component $\boldsymbol{\partial}\widetilde{\boldsymbol{\varphi}}$ of an obstacle type, subpotential subdiffferential [22]; and a linear potential strongly monotone component $\boldsymbol{\partial A}$.

Specifically, we may consider the following classical elliptic variational model problem related to a Hilbert space $\boldsymbol{V}$, with topological dual and corresponding pivot spaces, $\boldsymbol{V}^*$ and $\boldsymbol{H}$; i.e. $\boldsymbol{V}\subset\boldsymbol{H}\subset\boldsymbol{V}^*$ with embeddings being continuous and dense.

$$(\boldsymbol{\mathcal{P}})\begin{cases}\text{Given } \boldsymbol{f}^*\in\boldsymbol{V}^*,\ \text{Find } \boldsymbol{v}\in\boldsymbol{V}:\\ \boldsymbol{\partial Av}+\boldsymbol{\partial Gv}=\boldsymbol{f}^*,\ \ \text{in } \boldsymbol{V}^*\,,\end{cases}$$

where $G:\boldsymbol{V}\to\Re\cup\{+\infty\}$ is a proper lower semicontinuous convex functional with effective domain the nonempty closed convex subset $\boldsymbol{K}\subset\boldsymbol{V}$.

Then, as it is well known, a sufficient condition for variational existence and uniqueness solvability to problem $(\boldsymbol{\mathcal{P}})$ is precisely the qualifying monotonicity operator property

$$\boldsymbol{C}_{\partial A}\quad \boldsymbol{\partial A}:\boldsymbol{V}\to\mathbf{2}^{V^*},\ \text{is } \boldsymbol{K}-\text{strongly monotone},$$

result that naturally (cf. Theorem 2 Proof, below), can be validated via a fixed point characterization, in this case through the corresponding proximation operator

$\boldsymbol{Prox}_{\lambda G} : \boldsymbol{V} \rightarrow \boldsymbol{V}$, for any real $\lambda > 0$, defined by

$$\boldsymbol{w} \longmapsto \boldsymbol{\eta} = \boldsymbol{Prox}_{\lambda G}(\boldsymbol{w}) \text{ the solution in } \boldsymbol{K} \text{ of}$$
$$((\boldsymbol{w} - \boldsymbol{\eta}, \boldsymbol{v} - \boldsymbol{\eta}))_V \ \leq \ \lambda G(\boldsymbol{v}) - \lambda G(\boldsymbol{\eta}), \ \ \forall \boldsymbol{v} \in \boldsymbol{K}.$$

Consequently it can be concluded that problem $(\mathcal{P})$ states, in fact, a variational optimality condition of the minimization, optimization problem in $\boldsymbol{K}$ of the functional

$$\Phi(\boldsymbol{v}) = 1/2 \ |\boldsymbol{v} - \boldsymbol{w}\ |_V^2 + \lambda G(\boldsymbol{v}), \ \ \boldsymbol{v} \in \boldsymbol{K}.$$

Indeed, the mapping $\boldsymbol{v} \longmapsto ((\boldsymbol{v} - \boldsymbol{w}, \cdot))$ is the gradient of the continuous strictly convex functional $\boldsymbol{v} \longmapsto 1/2 \ |\boldsymbol{v} - \boldsymbol{w}\ |_V^2$ in $\boldsymbol{V}$. The solution of such a problem, which exists uniquely, is the proximity point of $\boldsymbol{w}$ relative to λG. Notice that when $G = I_K$ the solution then turns out to be the projection of $\boldsymbol{w}$ on $\boldsymbol{K}$.

3.1 Solvability of Macro-hybrid Mixed State System($\mathcal{MH}_\kappa$)

For the existence analysis of subpotential primal evolution macro-hybrid mixed state problem ($\mathcal{MH}_\kappa$), (cf. [3, Sect. 2]), we first establish its primal duality principle introducing the corresponding interior classical primal evolution compatibility condition

$$\left(\boldsymbol{C}_{\left(G, I_Q\right),\left(\Lambda, \pi_\Gamma\right)}\right) \quad \text{int}\, \boldsymbol{\mathcal{D}}\left(\left(G, I_Q\right)\right) \cap \boldsymbol{\mathcal{R}}\left(\left(\boldsymbol{\Lambda}, \boldsymbol{\pi}_\Gamma\right)\right) \neq \emptyset,$$

under which the compositional result [23]

$$\left(\boldsymbol{\Lambda}^T, \boldsymbol{\pi}_\Gamma^T\right)\left(\boldsymbol{\partial G}, \boldsymbol{\partial I}_Q\right)\circ\left(\boldsymbol{\Lambda}, \boldsymbol{\pi}_{\boldsymbol{\Gamma}}\right) = \boldsymbol{\partial}\Big(\left(G, I_Q\right) \circ \left(\boldsymbol{\Lambda}, \boldsymbol{\pi}_\Gamma\right)\Big) \tag{8}$$

holds true. Then, by convex dualization of the ($\mathcal{MH}_\kappa$)-dual inclusion, it follows that

$$\left(\boldsymbol{\Lambda}, \boldsymbol{\pi}_\Gamma\right)(\boldsymbol{v}, \boldsymbol{v}) - (\boldsymbol{g}, \boldsymbol{h}) \in \left(\boldsymbol{\partial G}^*, \boldsymbol{\partial I}_{Q^*}\right)(\boldsymbol{y}^*, \boldsymbol{\lambda}^*)$$
$$\Longleftrightarrow \ (\boldsymbol{y}^*, \boldsymbol{\lambda}^*) \in \left(\boldsymbol{\partial G}, \boldsymbol{\partial I}_Q\right)\Big(\left(\boldsymbol{\Lambda}, \boldsymbol{\pi}_\Gamma\right)(\boldsymbol{v}, \boldsymbol{v}) - (\boldsymbol{g}, \boldsymbol{h})\Big),$$

result that in conjunction with compositional dualization (8) leads to the evolution duality principle of the state multisystem.

Theorem 1 *Let compatibility condition* $\left(\boldsymbol{C}_{\left(G, I_Q\right),\left(\Lambda, \pi_\Gamma\right)}\right)$ *be fulfilled. Then macro-hybrid primal evolution mixed state problem* $\left(\mathcal{MH}_\kappa\right)$ *is solvable if, and only if,* *macro-hybrid primal evolution state problem*

$$
(\mathcal{P}_{MH_\kappa})\begin{cases} \text{Find } \boldsymbol{v} \in \mathcal{W}_{MH}: \\ \mathbf{0}_{\mathcal{V}^*_{MH}} \in \dfrac{d\boldsymbol{v}}{dt} + \boldsymbol{\partial}\widetilde{\boldsymbol{F}}\boldsymbol{v} + \boldsymbol{\partial}\Big(\big(G, I_Q\big)\circ\big(\boldsymbol{\Lambda}, \boldsymbol{\pi}_\Gamma\big)\Big)\big((\boldsymbol{v}, \boldsymbol{v}) - (\boldsymbol{w}_g, \boldsymbol{w}_h)\big) \\ \quad + \boldsymbol{B}^*\boldsymbol{\kappa} - \widetilde{\boldsymbol{f}^*}, \qquad\qquad \text{in } \mathcal{V}^*_{MH}, \\ \boldsymbol{v}(0) = \boldsymbol{v}_0, \end{cases}
$$

is solvable, where $(\boldsymbol{w}_g, \boldsymbol{w}_h) \in \mathcal{V}_{MH} \times \mathcal{V}_{MH}$ *is a fixed* $(\boldsymbol{\Lambda}, \boldsymbol{\pi}_\Gamma)$*-preimage of function* $(\boldsymbol{g}, \boldsymbol{h})$*. That is, if* $(\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*)) \in \mathcal{W}_{MH} \times (\mathcal{Y}^*_{MH} \times \mathcal{B}^*_\Gamma)$ *is a solution of mixed state problem* $(\mathcal{MH}_\kappa)$ *then primal function* $\boldsymbol{v}$ *is a solution of problem* $(\mathcal{P}_{MH_\kappa})$ *and, conversely, if* $\boldsymbol{v} \in \mathcal{W}_{MH}$ *is a solution of primal state problem* $(\mathcal{P}_{MH_\kappa})$ *then there is a dual function* $(\boldsymbol{y}^*, \boldsymbol{\lambda}^*) \in \boldsymbol{\partial}\big((G, I_Q)\big)\big((\boldsymbol{\Lambda}, \boldsymbol{\pi}_\Gamma)(\boldsymbol{v}, \boldsymbol{v}) - (\boldsymbol{g}, \boldsymbol{h})\big) \subset (\mathcal{Y}^*_{MH} \times \mathcal{B}^*_\Gamma)$*, such that* $(\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*))$ *is a solution of mixed state problem* $(\mathcal{MH}_\kappa)$*.*

Lastly, on the basis of Lemma 2 and Theorem 1, and the macro-hybrid monotonicity of the primal state operator

$$
(\boldsymbol{C}_{\partial\widetilde{F}})\begin{cases} \boldsymbol{\partial}\widetilde{\boldsymbol{F}}: \mathcal{V}_{MH} \to \mathbf{2}^{\mathcal{V}^*_{MH}} \text{ is strongly monotone; i.e., } \exists\, \alpha > 0: \\ \big\langle \widetilde{\boldsymbol{w}^*} - \widetilde{\boldsymbol{v}^*}, \widetilde{\boldsymbol{w}} - \widetilde{\boldsymbol{v}} \big\rangle_{\mathcal{V}_{MH}} \geq \alpha \|\widetilde{\boldsymbol{w}} - \widetilde{\boldsymbol{v}}\|^2_{\mathcal{V}_{MH}}, \\ \forall \widetilde{\boldsymbol{w}}, \widetilde{\boldsymbol{v}} \in \mathcal{V}_{MH},\ \widetilde{\boldsymbol{w}^*} \in \boldsymbol{\partial}\widetilde{\boldsymbol{F}}\widetilde{\boldsymbol{w}},\ \widetilde{\boldsymbol{v}^*} \in \boldsymbol{\partial}\widetilde{\boldsymbol{F}}\widetilde{\boldsymbol{v}}, \end{cases}
$$

the following solvability result can be established via a primal evolution resolvent fixed point existence analysis [3]. For completeness and due to the importance that this result plays in the context of our multidomain optimal control theory, we shall provide here the details of its validity.

Theorem 2 *Under the functional compatibility condition* $\Big(\boldsymbol{C}_{(G,I_Q),(\Lambda,\pi_\Gamma)}\Big)$*, and the monotonicity primal operator qualifying condition* $(\boldsymbol{C}_{\partial\widetilde{F}})$*, primal evolution macro-hybridized state multisystem* $(\mathcal{MH}_\kappa)$ *has a solution, with a unique primal component, for initial data as regular as*

$$
\boldsymbol{v}_0 \in \boldsymbol{V}_{MH} \subset \boldsymbol{H}_{MH}\,. \tag{9}
$$

Proof Taking into account regularity condition (9), we consider the auxiliary state primal evolution inclusion, with an homogeneous initial data,

$$
(\widetilde{\mathcal{P}}_{MH_\kappa})\begin{cases} \text{Find } \boldsymbol{s} \in \mathcal{W}_{MH}: \\ \mathbf{0}_{\mathcal{V}^*_{MH}} \in \dfrac{d\boldsymbol{s}}{dt} + \boldsymbol{\partial}\widetilde{\boldsymbol{F}}\boldsymbol{s} + \boldsymbol{\partial}\Big(\big(G, I_Q\big)\circ\big(\boldsymbol{\Lambda}, \boldsymbol{\pi}_\Gamma\big)\Big)\big((\boldsymbol{s}, \boldsymbol{s}) - (\boldsymbol{w}_g, \boldsymbol{w}_h)\big) \\ \quad + \boldsymbol{B}^*\boldsymbol{\kappa} - \widetilde{\boldsymbol{f}^*}, \qquad\qquad \text{in } \mathcal{V}^*_{MH}, \\ \boldsymbol{s}(0) = \boldsymbol{s}_0\,, \end{cases}
$$

whose solvability, for $\boldsymbol{s} = \boldsymbol{v} + \boldsymbol{v}_0$, results to be equivalent to that of primal state problem $(\mathcal{P}_{MH_\kappa})$. Further, we note that, importantly, its time derivative operator in the sense

$$\boldsymbol{d/dt} : \mathcal{D}(\boldsymbol{d/dt}) = \Big\{\boldsymbol{w} \in \boldsymbol{\mathcal{W}}_{MH} : \boldsymbol{w}(0) = \boldsymbol{0}\Big\} \subset \boldsymbol{\mathcal{V}}_{MH} \to \boldsymbol{\mathcal{V}}^*_{MH}, \tag{10}$$

is, in fact, a linear densely defined maximal monotone operator (cf. [24, Proposition 32.10]). Thereby, in accordance with [25], we can now apply a fixed-point subdifferential approach for the existence analysis. Indeed, due to the monotonicity of time derivative operator (10), qualifying condition $\big(\boldsymbol{C}_{\partial\widetilde{F}}\big)$ results to be equivalent to the strong monotonicity condition of the primal combined operator

$$\widetilde{\boldsymbol{\mathcal{A}}} = \boldsymbol{d/dt} + \boldsymbol{\partial}\widetilde{\boldsymbol{F}} : \boldsymbol{\mathcal{V}}_{MH} \to \boldsymbol{2}^{\mathcal{V}^*_{MH}} . \tag{11}$$

Moreover, operator $\widetilde{\boldsymbol{\mathcal{A}}}$ is maximal monotone, since int$\mathcal{D}(\boldsymbol{d/dt}) \cap \mathcal{D}(\boldsymbol{\partial}\widetilde{\boldsymbol{F}}) \neq \emptyset$. Then, for an m-linearly strongly monotone and a-Lipschitz operator $\boldsymbol{\mathcal{M}} : \boldsymbol{\mathcal{V}}_{MH} \to \boldsymbol{\mathcal{V}}^*_{MH}$, the primal state $(\widetilde{\boldsymbol{\mathcal{P}}}_{MH_\kappa})$-equation is $\boldsymbol{\mathcal{M}}$-preconditioned, augmented and exactly penalized with a parameter $r > 0$, for $\boldsymbol{s^*} \in \widetilde{\boldsymbol{\mathcal{A}}}\boldsymbol{s}$, by

$$\begin{aligned} &\boldsymbol{\mathcal{M}}(\boldsymbol{s},\boldsymbol{s}) - r\big(\boldsymbol{s}^* - \widetilde{\boldsymbol{f}^*}\big) \in \Big(\boldsymbol{\mathcal{M}} + r\boldsymbol{\partial}\big(\big(G, I_Q\big) \circ \big(\boldsymbol{\Lambda}_{w_{g,h}}, \boldsymbol{\pi}_\Gamma\big)\big)\Big)(\boldsymbol{s},\boldsymbol{s}) \\ &\Longleftrightarrow\ (\boldsymbol{s},\boldsymbol{s}) = \boldsymbol{F}^r_{s^*}(\boldsymbol{s},\boldsymbol{s}) \\ &\quad \equiv \boldsymbol{J}^r_{\mathcal{M},\partial\left(\left(G,I_Q\right)\circ\left(\Lambda_{w_{g,h}},\pi_\Gamma\right)\right)}\Big(\boldsymbol{\mathcal{M}}(\boldsymbol{s},\boldsymbol{s}) - r\big(\boldsymbol{s}^* - \widetilde{\boldsymbol{f}^*}\big)\Big). \end{aligned} \tag{12}$$

Here $\boldsymbol{J}^r_{\mathcal{M},\partial\left(\left(G,I_Q\right)\circ\left(\Lambda_{w_{g,h}},\pi_\Gamma\right)\right)} = \Big(\boldsymbol{\mathcal{M}} + r\boldsymbol{\partial}\big(\big(G, I_Q\big) \circ \big(\boldsymbol{\Lambda}_{w_{g,h}}, \boldsymbol{\pi}_\Gamma\big)\big)\Big)^{-1} : \boldsymbol{\mathcal{V}^*_{MH}}$ $\to \boldsymbol{\mathcal{V}_{MH}}$ is the $\boldsymbol{\mathcal{M}}$-resolvent operator of the maximal monotone composition operator $\boldsymbol{\partial}\Big(\big(G, I_Q\big) \circ \big(\boldsymbol{\Lambda}_{w_{g,h}}, \boldsymbol{\pi}_\Gamma\big)\Big) = \boldsymbol{\partial}\Big(\big(G, I_Q\big) \circ \big(\boldsymbol{\Lambda}, \boldsymbol{\pi}_\Gamma\big)\Big)\big((\cdot) - \boldsymbol{w}_{g,h}\big)$, which is a well defined $1/m$-firm contraction [26].
Thereby, auxiliary state primal evolution inclusion problem $(\widetilde{\boldsymbol{\mathcal{P}}}_{MH_\kappa})$ has the $\boldsymbol{\mathcal{M}}$-resolvent fixed-point problem characterization

$$(\widetilde{\widetilde{\boldsymbol{\mathcal{P}}}}_{MH_\kappa}) \begin{cases} \text{Find } \boldsymbol{s} \in \mathcal{D}\Big(\widetilde{\boldsymbol{\mathcal{A}}} + \boldsymbol{\partial}\Big(\big(G, I_Q\big) \circ \big(\Lambda_{w_g}, \boldsymbol{\pi}_\Gamma\big)\Big)\Big) \subset \boldsymbol{\mathcal{W}}_{MH} : \\ \text{for } \boldsymbol{s^*} \in \widetilde{\boldsymbol{\mathcal{A}}}\boldsymbol{s}, \\ (\boldsymbol{s},\boldsymbol{s}) = \boldsymbol{F}^r_{s^*}(\boldsymbol{s},\boldsymbol{s}), \end{cases}$$

which has a unique solution by the Banach fixed-point theorem, due to the $1/m$-firm contraction resolvent property that implies the contraction of fixed point operator

$$\boldsymbol{F}^r_{s^*} : \mathcal{D}\Big(\widetilde{\boldsymbol{\mathcal{A}}} + \boldsymbol{\partial}\Big(\big(G, I_Q\big) \circ \big(\Lambda_{w_g}, \boldsymbol{\pi}_\Gamma\big)\Big)\Big) \to \mathcal{D}\Big(\widetilde{\boldsymbol{\mathcal{A}}} + \boldsymbol{\partial}\Big(\big(G, I_Q\big) \circ \big(\Lambda_{w_g}, \boldsymbol{\pi}_\Gamma\big)\Big)\Big),$$

with contraction parameter $1/m(a - r\alpha) < 1$, for $r > (a - m)/\alpha \geq 0$. □

Now, we can conclude the multidomain state primal evolution mixed variational existence result of the theory, from the conjunction of Theorems 1 and 2.

Corollary 1 *Let primal functional compatibility condition* $\left(\boldsymbol{C}_{\left(G,I_Q\right),\left(\Lambda,\pi_\Gamma\right)}\right)$ *and monotonicity primal operator qualifying condition* $\left(\boldsymbol{C}_{\partial\widetilde{F}}\right)$ *be fulfilled. Then primal evolution macro-hybridized state system problem* $(\mathcal{MH}_\kappa)$ *is solvable, with a unique primal solution component, for initial condition data satisfying variational regularity* (9).

4 Multidomain Primal Evolution Macro-hybrid Mixed Transport Flow Mechanical System

In this section, we apply the multidomain evolution variational theory developed above, to mechanical Darcian transport flow processes in the subsurface, with intrinsic mass concentration and pressure controlling constraints, demonstrating its versatility.

4.1 Multidomain Macro-hybrid Mixed Variational Transport Local State System

For a transport model with implemented intrinsic mass concentration control mechanisms, applicable to underground compressible flow mechanical systems, in the setting of spatial multidomain decompositions and macro-hybrid mixed real functional frameworks of Sect. 2, we shall consider the following one.

Let $\{c_e\}$ be local mass concentration scalar fields, say, of distributed-chemical contaminants, and let $\{\boldsymbol{d}_e^*\} = -\{\boldsymbol{D}_{w_e^*}^* \; \boldsymbol{grad} \; c_e\}$ denote the corresponding vector flux field of a transport process driven by an underground compressible flow, with local velocity vector fields $\{\boldsymbol{w}_e^*\}$. Here $\{\boldsymbol{D}_{w_e^*}^*\}$ is the diffusion-dispersion tensor (cf., e.g., [27]). Further, let $\{s_{\alpha_e}^*\}$ be the local distributed intrinsic mass concentration control scalar fields, and let $\{\widehat{f_e^*}\}$ denote a given contaminant local source.

Then related to instantaneous material connected local parts $\mathcal{P}_{e_t}(x)$ of each subdomain Ω_e, $e = 1, 2, \ldots, E$, surrounding any point $x \in \Omega_e$ at a time $t \in]0, T[$, the Reynolds transport theorem establishes that, for local unit $\mathcal{P}_{e_t}$- $\boldsymbol{n}_e$boundary normal vectors, the following local mass balances must hold true (cf. e.g., [28]):

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{P}_{e_t}} c_e \, d\Omega_e &= \int_{\mathcal{P}_{e_t}} \frac{dc_e}{dt} + div(\{\boldsymbol{w}_e^* \, c_e) \, d\Omega_e \\ &= -\int_{\mathcal{P}_{e_t}} s_e^* \, d\Omega_e + \int_{\mathcal{P}_{e_t}} \{\widehat{f_e^*}\} \, d\Omega_e - \int_{\partial\mathcal{P}_{e_t}} \boldsymbol{d}_e^* \cdot \boldsymbol{n}_e \, d\partial\Omega_e \, , \end{aligned} \tag{13}$$

and by the divergence theorem applied to (13)-boundary term, and a continuous integral localization, the next local evolution mixed concentration-flux multidomain physical transport models are obtained:

$$\left.\begin{array}{l} \dfrac{d\{c_e\}}{dt} + \left[\boldsymbol{w}_e^*\cdot\boldsymbol{grad}_e\right]\{c_e\} + \{div_e \boldsymbol{w}_e^* c_e\} + \{div_e \boldsymbol{d}_e^*\} \\ \qquad = -\{s_e^*\} + \{\widehat{f}_e^*\}, \\ \{\boldsymbol{d}_e^*\} = -\left[\boldsymbol{D}_{w_e^*}^*\right]\{\boldsymbol{grad}_e c_e\}, \\ \{s_e^*\} \in [\partial\varphi_{\tau_e}]\{c_e\} \;\Leftrightarrow\; \{c_e\} \in \left[\partial\varphi_{\tau_e}^*\right]\{s_e^*\}, \end{array}\right\} \text{ in } \{\Omega_e\}\times]0,T[\,, \quad (14)$$

where $[\partial\varphi_{\tau_e}]$ stands as a subpotential maximal monotone subdifferential of primal distributed local control mechanisms, for the intrinsic concentration control,—the constraint of the transport processes,—with inverse control mechanisms $[\partial\varphi_{\tau_e}^*]$ ($\varphi_{\tau_e}^*$, the local conjugate functional of φ_{τ_e}) [22].

Here we are considering the classical macro-hybrid primal evolution transport formulation, whose continuity local transmission internal boundary constraints correspond to the dual normal diffusion-dispersion fluxes and the primal concentrations:

$$\begin{aligned} \{\boldsymbol{d}_e^*\cdot\boldsymbol{n}_e\} &= -\{\boldsymbol{d}_e^*\cdot\boldsymbol{n}_k\}, \\ \{c_{\alpha_e}\} &= \{c_k\}, \end{aligned} \tag{15}$$

across the internal boundary interfaces $\Gamma_{ek} = \Gamma_e \cap \Gamma_k$, $1 \le e < k \le E$.

4.1.1 Multidomain Macro-hybrid Mixed Variational Transport Local State System

Related to [16], the variational primal evolution formulation of localized macro-hybrid mixed constrained τ-transport model (13)-(15), as a state system governing the optimal control minimization problem ($\mathcal{O}_{MH}$) of the theory, is given as follows, adopting a simplified boldface generic notation of the previous sections.

$$(\mathcal{MH}_{\kappa_\tau}) \begin{cases} \text{Given } \widetilde{\boldsymbol{f}}_\tau^* \in \mathcal{R}(\boldsymbol{div}_\tau^T) \subset \mathcal{V}_{\tau_{MH}}^*,\ \widehat{\boldsymbol{c}}_0 \in \boldsymbol{H}_{MH_\tau}\ , \\ \text{find } \boldsymbol{c} \in \mathcal{W}_{MH_\tau} \text{ and } (\boldsymbol{d}^*, \boldsymbol{\lambda}_\tau^*) \in \mathcal{Y}_{MH_\tau}^* \times \mathcal{B}_{\Gamma_\tau}^*\ : \\ -\left(\boldsymbol{div}_\tau^T, \boldsymbol{\pi}_{\Gamma_\tau}^T\right)(\boldsymbol{d}^*, \boldsymbol{\lambda}_\tau^*) \\ \quad \in \dfrac{d\boldsymbol{c}}{dt} + \left(\boldsymbol{\partial}\widetilde{\boldsymbol{F}}_\tau + \boldsymbol{\partial\varphi}_\tau\right)\boldsymbol{c} + \boldsymbol{B}^*\boldsymbol{\kappa}_\tau - \widetilde{\boldsymbol{f}}_\tau^*, \qquad \text{in } \mathcal{V}_{MH_\tau}^*\ , \\ \quad \boldsymbol{c}(0) = \widehat{\boldsymbol{c}}_0\ ; \\ (\boldsymbol{div}_\tau, \boldsymbol{\pi}_{\Gamma_\tau})(\boldsymbol{c}, \boldsymbol{c}) \in \left(\boldsymbol{D}_u^{*-1}, \boldsymbol{\partial I}_{Q_\tau^*}\right)(\boldsymbol{d}^*, \boldsymbol{\lambda}_\tau^*), \text{ in } \mathcal{Y}_{MH_\tau} \times \mathcal{B}_{\Gamma_\tau}\ , \end{cases}$$

where dual unknown $\boldsymbol{\lambda}_\tau^* = \boldsymbol{\delta}_{\Gamma_\tau}^* \boldsymbol{d}^* \in \boldsymbol{Q}_\tau^* \subset \mathcal{B}_{\Gamma_\tau}^*$ corresponds to the transport macro-hybrid synchronizing internal boundary field (cf. (3)). Further, macro-hybrid primal subdifferential and right hand side terms correspond to

$$\begin{aligned} \boldsymbol{\partial}\widetilde{\boldsymbol{F}}_\tau &= \boldsymbol{w}^*\cdot\boldsymbol{grad}_\tau + \boldsymbol{div}_\tau \boldsymbol{w}^* + \boldsymbol{\partial}\left(I_{\widehat{C}} \circ \boldsymbol{\pi}_{\tau_D}\right) : \mathcal{V}_{MH_\tau} \to \boldsymbol{2}^{\mathcal{V}_{MH_\tau}^*}\ , \\ -\widetilde{\boldsymbol{f}}_\tau^* &= \boldsymbol{\pi}_{N_\tau}^T \widehat{\boldsymbol{d}}^* - \widehat{\boldsymbol{f}}^* \in \mathcal{V}_{MH_\tau}^*\ , \end{aligned} \tag{16}$$

with local essential Dirichlet $\boldsymbol{\pi}_{\Sigma_\tau} \boldsymbol{c} = \widehat{\boldsymbol{c}}$ and Neumann $\boldsymbol{\delta}^*_{\Sigma_\tau} \boldsymbol{d}^* = \widehat{\boldsymbol{d}}^*$ prescribed external boundary trace fields, where indicator functional $I_{\widehat{\boldsymbol{c}}}$ implements the primal Dirichlet condition.

In order to apply primal evolution macro-hybrid mixed fixed point existence Theorem 2, we assume that the transport local concentration initial data satisfies the regularity condition

$$\{c_{\alpha_{e0}}\} \in \boldsymbol{\mathcal{V}}_{MH_\tau} \subset \boldsymbol{H}_{MH_\tau} \ . \tag{17}$$

Theorem 3 *Under Theorem 2 corresponding transport interior compatibility local condition,* $(\boldsymbol{C}_{(D^{*-1}_{w^*}, \partial I_{Q^*_\tau})})$*, and qualifying strong monotonicity condition* $(\boldsymbol{C}_{\partial \widetilde{F}_\tau})$*, primal evolution transport state system* $(\boldsymbol{\mathcal{MH}}_{\kappa_\tau})$ *attains a multidomain macro-hybrid mixed solution with a unique primal concentration solution component.*

For this transport underground mechanical system, we recall that the primal constraints of intrinsic concentration control mechanisms $\boldsymbol{\partial\varphi}_\tau$ must be defined by strongly monotone maximal monotone operators [22].

4.2 Multidomain Primal Evolution Macro-hybrid Mixed Compressible Darcian Flow Mechanical System

For a compressible flow component of the underground mechanical process, we shall consider a constrained driving flow in the spatial multidomain, with a macro-hybrid decomposition $\Omega = \bigcup_{e=1}^{E} \{\Omega_e\} \subset \Re^n$ of the porous medium in the sense of Sect. 2. Let $\{\boldsymbol{u}^*_e\} = \{\phi_e \boldsymbol{u}^*_{a_e}\}$ be considered as the dual local fluid velocity vector field, defined in terms of the local porosity $\{\phi_e\}$ and the average fluid velocity $\{\boldsymbol{u}^*_{a_e}\}$ fields. We shall assume that the fluid is characterized by the local compressibility parameters

$$\Big\{\vartheta_\rho\Big\} = \Big\{\frac{\partial \rho_e}{\partial p_e}\Big\}, \tag{18}$$

where $\{p_e\}$ denotes the primal local scalar pressure field of the mixed flow model, and $\{\rho_e\} > \{0_e\}$ the local compressible fluid mass density (pressure dependent).

Let $\{\widehat{q}^*_e\}$ be a given local mass flow rate per unit mass, and let $\{s^*_e\}$ be the local distributed intrinsic pressure control dual field. Then the local mass balance principle that corresponds to the fluid mass density $\{\rho_e\}$ is stated via the Reynolds theorem as follows (cf. (13)), for $e = 1, 2, \ldots, E$,

$$\frac{d}{dt}\int_{\mathcal{P}_{e_t}} \rho_e \, d\Omega_e = \int_{\mathcal{P}_{e_t}} \frac{\partial \rho_e}{\partial t} + div_e(\rho_e \boldsymbol{u}^*_e) \, d\Omega_e = -\int_{\mathcal{P}_{e_t}} s^*_e + \int_{\mathcal{P}_{e_t}} \widehat{q}^*_e \rho_e \, d\Omega_e \ . \tag{19}$$

Thereby, by a continuous integral localization, and applying the expanded time differentiation

$$\frac{\partial \rho_e}{\partial t} = \frac{\partial \rho_e}{\partial p_e}\frac{\partial p_e}{\partial t} = \vartheta_{\rho_e}\frac{\partial p_e}{\partial t}, \tag{20}$$

the local equations system of compressible Darcian flow constitutivity and mass balance turns out to be:

$$\left.\begin{aligned} \vartheta_{\rho_e}\frac{\partial p_e}{\partial t} + div\, \boldsymbol{w}_e^* &= -s_e^* + \widehat{q}_e^*\ \rho_e\ , \\ \mu_e(\rho_e \boldsymbol{K}_e)^{-1}\boldsymbol{w}_e^* &= -\boldsymbol{grad}\ p_e + \rho_e \boldsymbol{g}, \\ s_e^* \in \partial\varphi_{f_e}(p_e) \Leftrightarrow p_e &\in \varphi_{f_e}(s_e^*), \end{aligned}\right\} \text{ in } \Omega_e \times]0, T[\ , \tag{21}$$

with given local flow physical parameters: $\{\mu_e\}$ and $\{\boldsymbol{K}_e\}$ of dynamic viscosity and invertible symmetric intrinsic permeability tensor. Here, for variational convenience, the mass flux rate field $\boldsymbol{w}_e^* = \rho_e u_e^*$ is utilized as the dual dependent variable, instead of the common compressible fluid velocity $\boldsymbol{u}_e^*$. Further, $\boldsymbol{g}$ denotes the gravity acceleration vector field.

Flow system (21) constitutes a local primal evolution mixed pressure-velocity constrained model, with a local primal pressure intrinsic control mechanism $\partial\varphi_e$, a subpotential maximal monotone subdifferential, with inverse $\partial\varphi_e^*$, subdifferential of the conjugate functional φ_e^* [22].

Regarding the interaction of the flow system with its underground exterior, we shall consider, for $e = 1, 2, \ldots, E$, prescribed local normal mass flux rates $\{\widehat{w}_{n_e}^*\}$ as Neumann natural boundary conditions and local pressures $\{\widehat{p}_e\}$ as Dirichlet essential boundary ones; i.e., $\boldsymbol{w}_e^* \cdot \boldsymbol{n}_e = \widehat{w}_{n_e}^*$ on $\partial\Omega_{N_e}$ and $p_e = \widehat{p}_e$ on $\partial\Omega_{D_e}$. Particularly, in the case of pure local Neumann boundary conditions, $\partial\Omega_{D_e} = \emptyset$, we assume that for conservation of mass the local compatibility conditions $\int_{\partial\Omega_e} \widehat{u}_{n_e}^*\, d\partial\Omega_e = \int_{\Omega_e} \widehat{q}_e^*\, d\Omega_e$ are satisfied. On the other hand, in relation with the spatial nonoverlapping decompositions of the underground porous medium, the continuity transmission internal boundary constraints are stated by the dual mass flux rate and primal pressure mechanical constraints:

$$\left.\begin{aligned} p_e &= p_{p_k}\ , \\ \boldsymbol{w}_e^* \cdot \boldsymbol{n}_e &= -\boldsymbol{w}_e^* \cdot \boldsymbol{n}_k\ , \end{aligned}\right\} \text{ on } \Gamma_{ek} \times]0, T[\ , \tag{22}$$

across the underground multidomain interfaces $\Gamma_{ek} = \Gamma_e \cap \Gamma_k$, $1 \leq e < k \leq E$.

4.2.1 Multidomain Primal Evolution Macro-hybrid Mixed Variational Compressible Darcian Flow State System

As for the previous transport case, on the basis of [16], the variational primal evolution formulation of localized macro-hybrid mixed pressure-flux rate constrained f-model (18)–(22), as a state system governing the optimal control minimization problem $(\mathcal{O}_{MH})$ of the theory, turns out to be the following variational version of the Darcian flow underground process, (utilizing once again the simplified boldface generic notation).

$$(\mathcal{MH}_{\kappa_f})\begin{cases} \text{Given } \widehat{q^*}\boldsymbol{\rho} \in \mathcal{R}(\boldsymbol{div}_f) \subset \mathcal{V}^*_{MH_f},\ \widehat{\boldsymbol{p}}_0 \in \boldsymbol{H}_{MH_f}\ , \\ \boldsymbol{\rho g} \in \mathcal{R}(\boldsymbol{div}_f^T) \subset \mathcal{Y}_{MH_f}\ , \\ \text{find } \boldsymbol{p} \in \mathcal{W}_{MH_f} \text{ and } (\boldsymbol{w}^*, \boldsymbol{\lambda}^*_f) \in \mathcal{Y}^*_{MH_f} \times \mathcal{B}^*_{\Gamma_f}\ : \\ -\Big(\boldsymbol{div}_f, \boldsymbol{\pi}^T_{\Gamma_f}\Big)(\boldsymbol{w}^*, \boldsymbol{\lambda}^*_f) \\ \quad \in \boldsymbol{\vartheta}_\rho \dfrac{\partial \boldsymbol{p}}{\partial t} + \Big(\boldsymbol{\partial} \widetilde{\boldsymbol{F}}_f + \boldsymbol{\partial\varphi}_f\Big)\boldsymbol{p} + \boldsymbol{B}^*\boldsymbol{\kappa}_f - \widetilde{\boldsymbol{f}^*_f}\ , \qquad \text{in } \mathcal{V}^*_{MH_f}\ , \\ \quad \boldsymbol{p}(0) = \widehat{\boldsymbol{p}}_0\ ; \\ \Big(\boldsymbol{div}^T_f, \boldsymbol{\pi}_{\Gamma_f}\Big)(\boldsymbol{p}, \boldsymbol{p}) \in \Big(\boldsymbol{\mu}(\boldsymbol{\rho K})^{-1}, \boldsymbol{\partial I}_{Q^*_f}\Big)(\boldsymbol{w}^*, \boldsymbol{\lambda}^*_f) - (\boldsymbol{\rho g}, \boldsymbol{0}_{\mathcal{B}_{MH_f}}), \\ \qquad\qquad \text{in } \mathcal{Y}_{MH_f} \times \mathcal{B}_{\Gamma_f}\ . \end{cases}$$

Here dual unknown $\boldsymbol{\lambda}^*_f = \boldsymbol{\delta}^*_{\Gamma_f}\boldsymbol{w}^* \in \boldsymbol{Q}^*_\tau \subset \mathcal{B}^*_{\Gamma_\tau}$ stands as the flow macro-hybrid synchronizing internal boundary field, (cf. (3)). Also macro-hybrid primal pressure flow subdifferential and right hand side term correspond to

$$\begin{aligned} \boldsymbol{\partial}\widetilde{\boldsymbol{F}}_f &= \boldsymbol{\partial}(I_{\widehat{p}} \circ \boldsymbol{\pi}_{fD})\boldsymbol{p} : \mathcal{V}_{MH_f} \to \boldsymbol{2}^{\mathcal{V}^*_{MH_f}}, \\ -\widetilde{\boldsymbol{f}^*_f} &= -\widehat{\boldsymbol{q}^*}\boldsymbol{\rho} + \boldsymbol{\pi}^T_{N_f}\boldsymbol{w}^* \in \mathcal{V}^*_{MH_f}\ , \end{aligned} \tag{23}$$

where the local essential Dirichlet $\boldsymbol{\pi}_{\Sigma_f}\boldsymbol{p} = \widehat{\boldsymbol{p}}$ and Neumann $\boldsymbol{\delta}^*_{\Sigma_f}\boldsymbol{w}^* = \widehat{\boldsymbol{w}^*}$, prescribed external boundary trace fields, indicator functional $I_{\widehat{\boldsymbol{p}}}$ implementing the primal Dirichlet condition.

Regarding the abstract primal evolution macro-hybrid mixed state problem $(\mathcal{MH}_\kappa)$ of Sect. 3, as operator identifications: the primal coupling operator $\boldsymbol{\Lambda}_f = \boldsymbol{div}^T_f \in \mathcal{L}(\mathcal{V}_{MH_f}, \mathcal{Y}_{MH_f})$ with negative transpose $-\boldsymbol{\Lambda}^T_f = -\boldsymbol{div}_f \in \mathcal{L}(\mathcal{Y}^*_{MH_f}, \mathcal{V}^*_{MH_f})$; and the primal subdifferential $\boldsymbol{\partial F}_f = \boldsymbol{\partial\varphi}_f$, the dual pressure maximal monotone control mechanism. Also, corresponding dual subdifferential $\boldsymbol{\partial G}^*_f$ is given by the operator $\boldsymbol{\mu}(\boldsymbol{\rho K})^{-1} : \mathcal{Y}^*_{MH_f} \to \boldsymbol{2}^{\mathcal{Y}_{MH_f}}$.

In order to apply primal evolution macro-hybrid mixed fixed point existence Theorem 2, we assume that the subsurface flow local pressure initial data satisfies the regularity

$$\widehat{\boldsymbol{p}}_0 \in \boldsymbol{V}_{MH_f} \subset \boldsymbol{H}_{MH_f}\ . \tag{24}$$

Hence, a primal compressible flow solvability result is as follows.

Theorem 4 *Under dual corresponding interior local compatibility condition* $\big(\boldsymbol{C}_{(\partial G^*, \partial I_{Q^*})}\big)$ *and qualifying strong monotonicity condition* $(\boldsymbol{C}_{\partial \widetilde{F}_f})$ *of Theorem 2, primal state evolution compressible Darcian flow constrained problem* $(\mathcal{MH}_{\kappa_f})$ *has a macro-hybrid mixed solution with a unique primal pressure component.*

Lastly, for this compressible Darcian flow case, we emphasize that the primal constraints of intrinsic pressure control mechanisms $\partial\varphi_f$ should be modeled by strongly monotone maximal monotone operators [22].

5 Perturbation Conjugate Duality Evolution Optimal Control

In this section we present the multidomain, macro-hybrid mixed perturbation conjugate duality optimal control theory, an extension of the evolution mixed set-value variational theory [1].

Specifically, once the perturbation theory is stated through the corresponding minimization primal, maximization dual and Lagrangian minimax optimization problems, the corresponding multidomain macro-hybrid mixed optimality condition will be established, whose existence analysis via a primal resolvent stationary type fixed point characterization, will be treated in the next section, Sect. 6, performing then the whole optimization analysis of the theory in the last Sect. 7.

In accordance with the perturbation conjugate duality evolution mixed method [1], from the $(\mathcal{MH}_\kappa)$-primal evolution state inclusion, the primal macro-hybrid mixed state-control operator $\mathcal{T}_{MH} : \mathcal{W}_{MH} \times (\mathcal{Y}^*_{MH} \times \mathcal{B}^*_\Gamma) \times \mathcal{C}_{MH} \to \mathcal{V}^*_{MH}$ is given by

$$\mathcal{T}_{MH}(\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), \boldsymbol{\eta}) = \frac{d\boldsymbol{w}}{dt} + \boldsymbol{w}^*_w + \Big(\boldsymbol{\Lambda}^T, \boldsymbol{\pi}^T_\Gamma\Big)(\boldsymbol{x}^*, \boldsymbol{\chi}^*) + \boldsymbol{B}^*\eta, \qquad \boldsymbol{w}^*_w \in \partial\widetilde{\boldsymbol{F}}\boldsymbol{w}, \tag{25}$$

whose closed convex constraint domain turns out to be

$$\mathcal{M}_{MH} = \Big\{(\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), \boldsymbol{\eta}) \in \mathcal{W}_{MH} \times (\mathcal{Y}^*_{MH} \times \mathcal{B}^*_\Gamma) \times \mathcal{C}_{MH} : \boldsymbol{w} \in \mathcal{D}\big(\widetilde{F}\big),\ (\boldsymbol{x}^*, \boldsymbol{\chi}^*) \in \mathcal{D}\big(G^*\big) \times \boldsymbol{Q}^*,\ \boldsymbol{\eta} \in \mathcal{C}_{MH_{ad}}\Big\}. \tag{26}$$

Then, assuming that there is a closed subspace $\mathcal{Q}^*_{MH} \subset \mathcal{V}^*_{MH}$ of macro-hybrid perturbations such that

$$(\boldsymbol{C}_{\mathcal{T}_{MH}}) \quad \mathcal{Q}^*_{MH} \subset \Re_+(\mathcal{T}_{MH}\big(\mathcal{M}_{MH}\big) - \widetilde{\boldsymbol{f}}^*),$$

and introducing the closed convex subset $\mathcal{K}_{MH} \subset \big(\mathcal{W}_{MH} \times (\mathcal{Y}^*_{MH}, \mathcal{B}^*_\Gamma) \times \mathcal{C}_{MH}\big) \times \mathcal{Q}^*_{MH}$,

$$\mathcal{K}_{MH} = \Big\{((\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), \boldsymbol{\eta}), \boldsymbol{q}^*) \in \mathcal{M}_{MH} \times \mathcal{Q}^*_{MH} : \boldsymbol{q}^* = \mathcal{T}_{MH}\big(\mathcal{M}_{MH}\big) - \widetilde{\boldsymbol{f}^*}\Big\}, \tag{27}$$

the perturbation functional $\mathcal{S}_{MH} : (\boldsymbol{\mathcal{W}}_{MH} \times \big(\boldsymbol{\mathcal{Y}}^*_{MH}, \boldsymbol{\mathcal{B}}^*_{\Gamma}\big) \times \boldsymbol{\mathcal{C}}_{MH}) \times \boldsymbol{\mathcal{Q}}^*_{MH} \to \Re \cup \{+\infty\}$, proper convex and lower semicontinuous, is naturally defined as

$$\begin{aligned} &\mathcal{S}_{MH}\big((\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), \boldsymbol{\eta}), \boldsymbol{q}^*\big) \\ &\quad = J_{MH}\big(\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), \boldsymbol{\eta}\big) + I_{\mathcal{K}_{MH}}\big((\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), \boldsymbol{\eta}), \boldsymbol{q}^*\big), \end{aligned} \tag{28}$$

with an associated marginal or infimal value convex functional $\mu^*_{MH} : \boldsymbol{\mathcal{Q}}^*_{MH} \to \Re \cup \{+\infty\}$,

$$\mu^*_{MH}(\boldsymbol{q}^*) = \inf_{(w,(x^*,\chi^*),\eta)\in(\mathcal{W}_{MH}\times(\mathcal{Y}^*_{MH},\mathcal{B}^*_{\Gamma})\times\mathcal{C}_{MH}} \mathcal{S}_{MH}\big((\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), \boldsymbol{\eta}), \boldsymbol{q}^*\big). \tag{29}$$

Thereby, the primal, dual and Lagrangian mixed optimization problems of the theory, in a multidomain macro-hybrid perturbation sense, following the duality theory [23] (cf. [1]), can be defined as follows:

Firstly, the optimal control $(\boldsymbol{\mathcal{O}}_{MH})$-subpotential $J_{MH} : \boldsymbol{\mathcal{W}}_{MH} \times (\boldsymbol{\mathcal{Y}}^*_{MH}, \boldsymbol{\mathcal{B}}^*_{\Gamma}) \times \boldsymbol{\mathcal{C}}_{MH} \to \Re \cup \{+\infty\}$ corresponds to a zero perturbation; i.e.,

$$J_{MH}\big(\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), \boldsymbol{\eta}\big) = \mathcal{S}_{MH}\big((\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), \boldsymbol{\eta}), \mathbf{0}_{\mathcal{Q}^*_{MH}}\big), \tag{30}$$

with a perturbed primal evolution macro-hybrid mixed optimal control problem given by

$$(\widetilde{\boldsymbol{\mathcal{OC}}}_{MH}) \begin{cases} \text{Find } \big(\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}\big) \in \boldsymbol{\mathcal{M}}_{MH} : \\ \mathcal{S}_{MH}\big((\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}), \mathbf{0}_{\mathcal{Q}^*_{MH}}\big) \leq \mathcal{S}_{MH}\big((\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), \boldsymbol{\eta}), \mathbf{0}_{\mathcal{Q}^*_{MH}}\big), \\ \qquad \forall (\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), \boldsymbol{\eta}) \in (\boldsymbol{\mathcal{W}}_{MH} \times (\boldsymbol{\mathcal{Y}}^*_{MH}, \boldsymbol{\mathcal{B}}^*_{\Gamma})) \times \boldsymbol{\mathcal{C}}_{MH} . \end{cases}$$

Secondly, the perturbed dual convex functional is defined by $\pi_{MH} : \boldsymbol{\mathcal{Q}}_{MH} \to \Re \cup \{+\infty\}$ on the primal constraint qualifying closed subspace $\boldsymbol{\mathcal{Q}}_{MH} \subset \boldsymbol{\mathcal{V}}_{MH}$, via the perturbation conjugate functional $\mathcal{S}^*_{MH} : (\boldsymbol{\mathcal{W}}^*_{MH} \times (\boldsymbol{\mathcal{Y}}_{MH}, \boldsymbol{\mathcal{B}}_{\Gamma}) \times \boldsymbol{\mathcal{C}}^*_{MH}) \times \boldsymbol{\mathcal{Q}}_{MH} \to \Re \cup \{+\infty\}$, for $\boldsymbol{q} \in \boldsymbol{\mathcal{Q}}_{MH}$, is given by

$$\pi_{MH}(\boldsymbol{q}) = \mathcal{S}^*_{MH}\big((\mathbf{0}_{\mathcal{V}^*_{MH}}, (\mathbf{0}_{\mathcal{Y}_{MH}}, \mathbf{0}_{\mathcal{B}_{\Gamma}}), \mathbf{0}_{\mathcal{C}^*_{MH}}), \boldsymbol{q}\big), \tag{31}$$

which results to be the conjugate of marginal functional (29); i.e, $\pi_{MH} = \mu_{MH}$. Then the perturbation dual maximization problem is expressed as

$$(\widetilde{\boldsymbol{\mathcal{OC}}}^*_{MH}) \begin{cases} \text{Find } \boldsymbol{p} \in \boldsymbol{\mathcal{D}}(\pi_{MH}) : \\ -\mathcal{S}^*_{MH}\big((\mathbf{0}_{\mathcal{V}^*_{MH}}, (\mathbf{0}_{\mathcal{Y}_{MH}}, \mathbf{0}_{\mathcal{B}_{\Gamma}}), \mathbf{0}_{\mathcal{C}^*_{MH}}), \boldsymbol{p}\big) \\ \quad \geq -\mathcal{S}^*_{MH}((\mathbf{0}_{\mathcal{V}^*_{MH}}, \big(\mathbf{0}_{\mathcal{Y}_{MH}}, \mathbf{0}_{\mathcal{B}_{\Gamma}}), \mathbf{0}_{\mathcal{C}^*_{MH}}), \boldsymbol{q}\big) \\ \qquad \forall \boldsymbol{q} \in \boldsymbol{\mathcal{Q}}_{MH} . \end{cases}$$

And thirdly, the perturbed convex-concave dual macro-hybrid mixed Lagrangian $\mathcal{L}_{MH} : (\boldsymbol{\mathcal{W}}_{MH} \times (\boldsymbol{\mathcal{Y}}^*_{MH}, \boldsymbol{\mathcal{B}}^*_{MH}) \times \boldsymbol{\mathcal{C}}_{MH} \times \boldsymbol{\mathcal{Q}}_{MH} \to \Re \cup \{+\infty\}$ of the theory is defined by

$$\begin{aligned}
&\mathcal{L}_{MH}((\boldsymbol{w},(\boldsymbol{x}^*,\boldsymbol{\chi}^*),\boldsymbol{\eta}),\boldsymbol{p}) = -\mathcal{S}^*_{MH_{(w,(x^*,\chi^*),\eta)}}(\boldsymbol{p})\\
&= \begin{cases} J_{MH}(\boldsymbol{w},(\boldsymbol{x}^*,\boldsymbol{\chi}^*),\boldsymbol{\eta}) - \big\langle \boldsymbol{\mathcal{T}}_{MH}(\boldsymbol{w},(\boldsymbol{x}^*,\boldsymbol{\chi}^*),\boldsymbol{\eta}) - \widetilde{\boldsymbol{f}}^*, \\ \quad \boldsymbol{p}\big\rangle_{\mathcal{Q}^*,\mathcal{Q}}, \qquad \text{if } (\boldsymbol{w},(\boldsymbol{x}^*,\boldsymbol{\chi}^*),\boldsymbol{\eta}) \in \boldsymbol{\mathcal{D}}_{MH}\ , \\ +\infty, \qquad \text{if } (\boldsymbol{w},(\boldsymbol{x}^*,\boldsymbol{\chi}^*)),\boldsymbol{\eta}) \notin \boldsymbol{\mathcal{D}}_{MH}\ , \end{cases}\\
&\boldsymbol{\mathcal{D}}_{MH} = \Big\{(\boldsymbol{w},(\boldsymbol{x}^*,\boldsymbol{\chi}^*),\boldsymbol{\eta}) \in \boldsymbol{\mathcal{M}}_{MH} : \boldsymbol{\mathcal{T}}_{MH}(\boldsymbol{\mathcal{M}}_{MH}) - \widetilde{\boldsymbol{f}}^* \subset \boldsymbol{\mathcal{Q}}^*_{MH}\Big\},
\end{aligned} \tag{32}$$

where $\mathcal{S}^*_{MH_{(w,x^*,\chi^*),\eta)}} : \boldsymbol{\mathcal{Q}}_{MH} \rightarrow \Re \cup \{+\infty\}$ is the conjugate of functional $\mathcal{S}_{MH_{(w,(x^*,\chi^*),\eta)}} = \mathcal{S}_{MH}(\boldsymbol{w},(\boldsymbol{x}^*,\boldsymbol{\chi}^*),\boldsymbol{\eta}),\cdot) : \boldsymbol{\mathcal{Q}}^*_{MH} \rightarrow \Re\cup\{+\infty\}$, for $(\boldsymbol{w},(\boldsymbol{x}^*,\boldsymbol{\chi}^*),\boldsymbol{\eta}) \in \boldsymbol{\mathcal{W}}_{MH} \times (\boldsymbol{\mathcal{Y}}^*_{MH},\boldsymbol{\mathcal{B}}^*_{\Gamma}) \times \boldsymbol{\mathcal{C}}_{MH}$. Further, $\boldsymbol{\mathcal{D}}_{MH}$ corresponds to the projection of perturbation constraint set $\boldsymbol{\mathcal{K}}_{MH}$ on $(\boldsymbol{\mathcal{W}}_{MH} \times (\boldsymbol{\mathcal{Y}}^*_{MH},\boldsymbol{\mathcal{B}}^*_{MH}) \times \boldsymbol{\mathcal{C}}_{MH})$. Consequently, the macro-hybrid mixed perturbed $\mathcal{L}_{MH}$-minimax problem turns out to be

$$(\widetilde{\boldsymbol{\mathcal{MOC}}_{MH}}) \begin{cases} \text{Find } ((\boldsymbol{v},(\boldsymbol{y}^*,\boldsymbol{\lambda}^*),\boldsymbol{\kappa}),\boldsymbol{p}) \in \boldsymbol{\mathcal{D}}(\mathcal{L}_{MH}) : \\ \mathcal{L}_{MH}((\boldsymbol{v},(\boldsymbol{y}^*,\boldsymbol{\lambda}^*),\boldsymbol{\kappa}),\boldsymbol{q}) \\ \quad \leq \mathcal{L}_{MH}((\boldsymbol{v},(\boldsymbol{y}^*,\boldsymbol{\lambda}^*),\boldsymbol{\kappa}),\boldsymbol{p}) \\ \quad \leq \mathcal{L}_{MH}((\boldsymbol{w},(\boldsymbol{x}^*,\boldsymbol{\chi}^*),\boldsymbol{\eta}),\boldsymbol{p}), \\ \forall\ ((\boldsymbol{w},(\boldsymbol{x}^*,\boldsymbol{\chi}^*),\boldsymbol{\eta}),\boldsymbol{q}) \\ \in (\boldsymbol{\mathcal{W}}_{MH} \times (\boldsymbol{\mathcal{Y}}^*_{MH},\boldsymbol{\mathcal{B}}^*_{\Gamma}) \times \boldsymbol{\mathcal{C}}_{\boldsymbol{MH}}) \times \boldsymbol{\mathcal{Q}}_{MH}\ . \end{cases}$$

Next, we lastly determine a new multidomain macro-hybrid mixed optimality condition for this perturbed Lagrangian problem (cf. [8]).

Theorem 5 *Let* $((\boldsymbol{v},(\boldsymbol{y}^*,\boldsymbol{\lambda}^*),\boldsymbol{\kappa}),\boldsymbol{p}) \in \boldsymbol{\mathcal{D}}_{MH}$ *be a solution of perturbed minimax problem* $\widetilde{\boldsymbol{\mathcal{MOC}}^*_{MH}}$*), then as an state control-perturbation function,* $((\boldsymbol{v},(\boldsymbol{y}^*,\boldsymbol{\lambda}^*),\boldsymbol{\kappa}),\boldsymbol{p})$ *solves the macro-hybrid dual mixed problem*

$$(\boldsymbol{\mathcal{MOC}}_{MH}) \begin{cases} \textit{Find } (\boldsymbol{v},(y^*,\boldsymbol{\lambda}^*),\boldsymbol{\kappa}) \in \boldsymbol{\mathcal{M}}_{MH} \textit{ and } \boldsymbol{p} \in \boldsymbol{\mathcal{Q}}_{MH} : \\ \boldsymbol{\mathcal{T}}^T_{MH}\boldsymbol{p} \in \boldsymbol{\partial J}_{MH}\big(\boldsymbol{v},(\boldsymbol{y}^*,\boldsymbol{\lambda}^*),\boldsymbol{\kappa}\big), \\ \qquad\qquad \textit{in } \boldsymbol{\mathcal{W}}^*_{MH} \times (\boldsymbol{\mathcal{Y}}_{MH},\boldsymbol{\mathcal{B}}_{\Gamma}) \times \boldsymbol{\mathcal{C}}^*_{MH}\ , \\ -\boldsymbol{\mathcal{T}}_{MH}\big(\boldsymbol{v},(\boldsymbol{y}^*,\boldsymbol{\lambda}^*),\boldsymbol{\kappa}\big) \in \boldsymbol{\partial 0}_{\mathcal{Q}_{MH}}\boldsymbol{p} - \widetilde{\boldsymbol{f}}^*,\ \textit{in } \boldsymbol{\mathcal{Q}}^*_{MH}\ , \end{cases}$$

a variational optimality condition of Lagrangian perturbed minimax optimization problem $(\widetilde{\boldsymbol{\mathcal{MOC}}^*_{MH}})$.

Proof The Lagrangian functional $\mathcal{L}_{MH}$ value at $((\boldsymbol{w},(\boldsymbol{x}^*,\boldsymbol{\chi}^*),\boldsymbol{\eta}),\boldsymbol{q}) \in \boldsymbol{\mathcal{D}}_{MH}$, turns out to be $\mathcal{L}_{MH}((\boldsymbol{w},(\boldsymbol{x}^*,\boldsymbol{\chi}^*),\boldsymbol{\eta}),\boldsymbol{q}) = J_{MH}(\boldsymbol{w},(\boldsymbol{x}^*,\boldsymbol{\chi}^*),\boldsymbol{\eta}) - \big\langle\boldsymbol{\mathcal{T}}_{MH}(\boldsymbol{w},(\boldsymbol{x}^*,\boldsymbol{\chi}^*),\boldsymbol{\eta}) - \widetilde{\boldsymbol{f}}^*,\boldsymbol{q}\big\rangle_{\mathcal{V}^*_{MH},\mathcal{V}_{MH}}$, and from the dual inclusion in $\boldsymbol{\mathcal{Q}}^*_{MH}$ of mixed problem $(\boldsymbol{\mathcal{MOC}}_{MH})$, $\boldsymbol{\mathcal{T}}_{MH}(\boldsymbol{w},(\boldsymbol{y}^*,\boldsymbol{\chi}^*),\boldsymbol{\eta}) = \widetilde{\boldsymbol{f}}^*$. Consequently, $\mathcal{L}_{MH}((\boldsymbol{w},(\boldsymbol{x}^*,\boldsymbol{\chi}^*),\boldsymbol{\eta}),\boldsymbol{q}) = J_{MH}((\boldsymbol{w},(\boldsymbol{x}^*,\boldsymbol{\chi}^*)),\boldsymbol{\eta})$, and multidomain macro-hybrid dual mixed system

$(\mathcal{MOC}_{MH})$ states a variational optimality condition for Lagrangian $\mathcal{L}_{MH}$-perturbed optimization $(\widetilde{\mathcal{MOC}^*_{MH}})$ problem. □

5.1 Multidomain Optimal Control of Transport State System

For transport macro-hybrid mixed state system $(\mathcal{MH}_{\kappa_\tau})$, formulated in Sect. 4.1, the cost or objective macro-hybrid mixed functional $J_{MH} : \mathcal{W}_{MH} \times (\mathcal{Y}^*_{MH}, \mathcal{B}^*_\Gamma) \times \mathcal{C}_{MH} \to \Re \cup \{+\infty\}$ of the theory is related to the local mixed mechanical state fields $\boldsymbol{c} \in \mathcal{W}_{MH_\tau}$ and $(\boldsymbol{d}^*, \boldsymbol{\lambda}^*) \in \mathcal{Y}^*_{MH_\tau} \times \mathcal{B}^*_{\Gamma_\tau}$, of primal mass concentration and dual diffusion-dispersion flux, and syncronizing interior boundary.

In this case, for proposed optimal local target profiles of the primal concentration field $\widetilde{\boldsymbol{c}} \in \mathcal{W}_{MH_\tau}$, the dual diffusion-dispersion and syncronization interior boundary fields $(\widetilde{\boldsymbol{d}}^*, \widetilde{\boldsymbol{\lambda}}^*) \in \mathcal{Y}^*_{MH_\tau} \times \mathcal{B}^*_{\Gamma_\tau}$, the optimal control of the transport state system may be implemented in terms of instantaneous convex cost functionals defined by

$$\begin{aligned} g_{1_\tau}(t; \boldsymbol{c}(t)) &= w_{g_{1_\tau}}(t)\ \tfrac{1}{2}\|\boldsymbol{c}(t) - \widetilde{\boldsymbol{c}}(t)\|^2_{\mathcal{W}_{MH_\tau}}, \\ g_{2_\tau}(t; \boldsymbol{d}^*(t)) &= w_{g_{2_\tau}}(t)\ \tfrac{1}{2}\|\boldsymbol{d}^*(t) - \widetilde{\boldsymbol{d}}^*(t)\|^2_{\mathcal{Y}^*_{MH_\tau}}, \\ g_{3_\tau}(t; \boldsymbol{\lambda}^*(t)) &= w_{g_{3_\tau}}(t)\ \tfrac{1}{2}\|\boldsymbol{\lambda}^*(t) - \widetilde{\boldsymbol{\lambda}}^*(t)\|^2_{\mathcal{B}^*_{\Gamma_\tau}}, \end{aligned} \tag{33}$$

and further, for a specific local $\boldsymbol{\kappa}$-optimal control field on the Hilbert space $\mathcal{C}_{MH_\tau} = \mathcal{H}_{MH_\tau} \equiv L^2(]0, T[; \boldsymbol{L}^2_{MH})$, by the convex functional

$$j_\tau(t; \boldsymbol{\kappa}(t)) = w_{j_\tau}(t)\ \tfrac{1}{2}\|\boldsymbol{\kappa}(t)\|^2_{\mathcal{C}_{MH_\tau}}\ ; \tag{34}$$

where the weight coefficients $w_{g_{1_\tau}}$, $w_{g_{2_\tau}}$, $w_{g_{3_\tau}}$ and w_{j_τ}, are given bounded and strictly positive $L^\infty]0, T[$-functions. Also, the set $\mathcal{C}_{MH_{cd}} \subset \mathcal{C}_{MH}$ of admissible controls may be of an obstacle constraint model type [22].

Consequently, in this manner, a specific transport cost functional would be defined in $\mathcal{W}_{MH_\tau} \times (\mathcal{Y}^*_{MH_\tau} \times \mathcal{B}^*_{\Gamma_\tau}) \times \mathcal{C}_{MH_\tau}$ by the continuous convex functional

$$\begin{aligned} &\widetilde{J}_{MH_\tau}(\boldsymbol{c}, (\boldsymbol{d}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}) \\ &\quad = w_{g_{1_\tau}}(t)\ \tfrac{1}{2}\|\boldsymbol{c}(t) - \widetilde{\boldsymbol{c}}(t)\|^2_{\mathcal{W}_{MH_\tau}} + w_{g_{2_\tau}}(t)\ \tfrac{1}{2}\|\boldsymbol{d}^*(t) - \widetilde{\boldsymbol{d}}^*(t)\|^2_{\mathcal{Y}^*_{MH_\tau}} \\ &\quad + w_{g_{3_\tau}}(t)\ \tfrac{1}{2}\|\boldsymbol{\lambda}^*(t) - \widetilde{\boldsymbol{\lambda}}^*(t)\|^2_{\mathcal{B}^*_{\Gamma_\tau}} + w_{j_\tau}(t)\ \tfrac{1}{2}\|\boldsymbol{\kappa}(t)\|^2_{\mathcal{C}_{MH_\tau}}, \end{aligned} \tag{35}$$

such that, for the validity of Corollary 2—of next Sect. 6,—that will determine the solvability of the optimality macro-hybrid mixed problem $(\mathcal{MOC}^*_{MH})$, its own transport objective maximal monotone subdifferential operator $\partial \widetilde{J}_{MH_\tau}$ should be strongly monotone.

Concerning the primal macro-hybrid mixed state-control operator $\mathcal{T}_{MH} : \mathcal{W}_{MH} \times (\mathcal{Y}^*_{MH} \times \mathcal{B}^*_{MH}) \times \mathcal{C}_{MH} \to \mathcal{V}^*_{MH}$ of the theory, (25), its transport model version is given, for $c_c^* \in \big(\partial \widetilde{F}_\tau + \partial \varphi_\tau\big) c$, by

$$\mathcal{T}_{\tau_{MH}}(c, (d^*, \lambda^*), \kappa) \;=\; \frac{dc}{dt} + c_c^* + \Big(div_\tau^T, \pi_{\Gamma_\tau}^T\Big)(d^*, \lambda^*) + B^* \kappa_\tau \,. \tag{36}$$

5.2 Multidomain Optimal Control of Flow State System

In the case of flow state system $(\mathcal{MH}_{f_\kappa})$ of Sect. 4.2, governing optimal control problem $(\mathcal{O}_{MH})$, its control is realized by the cost or objective macro-hybrid mixed functional $J_{MH} : \mathcal{W}_{MH} \times (\mathcal{Y}_{MH}^* \times \mathcal{B}_\Gamma^*) \times \mathcal{C}_{MH} \to \Re \cup \{+\infty\}$, related to the local mechanical state fields: $p \in \mathcal{W}_{f_{MH}}$ and $(w^*, \lambda_f^*) \in \mathcal{Y}_{f_{MH}}^* \times \mathcal{B}_{f_\Gamma}^*$, of multidomain primal pressure, dual mass flux rate and internal boundary synchronizing fields.

Hence, considering the control of the system in accordance with optimal local target profiles of primal pressure $\widetilde{p}$, dual mass flux rate $\widetilde{w^*}$ and internal boundary synchronizing $\widetilde{\lambda_f^*}$ fields, objective functional J_{MH} would be implemented in terms of instantaneous convex state-control functionals, defined by

$$\begin{aligned} g_{1_f}(t; p(t)) &= w_{g_{1_f}}(t)\ \tfrac{1}{2}\|p(t) - \widetilde{p}(t)\|^2_{\mathcal{V}_{MH_f}}\ , \\ g_{2_f}(t; w^*(t)) &= w_{g_{2_f}}(t)\ \tfrac{1}{2}\|w^*(t) - \widetilde{w^*}(t)\}\|^2_{\mathcal{Y}^*_{MH_f}}\ , \\ g_{3_f}(t; \lambda_f^*(t)) &= w_{g_{3_f}}(t)\ \tfrac{1}{2}\|\lambda_f^*(t) - \widetilde{\lambda_f^*}(t)\}\|^2_{\mathcal{Y}^*_{MH_f}}\ , \\ j_f(t; \kappa(t)) &= w_{j_f}(t)\ \tfrac{1}{2}\|\kappa(t)\|^2_{\mathcal{H}_{MH_f}}\ ; \end{aligned} \tag{37}$$

the latter functional term, for a specific local κ-optimal control field from the Hilbert space $\mathcal{C}_{MH} = \mathcal{H}_{MH_f} \equiv L^2(]0,T[; L^2_{MH})$. Here, as in the previous transport case, the weight coefficients $w_{g_{1_f}}$, $w_{g_{2_f}}$, $w_{g_{3_f}}$ and w_{j_f}, are assumed to be bounded and strictly positive $L^\infty]0,T[$-functions.

In such a manner, a flow cost functional defined in $\mathcal{W}_{MH_f} \times (\mathcal{Y}^*_{MH_f} \times \mathcal{B}^*_{\Gamma_f}) \times \mathcal{C}_{MH_f}$, would turn out to be

$$\begin{aligned} &\widetilde{J}_{MH_f}((p, (w^*, \lambda_f^*)), \kappa) \\ &\quad = w_{g_{1_f}}(t)\tfrac{1}{2}\|p(t) - \widetilde{p}(t)\|^2_{\mathcal{V}_{MH_f}} +\ w_{g_{2_f}}(t)\tfrac{1}{2}\|w^*(t) - \widetilde{w^*}(t)\}\|^2_{\mathcal{Y}^*_{MH_f}} \\ &\quad + w_{g_{3_f}}(t)\tfrac{1}{2}\|\lambda_f^*(t) - \widetilde{\lambda_f^*}(t)\}\|^2_{\mathcal{Y}^*_{MH_f}} +\ w_{j_f}(t)\ \tfrac{1}{2}\|\kappa(t)\|^2_{\mathcal{H}_{MH_f}}\ , \end{aligned} \tag{38}$$

a continuous and convex functional. Further, in this flow case, for the solvability of corresponding optimality macro-hybrid mixed problem $(\widetilde{\mathcal{MOC}}_{MH})$, objective monotone subdifferential operator $\partial \widetilde{J}_{MH_f}$ should satisfy the strong monotonicity condition of Corollary 2, to be established in Sect. 6 below.

Lastly, on the other hand, the primal macro-hybrid mixed flow state-control operator $\mathcal{T}_{MH} : (\mathcal{V}_{MH_f} \times \mathcal{X}^*_{MH_f}) \times \mathcal{C}_{MH_f} \to \mathcal{Y}_{MH_f}$ of the theory, (25), is identified in terms of $\boldsymbol{p}^*_f \in \left(\boldsymbol{\partial} \widetilde{\boldsymbol{F}}_f + \boldsymbol{\partial}\boldsymbol{\varphi}_f\right)\boldsymbol{p}$ by

$$\mathcal{T}^*_{MH_f}(\boldsymbol{p}, (\boldsymbol{w}^*, \boldsymbol{\lambda}^*_f), \boldsymbol{\kappa}) = \boldsymbol{\vartheta}_\rho \frac{\partial \boldsymbol{p}}{\partial t} + \boldsymbol{p}^*_f + \left(\boldsymbol{div}_f, \boldsymbol{\pi}^T_{\Gamma_f}\right)(\boldsymbol{w}^*, \boldsymbol{\lambda}^*_f) + \boldsymbol{B}^*\boldsymbol{\kappa}_f\,. \tag{39}$$

6 Existence Analysis of Optimality Mixed System ($\mathcal{MOC}_{MH}$)

For the solvability analysis of multidomain macro-hybrid mixed optimality problem ($\mathcal{MOC}_{MH}$) of Theorem 5, we next apply a resolvent fixed point stationary type method [3].

We first establish the primal duality principle of the optimality system, (cf. [26, Sect. 2]), introducing the dual state-control compatibility condition

$$(\widetilde{\boldsymbol{C}}_{-\mathcal{T}_{MH}}) \quad -\mathcal{T}_{MH} : \mathcal{W}_{MH} \times (\mathcal{Y}^*_{MH}, \mathcal{B}^*_\Gamma) \times \mathcal{C}_{MH} \to \mathcal{Q}^*_{MH}, \quad \text{is surjective,}$$

that permits the composition dualizaion of its dual inclusion.

Theorem 6 *Under compatibility condition* $(\widetilde{\boldsymbol{C}}_{-\mathcal{T}_{MH}})$*, the macro-hybrid mixed optimality problem* ($\mathcal{MOC}_{MH}$) *solvability, is equivalent to that of its macro-hybrid primal optimality problem*

$$(\mathcal{P}_{MH}) \begin{cases} \text{Find } ((\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*)), \boldsymbol{\kappa}) \in \mathcal{M}_{MH} : \\ \boldsymbol{0} \in \boldsymbol{\partial}\boldsymbol{J}_{MH}(\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}) + \boldsymbol{\partial}\left(I_{\left\{0_{\mathcal{Q}^*_{MH}}\right\}} \circ (-\mathcal{T}_{MH})\right) \\ \quad \left((\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}) - (\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\eta}^*), \boldsymbol{\nu})_{-\widetilde{\boldsymbol{f}}^*}\right), \\ \qquad\qquad \text{in } \mathcal{W}^*_{MH} \times (\mathcal{Y}_{MH}, \mathcal{B}_\Gamma) \times \mathcal{C}^*_{MH}\,, \end{cases}$$

where $(\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\eta}^*, \boldsymbol{\nu})_{-\widetilde{\boldsymbol{f}}^*} \in \mathcal{W}_{MH} \times (\mathcal{Y}^*_{MH}, \mathcal{B}^*_\Gamma) \times \mathcal{C}_{MH}$ *is a* $-\mathcal{T}_{MH}$*-preimage of functional* $-\widetilde{\boldsymbol{f}}^*$*. That is, if* $((\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}), \boldsymbol{p}) \in \mathcal{M}_{MH} \times \mathcal{Q}_{MH}$ *is a solution of mixed optimality problem* ($\mathcal{MOC}_{MH}$) *then* $(\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa})$ *is a solution of primal optimality problem* ($\mathcal{P}_{MH}$)*. Conversely, if* $(\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}) \in \mathcal{M}_{MH}$ *is a solution of problem* ($\mathcal{P}_{MH}$) *then there is a dual perturbation functional* $\boldsymbol{p} \in \boldsymbol{\partial}\boldsymbol{I}_{\{0_{\mathcal{Q}^*_{MH}}\}}\Big(-\mathcal{T}_{MH}\big((\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}) + -\widetilde{\boldsymbol{f}}^*\big)\Big) \subset \mathcal{Q}_{MH}$ *such that* $((\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}), \boldsymbol{p})$ *is a solution of mixed optimality problem* ($\mathcal{MOC}_{MH}$)*.*

Proof Indeed, under dual state-control condition $(\widetilde{\boldsymbol{C}}_{-\mathcal{T}_{MH}})$, the principle is a direct necessary result via the convex dualization of problem ($\mathcal{MOC}_{MH}$)-dual inclusion followed by its corresponding composition dualization (cf. Lemma 1 Proof):

$$\begin{gathered} -\mathcal{T}_{MH}\big(\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}\big) \in \boldsymbol{\partial}\boldsymbol{0}_{\mathcal{Q}_{MH}}\boldsymbol{p} - \widetilde{\boldsymbol{f}}^* \\ \Longleftrightarrow \\ -\mathcal{T}^T_{MH}\boldsymbol{p} \in \boldsymbol{\partial}\Big(I_{\{0_{\mathcal{Q}^*_{MH}}\}} \circ (-\mathcal{T}_{MH})\Big)\Big((\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}) - (\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\eta}^*), \boldsymbol{\nu})_{-\widetilde{\boldsymbol{f}}^*}\Big). \end{gathered} \tag{40}$$

For the sufficiency, let $((\boldsymbol{v}, (y^*, \boldsymbol{\lambda}^*)), \boldsymbol{\kappa}) \in \boldsymbol{\mathcal{M}}_{MH} \subset \boldsymbol{\mathcal{W}}_{MH} \times (\boldsymbol{\mathcal{Y}}^*_{MH}, \boldsymbol{\mathcal{B}}^*_{MH}) \times \boldsymbol{\mathcal{C}}_{MH}$ be a solution of primal optimality problem $(\boldsymbol{\mathcal{P}}_{MH})$. Then there is a functional $(\boldsymbol{w}^*, (\boldsymbol{x}, \boldsymbol{\chi}), \boldsymbol{\eta}^*) \in \boldsymbol{\partial J}_{MH}\big((\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*)), \boldsymbol{\kappa}\big)$ such that $-(\boldsymbol{w}^*\ (\boldsymbol{x}, \boldsymbol{\chi}), \boldsymbol{\eta}^*) - \widetilde{\boldsymbol{f}}^* \in \boldsymbol{\partial}\Big(I_{\{0_{\mathcal{Q}^*_{MH}}\}} \circ (-\boldsymbol{\mathcal{T}}_{\boldsymbol{MH}})\Big)\ \Big((\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*)) - (\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\eta}^*), \boldsymbol{v})_{-\widetilde{f}^*}\Big)$. Taking variations for the variational inequality of this last subdifferential inclusion of the kind $(\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), +\boldsymbol{\eta}) = \pm(\boldsymbol{v}, (\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), \boldsymbol{\eta})_0, + \Big((\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*))\ -\big(\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\eta}^*), \boldsymbol{v})_{-\widetilde{f}^*}\Big) \in \boldsymbol{\mathcal{D}}\Big(I_{\{0_{\mathcal{Q}^*_{MH}}\}} \circ (-\boldsymbol{\mathcal{T}}_{MH})\Big)$, for any $(\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), \boldsymbol{\eta})_0$ in the negative state-control kernel $\boldsymbol{\mathcal{N}}(-\boldsymbol{\mathcal{T}}_{MH})$, it follows that $\big\langle -(\boldsymbol{w}^*,\ (\boldsymbol{x}, \boldsymbol{\chi}), \boldsymbol{\eta}^*), (\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), \boldsymbol{\eta})_0\big\rangle_{\mathcal{M}_{MH}} = 0$. That is, $(-(\boldsymbol{w}^*,\ (\boldsymbol{x}, \boldsymbol{\chi}), \boldsymbol{\eta}^*))$ belongs to the polar subspace $(\boldsymbol{\mathcal{N}}(-\boldsymbol{\mathcal{T}}_{MH}))^\circ$. Now, since condition $(\widetilde{\boldsymbol{C}}_{-\mathcal{T}_{MH}})$ implies the closure of the range $\boldsymbol{\mathcal{R}}(-\boldsymbol{\mathcal{T}}_{MH})$, from the Closed Range Theorem $\boldsymbol{\mathcal{N}}(-\boldsymbol{\mathcal{T}}_{MH})^\circ = \boldsymbol{\mathcal{R}}(-\boldsymbol{\mathcal{T}}^T_{MH})$. Therefore, there is a functional $\boldsymbol{p} \in \boldsymbol{\mathcal{Q}}_{MH}$ such that $(\boldsymbol{w}^*, (\boldsymbol{x}, \boldsymbol{\chi}),\ \boldsymbol{\eta}^*) = -\boldsymbol{\mathcal{T}}^T_{MH}\boldsymbol{p} - \widetilde{\boldsymbol{f}}^*$, and applying composition dualization (40), $((\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}), \boldsymbol{p})$ is a solution of mixed optimality problem $(\boldsymbol{\mathcal{MOC}}_{MH})$. □

Next, introducing the macro-hybrid objective subdifferential monotonicity qualifying condition

$$(\boldsymbol{C}_{\partial J_{MH}}) \begin{cases} \boldsymbol{\partial J}_{MH} : \boldsymbol{\mathcal{U}}_{MH} \to \boldsymbol{\mathcal{U}}^*_{MH} \text{ is strongly monotone; i.e., } \exists\, \alpha > 0 : \\ \big\langle \widetilde{\boldsymbol{v}}^* - \boldsymbol{v}^*, \widetilde{\boldsymbol{v}}_{\mathcal{U}_{MH}} - \boldsymbol{v}_{\mathcal{U}_{MH}} \big\rangle_{\mathcal{U}_{MH}} \geq \alpha \|\widetilde{\boldsymbol{v}}_{\mathcal{U}_{MH}} - \boldsymbol{v}_{\mathcal{U}_{MH}}\|^2_{\mathcal{U}_{MH}}, \\ \forall\, \big(\widetilde{\boldsymbol{v}}_{\mathcal{U}_{MH}} = ((\widetilde{\boldsymbol{v}}, \widetilde{\boldsymbol{q}}^*), \widetilde{\boldsymbol{\eta}}), \boldsymbol{v}_{\mathcal{U}_{MH}} = ((\boldsymbol{v}, \boldsymbol{q}^*), \boldsymbol{\eta})\big) \in \boldsymbol{\mathcal{U}}_{MH}, \\ \widetilde{\boldsymbol{v}}^* \in \boldsymbol{\partial J}_{MH}\widetilde{\boldsymbol{v}}_{\mathcal{U}_{MH}},\ \boldsymbol{v}^* \in \boldsymbol{\partial J}_{MH}\boldsymbol{v}_{\mathcal{U}_{MH}}, \end{cases}$$

where $\boldsymbol{\mathcal{U}}_{MH} = \boldsymbol{\mathcal{W}}_{MH} \times (\boldsymbol{\mathcal{Y}}^*_{MH} \times \boldsymbol{\mathcal{B}}^*_{\Gamma}) \times \boldsymbol{\mathcal{C}}_{MH}$ stands as the variational macro-hybrid state-control space. The validity of the next fixed point solvability result can be demonstrated in accordance with [3, Sect. 4.2].

Theorem 7 *Let macro-hybrid strong monotonicity condition $(\widetilde{\boldsymbol{C}}_{\partial J_{MH}})$ be fulfilled, then macro-hybrid primal optimality problem $(\boldsymbol{\mathcal{P}}_{MH})$ has a unique solution.*

Proof Given an m-strongly monotone and a-Lipschitz continuous preconditioner operator $\boldsymbol{M}_{MH} : \boldsymbol{\mathcal{U}}_{MH} \to \boldsymbol{\mathcal{U}}^*_{MH}$, and an exact penalization parameter $r > 0$, the primal optimality $(\boldsymbol{\mathcal{P}}_{MH})$-equation is characterized as an augmented preconditioned and exactly penalized one introducing the state-control fields

$$\widetilde{\boldsymbol{v}} = \big(\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}\big)\,, \text{ and } \boldsymbol{v^*} = \big(\widetilde{\boldsymbol{v}^*}, (\boldsymbol{y}, \boldsymbol{\lambda}), \boldsymbol{\kappa^*}\big)\,,$$

for $\widetilde{\boldsymbol{v}^*} \in \boldsymbol{\partial J}_{MH}\ \widetilde{\boldsymbol{v}}$, by

$$\begin{aligned} &\boldsymbol{M}_{MH}\ \widetilde{\boldsymbol{v}} - r\widetilde{\boldsymbol{v}^*} \in \big(\boldsymbol{M}_{MH} + r\boldsymbol{\partial}\big(I_{\{0_{\mathcal{Q}_{MH}}\}} \circ (-\boldsymbol{\mathcal{T}}_{MH_{-\widetilde{f}^*}})\big)\big)\widetilde{\boldsymbol{v}} \\ &\Longleftrightarrow \widetilde{\boldsymbol{v}} = \boldsymbol{F}^r_{\widetilde{v}^*}\ \widetilde{\boldsymbol{v}} \equiv \boldsymbol{J}^r_{M_{MH}, \partial(I_{\{0_{\mathcal{Q}_{MH}}\}} \circ (-\mathcal{T}_{MH_{-\widetilde{f}^*}}))}\big(\boldsymbol{M}_{MH}\ \widetilde{\boldsymbol{v}} - r\widetilde{\boldsymbol{v}^*}\big), \end{aligned} \tag{41}$$

where the state-control operator notation $-\mathcal{T}_{MH\_\widetilde{f}^*}\ \widetilde{\boldsymbol{v}} = -\mathcal{T}_{MH}(\widetilde{\boldsymbol{v}} - \widetilde{\boldsymbol{w}}_{-\widetilde{f}^*})$ has been utilized. Further, $\boldsymbol{J}^r_{\boldsymbol{M}_{MH},\partial(I_{\{0_{\mathcal{Q}_{MH}}\}}\circ(-\mathcal{T}_{MH\_\widetilde{f}^*}))} = \Big(\boldsymbol{M}_{MH} + r\boldsymbol{\partial}\ \big(I_{\{0_{\mathcal{Q}_{MH}}\}} \circ(-\mathcal{T}_{MH\_\widetilde{f}^*})\big)\Big)^{-1} : \boldsymbol{\mathcal{U}}^*_{MH} \to \boldsymbol{\mathcal{U}}_{MH}$, is the $\boldsymbol{M}_{MH}$-resolvent operator of the subpotential subdifferential $\boldsymbol{\partial}\big(I_{\{0_{\mathcal{Q}_{MH}}\}} \circ (-\mathcal{T}_{MH\_\widetilde{f}^*})\big)$, a well defined $1/m$-firm contraction [26]. Thereby, the primal optimality problem $(\boldsymbol{\mathcal{P}}_{MH})$ is characterized by the $\boldsymbol{M}_{MH}$-resolvent fixed point problem,

$$(\widetilde{\boldsymbol{\mathcal{P}}}_{MH}) \begin{cases} \text{Find } \widetilde{\boldsymbol{v}} \in \boldsymbol{\mathcal{D}}\big(\partial(I_{\{0_{\mathcal{Q}_{MH}}\}} \circ (-\mathcal{T}_{MH\_\widetilde{f}^*}))\big) : \text{for } \widetilde{\boldsymbol{v}^*} \in \boldsymbol{\partial J}_{MH}\ \widetilde{\boldsymbol{v}}, \\ \widetilde{\boldsymbol{v}} = \boldsymbol{F}^r_{\widetilde{v}^*}\ \widetilde{\boldsymbol{v}}, \end{cases}$$

and by the Banach fixed-point theorem the desired existence result is concluded, due to the fact that the $1/m$-firm contraction resolvent property implies the contraction of fixed point operator $\boldsymbol{F}^r_{\widetilde{v}^*} : \boldsymbol{\mathcal{D}}\big(\partial(I_{\{0_{\mathcal{Q}_{MH}}\}} \circ (-\mathcal{T}_{MH\_\widetilde{f}^*}))\big) \to \boldsymbol{\mathcal{D}}\big(\partial(I_{\{0_{\mathcal{Q}_{MH}}\}} \circ (-\mathcal{T}_{MH\_\widetilde{f}^*}))\big)$ for $r > (a-m)/\alpha \geq 0$, with contraction parameter $1/m(a - r\alpha) < 1$ [25]. □

Therefore, from the composition duality principle of Theorem 6, and the primal existence result of Theorem 7, the macro-hybrid mixed optimality variational solvability is finally achieved.

Corollary 2 *Under state-control compatibility condition* $(\boldsymbol{C}_{-\mathcal{T}_{MH}})$ *and qualifying condition* $(\boldsymbol{C}_{\partial J_{MH}})$*, optimality macro-hybrid mixed problem* $(\boldsymbol{\mathcal{MOC}}_{MH})$ *attains a solution, with a unique primal component.*

6.1 Macro-hybrid Mixed Proximal Penalty-Duality Algorithms

In this subsection we introduce two- and three-field proximal penalty-duality algorithms, of a stationary type, for the resolution of multidomain macro-hybrid primal mixed optimality problem $(\boldsymbol{\mathcal{MOC}}_{MH})$ [26, 29, 30], which follow via preconditioned and augmented variational reformulations, with exact penalizations, characterized in terms of resolvent and proximation operators. Such preconditioned iterative penalty-duallity procedures have demonstrated to be of the most effective schemes in mechanics, for parallel computing, of multidomain mixed variational inclusions associated to optimization problems.

6.1.1 Two-field Algorithm

In accordance with [1, Sect. 5.1.1], in terms of a linear symmetric $\boldsymbol{M} : \boldsymbol{\mathcal{Q}} \to \boldsymbol{\mathcal{Q}}^*$ preconditioner, continuous and m-strongly monotone, with inverse $\boldsymbol{M}^{-1} : \boldsymbol{\mathcal{Q}}^* \to \boldsymbol{\mathcal{Q}}$, and an exact penalization parameter $r > 0$, a primal augmented regularization of mixed optimality problem ($\mathcal{MOC}_{MH}$) is given by

$$\left(\mathcal{MOC}^r_{MH_{M-1}}\right)\begin{cases} \text{Find } (\boldsymbol{v},(\boldsymbol{y}^*,\boldsymbol{\lambda}^*),\boldsymbol{\kappa}) \in \mathcal{M}_{MH} \text{ and } \boldsymbol{p} \in \mathcal{Q}_{MH} : \\ \mathcal{T}^T_{MH}\boldsymbol{p} \in \left(\partial \boldsymbol{J}_{MH} + r^*\mathcal{T}^T_{MH}\boldsymbol{M}^{-1}\mathcal{T}_{MH}\right)\left(\boldsymbol{v},(\boldsymbol{y}^*,\boldsymbol{\lambda}^*),\boldsymbol{\kappa}\right) \\ \quad -r\mathcal{T}^T_{MH}\boldsymbol{M}^{-1}\widetilde{\boldsymbol{f}^*}, \\ \qquad\qquad \text{in } \mathcal{W}^*_{MH} \times (\mathcal{Y}_{MH},\mathcal{B}_\Gamma) \times \mathcal{C}^*_{MH}, \\ \boldsymbol{p} = \boldsymbol{p} - r\boldsymbol{M}^{-1}\mathcal{T}_{MH}\left(\boldsymbol{v},(\boldsymbol{y}^*,\boldsymbol{\lambda}^*),\boldsymbol{\kappa}\right) + r\boldsymbol{M}^{-1}\widetilde{\boldsymbol{f}^*}, \\ \qquad\qquad \text{in } \mathcal{Q}_{MH}. \end{cases}$$

Thereby, a proper associated macro-hybrid primal mixed penalty-duality two-field iterative algorithm is the following one.

Algorithm $\mathbf{I}_{MH}$
Given $\boldsymbol{p}^0 \in \mathcal{Q}_{MH}$, known $\boldsymbol{p}^m$, $m \geq 0$,
calculate $(\boldsymbol{v}^{m+1},(\boldsymbol{y}^{*m+1},\boldsymbol{\lambda}^{*m+1}),\boldsymbol{\kappa}^{m+1})$ and $\boldsymbol{p}^{m+1}$:

$$\begin{aligned} &\mathcal{T}^T_{MH}\boldsymbol{p}^m \in \left(\partial \boldsymbol{J}_{MH} + r^*\mathcal{T}^T_{MH}\boldsymbol{M}^{-1}\mathcal{T}_{MH}\right)\left(\boldsymbol{v}^{m+1},(\boldsymbol{y}^{*m+1},\boldsymbol{\lambda}^{*m+1}),\boldsymbol{\kappa}^{m+1}\right) \\ &\quad -r^*\mathcal{T}^{\boldsymbol{T}}{}_{MH}\boldsymbol{M}^{-1}\widetilde{\boldsymbol{f}^*}, \qquad \text{in } \mathcal{W}^*_{MH} \times (\mathcal{Y}_{MH},\mathcal{B}_\Gamma) \times \mathcal{C}^*_{MH}, \\ &\boldsymbol{p}^{m+1} = \boldsymbol{p}^m - r^*\boldsymbol{M}^{-1}\mathcal{T}_{MH}\left(\boldsymbol{v}^{m+1},(\boldsymbol{y}^{*m+1},\boldsymbol{\lambda}^{*m+1}),\boldsymbol{\kappa}^{m+1}\right) + r^*\boldsymbol{M}^{-1}\widetilde{\boldsymbol{f}^*}, \\ &\qquad\qquad \text{in } \mathcal{Q}_{MH}. \end{aligned}$$

6.1.2 Three-field Algorithm

On the other hand, from a three-field variational version of duality problem ($\mathcal{MOC}^r_{MH_{M-1}}$) (cf. [1, Sect. 5.1.2]), an additional primal mixed proximation penalty-duality algorithm can be produced. Indeed, introducing its dual coupling field

$$\boldsymbol{\chi}^* = -\mathcal{T}_{MH}(\boldsymbol{v},(\boldsymbol{y}^*,\boldsymbol{\lambda}^*) \in \mathcal{Q}^*_{MH}, \tag{42}$$

being regarded as the new dual component, after dualizing the dual equation of the mixed system a three-field version is concluded. That is, the following precondition augmented and exactly penalized equivalent version of the mixed optimality problem is achieved.

$$(\mathcal{MOC}^r_{\widetilde{MH}_{M-1}})\begin{cases} \text{Find } (\boldsymbol{v},(\boldsymbol{y}^*,\boldsymbol{\lambda}^*),\boldsymbol{\kappa}) \in \mathcal{M}_{MH},\ \boldsymbol{\chi}^* \in \mathcal{Q}^*_{MH}, \\ \text{and } \boldsymbol{p} \in \mathcal{Q}_{MH} : \\ \mathcal{T}^{\boldsymbol{T}}{}_{MH}(\boldsymbol{p} - r\boldsymbol{M}^{-1}\boldsymbol{\chi}^*) \\ \quad \in \left(\partial \boldsymbol{J}_{MH} + r\mathcal{T}^T_{MH}\boldsymbol{M}^{-1}\mathcal{T}_{MH}\right)\left(\boldsymbol{v},(\boldsymbol{y}^*,\boldsymbol{\lambda}^*),\boldsymbol{\kappa}\right), \\ \qquad\qquad \text{in } \mathcal{W}^*_{MH} \times (\mathcal{Y}_{MH},\mathcal{B}_\Gamma) \times \mathcal{C}^*_{MH}, \\ \boldsymbol{p} - r\boldsymbol{M}^{-1}\mathcal{T}_{MH}\left(\boldsymbol{v},(\boldsymbol{y}^*,\boldsymbol{\lambda}^*),\boldsymbol{\kappa}\right) \\ \quad \in \partial \boldsymbol{I}_{0_{Q^*}}(\boldsymbol{\chi}^* + \widetilde{\boldsymbol{f}^*}) + r\boldsymbol{M}^{-1}\boldsymbol{\chi}^*, \qquad \text{in } \mathcal{Q}_{MH}, \\ \boldsymbol{p} = \boldsymbol{p} - r\boldsymbol{M}^{-1}\left(\mathcal{T}_{MH}\left(\boldsymbol{v},(\boldsymbol{y}^*,\boldsymbol{\lambda}^*),\boldsymbol{\kappa}\right) + \boldsymbol{\chi}^*\right), \text{ in } \mathcal{Q}_{MH}. \end{cases}$$

Consequently, a natural macro-hybrid primal mixed proximal-point iterative three-field algorithm is given by

Algorithm II$_{MH}$

Given $\boldsymbol{p}^0 \in \mathcal{Q}_{MH}$, known $\boldsymbol{p}^m$, $m \geq 0$,
calculate $\big((\boldsymbol{v}^{m+1}, \boldsymbol{y}^{*m+1}), \boldsymbol{\kappa}^{m+1}), \boldsymbol{\chi}^{*m+1}\big)$ and $\boldsymbol{p}^{m+1}$:

$$\begin{aligned}
&\boldsymbol{\mathcal{T}}_{MH}^{T}\big(\boldsymbol{p}^m - r\boldsymbol{M}^{-1}\boldsymbol{\chi}^{*m+1}\big)\\
&\quad \in \Big(\boldsymbol{\partial J}_{MH} + r\boldsymbol{\mathcal{T}}_{MH}^{T}\boldsymbol{M}^{-1}\boldsymbol{\mathcal{T}}_{MH}\Big)\big(\boldsymbol{v}^{m+1}, (\boldsymbol{y}^{*m+1}, \boldsymbol{\lambda}^{*m+1}), \boldsymbol{\kappa}^{m+1}\big),\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \boldsymbol{\mathcal{W}}_{MH}^{*} \times (\boldsymbol{\mathcal{Y}}_{MH}, \boldsymbol{\mathcal{B}}_{\Gamma}) \times \boldsymbol{\mathcal{C}}_{MH}^{*}\ ,\\
&\boldsymbol{p}^m - r\boldsymbol{M}^{-1}\boldsymbol{\mathcal{T}}_{MH}\big(\boldsymbol{v}^{m+1}, (\boldsymbol{y}^{*m+1}, \boldsymbol{\lambda}^{*m+1}), \boldsymbol{\kappa}^{m+1}\big)\\
&\quad \in \boldsymbol{\partial I}_{0_{Q^*}}(\boldsymbol{\chi}^{*m+1} + \widetilde{\boldsymbol{f}^{*}}) + r\boldsymbol{M}^{-1}\boldsymbol{\chi}^{*m+1}\ , \qquad\qquad \text{in } \mathcal{Q}_{MH}\ ,\\
&\boldsymbol{p}^{m+1} = \boldsymbol{p}^m - r\boldsymbol{M}^{-1}\big(\boldsymbol{\mathcal{T}}_{MH}\big(\boldsymbol{v}^{m+1}, (\boldsymbol{y}^{*m+1}, \boldsymbol{\lambda}^{*}), \boldsymbol{\kappa}^{m+1}\big) + \boldsymbol{\chi}^{*m+1}\big), \text{ in } \mathcal{Q}_{MH}\ .
\end{aligned}$$

A relevant interpretation, in the context of evolutionary systems, of these stationary variational proximal penalty-duality algorithms (cf. [30], Sect. 5), is that variational proximation time marching schemes under appropriate maximal monotonicity conditions, evolve as their step times $m \to \infty$ to corresponding stationary states limits. Thereby, optimal states $(\boldsymbol{v}, (y^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}) \in \boldsymbol{\mathcal{M}}_{MH}$ of macro-hybrid mixed optimality problem $(\boldsymbol{\mathcal{M}OC}_{MH})$ may be regarded as stationary states limits, for instance of a primal semi-implicit Euler time marching scheme, like **ALG0** analyzed in ([30], Sect. 5.1).

For the convergence analysis of these optimality proximal-point, macro-hybrid mixed penalty-duality iterative algorithms, we refer to [31, Sect. 4], where additionally some important algorithmic implementation variational results are treated.

Lastly, at this algorithmic stage of the study, we state that some alternative modern important local optimization splitting methods for multidomain structured monotone inclusions, are the following [32–37]. Specifically, in paper [32] a primal-dual parallel proximal splitting method is proposed and analyzed for domain decomposition of linear and nonlinear PDE's problems, via local coupling subdomains minimization energy interface functions; several results concerning the solution of monotone inclusion problems by splitting methods are outlined in study [33], for optimality conditions related to convex optimization problems where the sum of maximally monotone operators is involved, presenting some important contributions of their algorithmic realizations; paper [34] is concerned with structured coupled monotone inclusions in Hilbert spaces, analyzing the asymptotic behavior of a general primal-dual splitting solving algorithm, most steps of which can be executed in parallel. In paper [35] two different primal-dual splitting algorithms for solving structured monotone inclusions are proposed: the preconditioned forward-backward splitting algorithm and the forward-backward-half-forward splitting algorithm, both being calculated with a simple framework where single-valued operators are processed via explicit steps and set-valued operators are computed by their resolvents; rapidly convergent forward-backward algorithms for computing zeroes of the sum of finitely many maximally monotone operators, are developed in paper [36], incorporating an inertial term, a constant relaxation factor, and a correction term, for general monotone inclusions, specifically with a fast primal-dual algorithm for solving convex-concave saddle point problems. Finally, work [37] proposes two iterative penalty schemes for solving structured monotone inclusion problems, with backward steps for a set-valued operator and a single forward step for a single-valued operator.

7 Multidomain Macro-hybrid Mixed Optimization Existence Analysis

In this last section we proceed to determine the perturbed primal, dual and Lagrangian mixed optimization results, complementary to those established in Sect. 6, completing the proposed multidomain macro-hybrid primal evolution optimal control theory of the present study.

Let us first consider the classical duality principle, from the perturbation conjugate duality theory [23, 38],

Lemma 3 *Primal and dual solutions* $((\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}) \in \boldsymbol{\mathcal{D}}(J_{MH})$ *and* $\boldsymbol{p} \in \boldsymbol{\mathcal{D}}(\pi_{MH})$*, to minimization primal problem* $(\widetilde{\boldsymbol{\mathcal{OC}}}_{MH})$ *and maximization dual problem* $(\widetilde{\boldsymbol{\mathcal{OC}}}^*_{MH})$*, are such that*

$$inf(\widetilde{\boldsymbol{\mathcal{OC}}}_{MH}) = sup(\widetilde{\boldsymbol{\mathcal{OC}}}^*_{MH}),$$

if, and only if, $((\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}), \boldsymbol{p}) \in \boldsymbol{\mathcal{D}}(\mathcal{L}_{MH})$ *is a solution to minimax Lagrangian mixed problem* $(\widetilde{\boldsymbol{\mathcal{MOC}}}^*_{MH})$*;*

as well as the additional conjugate duality result [2, 39, 40] (cf. [40, Proposition 3.1]), which taking into account that the marginal domain $\boldsymbol{\mathcal{D}}(\mu^*_{MH})$ is the projection of perturbation functional domain $\boldsymbol{\mathcal{D}}(\mathcal{S}_{MH})$, on the closed subspace $\boldsymbol{\mathcal{Q}}^*$ of perturbations, states the following

Lemma 4 *Let the marginal domain* $\boldsymbol{\mathcal{D}}(\mu^*_{MH})$ *be such that* $\Re_+\boldsymbol{\mathcal{D}}(\mu^*_{MH})$ *is a closed subspace, and let* $\mu^*_{MH}(0)$ *be finite. Then*

$$\begin{aligned} &\inf_{((w,(x^*,\chi^*),\eta)\in(\mathcal{W}_{MH}\times(\mathcal{Y}^*_{MH},\mathcal{B}^*_{\Gamma})\times\mathcal{C}_{MH}} \boldsymbol{\mathcal{S}}_{MH}\Big(((\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), \boldsymbol{\eta}), \mathbf{0}_{\mathcal{Q}_{MH}})\Big) \\ &= \max_{q\in\mathcal{Q}_{MH}} -\boldsymbol{\mathcal{S}}^*_{MH}\Big(((\mathbf{0}_{\mathcal{W}_{MH}}, (\mathbf{0}_{\mathcal{Y}^*_{MH}}, \mathbf{0}_{\mathcal{B}^*_{\Gamma}}), \mathbf{0}_{\mathcal{C}_{MH}}), \boldsymbol{q})\Big). \end{aligned} \tag{43}$$

As a third new duality result, complementary to those of Lemmas 3 and 4, we next establish a fundamental result, that at a first instance permits the implementation of duality result (43) in the sense of [8], and then leading to the conclusive central optimality results of the theory.

Lemma 5 *Primal and dual optimization solutions* $(\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}) \in \boldsymbol{\mathcal{D}}(J_{MH})$ *and* $\boldsymbol{p} \in \boldsymbol{\mathcal{D}}(\pi_{MH})$*, of respective perturbed minimization problem* $(\widetilde{\boldsymbol{\mathcal{OC}}}_{MH})$ *and maximization problem* $(\widetilde{\boldsymbol{\mathcal{OC}}}^*_{MH})$*, are such that*

$$inf(\widetilde{\boldsymbol{\mathcal{OC}}}_{MH}) = sup(\widetilde{\boldsymbol{\mathcal{OC}}}^*_{MH}).$$

Proof For any perturbation $\boldsymbol{q}^* \in \boldsymbol{\mathcal{Q}}^*_{MH}$ and positive real number $\lambda \in \Re_+$, in accordance with qualifying condition ($\boldsymbol{C}_{\mathcal{T}_{MH}}$) (of Sect. 5), there is an admissible mixed state-control $(\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), \boldsymbol{\eta}) \in \boldsymbol{\mathcal{M}}_{MH}$ for which $\lambda^{-1}\boldsymbol{q}^* \in \boldsymbol{\mathcal{T}}_{MH}\big(\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), \boldsymbol{\eta}\big) - \widetilde{\boldsymbol{f}}^*$. Then $((\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), \boldsymbol{\eta}), \lambda^{-1}\boldsymbol{q}^*) \in \boldsymbol{\mathcal{K}}_{MH}$ and $\mathcal{S}_{MH}\ \big((\boldsymbol{w}, \boldsymbol{x}^*), \boldsymbol{\eta}),\ \lambda^{-1}\boldsymbol{q}^*\big) = J_{MH}\big(\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*), \boldsymbol{\eta}\big) < \infty$. Further, in accordance with marginal functional definition (29), it is concluded that $\lambda^{-1}\boldsymbol{q}^* \in \boldsymbol{\mathcal{D}}(\mu^*_{MH})$ and $\boldsymbol{\mathcal{D}}(\mu^*_{MH}) = \boldsymbol{\mathcal{T}}_{MH}\big(\boldsymbol{\mathcal{M}}_{MH}\big) - \widetilde{\boldsymbol{f}}^*$. Consequently $\boldsymbol{\mathcal{Q}}^*_{MH} = \Re_+\boldsymbol{\mathcal{D}}(\mu^*_{MH})$ and the conjugate duality result (43) holds true. Therefore, from the perturbation definitions of functionals J_{MH} and π_{MH}, (32) and (31), the desired result is concluded,

$$\begin{aligned}\inf_{((w,(x^*,\chi^*)),\eta)\in\mathcal{W}_{MH}\times(\mathcal{Y}^*_{MH},\mathcal{B}^*_{\Gamma})\times\mathcal{C}_{MH}} & \boldsymbol{J}_{MH}\Big(((\boldsymbol{w}, (\boldsymbol{x}^*, \boldsymbol{\chi}^*)), \boldsymbol{\eta})\Big) \\ &= \max_{q\in\mathcal{Q}_{MH}} (-\boldsymbol{\pi}_{MH}\, \boldsymbol{q}).\end{aligned}$$

□

Finally, let us conclude with the following new multidomain macro-hybrid optimality results:

The equivalent variational solvability of perturbed macro-hybrid optimization mixed problem $(\widetilde{\boldsymbol{\mathcal{OC}}}_{MH})$-$(\widetilde{\boldsymbol{\mathcal{OC}}}^*_{MH})$ and Lagrangian problem $(\widetilde{\boldsymbol{\mathcal{MOC}}}^*_{MH})$ is achieved from Lemmas 3 and 5.

Corollary 3 $(\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}) \in \boldsymbol{\mathcal{D}}(J_{MH})$ *and* $\boldsymbol{p} \in \boldsymbol{\mathcal{D}}(\pi_{MH})$ *are solutions to primal minimization* $(\widetilde{\boldsymbol{\mathcal{OC}}}_{MH})$ *and dual maximization problem* $(\widetilde{\boldsymbol{\mathcal{OC}}}^*_{MH})$*, respectively; if, and only if,* $((\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}), \boldsymbol{p}) \in \boldsymbol{\mathcal{D}}(\mathcal{L}_{MH})$ *is a solution to perturbed minimax Lagrangian problem* $((\widetilde{\boldsymbol{\mathcal{MOC}}}^*_{MH}))$.

Further, from the macro-hybrid mixed optimality condition of Theorem 5, $(\boldsymbol{\mathcal{MOC}}_{MH})$, and Corollary 2 on its solvability result, the J_{MH}-solution of the minimization optimal control problem $(\boldsymbol{\mathcal{O}}_{MH})$ is achieved.

Theorem 8 *Let state-control surjectivity compatibility condition* $(\boldsymbol{C}_{-\mathcal{T}_{MH}})$ *and qualifying monotonicity condition* $(\boldsymbol{C}_{\partial J_{MH}})$ *be fulfilled, then multidomain macro-hybrid optimal control problem* $(\boldsymbol{\mathcal{O}}_{MH})$ *possesses an admissible solution.*

Consequently, from this Theorem 8 result, in conjunction with the solvability of macro-hybrid dual mixed state system $(\boldsymbol{\mathcal{MH}}_{\kappa})$ determined by Theorem 2, in Section 3.1, the existence of an state $(\boldsymbol{\mathcal{MH}}_{\kappa})$ and control $(\boldsymbol{\mathcal{O}}_{MH})$ coupled solution is finally concluded.

Corollary 4 *Under Theorem 8 compatibility and qualifying conditions, as well as primal macro-hybrid strong monotonicity condition* $\big(\boldsymbol{C}_{\partial\widetilde{F}}\big)$*, there exists an optimal control pair solution* $(\boldsymbol{v}, (\boldsymbol{y}^*, \boldsymbol{\lambda}^*), \boldsymbol{\kappa}) \in \boldsymbol{\mathcal{M}}_{MH} \subset (\boldsymbol{\mathcal{W}}_{MH} \times (\boldsymbol{\mathcal{Y}}^*_{MH}, \boldsymbol{\mathcal{B}}^*_{\Gamma}) \times \boldsymbol{\mathcal{C}}_{MH})$*, to variational coupled multidomain macro-hybrid primal mixed state-control problem* $(\boldsymbol{\mathcal{MH}}_{\kappa})$-$(\boldsymbol{\mathcal{O}}_{MH})$.

8 Conclusions

Once the evolution macro-hybrid mixed, variational reflexive Banach functional framework of the study was stated, the solvability of the multidomain subpotential primal evolution mixed state governing system, was demonstrated as a resolvent fixed point characterization result, under a strongly monotone qualifying condition. Then the analysis has been illustrated, in the sequel, through the variational implementation of an underground transport flow mechanical process, with intrinsic control constraints. Next the perturbation conjugate duality optimal control, of the multidomain evolution mixed theory was presented, establishing its own Lagrangian mixed variational optimality condition, as a new result, showing its applicability to the multidomain optimal control of the mechanical transport flow process. Then the mixed optimality fixed point solvability was proved, concluding with the optimality two- and three-field macro-hybrid mixed proximal penalty-duality algorithms. Finally, an innovating optimization existence analysis was performed, determining the solvability of the minimization optimal control problem of the theory, as well as the corresponding variational solution existence of the coupled macro-hybrid mixed pair state-control system.

Funding The authors declare that no funds or grants, were received during the preparation of this manuscript.

Declarations

Competing Interests The authors declare that they have no conflict of interest.

References

1. Alduncin, G.: Optimal control of evolution mixed variational inclusions. Appl. Math. Optim. **68**, 445–473 (2013)
2. Azé, D., Bolintinéanu, S.: Optimality conditions for constrained convex parabolic control problems via duality. J. Convex Anal. **7**, 1–17 (2000)
3. Alduncin, G.: Fixed-point variational existence analysis of evolution mixed inclusions. Int. J. Math. Anal. (Ruse) **8**, 1833–1846 (2014)
4. Alduncin, G.: Optimal control of nonlinear transport-flow mixed variational problems. Numer. Funct. Anal. Optim. **36**, 305–328 (2015)
5. Alduncin, G.: Optimal control of evolution macro-hybrid mixed variational inclusions. J. Math. Res. **9**, 124–143 (2017)
6. Alduncin, G.: Optimal control of elastoviscoplastic macro-hybrid mixed variational problems. Set-Valued Var. Anal. **27**, 1–18 (2019)
7. Alduncin, G.: Optimality condition analysis of variational macro-hybrid mixed inclusions. J. Math. Anal. Appl. **475**, 313–336 (2019)

8. Alduncin, G.: Macro-hybrid dual optimal control of transport flow through elastoviscoplastic porous media. Acta Appl. Math. **178**, 1–24 (2022)
9. Chen, Q.: Optimal control for semilinear evolutionary variational bilateral problem. J. Math. Anal. Appl. **277**, 303–323 (2003)
10. Capatina, A., Timofte, C.: An optimal control problem governed by implicit evolution quasi-variational inequalities. Ann. Univ. Bucharest (mathematical series) **4**(LXII), 157–166 (2013)
11. Liu, Z.H., Migórski, S.: Analysis and control of differential inclusions with anti-periodic conditions. Proc. R. Soc. Edinburgh **144**, 591–602 (2014)
12. Migórski, S.: Optimal control of history-dependent evolution inclusions with applications to frictional contact. J. Optim. Theory Appl. **185**, 574–596 (2020)
13. Zhenhai, L., Migórski, S., Biao, Z.: Existence results and optimal control for a class of quasi mixed equilibrium problems involving the (f, g, h) quasimonotonicity. Appl. Math. Optim. **79**, 257–277 (2019)
14. Sofonea, M., Bollati, J., Tarzia, D.A.: Optimal control of differential quasivariational inequalities with applications in contact mechanics. J. Math. Anal. Appl. **493**, 124–567 (2021)
15. Alduncin, G.: Macro-hybrid variational formulations of constrained boundary value problems. Numer. Funct. Anal. Optim. **28**, 751–774 (2007)
16. Alduncin, G.: Composition duality methods for evolution mixed variational inclusions. Nonlinear Anal. Hybrid Syst **1**, 336–363 (2007)
17. Alduncin, G.: Primal and dual evolution macro-hybrid mixed variational inclusions. Int. J. Math. Anal. (Ruse) **5**, 1631–1664 (2011)
18. Girault, V., Raviart, P.-A.: Finite Element Methods for Navier–Stokes Equations. Springer, Berlin (1986)
19. Lions, J.-L.: Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires. Dunod/Gauthier-Villars, Paris (1969)
20. Alduncin, G.: Evolution mixed variational inclusions with optimal control. J. Math. Res. **4**, 64–79 (2012)
21. Barbu, V.: Nonlineat Differential Equations of Monotone Types in Banach Spaces. Springer, New York (2010)
22. Duvaut, G., Lions, J.-L.: Les Inéquations en Méchanique et en Physique. Dunod, Paris (1972)
23. Ekeland, I., Temam, R.: Analyse Convexe et Problèmes Variationnels. Dunod/Gauthier-Villars, Paris (1974)
24. Zeidler, E.: Nonlinear Functional Analisis and its Applications, II/B. Nonlinear Monotone Operators. Springer, New York (1990)
25. Alduncin, G.: Variational formulations of nonlinear constrained boundary value problems. Nonlinear Anal. **72**, 2639–2644 (2010)
26. Alduncin, G.: Composition duality methods for mixed variational inclusions. Appl. Math. Optim. **52**, 311–348 (2005)
27. de Marsil, G.: Quantitative Hydrogeology for Engineers. Academic Press, Boca Raton (1986)
28. Gurtin, M., Fried, E., Anand, L.: The Mechanics and Thermodynamics. Cambridge University Press, Cambridge (2010)
29. Gabay, D.: Application de la méthode des multiplicateurs aux inéquations variationnelles. In: Fortin, M., Glowinski, R. (eds.) Méthodes de Lagrangien Augmenté, pp. 279–307. Dunod-Bordas, Paris (1982)
30. Alduncin, G.: On Gabay's algorithms for mixed variational inequalities. Appl. Math. Optim. **35**, 21–24 (1997)
31. Alduncin, G.: Proximal penalty-duality algorithms for mixed optimality conditions. J. Fixed Point Theory Appl. **19**, 1775–1791 (2017)
32. Attouch, H., Briceño-Arias, L.M., Combettes, P.L.: A strongly convergent primal dual method for nonoverlapping domain decomposition. Numer. Math. **133**, 443–470 (2016)
33. Csetnek, E. R.: Recent advances in splitting methods for monotone inclusions and nonsmooth optimization problems. Habilitation Thesis, Faculty Math. Univ. Vienna, 1–173 (2016)
34. Combettes, P.L.: Systems of structured monotone inclusions: duality, algorithms, and applications. SIAM J. Optim. **23**, 2420–2447 (2013)
35. Jinjian, C., Xingyu, L., Yuchao, T., Qiaoli, D.: Primal-dual splitting algorithms for solving structured monotone inclusion with applications. Symmetry **13**, 1–23 (2021)
36. Maingé, P.-E.: Fast convergence of generalized forward-backward algorithms for structured monotone inclusions. J. Convex Anal. **29**, 893–920 (2022)

37. Banert, S., Bot, R.I.: Backward penalty schemes for monotone inclusion problems. J. Optim. Theory Appl. **166**, 930–948 (2015)
38. Rockafellar, R.T.: Conjugate Duality and Optimization. SIAM, Philadelphia (1974)
39. Simons, S.: The occasional distributivity of $\bullet$ over $\overset{+}{e}$ and the change of variables for conjugate functions. Nonlinear Anal. **14**, 1111–1120 (1990)
40. Jeyakumar, V.: Duality and infinite dimensional optimization. Nonlinear Anal. **15**, 1111–1122 (1990)