Applied Mathematics & Optimization (2023) 88:4
https://doi.org/10.1007/s00245-023-09974-4

Multi-Peak Solutions for Coupled Nonlinear Schrödinger Systems in Low Dimensions

Maoding Zhen[1] · Binlin Zhang[2] · Vicenţiu D. Rădulescu[3,4,5]

Accepted: 1 February 2023 / Published online: 10 April 2023
© The Author(s) 2023

Abstract

In this paper, we construct the solutions to the following nonlinear Schrödinger system

$$\begin{cases} -\epsilon^2 \Delta u + P(x)u = \mu_1 u^p + \beta u^{\frac{p-1}{2}} v^{\frac{p+1}{2}} & \text{in } \mathbb{R}^N, \\ -\epsilon^2 \Delta v + Q(x)v = \mu_2 v^p + \beta u^{\frac{p+1}{2}} v^{\frac{p-1}{2}} & \text{in } \mathbb{R}^N, \end{cases}$$

where $3 < p < +\infty$, $N \in \{1, 2\}$, $\epsilon > 0$ is a small parameter, the potentials P, Q satisfy $0 < P_0 \leq P(x) \leq P_1$ and $Q(x)$ satisfies $0 < Q_0 \leq Q(x) \leq Q_1$. We construct the solution for attractive and repulsive cases. When x_0 is a local maximum point of the potentials P and Q and $P(x_0) = Q(x_0)$, we construct k spikes concentrating near the local maximum point x_0. When x_0 is a local maximum point of P and $\overline{x}_0$ is a local maximum point of Q, we construct k spikes of u concentrating at the local maximum point x_0 and m spikes of v concentrating at the local maximum point $\overline{x}_0$ when $x_0 \neq \overline{x}_0$. This paper extends the main results established by Peng and Wang

✉ Vicenţiu D. Rădulescu
radulescu@inf.ucv.ro

Maoding Zhen
maodingzhen@163.com

Binlin Zhang
zhangbinlin2012@163.com

[1] School of Mathematics, Hefei University of Technology, Hefei 230009, People's Republic of China

[2] College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, People's Republic of China

[3] Faculty of Applied Mathematics, AGH University of Science and Technology, 30-059 Kraków, Poland

[4] Faculty of Electrical Engineering and Communication, Brno University of Technology, Technická 3058/10, 61600 Brno, Czech Republic

[5] Department of Mathematics, University of Craiova, Street A.I. Cuza No. 13, 200585 Craiova, Romania

(Arch Ration Mech Anal 208:305–339, 2013) and Peng and Pi (Discrete Contin Dyn Syst 36:2205–2227, 2016), where the authors considered the case $N = 3$, $p = 3$.

Keywords Nonlinear Schrödinger system · Lyapunov–Schmidt reduction · Singularity · Perturbation

Mathematics Subject Classification 35B40 · 35B20 · 35J47 · 35Q40

1 Introduction and Main Results

In this paper, we construct the solutions for the following nonlinear Schrödinger system

$$\begin{cases} -\epsilon^2 \Delta u + P(x)u = \mu_1 u^p + \beta u^{\frac{p-1}{2}} v^{\frac{p+1}{2}} & \text{in } \mathbb{R}^N, \\ -\epsilon^2 \Delta v + Q(x)v = \mu_2 v^p + \beta u^{\frac{p+1}{2}} v^{\frac{p-1}{2}} & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\epsilon > 0$ is a small parameter, $5 \le p < +\infty$, $N \in \{1, 2\}$, and the potentials P, Q satisfy $0 < P_0 \le P(x) \le P_1$, $Q(x)$, respectively $0 < Q_0 \le Q(x) \le Q_1$.

The use of the Lyapunov–Schmidt reduction method to construct solutions for the nonlinear Schrödinger equation attracted much attention in the last decade, starting from the pioneering contribution by Floer and Weinstein [11]. Noussair and Yan [21] considered multi-peak solutions for the following problem

$$\begin{cases} -\epsilon^2 \Delta u + u = Q(x)|u|^{q-2}u, \quad x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.2}$$

when x_0 is a local maximum point of $Q(x)$ and ϵ is sufficiently small, they proved that for each positive integer k, problem (1.2) has a positive solution with k-peaks concentrating near x_0. Wei and Yan [32] studied the following nonlinear Schroödinger equation

$$\begin{cases} -\Delta u + V(|x|)u = u^p, \ u > 0 \ \ x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.3}$$

when $1 < p < (N + 2)/(N - 2)$ and $V(|x|)$ is positive function with following expansion

$$V(|x|) = V_0 + \frac{a}{r^m} + O\left(\frac{1}{r^{m+\theta}}\right) \text{ as } r \to +\infty.$$

They proved that problem (1.3) has infinitely many non-radial positive solutions, whose energy can be made arbitrarily large. For more results about the nonlinear Schrödinger equation, we refer the reader to [1, 6, 8–10, 12] and the references therein.

Inspired by the work of [21, 32], when $N = 3,\ p = 3,\ \epsilon = 1$, Peng and Wang [23] considered system (1.1), where $P(x)$ and $Q(x)$ satisfy the following hypotheses:

(**P**) There are constants $a \in \mathbb{R}$, $m > 1$, and $\theta > 0$ such that as $r \to +\infty$

$$P(|x|) = 1 + \frac{a}{|x|^m} + O\left(\frac{1}{|x|^{m+\theta}}\right), \quad \text{as} \quad |x| \to \infty. \tag{H1}$$

(**Q**) There are constants $b \in \mathbb{R}$, $n > 1$, and $\epsilon > 0$ such that as $r \to +\infty$

$$Q(|x|) = 1 + \frac{b}{|x|^n} + O\left(\frac{1}{|x|^{n+\epsilon}}\right), \quad \text{as} \quad |x| \to \infty. \tag{H2}$$

By using the number of the bumps of the solutions as a parameter, for the repulsive case, they constructed non-radial positive vector solutions of segregated type, for the attractive case, they constructed non-radial positive vector solutions of synchronized type. When $N = 3$, $p = 3$, Peng and Pi [22] constructed k interacting spikes for u near the local maximum point x_0 of $P(x)$ and m interacting spikes for v near the local maximum point $\overline{x}_0$ of $Q(x)$, respectively, when $x_0 \neq \overline{x}_0$. Tang and Xie in [27] constructed synchronized positive vector solutions for ϵ small. Since it seems to be difficult to provide a complete list of references, we just refer the readers to [2–4, 7, 13, 14, 17–20, 22–26, 28–30, 33, 34] and the references therein.

In this paper, we have been inspired by the analysis developed in [21–23, 31, 32] for scalar nonlinear elliptic equations (systems), in particular, by the ideas introduced by Noussair and Yan [21] to deal with nonlinear elliptic equations. Compared with the single scalar equation, we encounter some new difficulties in estimates due to the nonlinear coupling. Firstly, we need to establish non-degenerate results for the solutions of the coupled system, which will be used to prove the invertibility of the operator for the repulsive case. The difficulty is that we need to give an exact integral estimate for the coupled term and the system is more complicated than single equations. We point out that the sign of β has great influence on the structure of the solutions. Roughly speaking, for the repulsive case, the solutions are small perturbations of (U, V), where (U, V) are scaling and translation of the solution of $-\Delta u + \lambda u = u^p$. For the attractive case, the solutions are small perturbations of (U_μ, U_ν), where U_μ are scaling and translation of the solution of $-\Delta u + \lambda u = \mu u^p$ and U_ν are scaling and translation of the solution of $-\Delta u + \lambda u = \nu u^p$.

In this paper, we examine how potentials and the interspecies scattering length β influence the structure of solutions to problem (1.1), which improves the results of [19, 22] for least energy solutions. We study the existence of high energy solutions to problem (1.1) and provide not only the locations of spikes, but also much finer information on the interaction of spikes. Furthermore, we prove the attractive phenomenon for $\beta < 0$ and the repulsive phenomenon for $\beta > 0$.

Define the function space

$$\mathcal{H} = \left\{(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} P(x)u^2 dx < +\infty, \int_{\mathbb{R}^N} Q(x)v^2 dx < +\infty\right\}$$

endowed with the following norm:

$$\|(u,v)\|^2 = \langle (u,v),(u,v)\rangle = \|u\|^2_{\epsilon,P} + \|v\|^2_{\epsilon,Q}, \ \forall\,(u,v)\in\mathcal{H},$$

where $H^1 = H^1(\mathbb{R}^N)$ is the usual Sobolev space,

$$\|u\|_{\epsilon,P} = \langle u,u\rangle_{\epsilon,P} = \int_{\mathbb{R}^N}\left(\epsilon^2|\nabla u|^2 + P(x)u^2\right)dx$$

and

$$\|v\|_{\epsilon,P} = \langle v,v\rangle_{\epsilon,Q} = \int_{\mathbb{R}^N}\left(\epsilon^2|\nabla v|^2 + Q(x)v^2\right)dx.$$

Define

$$\mathbf{E}_\epsilon = \left\{(\varphi,\psi)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N), \left\langle(\varphi,\psi), \left(\frac{\partial U_{\epsilon,x_{\epsilon,j}}}{\partial y_i}, \frac{\partial V_{\epsilon,x_{\epsilon,j}}}{\partial y_i}\right)\right\rangle_\epsilon = 0 \quad j=1,2\cdots k;\ \ i=1,2\cdots N\right\},$$

where

$$\langle (u,v),(g,h)\rangle = \int_{\mathbb{R}^N}(\epsilon^2\nabla u\nabla g + P(x)ug + \epsilon^2\nabla v\nabla h + Q(x)vh)dx.$$

Consider the following system

$$\begin{cases} -\Delta u + \lambda u = \mu_1 u^p + \beta u^{\frac{p-1}{2}} v^{\frac{p+1}{2}} & \text{in } \mathbb{R}^N,\\ -\Delta v + \lambda v = \mu_2 v^p + \beta u^{\frac{p+1}{2}} v^{\frac{p-1}{2}} & \text{in } \mathbb{R}^N. \end{cases} \tag{1.4}$$

Let W be the unique solution of

$$\begin{cases} -\Delta u + \lambda u = u^p, \ \text{in } \mathbb{R}^N,\\ u>0 \text{ in } \mathbb{R}^N, u(x)\to 0 \text{ as } |x|\to+\infty. \end{cases} \tag{1.5}$$

Then

$$(U,V) = (k_1 W, \tau_0 k_1 W)$$

in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ is a solution of (1.4), where $\lambda := P(x_0) = Q(x_0)$ and τ_0 satisfies

$$\mu_1 + \beta\tau_0^{\frac{p+1}{2}} - \mu_2\tau_0^{p-1} - \beta\tau_0^{\frac{p-3}{2}} = 0,\ k_1^{p-1} = \left(\mu_1 + \beta\tau_0^{\frac{p+1}{2}}\right)^{-1}. \tag{1.6}$$

Let

$$\left(U_{\epsilon,x_{j,\epsilon}}(x), V_{\epsilon,x_{j,\epsilon}}(x)\right) = \left(U\left(\frac{x-x_{j,\epsilon}}{\epsilon}\right), V\left(\frac{x-x_{j,\epsilon}}{\epsilon}\right)\right).$$

In the sequel, we will use $\left(U_{\epsilon,x_{j,\epsilon}}(x), V_{\epsilon,x_{j,\epsilon}}(x)\right)$ to build up the solutions of (1.1).

To show our main results, we first recall some known results from [15]. In the following, let

$$2^* = \begin{cases} +\infty, & N = 1, 2, \\ \frac{2N}{N-2}, & N \geq 3. \end{cases}$$

Proposition 1.1 *Suppose that* $1 < p < 2^* - 1,\ \mu_1 > \mu_2 > 0,\ \beta > 0$ *and one of the following conditions holds*

(A_1) $3 < p < 2^* - 1,\ 0 < \beta \leq (\frac{p-1}{2})\mu_1$;

(A_2) $3 < p < 2^* - 1,\ \mu_1 \geq \frac{\mu_2}{2}(\frac{p+1}{p-1})^{\frac{p-1}{2}},\ \beta > (\frac{p-1}{2})\mu_1$;

(A_3) $3 < p < 2^* - 1,\ \mu_1 < \frac{\mu_2}{2}(\frac{p+1}{p-1})^{\frac{p-1}{2}},\ (\frac{p-1}{2})\mu_1 \leq \beta \leq \beta_0$ *or* $\beta \geq \max\{\beta_1, (\frac{p-1}{2})\mu_1\}$.

Then problem (1.4) *has s positive solution* $(U, V) = (k_1 W, \tau_0 k_1 W)$ *in* $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ *which is non-degenerate, where* τ_0 *satisfies*

$$\mu_1 + \beta\tau_0^{\frac{p+1}{2}} - \mu_2\tau_0^{p-1} - \beta\tau_0^{\frac{p-3}{2}} = 0,\ k_1^{p-1} = \left(\mu_1 + \beta\tau_0^{\frac{p+1}{2}}\right)^{-1}.$$

Proposition 1.2 *Suppose* $1 < p < 2^* - 1,\ \mu_1 > \mu_2 > 0,\ \beta < 0$. *Then there exists a decreasing sequence* $\{\beta_k\} \subset (-\sqrt{\mu_1\mu_2}, 0)$ *such that for* $\beta \in (-\sqrt{\mu_1\mu_2}, 0)\backslash\{\beta_k\}$, *problem* (1.4) *has s positive solution* $(U, V) = (k_1 W, \tau_0 k_1 W)$ *in* $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ *which is non-degenerate, where* τ_0 *satisfies*

$$\mu_1 + \beta\tau_0^{\frac{p+1}{2}} - \mu_2\tau_0^{p-1} - \beta\tau_0^{\frac{p-3}{2}} = 0,\ k_1^{p-1} = \left(\mu_1 + \beta\tau_0^{\frac{p+1}{2}}\right)^{-1}.$$

Remark 1.1 Although the authors in [15] dealt with the existence and the non-degeneracy of positive solutions for the fractional Schrödinger system, the main results are still true for the classical Schrödinger system with $s = 1$.

The main results in this paper are stated in what follows.

Theorem 1.1 *Assume the conditions in Propositions 1.1 or 1.2. If* x_0 *is a local maximum point of* $P(x)$, $Q(x)$ *and* $P(x_0) = Q(x_0)$, *then there exists* $\epsilon_0 > 0$ *such that for any* $\epsilon \in (0, \epsilon_0]$, *problem* (1.1) *has a solution of the form*

$$u_\epsilon = \sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon}} + \varphi_\epsilon,\quad v_\epsilon = \sum_{j=1}^{k} V_{\epsilon,x_{j,\epsilon}} + \psi_\epsilon,$$

for some $x_{j,\epsilon} \in B_\delta(x_0)$ *and* $\|(\varphi_\epsilon, \psi_\epsilon)\|_\epsilon = O(\epsilon^{\frac{N}{2}+1})$. *Moreover, as* $\epsilon \to 0$, $x_{j,\epsilon} \to x_0$, $\frac{|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon} \to +\infty$ *if* $i \neq j$.

Set $\lambda = P(x_0)$, $\overline{\lambda} = Q(\overline{x}_0)$, it is easy to see that $U_{\lambda,\mu} = \lambda^{\frac{1}{p-1}} \mu^{-\frac{1}{p-1}} W(\sqrt{\lambda}x)$ is a solution of

$$\begin{cases} -\Delta u + \lambda u = \mu u^p, \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, u(x) \to 0 \text{ as } |x| \to +\infty, \end{cases}$$

and $U_{\overline{\lambda},\nu} = \overline{\lambda}^{\frac{1}{p-1}} \nu^{-\frac{1}{p-1}} W(\sqrt{\overline{\lambda}}x)$ is a solution of

$$\begin{cases} -\Delta u + \overline{\lambda} u = \nu u^p, \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, u(x) \to 0 \text{ as } |x| \to +\infty. \end{cases}$$

Let

$$\left(U_{\epsilon,x_j,\mu}(x), U_{\epsilon,z_j,\nu}(x)\right) = \left(U_{\lambda_j,\mu}\left(\frac{x-x_j}{\epsilon}\right), U_{\overline{\lambda}_j,\nu}\left(\frac{x-z_j}{\epsilon}\right)\right),$$

where $x_j \in B_\delta(x_0)$, $z_j \in B_\delta(\overline{x}_0)$,

$$\begin{aligned} \widetilde{\mathbf{E}}_\epsilon = \Bigg\{ &(\varphi, \psi) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N), \left\langle \varphi, \frac{\partial U_{\epsilon,x_j,\mu}}{\partial x_{j,l}} \right\rangle_\epsilon = 0, \\ &\left\langle \psi, \frac{\partial U_{\epsilon,z_j,\nu}}{\partial z_{j,l}} \right\rangle_\epsilon = 0, \ j = 1, 2 \ldots k; \ \ l = 1, 2 \ldots N \Bigg\}. \end{aligned}$$

We will use $\left(U_{\epsilon,x_j,\mu}(x), U_{\epsilon,z_j,\nu}(x)\right)$ to build up the solutions of problem (1.1).

Theorem 1.2 *Suppose that* x_0 *is a local maximum point of* $P(x)$ *and* $\overline{x}_0$ *is a local maximum point of* $Q(x)$, *with* $x_0 \neq \overline{x}_0$. *Then there exists* $\beta^* > 0$ *depending on* x_0 *and* $\overline{x}_0$ *such that for all* $\beta < \beta^*$, *there exists* $\epsilon_0 > 0$ *such that for any* $\epsilon \in (0, \epsilon_0]$, *problem* (1.1) *has a solution of the form*

$$u_\epsilon = \sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon},\mu}(x) + \overline{\varphi}_\epsilon, \quad v_\epsilon = \sum_{j=1}^{m} U_{\epsilon,z_{j,\epsilon},\nu}(x) + \overline{\psi}_\epsilon$$

for some $x_{j,\epsilon} \in B_\delta(x_0)$, $z_{j,\epsilon} \in B_\delta(\overline{x}_0)$, $\|(\overline{\varphi}_\epsilon, \overline{\psi}_\epsilon)\|_\epsilon = O(\epsilon^{\frac{N}{2}+1})$. *Moreover, as* $\epsilon \to 0$, $x_{j,\epsilon} \to x_0$, $\frac{|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon} \to +\infty$ *if* $i \neq j$, $i = 1, 2, \ldots k$, *and* $z_{j,\epsilon} \to \overline{x}_0$, $\frac{|z_{i,\epsilon}-z_{j,\epsilon}|}{\epsilon} \to +\infty$ $i \neq j$, $i = 1, 2, \ldots m$.

Remark 1.2 The conditions in Theorem 1.1 ensure the existence and the non-degeneracy of positive solutions (U, V) to the related limit problem (1.4), hence the reduction procedure can be carried out successfully. The condition $\beta < \beta^*$ in

Theorem 1.2 is necessary to prove that $Q_\epsilon B_\epsilon$ (see (2.5)) is invertible and the inverse operator is bounded.

Remark 1.3 Compared with the well-studied case $N = 3,\ p = 3$, in order to get an accurate error estimate and use the Contraction Mapping Theorem to prove that problem (2.54) has a unique solution, our situation is much more complicated. In this sense, we complement the main results established by Peng and Wang (Arch Ration Mech Anal 2013) and Peng and Pi (Discrete Contin Dyn Syst 2016), where the authors considered the case $N = 3$, $p = 3$.

The paper is organized as follows. In Sect. 2, we introduce some preliminaries that will be used to prove Theorems 1.1–1.2. In Sect. 3, we prove Theorem 1.1. In Sect. 4, we prove Theorem 1.2. Finally, we give some elementary computations in Appendix A.

Throughout this paper, $C, C_i, i = 1, 2, \ldots$ will always denote various generic positive constants, while $O(t)$ and $o(t)$ denote $C_1 \le \frac{|O(t)|}{|t|} \le C_2$ and $\frac{|o(t)|}{|t|} \to 0$ as $t \to 0$, respectively.

2 Preliminary Results

We first give the definition of multi-peak solutions of system (1.1).

Definition 2.1 Let $k \in \mathbb{N}, 1 \le j \le k$. We say that (u_ϵ, v_ϵ) is k-peak solutions of system (1.1) concentrated at $\{x_1, x_2, \ldots, x_k,\}$ if (u_ϵ, v_ϵ) satisfies the following properties.

(i) (u_ϵ, v_ϵ) has k local maximum points $x_{j,\epsilon} \in \mathbb{R}^N$, $j = 1, 2, \ldots, k$ satisfying

$$x_{j,\epsilon} \to x_j \text{ as } \epsilon \to 0 \text{ for each } j.$$

(ii) For any given $\tau > 0$, there exists $R \gg 1$, such that

$$|u_\epsilon(x)| \le \tau,\ |v_\epsilon(x)| \le \tau \text{ for } x \in \mathbb{R}^N \setminus \cup_j^k B_{R\epsilon}(x_{j,\epsilon}).$$

(iii) There exists $C > 0$ such that

$$\int_{\mathbb{R}^N} \epsilon^2(|\nabla u_\epsilon|^2 + |\nabla v_\epsilon|^2) + u_\epsilon^2 + v_\epsilon^2 \le C\epsilon^N.$$

Let x_0 be the local maximum points of $P(x)$, $Q(x)$ and $P(x_0) = Q(x_0)$. We want to construct a solution (u_ϵ, v_ϵ) of the following form

$$u_\epsilon = \sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}} + \varphi_\epsilon,\quad v_\epsilon = \sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}} + \psi_\epsilon$$

where $x_{j,\epsilon} \to x_0$ and $\|(\varphi_\epsilon, \psi_\epsilon)\|^2 = o(\epsilon^N)$ as $\epsilon \to 0$. Then, $(\varphi_\epsilon, \psi_\epsilon)$ satisfies the following equation

$$\begin{cases} B_\epsilon(\varphi_\epsilon, \psi_\epsilon) + l_\epsilon = R_\epsilon(\varphi_\epsilon, \psi_\epsilon), \ x \in \mathbb{R}^N, \\ (\varphi_\epsilon, \psi_\epsilon) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N), \end{cases} \tag{2.1}$$

where B_ϵ is a bounded linear operator in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, defined by

$$\begin{aligned}
&\langle B_\epsilon(\varphi_\epsilon, \psi_\epsilon), (g, h)\rangle \\
&= \int_{\mathbb{R}^N} \left(\epsilon^2 \nabla\varphi_\epsilon \nabla g + P(x)\varphi_\epsilon g - p\mu \left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}} \right)^{p-1} \varphi_\epsilon g \right) dx \\
&\quad + \int_{\mathbb{R}^N} \left(\epsilon^2 \nabla\psi_\epsilon \nabla h + Q(x)\psi_\epsilon h - p\nu \left(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}} \right)^{p-1} \psi_\epsilon h \right) dx \\
&\quad - \beta \int_{\mathbb{R}^N} \left(\frac{p-1}{2} \left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}} \right)^{\frac{p-3}{2}} \left(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}} \right)^{\frac{p+1}{2}} \varphi_\epsilon g \right. \\
&\quad \left. + \frac{p-1}{2} \left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}} \right)^{\frac{p+1}{2}} \left(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}} \right)^{\frac{p-3}{2}} \psi_\epsilon h \right) dx \\
&\quad - \beta \int_{\mathbb{R}^N} \left(\frac{p+1}{2} \left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}} \right)^{\frac{p-1}{2}} \left(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}} \right)^{\frac{p-1}{2}} \psi_\epsilon g \right. \\
&\quad \left. + \frac{p+1}{2} \left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}} \right)^{\frac{p-1}{2}} \left(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}} \right)^{\frac{p-1}{2}} \varphi_\epsilon h \right) dx
\end{aligned} \tag{2.2}$$

for all $(g, h) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. $l_\epsilon \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ satisfying

$$\begin{aligned}
&\langle l_\epsilon, (g, h)\rangle \\
&= \sum_{j=1}^k \int_{\mathbb{R}^N} (P(x) - \lambda) U_{\epsilon,x_{j,\epsilon}} g dx + \mu \int_{\mathbb{R}^N} (\sum_{j=1}^k U^p_{\epsilon,x_{j,\epsilon}} - \left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}} \right)^p g dx \\
&\quad + \sum_{j=1}^k \int_{\mathbb{R}^N} (Q(x) - \lambda) V_{\epsilon,x_{j,\epsilon}} h dx + \nu \int_{\mathbb{R}^N} (\sum_{j=1}^k V^p_{\epsilon,x_{j,\epsilon}} - \left(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}} \right)^p h dx
\end{aligned}$$

$$+\beta\int_{\mathbb{R}^N}\left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}}^{\frac{p-1}{2}}V_{\epsilon,x_{j,\epsilon}}^{\frac{p+1}{2}}-\left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-1}{2}}\left(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p+1}{2}}\right)gdx$$
$$+\beta\int_{\mathbb{R}^N}\left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}}^{\frac{p+1}{2}}V_{\epsilon,x_{j,\epsilon}}^{\frac{p-1}{2}}-\left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p+1}{2}}\left(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-1}{2}}\right)hdx, \tag{2.3}$$

for all $(g,h)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$.

$$\begin{aligned}
&\langle R_\epsilon(\varphi_\epsilon,\psi_\epsilon),(g,h)\rangle\\
&=\int_{\mathbb{R}^N}\left(\mu\left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}}+\varphi_\epsilon\right)^p+\beta\left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}}+\varphi_\epsilon\right)^{\frac{p-1}{2}}\left(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}}+\psi_\epsilon\right)^{\frac{p+1}{2}}\right)gdx\\
&\quad+\int_{\mathbb{R}^N}\left(\nu\left(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}}+\psi_\epsilon\right)^p+\beta\left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}}+\varphi_\epsilon\right)^{\frac{p+1}{2}}\left(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}}+\psi_\epsilon\right)^{\frac{p-1}{2}}\right)hdx\\
&\quad-\mu\int_{\mathbb{R}^N}\left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}}\right)^p gdx-\beta\int_{\mathbb{R}^N}\left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-1}{2}}\left(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p+1}{2}}gdx\\
&\quad-\nu\int_{\mathbb{R}^N}\left(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}}\right)^p h-\beta\int_{\mathbb{R}^N}\left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p+1}{2}}\left(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-1}{2}}hdx\\
&\quad-\beta\int_{\mathbb{R}^N}\frac{p-1}{2}\left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-3}{2}}\left(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p+1}{2}}\varphi_\epsilon gdx\\
&\quad-\beta\int_{\mathbb{R}^N}\frac{p-1}{2}\left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p+1}{2}}\left(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-3}{2}}\psi_\epsilon hdx\\
&\quad-\beta\int_{\mathbb{R}^N}\frac{p+1}{2}\left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-1}{2}}\left(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-1}{2}}\psi_\epsilon gdx\\
&\quad-\beta\int_{\mathbb{R}^N}\frac{p+1}{2}\left(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-1}{2}}\left(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-1}{2}}\varphi_\epsilon hdx,
\end{aligned} \tag{2.4}$$

for all $(g,h)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$.

From [8, 9], we have following estimates.

Lemma 2.1 *For any $\alpha>0$, $\beta>0$ and $l\neq j$, there exists a constant $\tau>0$ such that*

$$\int_{\mathbb{R}^N} U^{\alpha}_{\epsilon,x_{j,\epsilon}} U^{\beta}_{\epsilon,x_{l,\epsilon}} dx \leq C\epsilon^N e^{-\tau\frac{|x_{l,\epsilon}-x_{j,\epsilon}|}{\epsilon}}, \int_{\mathbb{R}^N} V^{\alpha}_{\epsilon,x_{j,\epsilon}} V^{\beta}_{\epsilon,x_{l,\epsilon}} dx \leq C\epsilon^N e^{-\tau\frac{|x_{l,\epsilon}-x_{j,\epsilon}|}{\epsilon}},$$

$$\int_{\mathbb{R}^N} U^{\alpha}_{\epsilon,x_{j,\epsilon}} V^{\beta}_{\epsilon,x_{l,,\epsilon}} dx \leq C\epsilon^N e^{-\tau\frac{|x_{l,\epsilon}-x_{j,\epsilon}|}{\epsilon}}, \int_{\mathbb{R}^N} V^{\alpha}_{\epsilon,x_{j,\epsilon}} U^{\beta}_{\epsilon,x_{l,\epsilon}} dx \leq C\epsilon^N e^{-\tau\frac{|x_{l,\epsilon}-x_{j,\epsilon}|}{\epsilon}}.$$

For any $(g, h) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, we define the projection Q_ϵ from $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ to $\mathbf{E}_\epsilon$ as follows:

$$Q_\epsilon(u, v) = (u, v) - \sum_{j=1}^{k}\sum_{i=1}^{N} b_{\epsilon,i,j}\left(\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i}, \frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i}\right), \tag{2.5}$$

where $b_{\epsilon,i,j}$ is chosen in such a way that

$$\left\langle Q_\epsilon(u, v), (\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i}, \frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i})\right\rangle_\epsilon = 0,\ j = 1, 2\cdots k; i = 1, 2\cdots N.$$

Therefore $b_{\epsilon,i,j}$ is determined by the following equations:

$$\begin{aligned}
&\sum_{j=1}^{k}\sum_{i=1}^{N} b_{\epsilon,i,j}\left\langle\left(\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i}, \frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i}\right), \left(\frac{\partial U_{\epsilon,x_{m,\epsilon}}}{\partial x_l}, \frac{\partial V_{\epsilon,x_{m,\epsilon}}}{\partial x_l}\right)\right\rangle_\epsilon \\
&= \left\langle (u, v), \left(\frac{\partial U_{\epsilon,x_{m,\epsilon}}}{\partial x_l}, \frac{\partial V_{\epsilon,x_{m,\epsilon}}}{\partial x_l}\right)\right\rangle_\epsilon \\
&m = 1, 2\ldots k, l = 1, 2\ldots N.
\end{aligned} \tag{2.6}$$

We now prove that problem (2.6) is solvable. Since $(U_{\epsilon,x_{j,\epsilon}}, V_{\epsilon,x_{j,\epsilon}})$ satisfies

$$\begin{cases}
-\epsilon^2\Delta U_{\epsilon,x_{j,\epsilon}} + \lambda U_{\epsilon,x_{j,\epsilon}} = \mu U^p_{\epsilon,x_{j,\epsilon}} + \beta U^{\frac{p-1}{2}}_{\epsilon,x_{j,\epsilon}} V^{\frac{p+1}{2}}_{\epsilon,x_{j,\epsilon}} & \text{in } \mathbb{R}^N, \\
-\epsilon^2\Delta V_{\epsilon,x_{j,\epsilon}} + \lambda V_{\epsilon,x_{j,\epsilon}} = \nu V^p_{\epsilon,x_{j,\epsilon}} + \beta U^{\frac{p+1}{2}}_{\epsilon,x_{j,\epsilon}} V^{\frac{p-1}{2}}_{\epsilon,x_{j,\epsilon}} & \text{in } \mathbb{R}^N,
\end{cases}$$

one has

$$\begin{cases}
-\epsilon^2\Delta\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i} + \lambda\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i} = p\mu U^{p-1}_{\epsilon,x_{j,\epsilon}}\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i} \\
+\beta\frac{p-1}{2}U^{\frac{p-3}{2}}_{\epsilon,x_{j,\epsilon}} V^{\frac{p+1}{2}}_{\epsilon,x_{j,\epsilon}}\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i} + \beta\frac{p+1}{2}U^{\frac{p-1}{2}}_{\epsilon,x_{j,\epsilon}} V^{\frac{p-1}{2}}_{\epsilon,x_{j,\epsilon}}\frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i} & \text{in } \mathbb{R}^N, \\
-\epsilon^2\Delta\frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i} + \lambda\frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i} = p\nu V^{p-1}_{\epsilon,x_{j,\epsilon}}\frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i} \\
+\beta\frac{p+1}{2}U^{\frac{p-1}{2}}_{\epsilon,x_{j,\epsilon}} V^{\frac{p-1}{2}}_{\epsilon,x_{j,\epsilon}}\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i} + \beta\frac{p-1}{2}U^{\frac{p+1}{2}}_{\epsilon,x_{j,\epsilon}} V^{\frac{p-3}{2}}_{\epsilon,x_{j,\epsilon}}\frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i} & \text{in } \mathbb{R}^N.
\end{cases} \tag{2.7}$$

Therefore

$$
\begin{aligned}
&\left\langle (\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i}, \frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i}), (\varphi, \psi) \right\rangle_\epsilon \\
&= \int_{\mathbb{R}^N} \left(\epsilon^2 \nabla \frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i} \nabla \varphi + P(x) \frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i} \varphi + \epsilon^2 \nabla \frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i} \nabla \psi + Q(x) \frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i} \psi \right) dx \\
&= \int_{\mathbb{R}^N} \left((P(x) - \lambda) \frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i} \varphi + p\mu U_{\epsilon,x_{j,\epsilon}}^{p-1} \frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i} \varphi \right) \\
&+ \int_{\mathbb{R}^N} \left((Q(x) - \lambda) \frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i} \psi + p\nu V_{\epsilon,x_{j,\epsilon}}^{p-1} \frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i} \psi \right) \\
&+ \int_{\mathbb{R}^N} \left(\beta \frac{p-1}{2} U_{\epsilon,x_{j,\epsilon}}^{\frac{p-3}{2}} V_{\epsilon,x_{j,\epsilon}}^{\frac{p+1}{2}} \frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i} + \beta \frac{p+1}{2} U_{\epsilon,x_{j,\epsilon}}^{\frac{p-1}{2}} V_{\epsilon,x_{j,\epsilon}}^{\frac{p-1}{2}} \frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i} \right) \varphi dx \\
&+ \int_{\mathbb{R}^N} \left(\beta \frac{p+1}{2} U_{\epsilon,x_{j,\epsilon}}^{\frac{p-1}{2}} V_{\epsilon,x_{j,\epsilon}}^{\frac{p-1}{2}} \frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i} + \beta \frac{p-1}{2} U_{\epsilon,x_{j,\epsilon}}^{\frac{p+1}{2}} V_{\epsilon,x_{j,\epsilon}}^{\frac{p-3}{2}} \frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i} \right) \psi dx.
\end{aligned} \tag{2.8}
$$

By Lemma 2.1, (2.8) and the symmetry of $U_{\epsilon,x_{j,\epsilon}}, V_{\epsilon,x_{j,\epsilon}}$, we have

$$
\left\langle \left(\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i}, \frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i} \right), \left(\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_h}, \frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_h} \right) \right\rangle_\epsilon = \delta_{h,i} \epsilon^{N-2} (c_j + o(1)), \tag{2.9}
$$

$$
\begin{aligned}
&\left\langle \left(\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i}, \frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i} \right), \left(\frac{\partial U_{\epsilon,x_{m,\epsilon}}}{\partial x_h}, \frac{\partial V_{\epsilon,x_{m,\epsilon}}}{\partial x_h} \right) \right\rangle_\epsilon \\
&= O \left(e^{-\frac{1}{2}\sqrt{\lambda} \frac{|x_m - x_j|}{\epsilon}} \right) \epsilon^{N-2}, \quad j \neq m,
\end{aligned} \tag{2.10}
$$

where $\delta_{h,i} = 0$ if $h \neq i$ and $\delta_{i,i} = 1$, $c_j > 0$ is a constant.

Hence (2.6) is solvable and we have the following estimate

$$
\begin{aligned}
|b_{\epsilon,i,j}| &\leq C\epsilon^{2-N} \sum_{j=1}^{k} \sum_{i=1}^{N} \left\langle (u, v), \left(\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i}, \frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i} \right) \right\rangle_\epsilon \\
&\leq C\epsilon^{2-N} \|(u, v)\|_\epsilon \| \left(\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i}, \frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i} \right) \|_\epsilon \leq C\epsilon^{-\frac{N}{2}+1} \|(u, v)\|_\epsilon.
\end{aligned} \tag{2.11}
$$

In order to prove the invertibility of the operator $Q_\epsilon B_\epsilon$, we use the following non-degenerate results for system (1.4), which can be found in [15].

Proposition 2.1 *Under the conditions of Propositions 1.1 or 1.2, the system (1.4) has a positive non-degenerate solution for* $(U_\lambda, V_\lambda) = (k_1 W, k_1 \tau_0 W)$ *in* $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ *in the sense that the kernel is given by span* $\{(\theta(\beta) \frac{\partial W}{\partial y_j}, \frac{\partial W}{\partial y_j}) \mid j = 1, 2, \cdots N\}$, *where* $\theta(\beta) \neq 0$, τ_0 *satisfies* $\mu_1 + \beta \tau_0^{\frac{p+1}{2}} - \mu_2 \tau_0^{p-1} - \beta \tau_0^{\frac{p-3}{2}} = 0$, $k_1^{p-1} = \left(\mu_1 + \beta \tau_0^{\frac{p+1}{2}} \right)^{-1}$.

Proposition 2.2 *Suppose* $1 < p < 2^* - 1,\ \mu_1 > \mu_2 > 0,\ \beta < 0$, *then there exists a decreasing sequence* $\{\beta_k\} \subset (-\sqrt{\mu_1\mu_2}, 0)$ *such that for* $\beta \in (-\sqrt{\mu_1\mu_2}, 0)\backslash\{\beta_k\}$, *the system* (1.4) *has a positive solution* $(U, V) = (k_1 W, \tau_0 k_1 W)$ *in* $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ *which is non-degenerate, where* τ_0 *satisfies*

$$\mu_1 + \beta\tau_0^{\frac{p+1}{2}} - \mu_2\tau_0^{p-1} - \beta\tau_0^{\frac{p-3}{2}} = 0, \quad k_1^{p-1} = \left(\mu_1 + \beta\tau_0^{\frac{p+1}{2}}\right)^{-1}.$$

In order to carry out the reduction arguments, we give following key lemma.

Lemma 2.2 *There exist* $\epsilon_0,\ \theta_0 > 0,\ \rho > 0$, *independent of* $x_j,\ j = 1, 2, \cdots, k$ *such that for any* $\epsilon \in (0, \epsilon_0]$ *and* $x_j \in B_{\theta_0}(x_0),\ Q_\epsilon B_\epsilon(\varphi_\epsilon, \psi_\epsilon)$ *is bijective in* $\mathbf{E}_\epsilon$. *Moreover, it holds*

$$\|Q_\epsilon B_\epsilon(\varphi_\epsilon, \psi_\epsilon)\|_\epsilon \geq \rho\|(\varphi_\epsilon, \psi_\epsilon)\|_\epsilon, \ \textit{for all } (\varphi_\epsilon, \psi_\epsilon) \in \mathbf{E}_\epsilon.$$

Proof Suppose by contradiction that there are $\epsilon_n \to 0,\ x_{\epsilon_n, j} \to x_0,\ (\varphi_n, \psi_n) \in \mathbf{E}_{\epsilon_n}$ such that

$$\|Q_{\epsilon_n} B_{\epsilon_n}(\varphi_n, \psi_n)\|_{\epsilon_n} \leq \frac{1}{n}\|(\varphi_n, \psi_n)\|_{\epsilon_n}, \ \text{for all } (\varphi_n, \psi_n) \in \mathbf{E}_{\epsilon_n}. \tag{2.12}$$

We assume $\|(\varphi_n, \psi_n)\|_{\epsilon_n}^2 = \epsilon_n^N$. By (2.12), for any $(g, h) \in \mathbf{E}_{\epsilon_n}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\epsilon_n^2 \nabla\varphi_n \nabla g + P(x)\varphi_n g - p\mu\left(\sum_{j=1}^k U_{\epsilon_n, x_{\epsilon_n, j}}\right)^{p-1} \varphi_n g\right) dx \\
&+ \int_{\mathbb{R}^N} \left(\epsilon_n^2 \nabla\psi_n \nabla h + Q(x)\psi_n h - p\nu\left(\sum_{j=1}^k V_{\epsilon_n, x_{\epsilon_n, j}}\right)^{p-1} \psi_n h\right) dx \\
&- \beta\int_{\mathbb{R}^N} \frac{p-1}{2}\left(\sum_{j=1}^k U_{\epsilon_n, x_{\epsilon_n, j}}\right)^{\frac{p-3}{2}} \left(\sum_{j=1}^k V_{\epsilon_n, x_{\epsilon_n, j}}\right)^{\frac{p+1}{2}} \varphi_n g dx \\
&- \beta\int_{\mathbb{R}^N} \frac{p-1}{2}\left(\sum_{j=1}^k U_{\epsilon_n, x_{\epsilon_n, j}}\right)^{\frac{p+1}{2}} \left(\sum_{j=1}^k V_{\epsilon_n, x_{\epsilon_n, j}}\right)^{\frac{p-3}{2}} \psi_n h dx \\
&- \beta\int_{\mathbb{R}^N} \frac{p+1}{2}\left(\sum_{j=1}^k U_{\epsilon_n, x_{\epsilon_n, j}}\right)^{\frac{p-1}{2}} \left(\sum_{j=1}^k V_{\epsilon_n, x_{\epsilon_n, j}}\right)^{\frac{p-1}{2}} \psi_n g dx \\
&- \beta\int_{\mathbb{R}^N} \frac{p+1}{2}\left(\sum_{j=1}^k U_{\epsilon_n, x_{\epsilon_n, j}}\right)^{\frac{p-1}{2}} \left(\sum_{j=1}^k V_{\epsilon_n, x_{\epsilon_n, j}}\right)^{\frac{p-1}{2}} \varphi_n h dx
\end{aligned}$$

$$
\begin{aligned}
&= \langle B_{\epsilon_n}(\varphi_n, \psi_n), (g,h)\rangle_{\epsilon_n} = \langle Q_{\epsilon_n} B_{\epsilon_n}(\varphi_n, \psi_n), (g,h)\rangle_{\epsilon_n} \\
&= o(1)\|(\varphi_n, \psi_n)\|_{\epsilon_n}\|(g,h)\|_{\epsilon_n} = o(\epsilon_n^{\frac{N}{2}})\|(g,h)\|_{\epsilon_n}.
\end{aligned} \tag{2.13}
$$

Taking $(g, h) = (\varphi_n, \psi_n)$ in (2.13), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \left(\epsilon_n^2 |\nabla \varphi_n|^2 + P(x)\varphi_n^2 - p\mu \left(\sum_{j=1}^k U_{\epsilon_n, x_{\epsilon_n,j}} \right)^{p-1} \varphi_n^2 \right) dx \\
&+ \int_{\mathbb{R}^N} \left(\epsilon_n^2 |\nabla \psi_n|^2 + Q(x)\psi_n^2 - p\nu \left(\sum_{j=1}^k V_{\epsilon_n, x_{\epsilon_n,j}} \right)^{p-1} \psi_n^2 \right) dx \\
&- \beta \int_{\mathbb{R}^N} \frac{p-1}{2} \left(\sum_{j=1}^k U_{\epsilon_n, x_{\epsilon_n,j}} \right)^{\frac{p-3}{2}} \left(\sum_{j=1}^k V_{\epsilon_n, x_{\epsilon_n,j}} \right)^{\frac{p+1}{2}} \varphi_n^2 dx \\
&- \beta \int_{\mathbb{R}^N} \frac{p-1}{2} \left(\sum_{j=1}^k U_{\epsilon_n, x_{\epsilon_n,j}} \right)^{\frac{p+1}{2}} \left(\sum_{j=1}^k V_{\epsilon_n, x_{\epsilon_n,j}} \right)^{\frac{p-3}{2}} \psi_n^2 dx \\
&- \beta \int_{\mathbb{R}^N} (p+1) \left(\sum_{j=1}^k U_{\epsilon_n, x_{\epsilon_n,j}} \right)^{\frac{p-1}{2}} \left(\sum_{j=1}^k V_{\epsilon_n, x_{\epsilon_n,j}} \right)^{\frac{p-1}{2}} \psi_n \varphi_n dx = o\left(\epsilon_n^N\right).
\end{aligned} \tag{2.14}
$$

By Hölder's inequality and Young's inequality, we have

$$
\begin{aligned}
&\beta(p+1) \int_{\mathbb{R}^N} \left(\sum_{j=1}^k U_{\epsilon_n, x_{\epsilon_n,j}} \right)^{\frac{p-1}{2}} \left(\sum_{j=1}^k V_{\epsilon_n, x_{\epsilon_n,j}} \right)^{\frac{p-1}{2}} \psi_n \varphi_n dx \\
&\le |\beta|(p+1) \left(\int_{\mathbb{R}^N} \left(\sum_{j=1}^k U_{\epsilon_n, x_{\epsilon_n,j}} \right)^{p-1} \varphi_n^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \left(\sum_{j=1}^k V_{\epsilon_n, x_{\epsilon_n,j}} \right)^{p-1} \psi_n^2 dx \right)^{\frac{1}{2}} \\
&\le |\beta| \frac{p+1}{2} \left(\int_{\mathbb{R}^N} \left(\sum_{j=1}^k U_{\epsilon_n, x_{\epsilon_n,j}} \right)^{p-1} \varphi_n^2 dx + \int_{\mathbb{R}^N} \left(\sum_{j=1}^k V_{\epsilon_n, x_{\epsilon_n,j}} \right)^{p-1} \psi_n^2 dx \right).
\end{aligned} \tag{2.15}
$$

Since $U_{\epsilon_n,x_{\epsilon_n,j}} = \frac{1}{\tau_0} V_{\epsilon_n,x_{\epsilon_n,j}}$, we have

$$\beta \frac{p-1}{2} \int_{\mathbb{R}^N} \left(\left(\sum_{j=1}^k U_{\epsilon_n,x_{\epsilon_n,j}} \right)^{\frac{p-3}{2}} \left(\sum_{j=1}^k V_{\epsilon_n,x_{\epsilon_n,j}} \right)^{\frac{p+1}{2}} \varphi_n^2 \right.$$
$$\left. + \left(\sum_{j=1}^k U_{\epsilon_n,x_{\epsilon_n,j}} \right)^{\frac{p+1}{2}} \left(\sum_{j=1}^k V_{\epsilon_n,x_{\epsilon_n,j}} \right)^{\frac{p-3}{2}} \psi_n^2 \right) dx$$
$$= \beta \frac{p-1}{2} \tau_0^{\frac{p+1}{2}} \int_{\mathbb{R}^N} \left(\sum_{j=1}^k U_{\epsilon_n,x_{\epsilon_n,j}} \right)^{p-1} \varphi_n^2 dx + \beta \frac{p-1}{2} \left(\frac{1}{\tau_0} \right)^{\frac{p+1}{2}}$$
$$\times \int_{\mathbb{R}^N} \left(\sum_{j=1}^k V_{\epsilon_n,x_{\epsilon_n,j}} \right)^{p-1} \psi_n^2 dx. \tag{2.16}$$

On the other hand, we can take a large $R > 0$ such that

$$\left(p\mu + |\beta| \frac{p+1}{2} + \beta \frac{p-1}{2} \tau_0^{\frac{p+1}{2}} \right) \left(\sum_{j=1}^k U_{\epsilon_n,x_{\epsilon_n,j}} \right)^{p-1}$$
$$\leq \frac{1}{2} P(x) \text{ in } \mathbb{R}^N \setminus \bigcup_{j=1}^k B_{\epsilon_n,R}(x_{\epsilon_n,j}),$$
$$\left(p\nu + |\beta| \frac{p+1}{2} + \beta \frac{p-1}{2} (\frac{1}{\tau_0})^{\frac{p+1}{2}} \right) \left(\sum_{j=1}^k V_{\epsilon_n,x_{\epsilon_n,j}} \right)^{p-1}$$
$$\leq \frac{1}{2} Q(x) \quad \text{in } \mathbb{R}^N \setminus \bigcup_{j=1}^k B_{\epsilon_n,R}(x_{\epsilon_n,j}). \tag{2.17}$$

Thus, combining relations (2.15), (2.16) and (2.17) with (2.14), we obtain

$$o(\epsilon_n^N) = \int_{\mathbb{R}^N} \left(\epsilon_n^2 |\nabla \varphi_n|^2 + P(x)\varphi_n^2 - p\mu \left(\sum_{j=1}^k U_{\epsilon_n,x_{\epsilon_n,j}} \right)^{p-1} \varphi_n^2 \right) dx$$
$$+ \int_{\mathbb{R}^N} \left(\epsilon_n^2 |\nabla \psi_n|^2 + Q(x)\psi_n^2 - p\nu \left(\sum_{j=1}^k V_{\epsilon_n,x_{\epsilon_n,j}} \right)^{p-1} \psi_n^2 \right) dx$$
$$- \beta \int_{\mathbb{R}^N} \frac{p-1}{2} \left(\sum_{j=1}^k U_{\epsilon_n,x_{\epsilon_n,j}} \right)^{\frac{p-3}{2}} \left(\sum_{j=1}^k V_{\epsilon_n,x_{\epsilon_n,j}} \right)^{\frac{p+1}{2}} \varphi_n^2$$

$$
\begin{aligned}
&- \beta \int_{\mathbb{R}^N} \frac{p-1}{2} \left(\sum_{j=1}^{k} U_{\epsilon_n, x_{\epsilon_n, j}} \right)^{\frac{p+1}{2}} \left(\sum_{j=1}^{k} V_{\epsilon_n, x_{\epsilon_n, j}} \right)^{\frac{p-3}{2}} \psi_n^2 \\
&- \beta \int_{\mathbb{R}^N} (p+1) \left(\sum_{j=1}^{k} U_{\epsilon_n, x_{\epsilon_n, j}} \right)^{\frac{p-1}{2}} \left(\sum_{j=1}^{k} V_{\epsilon_n, x_{\epsilon_n, j}} \right)^{\frac{p-1}{2}} \psi_n \varphi_n \\
\geq &\frac{1}{2} \|(\varphi_n, \psi_n)\|_{\epsilon_n}^2 - \left(p\mu + |\beta| \frac{p+1}{2} + \beta \frac{p-1}{2} \tau_0^{\frac{p+1}{2}} \right) \\
&\times \int_{\bigcup_{j=1}^k B_{\epsilon_n, R}(x_{\epsilon_n, j})} \left(\sum_{j=1}^{k} U_{\epsilon_n, x_{\epsilon_n, j}} \right)^{p-1} \varphi_n^2 dx \\
&- \left(p\nu + |\beta| \frac{p+1}{2} + \beta \frac{p-1}{2} \left(\frac{1}{\tau_0} \right)^{\frac{p+1}{2}} \right) \\
&\times \int_{\bigcup_{j=1}^k B_{\epsilon_n, R}(x_{\epsilon_n, j})} \left(\sum_{j=1}^{k} V_{\epsilon_n, x_{\epsilon_n, j}} \right)^{p-1} \psi_n^2 dx.
\end{aligned}
\tag{2.18}
$$

Therefore,

$$
\begin{aligned}
\epsilon_n^N \leq C &\int_{\bigcup_{j=1}^k B_{\epsilon_n, R}(x_{\epsilon_n, j})} \left(\sum_{j=1}^{k} U_{\epsilon_n, x_{\epsilon_n, j}} \right)^{p-1} \varphi_n^2 dx \\
&+ \int_{\bigcup_{j=1}^k B_{\epsilon_n, R}(x_{\epsilon_n, j})} \left(\sum_{j=1}^{k} V_{\epsilon_n, x_{\epsilon_n, j}} \right)^{p-1} \psi_n^2 dx + o\left(\epsilon_n^N\right) \\
\leq C &\sum_{j=1}^{k} \int_{B_{\epsilon_n, R}(x_{\epsilon_n, j})} \left(\varphi_n^2 + \psi_n^2 \right) dx + o\left(\epsilon_n^N\right).
\end{aligned}
\tag{2.19}
$$

If we can prove that

$$
\int_{B_{\epsilon_n, R}(x_{\epsilon_n, j})} (\varphi_n^2 + \psi_n^2) dx = o(\epsilon_n^N), \quad j = 1, 2, \cdots, k, \tag{2.20}
$$

we get a contradiction. For this purpose, we will discuss the local behaviors near each points $x_{\epsilon_n, m}$. We define

$$\widetilde{\varphi}_{n,m}(x) = \varphi_n(\epsilon_n x + x_{\epsilon_n,m}),$$

$$\widetilde{\psi}_{n,m}(x) = \psi_n(\epsilon_n x + x_{\epsilon_n,m}),$$

then

$$\begin{aligned}&\int_{\mathbb{R}^N} \Big(|\nabla\widetilde{\varphi}_{n,m}(x)|^2 + P(\epsilon_n x + x_{\epsilon_n,m})|\widetilde{\psi}_{n,m}(x)|^2 + |\nabla\widetilde{\varphi}_{n,m}(x)|^2 \\ &\quad + Q(\epsilon_n x + x_{\epsilon_n,m})|\widetilde{\psi}_{n,m}(x)|^2\Big)\,dx \\ &\quad = \epsilon_n^{-N}\int_{\mathbb{R}^N}(\epsilon_n^2|\nabla\varphi_n(x)|^2 + \lambda|\psi_n(x)|^2 + \epsilon_n^2|\nabla\varphi_n(x)|^2 \\ &\quad + \lambda|\psi_n(x)|^2)dx = o(1) \le C. \end{aligned} \tag{2.21}$$

Therefore,

$$(\widetilde{\varphi}_{n,m}, \widetilde{\psi}_{n,m}) \rightharpoonup (\varphi, \psi) \text{ weakly in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N),$$

$$(\widetilde{\varphi}_{n,m}, \widetilde{\psi}_{n,m}) \to (\varphi, \psi) \text{ strongly in } L^2_{loc}(\mathbb{R}^N) \times L^2_{loc}(\mathbb{R}^N).$$

Moreover, (φ, ψ) satisfies

$$\begin{aligned}&\int_{\mathbb{R}^N} \nabla\frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}\nabla\varphi + \lambda\frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}\varphi \\ &\quad + \int_{\mathbb{R}^N} \nabla\frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}\nabla\psi + \lambda\frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}\psi = 0, \quad l = 1, 2, \dots N. \end{aligned} \tag{2.22}$$

To prove (2.20), we only need to show that $(\varphi, \psi) = (0, 0)$. Remark that relation (2.13) holds just for $(g, h) \in \mathbf{E}_{\epsilon_n}$ not for all $(g, h) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. For $(g, h) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, we take

$$Q_{\epsilon_n}(g, h) = (g, h) - \sum_{j=1}^{k}\sum_{i=1}^{N} b_{\epsilon_n,i,j}\left(\frac{\partial U_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i}, \frac{\partial V_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i}\right) \in \mathbf{E}_{\epsilon_n}. \tag{2.23}$$

Then

$$b_{\epsilon_n,h,m} = \sum_{j=1}^{k}\sum_{i=1}^{N} a_{\epsilon_n,i,j}\left\langle (g, h), \left(\frac{\partial U_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i}, \frac{\partial V_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i}\right)\right\rangle_{\epsilon},$$

for some constant $a_{\epsilon,i,j}$. From (2.23), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}(\epsilon_n^2\nabla\varphi_n\nabla g + P(x)\varphi_n g - p\mu\left(\sum_{j=1}^{k}U_{\epsilon_n,x_{\epsilon_n,j}}\right)^{p-1}\varphi_n g)dx\\
&+\int_{\mathbb{R}^N}\left(\epsilon_n^2\nabla\psi_n\nabla h + Q(x)\psi_n h - p\nu\left(\sum_{j=1}^{k}V_{\epsilon_n,x_{\epsilon_n,j}}\right)^{p-1}\psi_n h\right)dx\\
&-\beta\int_{\mathbb{R}^N}\frac{p-1}{2}\left(\sum_{j=1}^{k}U_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p-3}{2}}\left(\sum_{j=1}^{k}V_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p+1}{2}}\varphi_n g dx\\
&-\beta\int_{\mathbb{R}^N}\frac{p-1}{2}\left(\sum_{j=1}^{k}U_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p+1}{2}}\left(\sum_{j=1}^{k}V_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p-3}{2}}\psi_n h dx\\
&-\beta\int_{\mathbb{R}^N}\frac{p+1}{2}\left(\sum_{j=1}^{k}U_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p-1}{2}}\left(\sum_{j=1}^{k}V_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p-1}{2}}\psi_n g dx\\
&-\beta\int_{\mathbb{R}^N}\frac{p+1}{2}\left(\sum_{j=1}^{k}U_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p-1}{2}}\left(\sum_{j=1}^{k}V_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p-1}{2}}\varphi_n h dx\\
&=\langle B_{\epsilon_n}(\varphi_n,\psi_n),(g,h)\rangle_{\epsilon_n} = \langle B_{\epsilon_n}(\varphi_n,\psi_n), Q_{\epsilon_n}(g,h)\rangle_{\epsilon_n}\\
&+\sum_{j=1}^{k}\sum_{i=1}^{N}b_{\epsilon_n,i,j}\left\langle B_{\epsilon_n}(\varphi_n,\psi_n),\left(\frac{\partial U_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i},\frac{\partial V_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i}\right)\right\rangle_{\epsilon_n}.
\end{aligned}\tag{2.24}$$

Since

$$\begin{aligned}
\langle B_{\epsilon_n}(\varphi_n,\psi_n), Q_{\epsilon_n}(g,h)\rangle_{\epsilon_n} &= \langle Q_{\epsilon_n}B_{\epsilon_n}(\varphi_n,\psi_n), Q_{\epsilon_n}(g,h)\rangle_{\epsilon_n}\\
&= o(1)\|(\varphi_n,\psi_n)\|_{\epsilon_n}\|Q_{\epsilon_n}(g,h)\|_{\epsilon_n} = o\left(\epsilon_n^{\frac{N}{2}}\right)\|(g,h)\|_{\epsilon_n},
\end{aligned}\tag{2.25}$$

$$\begin{aligned}
&\sum_{j=1}^{k}\sum_{i=1}^{N}b_{\epsilon_n,i,j}\left\langle B_{\epsilon_n}(\varphi_n,\psi_n),\left(\frac{\partial U_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i},\frac{\partial V_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i}\right)\right\rangle_{\epsilon_n}\\
&=\sum_{j=1}^{k}\sum_{i=1}^{N}a_{\epsilon,i,j}\left\langle (g,h),\left(\frac{\partial U_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i},\frac{\partial V_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i}\right)\right\rangle_{\epsilon_n}\\
&\times\left\langle Q_{\epsilon_n}B_{\epsilon_n}(\varphi_n,\psi_n),\left(\frac{\partial U_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i},\frac{\partial V_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i}\right)\right\rangle_{\epsilon_n}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{k}\sum_{i=1}^{N} c_{\epsilon,i,j}\left\langle (g,h),\left(\frac{\partial U_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i},\frac{\partial V_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i}\right)\right\rangle_{\epsilon_n}\\
&\times\left\langle (\frac{\partial U_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i},\frac{\partial V_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i}),(\frac{\partial U_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i},\frac{\partial V_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i})\right\rangle_{\epsilon_n}\\
&=\sum_{j=1}^{k}\sum_{i=1}^{N}\gamma_{n,i,j}\left\langle (g,h),\left(\frac{\partial U_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i},\frac{\partial V_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i}\right)\right\rangle_{\epsilon_n}. \qquad (2.26)
\end{aligned}
$$

Substitute (2.25), (2.26) into (2.24), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N}(\epsilon_n^2\nabla\varphi_n\nabla g+P(x)\varphi_n g-p\mu\left(\sum_{j=1}^{k}U_{\epsilon_n,x_{\epsilon_n,j}}\right)^{p-1}\varphi_n g)dx\\
&+\int_{\mathbb{R}^N}(\epsilon_n^2\nabla\psi_n\nabla h+Q(x)\psi_n h-p\nu\left(\sum_{j=1}^{k}V_{\epsilon_n,x_{\epsilon_n,j}}\right)^{p-1}\psi_n h)dx\\
&-\beta\int_{\mathbb{R}^N}\frac{p-1}{2}\left(\sum_{j=1}^{k}U_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p-3}{2}}\left(\sum_{j=1}^{k}V_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p+1}{2}}\varphi_n g dx\\
&-\beta\int_{\mathbb{R}^N}\frac{p-1}{2}\left(\sum_{j=1}^{k}U_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p+1}{2}}\left(\sum_{j=1}^{k}V_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p-3}{2}}\psi_n h dx\\
&-\beta\int_{\mathbb{R}^N}\frac{p+1}{2}\left(\sum_{j=1}^{k}U_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p-1}{2}}\left(\sum_{j=1}^{k}V_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p-1}{2}}\psi_n g dx\\
&-\beta\int_{\mathbb{R}^N}\frac{p+1}{2}\left(\sum_{j=1}^{k}U_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p-1}{2}}\left(\sum_{j=1}^{k}V_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p-1}{2}}\varphi_n h dx\\
&=o(\epsilon_n^{\frac{N}{2}})\|(g,h)\|_{\epsilon_n}+\sum_{j=1}^{k}\sum_{i=1}^{N}\gamma_{n,i,j}\left\langle (g,h),\left(\frac{\partial U_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i},\frac{\partial V_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i}\right)\right\rangle_{\epsilon_n}. \qquad (2.27)
\end{aligned}
$$

By (2.27) and $(\varphi_n,\psi_n)\in\mathbf{E}_{\epsilon_n}$, we can estimate $\gamma_{n,i,j}$ as following

$$
\begin{aligned}
&\sum_{j=1}^{k}\sum_{i=1}^{N}\gamma_{n,i,j}\left\langle\left(\frac{\partial U_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i},\frac{\partial V_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i}\right),\left(\frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l},\frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}\right)\right\rangle_{\epsilon_n}+o(\epsilon_n^{N-1})\\
&=-\int_{\mathbb{R}^N}p\mu\left(\sum_{j=1}^{k}U_{\epsilon_n,x_{\epsilon_n,j}}\right)^{p-1}\varphi_n\frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l})dx
\end{aligned}
$$

$$
\begin{aligned}
& - \int_{\mathbb{R}^N} p\nu \left(\sum_{j=1}^{k} V_{\epsilon_n,x_{\epsilon_n,j}}\right)^{p-1} \psi_n \frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l})dx \\
& - \beta \int_{\mathbb{R}^N} \frac{p-1}{2} (\sum_{j=1}^{k} U_{\epsilon_n,x_{\epsilon_n,j}})^{\frac{p-3}{2}} (\sum_{j=1}^{k} V_{\epsilon_n,x_{\epsilon_n,j}})^{\frac{p+1}{2}} \varphi_n \frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l} dx \\
& - \beta \int_{\mathbb{R}^N} \frac{p-1}{2} (\sum_{j=1}^{k} U_{\epsilon_n,x_{\epsilon_n,j}})^{\frac{p+1}{2}} (\sum_{j=1}^{k} V_{\epsilon_n,x_{\epsilon_n,j}})^{\frac{p-3}{2}} \psi_n \frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l} dx \\
& - \beta \int_{\mathbb{R}^N} \frac{p+1}{2} (\sum_{j=1}^{k} U_{\epsilon_n,x_{\epsilon_n,j}})^{\frac{p-1}{2}} (\sum_{j=1}^{k} V_{\epsilon_n,x_{\epsilon_n,j}})^{\frac{p-1}{2}} \psi_n \frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l} dx \\
& - \beta \int_{\mathbb{R}^N} \frac{p+1}{2} (\sum_{j=1}^{k} U_{\epsilon_n,x_{\epsilon_n,j}})^{\frac{p-1}{2}} (\sum_{j=1}^{k} V_{\epsilon_n,x_{\epsilon_n,j}})^{\frac{p-1}{2}} \varphi_n \frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l} dx \\
=: & A_1 + A_2 + A_3 + A_4 + A_5 + A_6. \qquad (2.28)
\end{aligned}
$$

On the other hand, from (2.8) and $(\varphi_n, \psi_n) \in \mathbf{E}_{\epsilon_n}$, we have

$$
\begin{aligned}
& \int_{\mathbb{R}^N} p\mu (U_{\epsilon_n,x_{\epsilon_n,m}})^{p-1} \varphi_n \frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l})dx + \int_{\mathbb{R}^N} p\nu (V_{\epsilon_n,x_{\epsilon_n,m}})^{p-1} \psi_n \frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l})dx \\
& + \beta \int_{\mathbb{R}^N} \frac{p-1}{2} (U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-3}{2}} (V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p+1}{2}} \varphi_n \frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l} \\
& + \beta \int_{\mathbb{R}^N} \frac{p-1}{2} (U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p+1}{2}} (V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-3}{2}} \psi_n \frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l} \\
& + \beta \int_{\mathbb{R}^N} \frac{p+1}{2} (U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}} (V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}} \psi_n \frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l} \\
& + \beta \int_{\mathbb{R}^N} \frac{p+1}{2} (U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}} (V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}} \varphi_n \frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l} \\
= & \int_{\mathbb{R}^N} \left((\lambda - P(x)) \frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l} \varphi_n + (\lambda - Q(x)) \frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l} \psi_n \right) \\
= & O\left(\left(\int_{\mathbb{R}^N} (\lambda - P(x))^2 \left(\frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l} \right)^2 \right)^{\frac{1}{2}} \right) \|(\varphi_n, \psi_n)\|_\epsilon \\
& + O\left(\left(\int_{\mathbb{R}^N} (\lambda - Q(x))^2 \left(\frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l} \right)^2 \right)^{\frac{1}{2}} \right) \|(\varphi_n, \psi_n)\|_\epsilon \\
= & \varepsilon^{\frac{N}{2}} O\left(\left(\int_{\mathbb{R}^N} (\lambda - P(\epsilon x + x_{\epsilon_n,m}))^2 \left(\frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}(\varepsilon x + x_{\epsilon_n,m})}{\partial x_l} \right)^2 \right)^{\frac{1}{2}} \right) \\
& \|(\varphi_n, \psi_n)\|_\epsilon
\end{aligned}
$$

$$+\varepsilon^{\frac{N}{2}} O\left(\left(\int_{\mathbb{R}^N} (\lambda - Q(\epsilon x + x_{\epsilon_n,m}))^2 \left(\frac{\partial V_{\epsilon_n, x_{\epsilon_n,m}}(\epsilon x + x_{\epsilon_n,m})}{\partial x_l}\right)^2\right)^{\frac{1}{2}}\right)$$
$$\|(\varphi_n, \psi_n)\|_\epsilon$$
$$= \epsilon_n^N O(|P(x_{\epsilon_n,m}) - \lambda| + |Q(x_{\epsilon_n,m}) - \lambda| + \epsilon_n) = o(\epsilon_n^{N-1}). \tag{2.29}$$

There is a constant $\sigma > 0$ such that

$$\left(\sum_{j=1}^k U_{\epsilon_n, x_{\epsilon_n,j}}\right)^{p-1} - \left(U_{\epsilon_n, x_{\epsilon_n,m}}\right)^{p-1} = O\left(\sum_{j\neq m}^k U^\sigma_{\epsilon_n, x_{\epsilon_n,j}}\right), \tag{2.30}$$

$$\left(\sum_{j=1}^k V_{\epsilon_n, x_{\epsilon_n,j}}\right)^{p-1} - \left(V_{\epsilon_n, x_{\epsilon_n,m}}\right)^{p-1}$$
$$= \tau_0^{p-1}\left(\left(\sum_{j=1}^k U_{\epsilon_n, x_{\epsilon_n,j}}\right)^{p-1} - \left(U_{\epsilon_n, x_{\epsilon_n,m}}\right)^{p-1}\right) = O\left(\sum_{j\neq m}^k U^\sigma_{\epsilon_n, x_{\epsilon_n,j}}\right), \tag{2.31}$$

$$\left(\sum_{j=1}^k U_{\epsilon_n, x_{\epsilon_n,j}}\right)^{\frac{p-3}{2}} \left(\sum_{j=1}^k V_{\epsilon_n, x_{\epsilon_n,j}}\right)^{\frac{p+1}{2}} - \left(U_{\epsilon_n, x_{\epsilon_n,j}}\right)^{\frac{p-3}{2}} \left(V_{\epsilon_n, x_{\epsilon_n,j}}\right)^{\frac{p+1}{2}}$$
$$= \tau_0^{\frac{p+1}{2}}\left(\left(\sum_{j=1}^k U_{\epsilon_n, x_{\epsilon_n,j}}\right)^{p-1} - \left(U_{\epsilon_n, x_{\epsilon_n,j}}\right)^{p-1}\right) = O\left(\sum_{j\neq m}^k U^\sigma_{\epsilon_n, x_{\epsilon_n,j}}\right), \tag{2.32}$$

$$\left(\sum_{j=1}^k U_{\epsilon_n, x_{\epsilon_n,j}}\right)^{\frac{p-1}{2}} \left(\sum_{j=1}^k V_{\epsilon_n, x_{\epsilon_n,j}}\right)^{\frac{p-1}{2}} - \left(U_{\epsilon_n, x_{\epsilon_n,j}}\right)^{\frac{p-1}{2}} \left(V_{\epsilon_n, x_{\epsilon_n,j}}\right)^{\frac{p-1}{2}}$$
$$= \tau_0^{\frac{p-1}{2}}\left(\left(\sum_{j=1}^k U_{\epsilon_n, x_{\epsilon_n,j}}\right)^{p-1} - \left(U_{\epsilon_n, x_{\epsilon_n,j}}\right)^{p-1}\right) = O\left(\sum_{j\neq m}^k U^\sigma_{\epsilon_n, x_{\epsilon_n,j}}\right). \tag{2.33}$$

By (2.30)–(2.33) and (2.29), we obtain

$$A_1 = -\int_{\mathbb{R}^N} p\mu\left(\left(\sum_{j=1}^k U_{\epsilon_n, x_{\epsilon_n,j}}\right)^{p-1} - \left(U_{\epsilon_n, x_{\epsilon_n,m}}\right)^{p-1}\right) \varphi_n \frac{\partial U_{\epsilon_n, x_{\epsilon_n,m}}}{\partial x_l})dx$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^N} p\mu(U_{\epsilon_n,x_{\epsilon_n,m}})^{p-1}\varphi_n \frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}dx \\
&= O(e^{-\frac{\tau}{\epsilon_n}})\|(\varphi_n,0)\|_{\epsilon_n} - \int_{\mathbb{R}^N} p\mu(U_{\epsilon_n,x_{\epsilon_n,m}})^{p-1}\varphi_n \frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}dx,
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= -\int_{\mathbb{R}^N} p\nu \left(\left(\sum_{j=1}^{k} V_{\epsilon_n,x_{\epsilon_n,j}} \right)^{p-1} - \left(V_{\epsilon_n,x_{\epsilon_n,m}}\right)^{p-1} \right) \psi_n \frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}dx \\
&\quad -\int_{\mathbb{R}^N} p\nu \left(V_{\epsilon_n,x_{\epsilon_n,m}}\right)^{p-1} \psi_n \frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}dx \\
&= O\left(e^{-\frac{\tau}{\epsilon_n}}\right)\|(0,\psi_n)\|_{\epsilon_n} - \int_{\mathbb{R}^N} p\nu(V_{\epsilon_n,x_{\epsilon_n,m}})^{p-1}\psi_n \frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}dx,
\end{aligned}
$$

$$
\begin{aligned}
A_3 &= -\frac{p-1}{2}\beta \int_{\mathbb{R}^N} \Big((\sum_{j=1}^{k} U_{\epsilon_n,x_{\epsilon_n,j}})^{\frac{p-3}{2}} (\sum_{j=1}^{k} V_{\epsilon_n,x_{\epsilon_n,j}})^{\frac{p+1}{2}} \\
&\quad - (U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-3}{2}} (V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p+1}{2}} \Big) \varphi_n \frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}dx \\
&\quad - \frac{p-1}{2}\beta \int_{\mathbb{R}^N} (U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-3}{2}} (V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p+1}{2}} \varphi_n \frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}dx \\
&= O(e^{-\frac{\tau}{\epsilon_n}})\|(\varphi_n,0)\|_{\epsilon_n} - \frac{p-1}{2}\beta \\
&\quad \times \int_{\mathbb{R}^N} (U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-3}{2}} (V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p+1}{2}} \varphi_n \frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}dx,
\end{aligned}
$$

$$
\begin{aligned}
A_4 &= -\frac{p-1}{2}\beta \int_{\mathbb{R}^N} \Big((\sum_{j=1}^{k} U_{\epsilon_n,x_{\epsilon_n,j}})^{\frac{p+1}{2}} (\sum_{j=1}^{k} V_{\epsilon_n,x_{\epsilon_n,j}})^{\frac{p-3}{2}} \\
&\quad - (U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p+1}{2}} (V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-3}{2}} \Big) \psi_n \frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}dx \\
&\quad - \frac{p-1}{2}\beta \int_{\mathbb{R}^N} (U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p+1}{2}} (V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-3}{2}} \psi_n \frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}dx \\
&= O(e^{-\frac{\tau}{\epsilon_n}})\|(0,\psi_n)\|_{\epsilon_n} - \frac{p-1}{2}\beta \\
&\quad \times \int_{\mathbb{R}^N} (U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p+1}{2}} (V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-3}{2}} \psi_n \frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}dx,
\end{aligned}
$$

$$
\begin{aligned}
A_5 &= -\frac{p+1}{2}\beta \int_{\mathbb{R}^N} \Big((\sum_{j=1}^{k} U_{\epsilon_n,x_{\epsilon_n,j}})^{\frac{p-1}{2}} (\sum_{j=1}^{k} V_{\epsilon_n,x_{\epsilon_n,j}})^{\frac{p-1}{2}} \\
&\quad - (U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}} (V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}} \Big) \psi_n \frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}dx
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{p+1}{2}\beta \int_{\mathbb{R}^N} (U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}} (V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}} \psi_n \frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l} dx \\
= & \, O(e^{-\frac{\tau}{\epsilon_n}}) \|(0, \psi_n)\|_{\epsilon_n} - \frac{p+1}{2}\beta \\
& \int_{\mathbb{R}^N} (U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}} (V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}} \psi_n \frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l} dx,
\end{aligned}
$$

$$
\begin{aligned}
A_6 = & -\frac{p+1}{2}\beta \int_{\mathbb{R}^N} \Bigg((\sum_{j=1}^{k} U_{\epsilon_n,x_{\epsilon_n,j}})^{\frac{p-1}{2}} (\sum_{j=1}^{k} V_{\epsilon_n,x_{\epsilon_n,j}})^{\frac{p-1}{2}} \\
& - (U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}} (V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}} \Bigg) \varphi_n \frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l} dx \\
& - \frac{p+1}{2}\beta \int_{\mathbb{R}^N} (U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}} (V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}} \varphi_n \frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l} dx \\
= & \, O(e^{-\frac{\tau}{\epsilon_n}}) \|(\varphi_n, 0)\|_{\epsilon_n} - \frac{p+1}{2}\beta \\
& \int_{\mathbb{R}^N} (U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}} (V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}} \varphi_n \frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l} dx.
\end{aligned}
$$

Combining (2.28), (2.29) with above A_1 to A_6, we have

$$
\sum_{j=1}^{k} \sum_{i=1}^{N} \gamma_{n,i,j} \left\langle \left(\frac{\partial U_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i}, \frac{\partial V_{\epsilon_n,x_{\epsilon_n,j}}}{\partial x_i} \right), \left(\frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}, \frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l} \right) \right\rangle_{\epsilon_n} = o\left(\epsilon_n^{N-1}\right). \tag{2.34}
$$

So, by (2.9) and (2.10), we have

$$
\gamma_{n,i,j} = o(\epsilon_n). \tag{2.35}
$$

Thus, (2.27) becomes

$$
\begin{aligned}
& \int_{\mathbb{R}^N} \left(\epsilon_n^2 \nabla \varphi_n \nabla g + P(x) \varphi_n g - p\mu \left(\sum_{j=1}^{k} U_{\epsilon_n,x_{\epsilon_n,j}} \right)^{p-1} \varphi_n g \right) dx \\
& + \int_{\mathbb{R}^N} \left(\epsilon_n^2 \nabla \psi_n \nabla h + Q(x) \psi_n h - p\nu \left(\sum_{j=1}^{k} V_{\epsilon_n,x_{\epsilon_n,j}} \right)^{p-1} \psi_n h \right) dx \\
& - \beta \int_{\mathbb{R}^N} \frac{p-1}{2} \left(\sum_{j=1}^{k} U_{\epsilon_n,x_{\epsilon_n,j}} \right)^{\frac{p-3}{2}} \left(\sum_{j=1}^{k} V_{\epsilon_n,x_{\epsilon_n,j}} \right)^{\frac{p+1}{2}} \varphi_n g dx
\end{aligned}
$$

$$
\begin{aligned}
&-\beta\int_{\mathbb{R}^N}\frac{p-1}{2}\left(\sum_{j=1}^{k}U_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p+1}{2}}\left(\sum_{j=1}^{k}V_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p-3}{2}}\psi_n h dx\\
&-\beta\int_{\mathbb{R}^N}\frac{p+1}{2}\left(\sum_{j=1}^{k}U_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p-1}{2}}\left(\sum_{j=1}^{k}V_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p-1}{2}}\psi_n g dx\\
&-\beta\int_{\mathbb{R}^N}\frac{p+1}{2}\left(\sum_{j=1}^{k}U_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p-1}{2}}\left(\sum_{j=1}^{k}V_{\epsilon_n,x_{\epsilon_n,j}}\right)^{\frac{p-1}{2}}\varphi_n h dx\\
&=\|(\varphi_n,\psi_n)\|_{\epsilon_n}\|(g,h)\|_{\epsilon_n},\ \forall\,(g,h)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N).
\end{aligned}\tag{2.36}
$$

For any $(g,h)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, we let $\widetilde{g}_n(x)=g(\frac{x-x_{\epsilon_n,m}}{\epsilon_n})$. Using (2.36), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\Bigg(\nabla\widetilde{\varphi}_{n,m}\nabla g+P(\epsilon_n x+x_{\epsilon_n,m})\widetilde{\varphi}_{n,m}g\\
&\quad-p\mu\left(\sum_{j=1}^{k}U_{\epsilon_n,x_{\epsilon_n,j}}(\epsilon_n x+x_{\epsilon_n,m})\right)^{p-1}\widetilde{\varphi}_{n,m}g\Bigg)dx\\
&+\int_{\mathbb{R}^N}\Bigg(\nabla\widetilde{\psi}_{n,m}\nabla h+Q(\epsilon_n x+x_{\epsilon_n,m})\widetilde{\psi}_{n,m}h\\
&\quad-p\nu\left(\sum_{j=1}^{k}V_{\epsilon_n,x_{\epsilon_n,j}}(\epsilon_n x+x_{\epsilon_n,m})\right)^{p-1}\widetilde{\psi}_{n,m}h\Bigg)dx\\
&-\beta\int_{\mathbb{R}^N}\frac{p-1}{2}\left((\sum_{j=1}^{k}U_{\epsilon_n,x_{\epsilon_n,j}}(\epsilon_n x+x_{\epsilon_n,m})\right)^{\frac{p-3}{2}}\\
&\quad\left(\sum_{j=1}^{k}V_{\epsilon_n,x_{\epsilon_n,j}}(\epsilon_n x+x_{\epsilon_n,m})\right)^{\frac{p+1}{2}}\widetilde{\varphi}_{n,m}g dx\beta\int_{\mathbb{R}^N}\frac{p-1}{2}\\
&\quad\left((\sum_{j=1}^{k}U_{\epsilon_n,x_{\epsilon_n,j}}(\epsilon_n x+x_{\epsilon_n,m})\right)^{\frac{p+1}{2}}\\
&\quad\left(\sum_{j=1}^{k}V_{\epsilon_n,x_{\epsilon_n,j}}(\epsilon_n x+x_{\epsilon_n,m})\right)^{\frac{p-3}{2}}\widetilde{\psi}_{n,m}h dx
\end{aligned}
$$

$$
\begin{aligned}
&-\beta\int_{\mathbb{R}^N}\frac{p+1}{2}\left((\sum_{j=1}^{k}U_{\epsilon_n,x_{\epsilon_n,j}}(\epsilon_n x+x_{\epsilon_n,m})\right)^{\frac{p-1}{2}}\\
&\left(\sum_{j=1}^{k}V_{\epsilon_n,x_{\epsilon_n,j}}(\epsilon_n x+x_{\epsilon_n,m})\right)^{\frac{p-1}{2}}\widetilde{\psi}_{n,m}g dx\\
&-\beta\int_{\mathbb{R}^N}\frac{p+1}{2}\\
&\left((\sum_{j=1}^{k}U_{\epsilon_n,x_{\epsilon_n,j}}(\epsilon_n x+x_{\epsilon_n,m})\right)^{\frac{p-1}{2}}\\
&\left(\sum_{j=1}^{k}V_{\epsilon_n,x_{\epsilon_n,j}}(\epsilon_n x+x_{\epsilon_n,m})\right)^{\frac{p-1}{2}}\widetilde{\varphi}_{n,m}h dx\\
&=\epsilon_n^{-N}\|(\varphi_n,\psi_n)\|_{\epsilon_n}\|(\widetilde{g}_n,\widetilde{h}_n)\|_{\epsilon_n}=o(1),\ \forall\,(g,h)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N).
\end{aligned}
\tag{2.37}
$$

Therefore, (φ,ψ) satisfies

$$
\begin{cases}
-\Delta\varphi+\lambda\varphi-p\mu U_\lambda(x)^{p-1}\varphi-\beta\frac{p-1}{2}U_\lambda(x)^{\frac{p-3}{2}}V_\lambda(x)^{\frac{p+1}{2}}\varphi\\
-\beta\frac{p+1}{2}U_\lambda(x)^{\frac{p-1}{2}}V_\lambda(x)^{\frac{p-1}{2}}\psi=0 \quad \text{in } \mathbb{R}^N,\\
-\Delta\psi+\lambda\psi-p\nu V_\lambda(x)^{p-1}\psi-\beta\frac{p-1}{2}U_\lambda(x)^{\frac{p+1}{2}}V_\lambda(x)^{\frac{p-3}{2}}\psi\\
-\beta\frac{p+1}{2}U_\lambda(x)^{\frac{p-1}{2}}V_\lambda(x)^{\frac{p-1}{2}}\varphi=0 \quad \text{in } \mathbb{R}^N.
\end{cases}
\tag{2.38}
$$

From proposition 2.1, the solution of (U_λ, V_λ) gives

$$
\varphi=\sum_{l=1}^{N}c_l\frac{\partial U_\lambda}{\partial x_l},\ \ \psi=\sum_{l=1}^{N}d_l\frac{\partial V_\lambda}{\partial x_l}.
\tag{2.39}
$$

On the other hand, from $(\varphi_n,\psi_n)\in\mathbf{E}_{\epsilon_n}$ and (2.29), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N}p\mu(U_{\epsilon_n,x_{\epsilon_n,m}})^{p-1}\varphi_n\frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}dx+\int_{\mathbb{R}^N}p\nu(V_{\epsilon_n,x_{\epsilon_n,m}})^{p-1}\psi_n\frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}dx\\
&\quad+\beta\int_{\mathbb{R}^N}\frac{p-1}{2}(U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-3}{2}}(V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p+1}{2}}\varphi_n\frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}dx\\
&\quad+\beta\int_{\mathbb{R}^N}\frac{p-1}{2}(U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p+1}{2}}(V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-3}{2}}\psi_n\frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}dx\\
&\quad+\beta\int_{\mathbb{R}^N}\frac{p+1}{2}(U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}}(V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}}\psi_n\frac{\partial U_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}dx\\
&\quad+\beta\int_{\mathbb{R}^N}\frac{p+1}{2}(U_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}}(V_{\epsilon_n,x_{\epsilon_n,m}})^{\frac{p-1}{2}}\varphi_n\frac{\partial V_{\epsilon_n,x_{\epsilon_n,m}}}{\partial x_l}dx
\end{aligned}
$$

$$= o(\epsilon_n^{N-1}). \tag{2.40}$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^N} p\mu U_\lambda^{p-1}\varphi\frac{\partial U_\lambda}{\partial x_l}dx \\
&+\int_{\mathbb{R}^N} p\nu V_\lambda^{p-1}\psi_n\frac{\partial V_\lambda}{\partial x_l}dx+\beta\int_{\mathbb{R}^N}\frac{p-1}{2}U_\lambda^{\frac{p-3}{2}}V_\lambda^{\frac{p+1}{2}}\varphi\frac{\partial U_\lambda}{\partial x_l}dx \\
&+\beta\int_{\mathbb{R}^N}\frac{p-1}{2}U_\lambda^{\frac{p+1}{2}}V_\lambda^{\frac{p-3}{2}}\psi\frac{\partial V_\lambda}{\partial x_l}dx+\beta\int_{\mathbb{R}^N}\frac{p+1}{2}U_\lambda^{\frac{p-1}{2}}V_\lambda^{\frac{p-1}{2}}\psi\frac{\partial U_\lambda}{\partial x_l}dx \\
&+\beta\int_{\mathbb{R}^N}\frac{p+1}{2}U_\lambda^{\frac{p-1}{2}}V_\lambda^{\frac{p-1}{2}}\varphi\frac{\partial V_\lambda}{\partial x_l}dx=0.
\end{aligned} \tag{2.41}$$

By (2.39), (2.41) and $U_\lambda = \frac{1}{\tau_0}V_\lambda$, we have

$$c_l = d_l = 0, \quad l = 1, 2, \ldots N.$$

Thus,

$$(\varphi, \psi) = (0, 0).$$

So, we have prove there exist $\epsilon_0, \theta_0 > 0$, $\rho > 0$, independent of x_j, $j = 1, 2, \cdots k$ such that for any $\epsilon \in (0, \epsilon_0]$ and $x_j \in B_{\theta_0}(y_0)$, $Q_\epsilon B_\epsilon(\varphi_\epsilon, \psi_\epsilon)$ is bijective in $\mathbf{E}_\epsilon$. Moreover, it holds

$$\|Q_\epsilon B_\epsilon(\varphi, \psi)\|_\epsilon \geq \rho\|(\varphi, \psi)\|_\epsilon, \qquad \forall\, (\varphi, \psi) \in \mathbf{E}_\epsilon.$$

Thus the proof is complete. □

Next, we give the error estimate for $\|l_\epsilon\|_\epsilon$ and $\|R_\epsilon(\varphi_\epsilon, \psi_\epsilon)\|_\epsilon$.

Lemma 2.3 *There is a constant $C > 0$ independent of ϵ, such that*

$$\begin{aligned}
\|l_\epsilon\|_\epsilon \leq C&\left(\epsilon^{\frac{N+2}{2}}+\epsilon^{\frac{N}{2}}\sum_{j=1}^k(|P(x_{j,\epsilon})-\lambda|+|Q(x_{j,\epsilon})-\lambda|)\right) \\
&+\epsilon^{\frac{N}{2}}\sum_{i\neq j}e^{-\frac{\sqrt{\alpha}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}.
\end{aligned}$$

Proof Observe that

$$\begin{aligned}
&\langle l_\epsilon, (g, h)\rangle \\
&=\sum_{j=1}^k\int_{\mathbb{R}^N}(P(x)-\lambda)U_{\epsilon,x_{j,\epsilon}}g\,dx+\mu\int_{\mathbb{R}^N}(\sum_{j=1}^k U^p_{\epsilon,x_{j,\epsilon}}-(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}})^p g\,dx
\end{aligned}$$

$$+\sum_{j=1}^{k}\int_{\mathbb{R}^N}(Q(x)-\lambda)V_{\epsilon,x_{j,\epsilon}}hdx+\nu\int_{\mathbb{R}^N}(\sum_{j=1}^{k}V^p_{\epsilon,x_{j,\epsilon}}-(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}})^p hdx$$
$$+\beta\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k}U^{\frac{p-1}{2}}_{\epsilon,x_{j,\epsilon}}V^{\frac{p+1}{2}}_{\epsilon,x_{j,\epsilon}}-(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}})^{\frac{p-1}{2}}(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}})^{\frac{p+1}{2}}\right)gdx$$
$$+\beta\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k}U^{\frac{p+1}{2}}_{\epsilon,x_{j,\epsilon}}V^{\frac{p-1}{2}}_{\epsilon,x_{j,\epsilon}}-(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}})^{\frac{p+1}{2}}(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}})^{\frac{p-1}{2}}\right)hdx.$$

Firstly, by Hölder's inequality, we have

$$\begin{aligned}
&\sum_{j=1}^{k}\int_{\mathbb{R}^N}(P(x)-\lambda)U_{\epsilon,x_{j,\epsilon}}gdx+\sum_{j=1}^{k}\int_{\mathbb{R}^N}(Q(x)-\lambda)V_{\epsilon,x_{j,\epsilon}}hdx\\
&\le C\sum_{j=1}^{k}\left(\int_{\mathbb{R}^N}(P(x)-\lambda)^2U^2_{\epsilon,x_{j,\epsilon}}dx\right)^{\frac12}\|g\|_\epsilon\\
&\quad+C\sum_{j=1}^{k}\left(\int_{\mathbb{R}^N}(Q(x)-\lambda)^2V^2_{\epsilon,x_{j,\epsilon}}dx\right)^{\frac12}\|h\|_\epsilon\\
&\le C\sum_{j=1}^{k}\left(\epsilon^N\int_{\mathbb{R}^N}(P(\epsilon x+x_{j,\epsilon})-\lambda)^2U^2_{\epsilon,x_{j,\epsilon}}(\epsilon x+x_{j,\epsilon})dx\right)^{\frac12}\|g\|_\epsilon\\
&\quad+C\sum_{j=1}^{k}\left(\epsilon^N\int_{\mathbb{R}^N}(Q(\epsilon x+x_{j,\epsilon})-\lambda)^2V^2_{\epsilon,x_{j,\epsilon}}(\epsilon x+x_{j,\epsilon})dx\right)^{\frac12}\|h\|_\epsilon\\
&\le C\sum_{j=1}^{k}\epsilon^{\frac N2}\left(\int_{\mathbb{R}^N}\left(\left(P(x_{j,\epsilon})-\lambda\right)+\epsilon^2|x|^2)^2U^2_{\epsilon,x_{j,\epsilon}}(\epsilon x+x_{j,\epsilon}\right)dx\right)^{\frac12}\|g\|_\epsilon\\
&\quad+C\sum_{j=1}^{k}\epsilon^{\frac N2}\left(\int_{\mathbb{R}^N}\left(\left(Q(x_{j,\epsilon})-\lambda\right)+\epsilon^2|x|^2)^2V^2_{\epsilon,x_{j,\epsilon}}(\epsilon x+x_{j,\epsilon}\right)dx\right)^{\frac12}\|h\|_\epsilon\\
&\le C\left(\epsilon^{\frac{N+2}{2}}+\sum_{j=1}^{k}\epsilon^{\frac N2}\left(|P(x_{j,\epsilon})-\lambda|+|Q(x_{j,\epsilon})-\lambda|\right)\right)\|(g,h)\|_\epsilon.
\end{aligned}\tag{2.42}$$

Since $U_{\epsilon,x_{j,\epsilon}}(x)\le Ce^{-\frac{\sqrt{\alpha}|x-x_{j,\epsilon}|}{\epsilon}}$, $V_{\epsilon,x_{j,\epsilon}}(x)\le Ce^{-\frac{\sqrt{\alpha}|x-x_{j,\epsilon}|}{\epsilon}}$, we have if $p\ge 3$, then

$$\left(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}}\right)^p-\sum_{j=1}^{k}U^p_{\epsilon,x_{j,\epsilon}}$$

$$
\begin{aligned}
&= p\sum_{i\neq j}^{k} U_{\epsilon,x_i}^{p-1} U_{\epsilon,x_{j,\epsilon}} + O\left(\sum_{i\neq j}^{k} U_{\epsilon,x_i}^{p-2} U_{\epsilon,x_j}^{2} + \sum_{i\neq j}^{k} U_{\epsilon,x_j}^{p-2}\right)\\
&= O\left(\sum_{i\neq j}^{k} e^{-\frac{\sqrt{\alpha}|x-x_{j,\epsilon}|}{\epsilon}}\right), \ \forall\, x\in B_\theta(x_{j,\epsilon}), \qquad (2.43)
\end{aligned}
$$

$$
\begin{aligned}
&\left(\sum_{j=1}^{k} V_{\epsilon,x_{j,\epsilon}}\right)^{p} - \sum_{j=1}^{k} V_{\epsilon,x_{j,\epsilon}}^{p} = p\sum_{i\neq j}^{k} V_{\epsilon,x_i}^{p-1} V_{\epsilon,x_{j,\epsilon}}\\
&\quad + O\left(\sum_{i\neq j}^{k} V_{\epsilon,x_i}^{p-2} V_{\epsilon,x_j}^{2} + \sum_{i\neq j}^{k} V_{\epsilon,x_j}^{p-2}\right)\\
&\quad = O\left(\sum_{i\neq j}^{k} e^{-\frac{\sqrt{\alpha}|x-x_{j,\epsilon}|}{\epsilon}}\right), \ \forall\, x\in B_\theta(x_{j,\epsilon}), \qquad (2.44)
\end{aligned}
$$

$$
\begin{aligned}
&\left(\sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-1}{2}} \left(\sum_{j=1}^{k} V_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p+1}{2}} - \sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon}}^{\frac{p-1}{2}} V_{\epsilon,x_{j,\epsilon}}^{\frac{p+1}{2}}\\
&= \left(\frac{b_\lambda}{a_\lambda}\right)^{\frac{p+1}{2}} \left((\sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon}})^{p} - \sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon}}^{p}\right)\\
&= O\left(\sum_{i\neq j}^{k} e^{-\frac{\sqrt{\alpha}|x-x_{j,\epsilon}|}{\epsilon}}\right), \ \forall\, x\in B_\theta(x_{j,\epsilon}), \qquad (2.45)
\end{aligned}
$$

$$
\begin{aligned}
&\left(\sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p+1}{2}} \left(\sum_{j=1}^{k} V_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-1}{2}} - \sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon}}^{\frac{p+1}{2}} V_{\epsilon,x_{j,\epsilon}}^{\frac{p-1}{2}}\\
&= \left(\frac{b_\lambda}{a_\lambda}\right)^{\frac{p-1}{2}} \left((\sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon}})^{p} - \sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon}}^{p}\right)\\
&= O\left(\sum_{i\neq j}^{k} e^{-\frac{\sqrt{\alpha}|x-x_{j,\epsilon}|}{\epsilon}}\right), \quad \forall\, x\in B_\theta(x_{j,\epsilon}). \qquad (2.46)
\end{aligned}
$$

From (2.43) to (2.46), we have

$$\begin{aligned}
&\mu\int_{\mathbb{R}^N}\left(\sum_{j=1}^k U^p_{\epsilon,x_{j,\epsilon}}-(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}})^p\right)gdx+\nu\int_{\mathbb{R}^N}\left(\sum_{j=1}^k V^p_{\epsilon,x_{j,\epsilon}}-(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}})^p\right)hdx\\
&\quad+\beta\int_{\mathbb{R}^N}\left(\sum_{j=1}^k U^{\frac{p-1}{2}}_{\epsilon,x_{j,\epsilon}}V^{\frac{p+1}{2}}_{\epsilon,x_{j,\epsilon}}-(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}})^{\frac{p-1}{2}}(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}})^{\frac{p+1}{2}}\right)gdx\\
&\quad+\beta\int_{\mathbb{R}^N}\left(\sum_{j=1}^k U^{\frac{p+1}{2}}_{\epsilon,x_{j,\epsilon}}V^{\frac{p-1}{2}}_{\epsilon,x_{j,\epsilon}}-(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}})^{\frac{p+1}{2}}(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}})^{\frac{p-1}{2}}\right)hdx\\
&\quad\le C\epsilon^{\frac{N}{2}}\sum_{i\neq j}e^{-\frac{\sqrt{\alpha}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}\|(g,h)\|_\epsilon.
\end{aligned}\tag{2.47}$$

So, by (2.42) and (2.47), we obtain

$$\begin{aligned}
\|l_\epsilon\|_\epsilon\le C\Bigg(&\epsilon^{\frac{N+2}{2}}+\epsilon^{\frac{N}{2}}\sum_{j=1}^k(|P(x_{j,\epsilon})-\lambda|+|Q(x_{j,\epsilon})-\lambda|)\Bigg)\\
&+\epsilon^{\frac{N}{2}}\sum_{i\neq j}e^{-\frac{\sqrt{\alpha}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}.
\end{aligned}$$

This completes the proof. □

Lemma 2.4 *There is a constant $C>0$ independent of ϵ, such that*

$$\|R_\epsilon(\varphi,\psi)\|_\epsilon\le C\left(\epsilon^{-\frac{N}{2}}\|(\varphi,\psi)\|^2_\epsilon+\epsilon^{-N}\|(\varphi,\psi)\|^3_\epsilon+\epsilon^{-\frac{3N}{2}}\|(\varphi,\psi)\|^4_\epsilon\right).$$

Proof Since

$$\langle R_\epsilon(\varphi,\psi),(g,h)\rangle=B_1+B_2+B_3-B_4-B_5,\tag{2.48}$$

where

$$B_1=\int_{\mathbb{R}^N}\left(\mu(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}}+\varphi)^p-\mu(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}})^p\right)gdx,$$

$$B_2=\int_{\mathbb{R}^N}\left(\nu(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}}+\psi)^p-\nu(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}})^p\right)hdx,$$

$$
\begin{aligned}
B_3 = &\int_{\mathbb{R}^N} \beta(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}} + \varphi)^{\frac{p-1}{2}} (\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}} + \psi)^{\frac{p+1}{2}} g dx \\
&+ \int_{\mathbb{R}^N} \beta(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}} + \varphi)^{\frac{p+1}{2}} (\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}} + \psi)^{\frac{p-1}{2}} h dx,
\end{aligned}
$$

$$
\begin{aligned}
B_4 = &\beta \int_{\mathbb{R}^N} \left((\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}})^{\frac{p-1}{2}} (\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}})^{\frac{p+1}{2}} \right) g dx \\
&+ \beta \int_{\mathbb{R}^N} \left((\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}})^{\frac{p+1}{2}} (\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}})^{\frac{p-1}{2}} \right) h dx,
\end{aligned}
$$

$$
\begin{aligned}
B_5 = \beta &\int_{\mathbb{R}^N} \frac{p-1}{2} (\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}})^{\frac{p-3}{2}} (\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}})^{\frac{p+1}{2}} \varphi g dx \\
&+ \beta \int_{\mathbb{R}^N} \frac{p-1}{2} (\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}})^{\frac{p+1}{2}} (\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}})^{\frac{p-3}{2}} \psi h dx \\
&+ \beta \int_{\mathbb{R}^N} \frac{p+1}{2} (\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}})^{\frac{p-1}{2}} (\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}})^{\frac{p-1}{2}} \psi g dx \\
&+ \beta \int_{\mathbb{R}^N} \frac{p+1}{2} (\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}})^{\frac{p-1}{2}} (\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}})^{\frac{p-1}{2}} \varphi h dx.
\end{aligned}
$$

For any ξ, we let $\widetilde{\xi}(y) = \xi(\epsilon y)$, then

$$
\begin{aligned}
\int_{\mathbb{R}^N} |\xi|^{p+1} dx = \epsilon^N \int_{\mathbb{R}^N} |\widetilde{\xi}|^{p+1} dx &\le C\epsilon^N \left(\int_{\mathbb{R}^N} (|\nabla \widetilde{\xi}|^2 + |\widetilde{\xi}|^2) dx \right)^{\frac{p+1}{2}} \\
&\le C\epsilon^{N(1-\frac{p+1}{2})} \left(\int_{\mathbb{R}^N} (\epsilon^2 |\nabla \xi|^2 + |\xi|^2) dx \right)^{\frac{p+1}{2}} \\
&\le C\epsilon^{N(1-\frac{p+1}{2})} \|\xi\|_\epsilon^{p+1}. \qquad (2.49)
\end{aligned}
$$

When $p \ge 3$, by (2.49) and the Hölder inequality, we have

$$
B_1 \le \mu \int_{\mathbb{R}^N} \left(p(p-1)(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}})^{p-2} \varphi^2 + o(|\varphi|^{p-1}) \right) g dx
$$

$$\leq C\left(\int_{\mathbb{R}^N}(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}})^{p+1}dx\right)^{\frac{p-2}{p+1}}\left(\int_{\mathbb{R}^N}|\varphi|^{p+1}dx\right)^{\frac{2}{p+1}}\left(\int_{\mathbb{R}^N}|g|^{p+1}dx\right)^{\frac{1}{p+1}}$$
$$\leq C\epsilon^{N\times\frac{p-2}{p+1}}\epsilon^{N(1-\frac{p+1}{2})\frac{3}{p+1}}\|\varphi\|_\epsilon^2\|g\|_\epsilon \leq C\epsilon^{-\frac{N}{2}}\|\varphi\|_\epsilon^2\|g\|_\epsilon. \tag{2.50}$$

Similarly, we have

$$B_2=\int_{\mathbb{R}^N}\left(\nu(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}}+\varphi)^p-\nu(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}})^p\right)hdx\leq C\epsilon^{-\frac{N}{2}}\|\psi\|_\epsilon^2\|h\|_\epsilon. \tag{2.51}$$

Thus

$$B_1+B_2\leq C\epsilon^{-\frac{N}{2}}\|(\varphi,\psi)\|_\epsilon^2\|(g,h)\|_\epsilon. \tag{2.52}$$

We expand B_3 as following

$$\begin{aligned}B_3&=\int_{\mathbb{R}^N}\beta(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}}+\varphi)^{\frac{p-1}{2}}(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}}+\psi)^{\frac{p+1}{2}}gdx\\&\quad+\int_{\mathbb{R}^N}\beta(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}}+\varphi)^{\frac{p+1}{2}}(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}}+\psi)^{\frac{p-1}{2}}hdx\\&=B_4+B_5+C_1+C_2+C_3+C_4+C_5+C_6+o\Big(\int_{\mathbb{R}^N}(\varphi^2+\psi^2)gdx\Big)\\&\quad+o\Big(\int_{\mathbb{R}^N}(\varphi^2+\psi^2)hdx\Big),\end{aligned} \tag{2.53}$$

where

$$\begin{aligned}C_1&=\beta\frac{p+1}{2}\frac{p-1}{2}\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-1}{2}}\left(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-3}{2}}\psi^2gdx\\&\quad+\beta\frac{p+1}{2}\frac{p-1}{2}\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-3}{2}}\left(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-1}{2}}\varphi^2hdx,\end{aligned}$$

$$C_2=\beta\frac{p+1}{2}\frac{p-1}{2}\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-3}{2}}(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}})^{\frac{p-1}{2}}\varphi\psi gdx$$

$$+\beta\frac{p+1}{2}\frac{p-1}{2}\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-1}{2}}(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}})^{\frac{p-3}{2}}\varphi\psi h dx,$$

$$C_3=\beta\frac{p+1}{2}(\frac{p-1}{2})^2\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-3}{2}}(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}})^{\frac{p-3}{2}}\varphi\psi^2 g dx$$
$$+\beta\frac{p+1}{2}(\frac{p-1}{2})^2\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-3}{2}}(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}})^{\frac{p-3}{2}}\varphi^2\psi h dx,$$

$$C_4=\beta\frac{p-1}{2}\frac{p-3}{2}\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-5}{2}}(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}})^{\frac{p+1}{2}}\varphi^2 g dx$$
$$+\beta\frac{p-1}{2}\frac{p-3}{2}\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p+1}{2}}(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}})^{\frac{p-5}{2}}\psi^2 h dx,$$

$$C_5=\beta\frac{p+1}{2}\frac{p-1}{2}\frac{p-3}{2}\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-5}{2}}(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}})^{\frac{p-1}{2}}\varphi^2\psi g dx$$
$$+\beta\frac{p+1}{2}\frac{p-1}{2}\frac{p-3}{2}\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-1}{2}}(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}})^{\frac{p-5}{2}}\varphi\psi^2 h dx,$$

$$C_6=\beta\frac{p+1}{2}\frac{p-3}{2}(\frac{p-1}{2})^2\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-5}{2}}(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}})^{\frac{p-3}{2}}\varphi^2\psi^2 g dx$$
$$+\beta\frac{p+1}{2}\frac{p-3}{2}(\frac{p-1}{2})^2\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}}\right)^{\frac{p-3}{2}}(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}})^{\frac{p-5}{2}}\varphi^2\psi^2 h dx.$$

Since $V_{\epsilon,x_{j,\epsilon}}=\tau_0 U_{\epsilon,x_{j,\epsilon}}$, we have

$$C_1\le C\left(\int_{\mathbb{R}^N}(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}})^{p+1}dx\right)^{\frac{p-2}{p+1}}\left(\int_{\mathbb{R}^N}|\psi|^{p+1}dx\right)^{\frac{2}{p+1}}\left(\int_{\mathbb{R}^N}|g|^{p+1}dx\right)^{\frac{1}{p+1}}$$

$$
\begin{aligned}
&+ C\left(\int_{\mathbb{R}^N}(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}})^{p+1}dx\right)^{\frac{p-2}{p+1}}\left(\int_{\mathbb{R}^N}|\varphi|^{p+1}dx\right)^{\frac{2}{p+1}}\left(\int_{\mathbb{R}^N}|h|^{p+1}dx\right)^{\frac{1}{p+1}}\\
&\le C\epsilon^{N\times\frac{p-2}{p+1}}\epsilon^{N(1-\frac{p+1}{2})\frac{3}{p+1}}\|(\varphi,\psi)\|_\epsilon^2\|(g,h)\|_\epsilon \le C\epsilon^{-\frac{N}{2}}\|(\varphi,\psi)\|_\epsilon^2\|(g,h)\|_\epsilon.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
C_2 &\le C\epsilon^{-\frac{N}{2}}\|(\varphi,\psi)\|_\epsilon^2\|(g,h)\|_\epsilon,\\
C_4 &\le C\epsilon^{-\frac{N}{2}}\|(\varphi,\psi)\|_\epsilon^2\|(g,h)\|_\epsilon.
\end{aligned}
$$

$$
\begin{aligned}
C_3 \le\ & C\left(\int_{\mathbb{R}^N}(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}})^{p+1}dx\right)^{\frac{p-3}{p+1}}\\
&\left(\int_{\mathbb{R}^N}|\psi|^{p+1}dx\right)^{\frac{2}{p+1}}\left(\int_{\mathbb{R}^N}|\varphi|^{p+1}dx\right)^{\frac{1}{p+1}}\left(\int_{\mathbb{R}^N}|g|^{p+1}\right)^{\frac{1}{p+1}}\\
&+ C\left(\int_{\mathbb{R}^N}(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}})^{p+1}dx\right)^{\frac{p-3}{p+1}}\\
&\left(\int_{\mathbb{R}^N}|\varphi|^{p+1}dx\right)^{\frac{2}{p+1}}\left(\int_{\mathbb{R}^N}|\psi|^{p+1}dx\right)^{\frac{1}{p+1}}\left(\int_{\mathbb{R}^N}|h|^{p+1}dx\right)^{\frac{1}{p+1}}\\
&\le C\epsilon^{-N}\|(\varphi,\psi)\|_\epsilon^3\|(g,h)\|_\epsilon.
\end{aligned}
$$

By the similar argument as C_3, we have

$$
C_5 \le C\epsilon^{-N}\|(\varphi,\psi)\|_\epsilon^3\|(g,h)\|_\epsilon.
$$

We also have

$$
\begin{aligned}
C_6 \le\ & C\left(\int_{\mathbb{R}^N}(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}})^{p+1}dx\right)^{\frac{p-4}{p+1}}\\
&\left(\int_{\mathbb{R}^N}|\psi|^{p+1}dx\right)^{\frac{2}{p+1}}\left(\int_{\mathbb{R}^N}|\varphi|^{p+1}dx\right)^{\frac{2}{p+1}}\left(\int_{\mathbb{R}^N}|g|^{p+1}dx\right)^{\frac{1}{p+1}}\\
&+ C\left(\int_{\mathbb{R}^N}(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}})^{p+1}dx\right)^{\frac{p-4}{p+1}}\\
&\left(\int_{\mathbb{R}^N}|\varphi|^{p+1}dx\right)^{\frac{2}{p+1}}\left(\int_{\mathbb{R}^N}|\psi|^{p+1}dx\right)^{\frac{2}{p+1}}\left(\int_{\mathbb{R}^N}|h|^{p+1}dx\right)^{\frac{1}{p+1}}
\end{aligned}
$$

$$\leq C\epsilon^{-\frac{3N}{2}}\|(\varphi,\psi)\|_\epsilon^4\|(g,h)\|_\epsilon.$$

Combining (2.50)–(2.53), the estimates for C_1–C_6 and (2.48), we obtain

$$\|R_\epsilon(\varphi,\psi)\|_\epsilon \leq C\left(\epsilon^{-\frac{N}{2}}\|(\varphi,\psi)\|_\epsilon^2+\epsilon^{-N}\|(\varphi,\psi)\|_\epsilon^3+\epsilon^{-\frac{3N}{2}}\|(\varphi,\psi)\|_\epsilon^4\right).$$

This finishes the proof. □

Now, we consider the following projection problem

$$Q_\epsilon B_\epsilon(\varphi,\psi)+Q_\epsilon l_\epsilon = Q_\epsilon R_\epsilon(\varphi,\psi), \tag{2.54}$$

by using the contraction mapping theorem, we give the following lemma.

Lemma 2.5 *There exists $\epsilon_0>0$, such that for any $\epsilon\in(0,\epsilon_0]$, $x_j\in B_\theta(x_0)$, then the problem (2.54) has a unique $(\varphi_\epsilon,\psi_\epsilon)\in\mathbf{E}_\epsilon$ and*

$$\begin{aligned}\|(\varphi_\epsilon,\psi_\epsilon)\|_\epsilon &\leq C\|l_\epsilon\|_\epsilon \leq C\left(\epsilon^{\frac{N+2}{2}}+\epsilon^{\frac{N}{2}}\sum_{j=1}^k(|P(x_{j,\epsilon})-\lambda|+|Q(x_{j,\epsilon})-\lambda|)\right)\\ &\quad+\epsilon^{\frac{N}{2}}\sum_{i\neq j}e^{-\frac{\sqrt{\alpha}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}.\end{aligned}$$

Proof From Lemma 2.2, we can rewrite (2.54) as follows:

$$(\varphi,\psi)=\mathbf{B}(\varphi,\psi):=(Q_\epsilon B_\epsilon)^{-1}Q_\epsilon l_\epsilon+(Q_\epsilon B_\epsilon)^{-1}Q_\epsilon R_\epsilon(\varphi,\psi).$$

By Lemmas 2.2 and 2.3, we have

$$\begin{aligned}&\|(Q_\epsilon B_\epsilon)^{-1}Q_\epsilon l_\epsilon\|_\epsilon \leq C\|Q_\epsilon l_\epsilon\|_\epsilon \leq C\|l_\epsilon\|_\epsilon\\ &\quad\leq C\left(\epsilon^{\frac{N+2}{2}}+\epsilon^{\frac{N}{2}}\sum_{j=1}^k(|P(x_{j,\epsilon})-\lambda|+|Q(x_{j,\epsilon})-\lambda|)\right)+\epsilon^{\frac{N}{2}}\sum_{i\neq j}e^{-\frac{\sqrt{\alpha}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}.\end{aligned} \tag{2.55}$$

Next, we will use the contraction mapping theorem in a ball whose radius is slightly bigger than $C\|l_\epsilon\|_\epsilon$. So we take

$$\begin{aligned}S:=\Bigg\{&(\varphi,\psi):(\varphi,\psi)\in\mathbf{E}_\epsilon,\\ &\|(\varphi,\psi)\|_\epsilon \leq C\left(\epsilon^{\frac{N+2}{2}-\tau}+\epsilon^{\frac{N}{2}}\sum_{j=1}^k(|P(x_{j,\epsilon})-\lambda|^{1-\tau}+|Q(x_{j,\epsilon})-\lambda|^{1-\tau})\right)\\ &+\epsilon^{\frac{N}{2}}\sum_{i\neq j}e^{-\frac{\sqrt{\alpha}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}\Bigg\},\end{aligned}$$

where $\tau > 0$ is a fixed small constant.

Step 1 B is a map from S to S. In fact, from Lemmas 2.2, 2.3 and 2.4, we have

$$
\begin{aligned}
&\|\mathbf{B}(\varphi,\psi)\|_\epsilon \le C\|l_\epsilon\|_\epsilon + C\|R_\epsilon(\varphi,\psi)\|_\epsilon\\
&\le C\left(\epsilon^{\frac{N+2}{2}-\tau} + \epsilon^{\frac{N}{2}}\sum_{j=1}^{k}(|P(x_{j,\epsilon})-\lambda|^{1-\tau} + |Q(x_{j,\epsilon})-\lambda|^{1-\tau})\right) + \epsilon^{\frac{N}{2}}\sum_{i\neq j} e^{-\frac{\sqrt{\alpha}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}\\
&\quad + C\left(\epsilon^{-\frac{N}{2}}\|(\varphi,\psi)\|_\epsilon^2 + \epsilon^{-N}\|(\varphi,\psi)\|_\epsilon^3 + \epsilon^{-\frac{3N}{2}}\|(\varphi,\psi)\|_\epsilon^4\right)\\
&\le C\left(\epsilon^{\frac{N+2}{2}-\tau} + \epsilon^{\frac{N}{2}}\sum_{j=1}^{k}(|P(x_{j,\epsilon})-\lambda|^{1-\tau} + |Q(x_{j,\epsilon})-\lambda|^{1-\tau})\right) + \epsilon^{\frac{N}{2}}\sum_{i\neq j} e^{-\frac{\sqrt{\alpha}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}\\
&\quad + C\left(\epsilon^{\frac{N}{2}+2(1-\tau)} + \epsilon^{\frac{N}{2}}\sum_{j=1}^{k}(|P(x_{j,\epsilon})-\lambda|^{2(1-\tau)} + |Q(x_{j,\epsilon})-\lambda|^{2(1-\tau)})\right)\\
&\quad + \epsilon^{\frac{N}{2}}\sum_{i\neq j} e^{-\frac{\sqrt{\alpha}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}\\
&\quad + C\left(\epsilon^{\frac{N}{2}+3(1-\tau)} + \epsilon^{\frac{N}{2}}\sum_{j=1}^{k}(|P(x_{j,\epsilon})-\lambda|^{3(1-\tau)} + |Q(x_{j,\epsilon})-\lambda|^{3(1-\tau)})\right)\\
&\quad + \epsilon^{\frac{N}{2}}\sum_{i\neq j} e^{-\frac{\sqrt{\alpha}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}\\
&\quad + C\left(\epsilon^{\frac{N}{2}+4(1-\tau)} + \epsilon^{\frac{N}{2}}\sum_{j=1}^{k}(|P(x_{j,\epsilon})-\lambda|^{4(1-\tau)} + |Q(x_{j,\epsilon})-\lambda|^{4(1-\tau)})\right)\\
&\quad + \epsilon^{\frac{N}{2}}\sum_{i\neq j} e^{-\frac{\sqrt{\alpha}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}\\
&\le C\left(\epsilon^{\frac{N+2}{2}-\tau} + \epsilon^{\frac{N}{2}}\sum_{j=1}^{k}(|P(x_{j,\epsilon})-\lambda|^{1-\tau} + |Q(x_{j,\epsilon})-\lambda|^{1-\tau})\right)\\
&\quad + \epsilon^{\frac{N}{2}}\sum_{i\neq j} e^{-\frac{\sqrt{\alpha}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}.
\end{aligned}
$$

Thus, **B** is a map from S to S.

Step 2 B is a contraction map. For any $(\varphi_1, \psi_1) \in S$, we have

$$
\begin{aligned}
\|(\varphi_1,\psi_1)\|_\epsilon \ &\le C\left(\epsilon^{\frac{N+2}{2}-\tau} + \epsilon^{\frac{N}{2}}\sum_{j=1}^{k}(|P(x_{j,\epsilon})-\lambda|^{1-\tau} + |Q(x_{j,\epsilon})-\lambda|^{1-\tau})\right)\\
&\quad + \epsilon^{\frac{N}{2}}\sum_{i\neq j} e^{-\frac{\sqrt{\alpha}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}},
\end{aligned}
$$

$$
\begin{aligned}
\|(\varphi_2, \psi_2)\|_\epsilon \ &\leq C \left(\epsilon^{\frac{N+2}{2}-\tau} + \epsilon^{\frac{N}{2}} \sum_{j=1}^{k} (|P(x_{j,\epsilon}) - \lambda|^{1-\tau} + |Q(x_{j,\epsilon}) - \lambda|^{1-\tau}) \right) \\
&\quad + \epsilon^{\frac{N}{2}} \sum_{i \neq j} e^{-\frac{\sqrt{\alpha}|x_{i,\epsilon} - x_{j,\epsilon}|}{\epsilon}}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\int_{\mathbb{R}^N} \langle R_\epsilon(\varphi_1, \psi_1) - R_\epsilon(\varphi_2, \psi_2), (g, h) \rangle dx &= D_1 + D_2 + D_3 - D_4 + D_5 \\
&\quad - D_6 - D_7 - D_8 - D_9 - D_{10}, \quad (2.56)
\end{aligned}
$$

where

$$
D_1 = \int_{\mathbb{R}^N} \left(\mu \left(\sum_{j=1}^{k} U_{\epsilon, x_{j,\epsilon}} + \varphi_1 \right)^p - \mu \left(\sum_{j=1}^{k} U_{\epsilon, x_{j,\epsilon}} + \varphi_2 \right)^p \right) g dx,
$$

$$
D_2 = \int_{\mathbb{R}^N} \left(\nu \left(\sum_{j=1}^{k} V_{\epsilon, x_{j,\epsilon}} + \psi_1 \right)^p - \nu \left(\sum_{j=1}^{k} V_{\epsilon, x_{j,\epsilon}} + \psi_2 \right)^p \right) h dx,
$$

$$
D_3 = \int_{\mathbb{R}^N} \beta \left(\sum_{j=1}^{k} U_{\epsilon, x_{j,\epsilon}} + \varphi_1 \right)^{\frac{p-1}{2}} \left(\sum_{j=1}^{k} V_{\epsilon, x_{j,\epsilon}} + \psi_1 \right)^{\frac{p+1}{2}} g dx,
$$

$$
D_4 = \int_{\mathbb{R}^N} \beta \left(\sum_{j=1}^{k} U_{\epsilon, x_{j,\epsilon}} + \varphi_2 \right)^{\frac{p-1}{2}} \left(\sum_{j=1}^{k} V_{\epsilon, x_{j,\epsilon}} + \psi_2 \right)^{\frac{p+1}{2}} g dx,
$$

$$
D_5 = \int_{\mathbb{R}^N} \beta \left(\sum_{j=1}^{k} U_{\epsilon, x_{j,\epsilon}} + \varphi_1 \right)^{\frac{p+1}{2}} \left(\sum_{j=1}^{k} V_{\epsilon, x_{j,\epsilon}} + \psi_1 \right)^{\frac{p-1}{2}} h dx,
$$

$$
D_6 = \int_{\mathbb{R}^N} \beta \left(\sum_{j=1}^{k} U_{\epsilon, x_{j,\epsilon}} + \varphi_2 \right)^{\frac{p+1}{2}} \left(\sum_{j=1}^{k} V_{\epsilon, x_{j,\epsilon}} + \psi_2 \right)^{\frac{p-1}{2}} h dx,
$$

$$D_7 = \beta \int_{\mathbb{R}^N} \frac{p-1}{2} \left(\sum_{j=1}^{k} U_{\epsilon, x_{j,\epsilon}} \right)^{\frac{p-3}{2}} \left(\sum_{j=1}^{k} V_{\epsilon, x_{j,\epsilon}} \right)^{\frac{p+1}{2}} (\varphi_1 - \varphi_2) g dx,$$

$$D_8 = \beta \int_{\mathbb{R}^N} \frac{p-1}{2} \left(\sum_{j=1}^{k} U_{\epsilon, x_{j,\epsilon}} \right)^{\frac{p+1}{2}} \left(\sum_{j=1}^{k} V_{\epsilon, x_{j,\epsilon}} \right)^{\frac{p-3}{2}} (\psi_1 - \psi_2) h dx,$$

$$D_9 = \beta \int_{\mathbb{R}^N} \frac{p+1}{2} \left(\sum_{j=1}^{k} U_{\epsilon, x_{j,\epsilon}} \right)^{\frac{p-1}{2}} \left(\sum_{j=1}^{k} V_{\epsilon, x_{j,\epsilon}} \right)^{\frac{p-1}{2}} (\psi_1 - \psi_2) g dx,$$

$$D_{10} = \beta \int_{\mathbb{R}^N} \frac{p+1}{2} \left(\sum_{j=1}^{k} U_{\epsilon, x_{j,\epsilon}} \right)^{\frac{p-1}{2}} \left(\sum_{j=1}^{k} V_{\epsilon, x_{j,\epsilon}} \right)^{\frac{p-1}{2}} (\varphi_1 - \varphi_2) h dx.$$

Since $p \geq 3$, we have

$$\begin{aligned}
D_1 &= \int_{\mathbb{R}^N} p\mu \left(\sum_{j=1}^{k} U_{\epsilon, x_{j,\epsilon}} + \varphi_1 + t(\varphi_1 - \varphi_2) \right)^{p-1} (\varphi_1 - \varphi_2) g dx \\
&\leq C \int_{\mathbb{R}^N} \left(\sum_{j=1}^{k} U_{\epsilon, x_{j,\epsilon}} \right)^{p-2} (|\varphi_1| + |\varphi_2|)(|\varphi_1 - \varphi_2|) g dx \\
&\leq C \left(\int_{\mathbb{R}^N} (\sum_{j=1}^{k} U_{\epsilon, x_{j,\epsilon}})^{p+1} dx \right)^{\frac{p-2}{p+1}} (\|\varphi_1\|_{L^{p+1}} + \|\varphi_2\|_{L^{p+1}}) \|\varphi_1 - \varphi_2\|_{L^{p+1}} \|g\|_{L^{p+1}} \\
&\leq C \epsilon^{N \times \frac{p-2}{p+1}} \epsilon^{N(1 - \frac{p+1}{2}) \times \frac{3}{p+1}} (\|\varphi_1\|_{L^{p+1}} + \|\varphi_2\|_{L^{p+1}}) \|\varphi_1 - \varphi_2\|_{L^{p+1}} \|g\|_{L^{p+1}} \\
&\leq \epsilon^{-\frac{N}{2}} (\|\varphi_1\|_\epsilon + \|\varphi_2\|_\epsilon) \|\varphi_1 - \varphi_2\|_\epsilon \|g\|_\epsilon.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
D_2 &= \int_{\mathbb{R}^N} \left(\nu(\sum_{j=1}^{k} V_{\epsilon, x_{j,\epsilon}} + \psi_1)^p - \nu(\sum_{j=1}^{k} V_{\epsilon, x_{j,\epsilon}} + \psi_2)^p \right) h dx \\
&\leq \epsilon^{-\frac{N}{2}} (\|\psi_1\|_\epsilon + \|\psi_2\|_\epsilon) \|\psi_1 - \psi_2\|_\epsilon \|h\|_\epsilon.
\end{aligned}$$

Thus,

$$D_1 + D_2 \le C\epsilon^{-\frac{N}{2}}(\|(\varphi_1, \psi_1)\|_\epsilon + \|(\varphi_2, \psi_2)\|_\epsilon)\|(\varphi_1 - \varphi_2, \psi_1 - \psi_2\|_\epsilon\|(g, h)\|_\epsilon. \tag{2.57}$$

We also have

$$\begin{aligned} D_3 - D_4 = &\int_{\mathbb{R}^N} \beta\frac{p-1}{2}(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}} + \varphi_2)^{\frac{p-3}{2}}(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}} + \psi_2)^{\frac{p+1}{2}}(\varphi_1 - \varphi_2)g dx \\ &+ \int_{\mathbb{R}^N} \beta\frac{p+1}{2}(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}} + \varphi_2)^{\frac{p-1}{2}}(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}} + \psi_2)^{\frac{p-1}{2}}(\psi_1 - \psi_2)g dx \\ &+ o(\int_{\mathbb{R}^N}(|\varphi_1 - \varphi_2|^2)g dx) + o(\int_{\mathbb{R}^N} o(|\psi_1 - \psi_2|^2)g dx) \\ := &E_1 + E_2 + o(\int_{\mathbb{R}^N}(|\varphi_1 - \varphi_2|^2)g dx) + o(\int_{\mathbb{R}^N} o(|\psi_1 - \psi_2|^2)g dx), \end{aligned} \tag{2.58}$$

$$\begin{aligned} D_5 - D_6 = &\int_{\mathbb{R}^N} \beta\frac{p+1}{2}(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}} + \varphi_2)^{\frac{p-1}{2}}(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}} + \psi_2)^{\frac{p-1}{2}}(\varphi_1 - \varphi_2)h dx \\ &+ \int_{\mathbb{R}^N} \beta\frac{p-1}{2}(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}} + \varphi_2)^{\frac{p+1}{2}}(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}} + \psi_2)^{\frac{p-3}{2}}(\psi_1 - \psi_2)h dx \\ &+ o(\int_{\mathbb{R}^N}(|\varphi_1 - \varphi_2|^2)h dx) + o(\int_{\mathbb{R}^N} o(|\psi_1 - \psi_2|^2)h dx) \\ := &E_3 + E_4 + o(\int_{\mathbb{R}^N}(|\varphi_1 - \varphi_2|^2)h dx) + o(\int_{\mathbb{R}^N} o(|\psi_1 - \psi_2|^2)h dx). \end{aligned} \tag{2.59}$$

Thus,

$$\begin{aligned} E_1 - D_7 = &\beta\frac{p-1}{2}\frac{p-3}{2}\int_{\mathbb{R}^N}(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}})^{\frac{p-5}{2}}(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}})^{\frac{p+1}{2}}\varphi_2(\varphi_1 - \varphi_2)g dx \\ &+ \beta\frac{p-1}{2}\frac{p+1}{2}\int_{\mathbb{R}^N}(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}})^{\frac{p-3}{2}}(\sum_{j=1}^k V_{\epsilon,x_{j,\epsilon}})^{\frac{p-1}{2}}\psi_2(\varphi_1 - \varphi_2)g dx \\ &+ o(\int_{\mathbb{R}^N}|\varphi_2|^2(\varphi_1 - \varphi_2)g dx) + o(\int_{\mathbb{R}^N}|\psi_2|^2(\varphi_1 - \varphi_2)g dx) \\ \le &C\left(\int_{\mathbb{R}^N}(\sum_{j=1}^k U_{\epsilon,x_{j,\epsilon}})^{p+1}\right)^{\frac{p-2}{p+1}} \\ &\times\left(\|\varphi_2\|_{L^{p+1}}\|\varphi_1 - \varphi_2\|_{L^{p+1}} + \|\psi_2\|_{L^{p+1}})\|\psi_1 - \psi_2\|_{L^{p+1}}\right)\|g\|_{L^{p+1}} \end{aligned}$$

$$\leq C\epsilon^{-\frac{N}{2}}\|\varphi_2,\psi_2)\|_\epsilon\|(\varphi_1-\varphi_2,\psi_1-\psi_2\|_\epsilon\|(g,h)\|_\epsilon. \tag{2.60}$$

Similarly, we have

$$E_2-D_9\leq C\epsilon^{-\frac{N}{2}}\|\varphi_2,\psi_2)\|_\epsilon\|(\varphi_1-\varphi_2,\psi_1-\psi_2\|_\epsilon\|(g,h)\|_\epsilon, \tag{2.61}$$

$$E_3-D_{10}\leq C\epsilon^{-\frac{N}{2}}\|\varphi_2,\psi_2)\|_\epsilon\|(\varphi_1-\varphi_2,\psi_1-\psi_2\|_\epsilon\|(g,h)\|_\epsilon, \tag{2.62}$$

$$E_4-D_8\leq C\epsilon^{-\frac{N}{2}}\|\varphi_2,\psi_2)\|_\epsilon\|(\varphi_1-\varphi_2,\psi_1-\psi_2\|_\epsilon\|(g,h)\|_\epsilon. \tag{2.63}$$

Combining (2.57)–(2.63) with (2.56), we have

$$\begin{aligned}&\int_{\mathbb{R}^N} R_\epsilon(\varphi_1,\psi_1)-R_\epsilon(\varphi_2,\psi_2),(g,h)\rangle dx\\&\quad\leq C\epsilon^{-\frac{N}{2}}\|\varphi_2,\psi_2)\|_\epsilon\|(\varphi_1-\varphi_2,\psi_1-\psi_2\|_\epsilon\|(g,h)\|_\epsilon.\end{aligned} \tag{2.64}$$

Thus,

$$\|\mathbf{B}(\varphi_1,\psi_1)-\mathbf{B}(\varphi_2,\psi_2)\|_\epsilon\leq\frac{1}{2}\|(\varphi_1,\psi_1)-(\varphi_2,\psi_2)\|_\epsilon.$$

So, $\mathbf{B}$ is a contraction map.

By the contraction mapping theorem, we conclude that for any $\epsilon\in(0,\epsilon_0]$, $x_j\in B_\theta(x_0)$, there is a $(\varphi_\epsilon,\psi_\epsilon)\in\mathbf{E}_\epsilon$ depending only on x_j and ϵ such that

$$(\varphi_\epsilon,\psi_\epsilon)=\mathbf{B}(\varphi_\epsilon,\psi_\epsilon).$$

Moreover, from Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}&\|(\varphi_\epsilon,\psi_\epsilon)\|_\epsilon\\&\quad=\|\mathbf{B}(\varphi_\epsilon,\psi_\epsilon)\|_\epsilon\leq C\|l_\epsilon\|_\epsilon+C\|R_\epsilon(\varphi,\psi)\|_\epsilon\\&\leq C\|l_\epsilon\|_\epsilon+C\left(\epsilon^{-\frac{N}{2}}\|(\varphi,\psi)|_\epsilon^2+\epsilon^{-N}\|(\varphi,\psi)|_\epsilon^3+\epsilon^{-\frac{3N}{2}}\|(\varphi,\psi)|_\epsilon^4\right)\\&\leq C\|l_\epsilon\|_\epsilon+C\left(\epsilon^{-\frac{N}{2}}\|(\varphi,\psi)|_\epsilon+\epsilon^{-N}\|(\varphi,\psi)|_\epsilon^2+\epsilon^{-\frac{3N}{2}}\|(\varphi,\psi)|_\epsilon^3\right)\|(\varphi,\psi)|_\epsilon\\&\leq C\left(\epsilon^{\frac{N+2}{2}}+\epsilon^{\frac{N}{2}}\sum_{j=1}^k(|P(x_{j,\epsilon})-\lambda|+|Q(x_{j,\epsilon})-\lambda|)\right)+\epsilon^{\frac{N}{2}}\sum_{i\neq j}e^{-\frac{\sqrt{\alpha}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}.\end{aligned}$$

As desired. ◻

Next, we solve equation (2.1). Since

$$\begin{aligned} Q_\epsilon B_\epsilon(\varphi,\psi)+Q_\epsilon l_\epsilon-Q_\epsilon R_\epsilon(\varphi,\psi) &= B_\epsilon(\varphi,\psi)+l_\epsilon-R_\epsilon(\varphi,\psi)\\ &\quad -\sum_{j=1}^{k}\sum_{i=1}^{N} b_{\epsilon,i,j}\left(\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i},\frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i}\right). \end{aligned}$$

From Lemma 2.5, we know the following equation

$$Q_\epsilon B_\epsilon(\varphi,\psi)+Q_\epsilon l_\epsilon=Q_\epsilon R_\epsilon(\varphi,\psi)$$

has a unique solution $(\varphi_\epsilon,\psi_\epsilon)$. So

$$B_\epsilon(\varphi_\epsilon,\psi_\epsilon)+l_\epsilon-R_\epsilon(\varphi_\epsilon,\psi_\epsilon)=\sum_{j=1}^{k}\sum_{i=1}^{N} b_{\epsilon,i,j}\left(\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i},\frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i}\right) \tag{2.65}$$

for some constant $b_{\epsilon,i,j}$. Next, we should to choose suitable x_j such that $b_{\epsilon,i,j}=0, i=1,2\ldots N.j=1,2,\ldots k$.

Firstly, it is easy to see that the right hand of (2.65) belongs to $\mathbf{E}_\epsilon$, if the left hand of of (2.65) belongs to $\mathbf{E}_\epsilon$, then the right hand of (2.65) must be zero.

Let

$$u_\epsilon=\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}}+\varphi_\epsilon,\quad v_\epsilon=\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}}+\psi_\epsilon,$$

then

$$\begin{aligned} &\langle B_\epsilon(\varphi_\epsilon,\psi_\epsilon)+l_\epsilon-R_\epsilon(\varphi_\epsilon,\psi_\epsilon),(g,h)\rangle_\epsilon\\ &=\int_{\mathbb{R}^N}(\epsilon^2\nabla u_\epsilon\nabla g+P(x)u_\epsilon g+\epsilon^2\nabla v_\epsilon\nabla v+Q(x)v_\epsilon h)dx\\ &\quad-\int_{\mathbb{R}^N}\left(\mu u_\epsilon^p g+\beta u_\epsilon^{\frac{p-1}{2}}v_\epsilon^{\frac{p+1}{2}}g+\nu v_\epsilon^p h+\beta u_\epsilon^{\frac{p+1}{2}}v_\epsilon^{\frac{p-1}{2}}h\right)dx \end{aligned}$$

for any $(g,h)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$.

Lemma 2.6 *Suppose that $x_{j,\epsilon}$ satisfies*

$$\begin{aligned} &\int_{\mathbb{R}^N}(\epsilon^2\nabla u_\epsilon\nabla\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i}+P(x)u_\epsilon\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i}+\epsilon^2\nabla v_\epsilon\nabla\frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i}+Q(x)v_\epsilon\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i})dx\\ &-\int_{\mathbb{R}^N}\left(\mu u_\epsilon^p\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i}+\beta u_\epsilon^{\frac{p-1}{2}}v_\epsilon^{\frac{p+1}{2}}\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i}+\nu v_\epsilon^p\frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i}+\beta u_\epsilon^{\frac{p+1}{2}}v_\epsilon^{\frac{p-1}{2}}\frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i}\right)\\ &dx=0 i=1,2,\ldots N,\ j=1,2,\ldots k. \end{aligned}\tag{2.66}$$

then

$$b_{\epsilon,i,j} = 0,\ i = 1, 2, \ldots N, j = 1, 2, \cdots k.$$

Proof If (2.66) holds, then

$$\sum_{m=1}^{k}\sum_{h=1}^{N} b_{\epsilon,h,m} \Big\langle \big(\frac{\partial U_{\epsilon,x_{j,\epsilon}}}{\partial x_i}, \frac{\partial V_{\epsilon,x_{j,\epsilon}}}{\partial x_i}\big), \big(\frac{\partial U_{\epsilon,x_m}}{\partial y_h}, \frac{\partial V_{\epsilon,x_m}}{\partial y_h}\big)\Big\rangle = 0,\quad i = 1, 2, \ldots N, j = 1, 2, \ldots k. \tag{2.67}$$

By (2.9) and (2.10), we obtain

$$b_{\epsilon,i,j} = 0,\ i = 1, 2, \ldots N, j = 1, 2, \ldots k.$$

□

3 Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1.

Proof In order to solve (2.65), we define a function as following

$$K(\mathbf{x}) = I\left(\sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon}} + \varphi_\epsilon, \sum_{j=1}^{k} V_{\epsilon,x_{j,\epsilon}} + \psi_\epsilon\right),$$

where

$$\begin{aligned} I(u, v) = &\frac{1}{2}\int_{\mathbb{R}^N} (\epsilon^2|\nabla u|^2 + P(x)u^2 + \epsilon^2|\nabla v|^2 + Q(x)v^2)dx \\ &- \frac{1}{p+1}\int_{\mathbb{R}^N} (\mu u^{p+1} + \nu u^{p+1} \\ &+ 2\beta u^{\frac{p+1}{2}} v^{\frac{p+1}{2}})dx,\ \forall\,(u, v) \in\ H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N). \end{aligned} \tag{3.1}$$

Then, from Lemmas 2.3, 2.4 and Proposition 5.1, there exists a small constant $\sigma > 0$ such that

$$\begin{aligned} K(\mathbf{x}) &= I\left(\sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon}}, \sum_{j=1}^{k} V_{\epsilon,x_{j,\epsilon}}\right) + \langle l_\epsilon, (\varphi_\epsilon, \psi_\epsilon)\rangle + O(\|l_\epsilon\|\|(\varphi_\epsilon, \psi_\epsilon)\| + \|\varphi_\epsilon, \psi_\epsilon\|^2) \\ &= I\left(\sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon}}, \sum_{j=1}^{k} V_{\epsilon,x_{j,\epsilon}}\right) + O\left(\epsilon^N\left((P(x_{j,\epsilon}) - \lambda)^2 + (Q(x_{j,\epsilon}) - \lambda)^2 + \epsilon^2\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= (\frac{1}{2} - \frac{1}{p+1})k\epsilon^N \lambda^{1-\frac{N}{2}} A - C\epsilon^N \left((\lambda - P(x_{j,\epsilon})) + (\lambda - Q(x_{j,\epsilon})) + \epsilon\right) \\
&\quad - C\epsilon^N \sum_{i\neq j}^{k} e^{-\frac{\sqrt{\lambda}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}} + O\left(\epsilon^N \sum_{i\neq j}^{k} e^{-\frac{2(\sqrt{\lambda}-\sigma)|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}\right) \\
&\quad + O\left(\epsilon^N \left((P(x_{j,\epsilon}) - \lambda)^2 + (Q(x_{j,\epsilon}) - \lambda)^2 + \epsilon^2\right)\right).
\end{aligned} \tag{3.2}
$$

We use the ideas introduced in [22] to consider the following maximizing problem

$$
\max_{x\in D} K(x),
$$

where

$$
D = \{\mathbf{x} : x_j \in \overline{B_\theta(x_0)},\ j = 1, 2, \ldots k,\ |x_m - x_j| \geq \theta\epsilon|\ln\epsilon|^{\frac{1}{2}},\ m \neq j\}.
$$

We prove that if $\max_{x\in D} K(x)$ is achieved by some $\mathbf{x}_\epsilon$ in $\overline{D}$, then $\mathbf{x}_\epsilon$ is an interior point of $\overline{D}$. Taking $\overline{x}_i = x_0 + \theta\epsilon|\ln\epsilon|e_j$, $(j = 1, 2, \ldots k)$, then $|\overline{x}_i - \overline{x}_j| = \theta\epsilon|\ln\epsilon|$, which means that $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k) \in \overline{D}$, thus, we have

$$
\begin{aligned}
&\left(\frac{1}{2} - \frac{1}{p+1}\right)k\epsilon^N \lambda^{1-\frac{N}{2}} A - C\epsilon^{N+1}|\ln\epsilon| \leq K(\overline{\mathbf{x}}) \leq K(\mathbf{x}_\epsilon) \\
&\leq \left(\frac{1}{2} - \frac{1}{p+1}\right)k\epsilon^N \lambda^{1-\frac{N}{2}} A - C\epsilon^N \\
&\left(\sum_{j=1}^{k}(\lambda - P(x_{j,\epsilon})) + \sum_{j=1}^{k}(\lambda - Q(x_{j,\epsilon})) + \epsilon\right) \\
&\quad - C\epsilon^N \sum_{i\neq j}^{k} e^{-\frac{\sqrt{\alpha}|x_{\epsilon,i}-x_{j,\epsilon}|}{\epsilon}}.
\end{aligned}
$$

Thus,

$$
\sum_{j=1}^{k}(\lambda - P(x_{j,\epsilon})) + \sum_{j=1}^{k}(\lambda - Q(x_{j,\epsilon})) + \epsilon + \sum_{i\neq j}^{k} e^{-\frac{\sqrt{\alpha}|x_{\epsilon,i}-x_{j,\epsilon}|}{\epsilon}} \leq C\epsilon|\ln\epsilon|,
$$

so

$$
\sum_{j=1}^{k}(\lambda - P(x_{j,\epsilon})) \leq C\epsilon|\ln\epsilon|, \quad \sum_{j=1}^{k}(\lambda - Q(x_{j,\epsilon})) \leq C\epsilon|\ln\epsilon|,
$$

$$
\frac{\sqrt{\alpha}|x_{\epsilon,i} - x_{j,\epsilon}|}{\epsilon} \geq C|\ln\epsilon| \geq |\ln\epsilon|^{\frac{1}{2}}.
$$

Therefore, $\mathbf{x}_\epsilon$ is an interior point of $\overline{D}$, which implies that

$$(u_\epsilon, v_\epsilon) = \left(\sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon}} + \varphi_\epsilon,\ \sum_{j=1}^{k} V_{\epsilon,x_{j,\epsilon}} + \psi_\epsilon\right)$$

is a critical point of $K(\mathbf{x})$. So, (1.1) has a solution of the form

$$u_\epsilon = \sum_{j=1}^{k} U_{\epsilon,x_\epsilon,j} + \varphi_\epsilon, \quad v_\epsilon = \sum_{j=1}^{k} V_{\epsilon,x_{j,\epsilon}} + \psi_\epsilon$$

for some $x_{j,\epsilon} \in B_\delta(x_0)$ and $\|(\varphi_\epsilon, \psi_\epsilon)\|_\epsilon = O(\epsilon^{\frac{N}{2}+1})$. □

4 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2.

Proof Set $\lambda = P(x_0)$, $\overline{\lambda} = Q(\overline{x}_0)$ it is easy to see that $U_{\lambda,\mu} = \lambda^{\frac{1}{p-1}} \mu^{-\frac{1}{p-1}} W(\sqrt{\lambda} x)$ is a solution of

$$\begin{cases} -\Delta u + \lambda u = \mu u^p, \text{ in } \mathbb{R}^N \\ u > 0 \text{ in } \mathbb{R}^N, u(x) \to 0 \text{ as } |x| \to +\infty, \end{cases}$$

and $U_{\overline{\lambda},\nu} = \overline{\lambda}^{\frac{1}{p-1}} \nu^{-\frac{1}{p-1}} W(\sqrt{\overline{\lambda}} x)$ is a solution of

$$\begin{cases} -\Delta u + \overline{\lambda} u = \nu u^p, \text{ in } \mathbb{R}^N \\ u > 0 \text{ in } \mathbb{R}^N, u(x) \to 0 \text{ as } |x| \to +\infty. \end{cases}$$

Let

$$\big(U_{\epsilon,x_j,\mu}(x), U_{\epsilon,z_j,\nu}(x)\big) = \left(U_{\lambda_j,\mu}\left(\frac{x - x_j}{\epsilon}\right), U_{\overline{\lambda}_j,\nu}\left(\frac{x - z_j}{\epsilon}\right)\right),$$

where $x_j \in B_\delta(x_0),\ z_j \in B_\delta(\overline{x}_0)$.

$$\begin{aligned} \widetilde{\mathbf{E}}_\epsilon = \Bigg\{ &(\varphi, \psi) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N), \left\langle \varphi, \frac{\partial U_{\epsilon,x_j,\mu}}{\partial x_{j,l}} \right\rangle_\epsilon = 0, \\ &\left\langle \psi, \frac{\partial U_{\epsilon,z_j,\nu}}{\partial z_{j,l}} \right\rangle_\epsilon = 0,\ j = 1, 2 \cdots k;\ \ l = 1, 2 \cdots N \Bigg\}. \end{aligned}$$

Let $\lambda_j = P(x_j)$, $\overline{\lambda}_j = Q(z_j)$ and x_0 is a local maximum point of $P(x)$ and $\overline{x}_0$ is a local maximum point of $Q(x)$, we want to construct a solution (u_ϵ, v_ϵ) of the following form

$$u_\epsilon = \sum_{j=1}^{k} U_{\epsilon,x_j,\mu} + \widetilde{\varphi}_\epsilon, \ \ v_\epsilon = \sum_{j=1}^{m} U_{\epsilon,z_j,\nu} + \widetilde{\psi}_\epsilon$$

where as $\epsilon \to 0, x_j \to x_0, \ \ z_j \to \overline{x}_0$ and $\|(\widetilde{\varphi}_\epsilon, \widetilde{\psi}_\epsilon)\|^2 = o(\epsilon^N)$. Then, $(\widetilde{\varphi}_\epsilon, \widetilde{\psi}_\epsilon)$ satisfies the following equation

$$\begin{cases} \overline{B}_\epsilon(\widetilde{\varphi}_\epsilon, \widetilde{\psi}_\epsilon) + l_\epsilon = R_\epsilon(\widetilde{\varphi}_\epsilon, \widetilde{\psi}_\epsilon), x \in \mathbb{R}^N, \\ (\widetilde{\varphi}_\epsilon, \widetilde{\psi}_\epsilon) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N). \end{cases}$$

where $\overline{B}_\epsilon$ is a bounded linear operator in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, defined by

$$\begin{aligned}
&\langle \overline{B}_\epsilon(\widetilde{\varphi}_\epsilon, \widetilde{\psi}_\epsilon), (g, h)\rangle \\
&= \int_{\mathbb{R}^N} \left(\epsilon^2 \nabla \widetilde{\varphi}_\epsilon \nabla g + P(x)\widetilde{\varphi}_\epsilon g - p\mu(\sum_{j=1}^{k} U_{\epsilon,x_j,\mu})^{p-1} \widetilde{\varphi}_\epsilon g \right) dx \\
&\quad + \int_{\mathbb{R}^N} \left(\epsilon^2 \nabla \widetilde{\psi}_\epsilon \nabla h + Q(x)\widetilde{\psi}_\epsilon h - p\nu(\sum_{j=1}^{m} U_{\epsilon,z_j,\nu})^{p-1} \widetilde{\psi}_\epsilon h \right) dx \\
&\quad - \beta \int_{\mathbb{R}^N} \frac{p-1}{2} (\sum_{j=1}^{k} U_{\epsilon,x_j,\mu})^{\frac{p-3}{2}} (\sum_{j=1}^{m} U_{\epsilon,z_j,\nu})^{\frac{p+1}{2}} \widetilde{\varphi}_\epsilon g \\
&+ \frac{p-1}{2} (\sum_{j=1}^{k} U_{\epsilon,x_j,\mu})^{\frac{p+1}{2}} (\sum_{j=1}^{m} U_{\epsilon,z_j,\nu})^{\frac{p-3}{2}} \widetilde{\psi}_\epsilon h \\
&\quad - \beta \int_{\mathbb{R}^N} \frac{p+1}{2} (\sum_{j=1}^{k} U_{\epsilon,x_j,\mu})^{\frac{p-1}{2}} (\sum_{j=1}^{m} U_{\epsilon,z_j,\nu})^{\frac{p-1}{2}} \widetilde{\psi}_\epsilon g \\
&+ \frac{p+1}{2} (\sum_{j=1}^{k} U_{\epsilon,x_j,\mu})^{\frac{p-1}{2}} (\sum_{j=1}^{m} U_{\epsilon,z_j,\nu})^{\frac{p-1}{2}} \widetilde{\varphi}_\epsilon h
\end{aligned}$$

for any $(g, h) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

$\bar{l}_\epsilon \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ satisfying

$$\begin{aligned}
\langle \bar{l}_\epsilon, (g, h)\rangle &= \sum_{j=1}^{k} \int_{\mathbb{R}^N} (P(x) - P(x^j)) U_{\epsilon,x_j,\mu} g dx \\
&\quad + \mu \int_{\mathbb{R}^N} \left(\sum_{j=1}^{k} U^p_{\epsilon,x_j,\mu} - (\sum_{j=1}^{k} U_{\epsilon,x_j,\mu})^p \right) g dx \\
&\quad + \sum_{j=1}^{k} \int_{\mathbb{R}^N} (Q(x) - Q(z_j)) U_{\epsilon,z_j,\nu} h dx
\end{aligned}$$

$$
\begin{aligned}
&+\nu\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k}U^p_{\epsilon,z_j,\nu}-(\sum_{j=1}^{m}U_{\epsilon,z_j,\nu})^p\right)hdx\\
&-\beta\int_{\mathbb{R}^N}\left((\sum_{j=1}^{k}U_{\epsilon,x_j,\mu})^{\frac{p-1}{2}}(\sum_{j=1}^{m}U_{\epsilon,z_j,\nu})^{\frac{p+1}{2}}\right)gdx\\
&-\beta\int_{\mathbb{R}^N}\left((\sum_{j=1}^{k}U_{\epsilon,x_j,\mu})^{\frac{p+1}{2}}(\sum_{j=1}^{m}U_{\epsilon,z_j,\nu})^{\frac{p-1}{2}}\right)hdx
\end{aligned}
$$

for any $(g,h)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$.

$$
\begin{aligned}
&\langle\overline{R}_\epsilon(\widetilde{\varphi}_\epsilon,\widetilde{\psi}_\epsilon),(g,h)\rangle\\
&=\int_{\mathbb{R}^N}\left(\mu(\sum_{j=1}^{k}U_{\epsilon,x_j,\mu}+\widetilde{\varphi}_\epsilon)^p+\beta(\sum_{j=1}^{k}U_{\epsilon,x_j,\mu}+\widetilde{\varphi}_\epsilon)^{\frac{p-1}{2}}(\sum_{j=1}^{m}U_{\epsilon,z_j,\nu}+\widetilde{\psi}_\epsilon)^{\frac{p+1}{2}}\right)gdx\\
&\quad+\int_{\mathbb{R}^N}\left(\nu(\sum_{j=1}^{m}U_{\epsilon,z_j,\nu}+\widetilde{\psi}_\epsilon)^p+\beta(\sum_{j=1}^{k}U_{\epsilon,x_j,\mu}+\widetilde{\varphi}_\epsilon)^{\frac{p+1}{2}}(\sum_{j=1}^{m}U_{\epsilon,z_j,\nu}+\widetilde{\psi}_\epsilon)^{\frac{p-1}{2}}\right)hdx\\
&\quad-\beta\int_{\mathbb{R}^N}\mu(\sum_{j=1}^{k}U_{\epsilon,x_j,\mu})^pg+\left((\sum_{j=1}^{k}U_{\epsilon,x_j,\mu})^{\frac{p-1}{2}}(\sum_{j=1}^{m}U_{\epsilon,z_j,\nu})^{\frac{p+1}{2}}\right)gdx\\
&\quad-\beta\int_{\mathbb{R}^N}\nu(\sum_{j=1}^{m}U_{\epsilon,z_j,\nu})^ph+\left((\sum_{j=1}^{k}U_{\epsilon,x_j,\mu})^{\frac{p+1}{2}}(\sum_{j=1}^{m}U_{\epsilon,z_j,\nu})^{\frac{p-1}{2}}\right)hdx\\
&\quad-\beta\int_{\mathbb{R}^N}\frac{p-1}{2}(\sum_{j=1}^{k}U_{\epsilon,x_j,\mu})^{\frac{p-3}{2}}(\sum_{j=1}^{m}U_{\epsilon,z_j,\nu})^{\frac{p+1}{2}}\widetilde{\varphi}_\epsilon gdx\\
&\quad-\beta\int_{\mathbb{R}^N}\frac{p-1}{2}(\sum_{j=1}^{k}U_{\epsilon,x_j,\mu})^{\frac{p+1}{2}}(\sum_{j=1}^{m}U_{\epsilon,z_j,\nu})^{\frac{p-3}{2}}\widetilde{\psi}_\epsilon hdx\\
&\quad-\beta\int_{\mathbb{R}^N}\frac{p+1}{2}(\sum_{j=1}^{k}U_{\epsilon,x_j,\mu})^{\frac{p-1}{2}}(\sum_{j=1}^{m}U_{\epsilon,z_j,\nu})^{\frac{p-1}{2}}\widetilde{\psi}_\epsilon gdx\\
&\quad-\beta\int_{\mathbb{R}^N}\frac{p+1}{2}(\sum_{j=1}^{k}U_{\epsilon,x_j,\mu})^{\frac{p-1}{2}}(\sum_{j=1}^{m}U_{\epsilon,z_j,\nu})^{\frac{p-1}{2}}\widetilde{\varphi}_\epsilon hdx
\end{aligned}
$$

for any $(g,h)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. □

Lemma 4.1 *There exist $\beta^*>0$ and $\epsilon_0,\theta_0>0,\rho>0$, independent of x_j, $j=1,2,\dots k$ and z_j, $j=1,2,\dots m$, such that for any $\epsilon\in(0,\epsilon_0]$, $x_j\in B_{\theta_0}(x_0)$ and $z_j\in B_{\theta_0}(\overline{x}_0)$, if $\beta<\beta^*$, then $Q_\epsilon\overline{B}_\epsilon(\widetilde{\varphi}_\epsilon,\widetilde{\psi}_\epsilon)$ is bijective in $\widetilde{\mathbf{E}}_\epsilon$. Moreover, it holds*

$$\|Q_\epsilon\overline{B}_\epsilon(\widetilde{\varphi}_\epsilon,\widetilde{\psi}_\epsilon)\|_\epsilon\geq\rho\|(\widetilde{\varphi}_\epsilon,\widetilde{\psi}_\epsilon)\|_\epsilon,\ \ \forall\,(\widetilde{\varphi}_\epsilon,\widetilde{\psi}_\epsilon)\in\widetilde{\mathbf{E}}_\epsilon.$$

Proof The proof is similar as the proof of Lemma 2.2. To get a contradiction, we take the projection to

$$\widetilde{\mathbf{E}}_\epsilon = \Big\{ (\varphi, \psi) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N), \Big\langle \varphi, \frac{\partial U_{\epsilon, x_j, \mu}}{\partial x_{j,l}} \Big\rangle_\epsilon = 0, \\ \Big\langle \psi, \frac{\partial U_{\epsilon, z_j, \nu}}{\partial z_{j,l}} \Big\rangle_\epsilon = 0,\ j = 1, 2 \cdots k;\ \ l = 1, 2 \cdots N \Big\}.$$

We need to prove $\varphi = 0$ and $\psi = 0$. When we prove $\varphi = 0$, we only need to set $h = 0, \varphi_n = 0$ after (2.23) in Lemma 2.2. So we will get φ satisfies $-\Delta\varphi + \lambda\varphi - p\mu U_r^{p-1} = 0$. By the non-degenerate of the solution of $-\Delta u + \lambda u = \mu u^p$, we have $\varphi = \sum_{i=1}^k c_i \frac{\partial U_{\mu,r}}{\partial y_i}$. Then, we can get $c_i = 0$, thus $\varphi = 0$. To show $\psi = 0$, we need to set $g = 0, \psi_n = 0$ after (2.23) in Lemma 2.2. So we will get that ψ satisfies $-\Delta\psi + \overline{\lambda}\psi - p\nu U_r^{p-1} = 0$. By the non-degenerate of the solution of $-\Delta u + \lambda u = \nu u^p$, we have $\psi = \sum_{i=1}^k d_i \frac{\partial U_{\nu,r}}{\partial y_i}$. Then, we can get $d_i = 0$, thus $\psi = 0$. The rest is similar to the proof of (3.8) in Lemma 3.2 in [23], we can get a contradiction when $\beta < \beta^*$. □

To carry out the contraction mapping theorem, we give the following error estimate.

Lemma 4.2 *There is a constant $C > 0$ independent of ϵ such that*

$$\|\bar{l}_\epsilon\|_\epsilon \le C \left(\epsilon^{\frac{N+2}{2}} + \epsilon^{\frac{N}{2}} \left(\sum_{i \neq j}^{k} (P(x_i) - P(x_j)) + \sum_{i \neq j}^{m} (Q(z_i) - Q(z_j)) \right) \right) \\ + C\epsilon^{\frac{N}{2}} \sum_{i=1}^{k} \sum_{j=1}^{m} e^{-\frac{\min\{\sqrt{\lambda_i}, \sqrt{\overline{\lambda}_j}\}|x_i - z_j|}{\epsilon}} + C\epsilon^{\frac{N}{2}} \sum_{i=1}^{k} \sum_{j=1}^{m} e^{-\frac{\min\{\sqrt{\lambda_i}, \sqrt{\lambda}_j\}|x_i - x_j|}{\epsilon}} \\ + C\epsilon^{\frac{N}{2}} \sum_{i=1}^{k} \sum_{j=1}^{m} e^{-\frac{\min\{\sqrt{\overline{\lambda}_i}, \sqrt{\lambda_j}\}|z_i - x_j|}{\epsilon}}.$$

Proof The proof is similar to that of Lemma 2.3. First, by the Hölder inequality, we have

$$\sum_{j=1}^{k} \int_{\mathbb{R}^N} (P(x) - P(x_j)) U_{\epsilon, x_j, \nu} g dx + \mu \int_{\mathbb{R}^N} \left(\sum_{j=1}^{k} U^p_{\epsilon, x_j, \mu} - (\sum_{j=1}^{k} U_{\epsilon, x_j, \mu})^p \right) g dx \\ + \sum_{j=1}^{k} \int_{\mathbb{R}^N} (Q(x) - Q(z_j)) U_{\epsilon, z_j, \nu} h dx + \nu \int_{\mathbb{R}^N} \left(\sum_{j=1}^{k} U^p_{\epsilon, z_j, \nu} - (\sum_{j=1}^{m} U_{\epsilon, z_j, \nu})^p \right) h dx \\ \le C \left(\epsilon^{\frac{N+2}{2}} + \epsilon^{\frac{N}{2}} \left(\sum_{i \neq j}^{k} (P(x_i) - P(x_j)) + \sum_{i \neq j}^{m} (Q(z_i) - Q(z_j)) \right) \right) \|(g, h)\|_\epsilon,$$

and

$$\beta \int_{\mathbb{R}^N} \left((\sum_{j=1}^{k} U_{\epsilon,x_j,\mu})^{\frac{p-1}{2}} (\sum_{j=1}^{m} U_{\epsilon,z_j,\nu})^{\frac{p+1}{2}} \right) g dx$$
$$+ \beta \int_{\mathbb{R}^N} \left((\sum_{j=1}^{k} U_{\epsilon,x_j,\mu})^{\frac{p+1}{2}} (\sum_{j=1}^{m} U_{\epsilon,z_j,\nu})^{\frac{p-1}{2}} \right) h dx$$
$$\leq \Bigg(C\epsilon^{\frac{N}{2}} \sum_{i=1}^{k}\sum_{j=1}^{m} e^{-\frac{\min\{\sqrt{\lambda_i},\sqrt{\bar{\lambda}_j}\}|x_i-z_j|}{\epsilon}} + C\epsilon^{\frac{N}{2}} \sum_{i=1}^{k}\sum_{j=1}^{m} e^{-\frac{\min\{\sqrt{\lambda_i},\sqrt{\lambda}_j\}|x_i-x_j|}{\epsilon}}$$
$$+ C\epsilon^{\frac{N}{2}} \sum_{i=1}^{k}\sum_{j=1}^{m} e^{-\frac{\min\{\sqrt{\bar{\lambda}_i},\sqrt{\lambda_j}\}|z_i-x_j|}{\epsilon}} \Bigg) \|(g,h)\|_\epsilon .$$

So, we obtain

$$\|\bar{l}_\epsilon\|_\epsilon \leq C \left(\epsilon^{\frac{N+2}{2}} + \epsilon^{\frac{N}{2}} \left(\sum_{i\neq j}^{k} (P(x_i) - P(x_j)) + \sum_{i\neq j}^{m} (Q(z_i) - Q(z_j)) \right) \right)$$
$$+ C\epsilon^{\frac{N}{2}} \sum_{i=1}^{k}\sum_{j=1}^{m} e^{-\frac{\min\{\sqrt{\lambda_i},\sqrt{\bar{\lambda}_j}\}|x_i-z_j|}{\epsilon}} + C\epsilon^{\frac{N}{2}} \sum_{i=1}^{k}\sum_{j=1}^{m} e^{-\frac{\min\{\sqrt{\lambda_i},\sqrt{\lambda}_j\}|x_i-x_j|}{\epsilon}}$$
$$+ C\epsilon^{\frac{N}{2}} \sum_{i=1}^{k}\sum_{j=1}^{m} e^{-\frac{\min\{\sqrt{\bar{\lambda}_i},\sqrt{\lambda_j}\}|z_i-x_j|}{\epsilon}} .$$

This proof is thus complete. □

Lemma 4.3 *There is a constant $C > 0$ independent of ϵ and a small constant $\sigma > 0$ such that*

$$\|\overline{R}_\epsilon(\varphi,\psi)\|_\epsilon \leq C \left(\epsilon^{-\frac{N}{2}} \|(\varphi,\psi)\|_\epsilon^2 + \epsilon^{-N} \|(\varphi,\psi)\|_\epsilon^3 + \epsilon^{-\frac{3N}{2}} \|(\varphi,\psi)\|_\epsilon^4 \right)$$
$$+ C\epsilon^N \sum_{i\neq j}^{k} e^{-\frac{\min\{\sqrt{\lambda_i},\sqrt{\lambda_j}\}|x_i-x_j|}{\epsilon}}$$
$$+ O\left(\epsilon^N \sum_{i\neq j}^{k} e^{-\frac{(2\min\{\sqrt{\lambda_i},\sqrt{\lambda_j}\}-\sigma)|x_i-x_j|}{\epsilon}} \right)$$
$$+ C\epsilon^N \sum_{i\neq j}^{m} e^{-\frac{\min\{\sqrt{\bar{\lambda}_i},\sqrt{\bar{\lambda}_j}\}|z_i-z_j|}{\epsilon}}$$
$$+ O\left(\epsilon^N \sum_{i\neq j}^{m} e^{-\frac{(2\min\{\sqrt{\bar{\lambda}_i},\sqrt{\bar{\lambda}_j}\}-\sigma)|z_i-z_j|}{\epsilon}} \right)$$

$$
+ C\epsilon^N \sum_{i=1}^{k} \sum_{j=1}^{m} e^{-\frac{\min\{\sqrt{\lambda_i}, \sqrt{\overline{\lambda}_j}\}|x_i - z_j|}{\epsilon}}
$$

$$
+ O\left(\epsilon^N \sum_{i=1}^{k} \sum_{j=1}^{m} e^{-\frac{(2\min\{\sqrt{\lambda_i}, \sqrt{\overline{\lambda}_j}\} - \sigma)|x_i - z_j|}{\epsilon}}\right).
$$

Proof The proof is similar to that of Lemma 2.4, we only need to small changes, so, we omit the details here. □

Based on the similar arguments as in Lemma 2.5, we have the following lemma.

Lemma 4.4 *There exists $\epsilon_0 > 0$, such that for any $\epsilon \in (0, \epsilon_0]$, $x_j \in B_\theta(x_0)$, $z_j \in B_\theta(\overline{x}_0)$, then (2.54) has a unique $(\overline{\varphi}_\epsilon, \overline{\psi}_\epsilon) \in \overline{\mathbf{E}}_\epsilon$ and*

$$
\begin{aligned}
&\|(\overline{\varphi}_\epsilon, \overline{\psi}_\epsilon)\|_\epsilon \le C\|\bar{l}_\epsilon\|_\epsilon \\
&\le C\left(\epsilon^{\frac{N+2}{2}} + \epsilon^{\frac{N}{2}}\left(\sum_{i\neq j}^{k}(P(x_i) - P(x_j)) + \sum_{i\neq j}^{m}(Q(z_i) - Q(z_j))\right)\right) \\
&\quad + C\epsilon^{\frac{N}{2}} \sum_{i=1}^{k}\sum_{j=1}^{m} e^{-\frac{\min\{\sqrt{\lambda_i}, \sqrt{\overline{\lambda}_j}\}|x_i - z_j|}{\epsilon}} + C\epsilon^{\frac{N}{2}} \sum_{i=1}^{k}\sum_{j=1}^{m} e^{-\frac{\min\{\sqrt{\lambda_i}, \sqrt{\lambda}_j\}|x_i - x_j|}{\epsilon}} \\
&\quad + C\epsilon^{\frac{N}{2}} \sum_{i=1}^{k}\sum_{j=1}^{m} e^{-\frac{\min\{\sqrt{\overline{\lambda}_i}, \sqrt{\lambda_j}\}|z_i - x_j|}{\epsilon}}.
\end{aligned}
$$

Proof of Theorem 1.2 In order to solve (2.65), we define a function as following

$$
\overline{K}(\mathbf{x}, \mathbf{z}) = I\left(\sum_{j=1}^{k} U_{\epsilon, x_j, \mu} + \overline{\varphi}_\epsilon, \sum_{j=1}^{k} U_{\epsilon, z_j, \nu} + \overline{\psi}_\epsilon\right)
$$

where $I(u, v)$ ia given by (3.1). From Lemma 4.1, 4.2, 4.3, 4.4 and Proposition 5.2, we have

$$
\begin{aligned}
\overline{K}(\mathbf{x}, \mathbf{z}) &= I\left(\sum_{j=1}^{k} U_{\epsilon, x_j, \mu}, \sum_{j=1}^{k} U_{\epsilon, z_j, \nu}\right) + \langle \bar{l}_\epsilon, (\varphi_\epsilon, \psi_\epsilon)\rangle \\
&\quad + O(\|\bar{l}_\epsilon\|\|(\overline{\varphi}_\epsilon, \overline{\psi}_\epsilon)\| + \|\overline{\varphi}_\epsilon, \overline{\psi}_\epsilon\|^2) \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right) A\left(\sum_{j=1}^{k} \epsilon^N \lambda_j^{\frac{p+1}{p-1} - \frac{N}{2}} \mu^{-\frac{2}{p-1}} + \sum_{j=1}^{m} \epsilon^N \overline{\lambda}_j^{\frac{p+1}{p-1} - \frac{N}{2}} \nu^{-\frac{2}{p-1}}\right) \\
&\quad - C\epsilon^N \sum_{i\neq j}^{k} e^{-\frac{\min\{\sqrt{\lambda_i}, \sqrt{\lambda_j}\}|x_i - x_j|}{\epsilon}} + O\left(\epsilon^N \sum_{i\neq j}^{k} e^{-\frac{(2\min\{\sqrt{\lambda_i}, \sqrt{\lambda_j}\} - \sigma)|x_i - x_j|}{\epsilon}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- C\epsilon^N \sum_{i\neq j}^{m} e^{-\frac{\min\{\sqrt{\overline{\lambda}_i},\sqrt{\overline{\lambda}_j}\}-\sigma)|z_i-z_j|}{\epsilon}} + O\left(\epsilon^N \sum_{i\neq j}^{m} e^{-\frac{(2\min\{\sqrt{\overline{\lambda}_i},\sqrt{\overline{\lambda}_j}\}-\sigma)|z_i-z_j|}{\epsilon}}\right)\\
&- C\epsilon^N \sum_{i=1}^{k}\sum_{j=1}^{m} e^{-\frac{\min\{\sqrt{\lambda_i},\sqrt{\overline{\lambda}_j}\}|x_i-z_j|}{\epsilon}} + O\left(\sum_{i=1}^{k}\sum_{j=1}^{m} e^{-\frac{(2\min\{\sqrt{\lambda_i},\sqrt{\overline{\lambda}_j}\}-\sigma)|x_i-z_j|}{\epsilon}}\right)\\
&+ O(\epsilon^{N+1}).
\end{aligned}
$$

Consider the following maximizing problem

$$\max_{(x,y)\in D_1\times D_2} \overline{K}(\mathbf{x},\mathbf{z}),$$

where

$$D_1 = \{\mathbf{x} : x_j \in \overline{B_\theta(x_0)},\ j = 1,2,\dots k,\ |x_i - x_j| \geq \theta\epsilon|\ln\epsilon|^{\frac{1}{2}},\ i \neq j\},$$

$$D_2 = \{\mathbf{z} : z_j \in \overline{B_\theta(\overline{x}_0)},\ j = 1,2,\dots m,\ |z_i - z_j| \geq \theta\epsilon|\ln\epsilon|^{\frac{1}{2}},\ i \neq j\}.$$

We prove that if $\max_{(x,y)\in D_1\times D_2} \overline{K}(\mathbf{x},\mathbf{z})$, is achieved by some $(\mathbf{x}_\epsilon,\mathbf{z}_\epsilon)$ in $D_1 \times D_2$, then $(\mathbf{x}_\epsilon,\mathbf{z}_\epsilon)$ is an interior point of $D_1 \times D_2$. Taking $\overline{x}_i = x_0 + \theta\epsilon\ln\epsilon|e_j, (j = 1,2,\dots k)$, $\overline{z}_i = \overline{x}_0 + \theta\epsilon\ln\epsilon|e_j, (j = 1,2,\dots m)$, then $|\overline{x}_i - \overline{x}_j| = \theta\epsilon|\ln\epsilon|$, $|\overline{z}_i - \overline{z}_j| = \theta\epsilon|\ln\epsilon|$, which means that $(\overline{\mathbf{x}} = (\overline{x}_1,\overline{x}_2,\dots,\overline{x}_k), \overline{\mathbf{z}} = (\overline{z}_1,\overline{z}_2,\dots,\overline{z}_m)) \in D_1 \times D_2$, thus, we have

$$
\begin{aligned}
&= \left(\frac{1}{2} - \frac{1}{p+1}\right) A \left(\sum_{j=1}^{k} \epsilon^N \lambda_j^{\frac{p+1}{p-1}-\frac{N}{2}} \mu^{-\frac{2}{p-1}} + \sum_{j=1}^{m} \epsilon^N \overline{\lambda}_j^{\frac{p+1}{p-1}-\frac{N}{2}} \nu^{-\frac{2}{p-1}}\right) - C\epsilon^{N+1}|\ln\epsilon|\\
&\leq \overline{K}(\mathbf{x}_\epsilon,\mathbf{z}_\epsilon) = (\frac{1}{2} - \frac{1}{p+1})A \left(\sum_{j=1}^{k} \epsilon^N \lambda_{j,\epsilon}^{\frac{p+1}{p-1}-\frac{N}{2}} \mu^{-\frac{2}{p-1}} + \sum_{j=1}^{m} \epsilon^N \overline{\lambda}_{j,\epsilon}^{\frac{p+1}{p-1}-\frac{N}{2}} \nu^{-\frac{2}{p-1}}\right)\\
&- C\epsilon^N \sum_{i\neq j}^{k} e^{-\frac{\min\{\sqrt{\lambda_{i,\epsilon}},\sqrt{\lambda_{j,\epsilon}}\}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}} + O\left(\epsilon^N \sum_{i\neq j}^{k} e^{-\frac{(2\min\{\sqrt{\lambda_{i,\epsilon}},\sqrt{\lambda_{j,\epsilon}}\}-\sigma)|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}\right)\\
&- C\epsilon^N \sum_{i\neq j}^{m} e^{-\frac{\min\{\sqrt{\overline{\lambda}_{i,\epsilon}},\sqrt{\overline{\lambda}_{j,\epsilon}}\}|z_{i,\epsilon}-z_{j,\epsilon}|}{\epsilon}} + O\left(\epsilon^N \sum_{i\neq j}^{m} e^{-\frac{(2\min\{\sqrt{\overline{\lambda}_{i,\epsilon}},\sqrt{\overline{\lambda}_{j,\epsilon}}\}-\sigma)|z_{i,\epsilon}-z_{j,\epsilon}|}{\epsilon}}\right).
\end{aligned}
$$

Thus, we can get

$$
\begin{aligned}
&\left(\frac{1}{2} - \frac{1}{p+1}\right) A\epsilon^N \left(\sum_{j=1}^{k} \left(\lambda_j^{\frac{p+1}{p-1}-\frac{N}{2}} - \lambda_{j,\epsilon}^{\frac{p+1}{p-1}-\frac{N}{2}}\right) \mu^{-\frac{2}{p-1}}\right.\\
&\qquad \left. + \sum_{j=1}^{m} \left(\overline{\lambda}_j^{\frac{p+1}{p-1}-\frac{N}{2}} - \overline{\lambda}_{j,\epsilon}^{\frac{p+1}{p-1}-\frac{N}{2}}\right) \nu^{-\frac{2}{p-1}}\right) \leq C\epsilon|\ln\epsilon|
\end{aligned}
$$

and

$$\sum_{i\neq j}^{m} e^{-\frac{\min\{\sqrt{\lambda_{i,\epsilon}},\sqrt{\lambda_{j,\epsilon}}\}|z_{i,\epsilon}-z_{j,\epsilon}|}{\epsilon}} \leq C\epsilon|\ln\epsilon|,$$

$$\sum_{i\neq j}^{k} e^{-\frac{\min\{\sqrt{\lambda_{i,\epsilon}},\sqrt{\lambda_{j,\epsilon}}\}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}} \leq C\epsilon|\ln\epsilon|,$$

which imply that

$$\sum_{i\neq j}^{m} \frac{|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon} \geq C|\ln\epsilon| \geq |\ln\epsilon|^{\frac{1}{2}}, \quad \sum_{i\neq j}^{k} \frac{|z_{i,\epsilon}-z_{j,\epsilon}|}{\epsilon} \geq C|\ln\epsilon| \geq |\ln\epsilon|^{\frac{1}{2}}.$$

Therefore, $(\mathbf{x}_\epsilon, \mathbf{z}_\epsilon)$ is an interior point of $D_1 \times D_2$, which implies that

$$(u_\epsilon, v_\epsilon) = \left(\sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon},\mu} + \overline{\varphi}_\epsilon, \sum_{j=1}^{k} U_{\epsilon,z_{j,\epsilon},\nu} + \overline{\psi}_\epsilon\right)$$

is a critical point of $\overline{K}(\mathbf{x}, \mathbf{z})$. So, (1.1) has a solution of the form

$$u_\epsilon = \sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon},\mu} + \overline{\varphi}_\epsilon, \quad v_\epsilon = \sum_{j=1}^{k} U_{\epsilon,z_{j,\epsilon},\nu} + \overline{\psi}_\epsilon$$

for some $x_{j,\epsilon} \in B_\theta(p_j), \;\; z_{\epsilon,j} \in B_\theta(\overline{p}_j)$ and $\|(\overline{\varphi}_\epsilon, \overline{\psi}_\epsilon)\|_\epsilon = O(\epsilon^{\frac{N}{2}+1})$. □

Author Contributions All authors contribute equally to this paper. All authors read and approved the final manuscript.

Funding Maoding Zhen was supported by the NSFC (Grant No. 12201167) and the Fundamental Research Funds for the Central University of China (Grant No. JZ2021HGTA0177, JZ2022HGQA0155). Binlin Zhang was supported by the National Natural Science Foundation of China (Grant No. 11871199), the Heilongjiang Province Postdoctoral Startup Foundation, PR China (Grant No. LBH-Q18109), and the Cultivation Project of Young and Innovative Talents in Universities of Shandong Province. The research of Vicenţiu D. Rădulescu was supported by a grant of the Romanian Ministry of Research, Innovation and Digitization, CNCS/CCCDI-UEFISCDI, project number PCE 137/2021, within PNCDI III.

Data Availability This paper has no associated data and material.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Appendix A: Error Estimate

Proposition 5.1 *There exist positive constants C_1, C_2, C_3 and C_4, such that*

$$I\left(\sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon}}, \sum_{j=1}^{k} V_{\epsilon,x_{j,\epsilon}}\right) = \left(\frac{1}{2}-\frac{1}{p+1}\right) k\epsilon^N \lambda^{1-\frac{N}{2}} A$$
$$+ C\epsilon^N \left(\sum_{j=1}^{k} \epsilon(P(x_{j,\epsilon})-\lambda) + \sum_{j=1}^{k} \epsilon(Q(x_{j,\epsilon})-\lambda) + \epsilon^2\right)$$
$$- C\epsilon^N \sum_{i\neq j}^{k} e^{-\frac{\sqrt{\lambda}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}} + O(\epsilon^N \sum_{i\neq j}^{k} e^{-\frac{2(\sqrt{\lambda}-\sigma)|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}),$$

where $A = (a_\lambda^2 + b_\lambda^2)\int_{\mathbb{R}^N} W^{p+1}dx$.

Proof By the elementary computations, we have

$$I\left(\sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon}}, \sum_{j=1}^{k} V_{\epsilon,x_{j,\epsilon}}\right)$$
$$= \left(\frac{1}{2}-\frac{1}{p+1}\right)\sum_{j=1}^{k}\int_{\mathbb{R}^N}(\mu U_{\epsilon,x_{j,\epsilon}}^{p+1} + \nu V_{\epsilon,x_{j,\epsilon}}^{p+1} + 2\beta U_{\epsilon,x_{j,\epsilon}}^{\frac{p+1}{2}} V_{\epsilon,x_{j,\epsilon}}^{\frac{p+1}{2}})dx$$
$$+ \frac{1}{2}\int_{\mathbb{R}^N}\sum_{j=1}^{k}\Big((P(x)-P(x_0))U_{\epsilon,x_{j,\epsilon}}^2 + (Q(x)-Q(x_0))V_{\epsilon,x_{j,\epsilon}}^2\Big)dx$$
$$+ \frac{1}{2}\int_{\mathbb{R}^N}\sum_{j\neq i}^{k}\Big((P(x)-P(x_0))U_{\epsilon,x_{j,\epsilon}}U_{\epsilon,x_{\epsilon,i}} + (Q(x)-Q(x_0))V_{\epsilon,x_{j,\epsilon}}V_{\epsilon,x_{\epsilon,i}}\Big)dx$$
$$- \frac{\mu}{p+1}\int_{\mathbb{R}^N}\left((\sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon}})^{p+1} - \sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon}}^{p+1} - \frac{p+1}{2}\sum_{j\neq i}^{k} U_{\epsilon,x_{j,\epsilon}}^{p} U_{\epsilon,x_{\epsilon,i}}\right)dx$$
$$- \frac{\nu}{p+1}\int_{\mathbb{R}^N}\left((\sum_{j=1}^{k} V_{\epsilon,x_{j,\epsilon}})^{p+1} - \sum_{j=1}^{k} V_{\epsilon,x_{j,\epsilon}}^{p+1} - \frac{p+1}{2}\sum_{j\neq i}^{k} V_{\epsilon,x_{j,\epsilon}}^{p} V_{\epsilon,x_{\epsilon,i}}\right)dx$$
$$- \frac{2\beta}{p+1}\int_{\mathbb{R}^N}\left((\sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon}})^{\frac{p+1}{2}}(\sum_{j=1}^{k} V_{\epsilon,x_{j,\epsilon}})^{\frac{p+1}{2}} - \sum_{j=1}^{k} U_{\epsilon,x_{j,\epsilon}}^{\frac{p+1}{2}} V_{\epsilon,x_{j,\epsilon}}^{\frac{p+1}{2}}\right)dx$$
$$+ \frac{2\beta}{p+1}\int_{\mathbb{R}^N}\frac{p+1}{4}\left(\sum_{j\neq i}^{k} U_{\epsilon,x_{j,\epsilon}}^{\frac{p-1}{2}} V_{\epsilon,x_{\epsilon,i}}^{\frac{p+1}{2}} U_{\epsilon,x_{\epsilon,i}} + \sum_{j\neq i}^{k} U_{\epsilon,x_{j,\epsilon}}^{\frac{p+1}{2}} V_{\epsilon,x_{\epsilon,i}}^{\frac{p-1}{2}} V_{\epsilon,x_{\epsilon,i}}\right)dx.$$

By the definition of $U_{\epsilon,x_{j,\epsilon}}$ and $V_{\epsilon,x_{j,\epsilon}}$, we get

$$\left(\frac{1}{2}-\frac{1}{p+1}\right)\sum_{j=1}^{k}\int_{\mathbb{R}^N}(\mu U_{\epsilon,x_{j,\epsilon}}^{p+1}+\nu V_{\epsilon,x_{j,\epsilon}}^{p+1}+2\beta U_{\epsilon,x_{j,\epsilon}}^{\frac{p+1}{2}}V_{\epsilon,x_{j,\epsilon}}^{\frac{p+1}{2}})dx$$
$$=\left(\frac{1}{2}-\frac{1}{p+1}\right)k\epsilon^N\lambda^{1-\frac{N}{2}}(a_\lambda^2+b_\lambda^2)\int_{\mathbb{R}^N}W^{p+1}dx.$$

From (2.42), we know

$$\frac{1}{2}\int_{\mathbb{R}^N}\sum_{j=1}^{k}\Big((P(x)-P(x_0))U_{\epsilon,x_{j,\epsilon}}^2+(Q(x)-Q(x_0))V_{\epsilon,x_{j,\epsilon}}^2\Big)dx$$
$$+\frac{1}{2}\int_{\mathbb{R}^N}\sum_{j\neq i}^{k}\Big((P(x)-P(x_0))U_{\epsilon,x_{j,\epsilon}}U_{\epsilon,x_{\epsilon,i}}+(Q(x)-Q(x_0))V_{\epsilon,x_{j,\epsilon}}V_{\epsilon,x_{\epsilon,i}}\Big)dx$$
$$=C\epsilon^N\left(\sum_{j=1}^{k}\epsilon(P(x_{j,\epsilon})-\lambda)+\sum_{j=1}^{k}\epsilon(Q(x_{j,\epsilon})-\lambda)+\epsilon^2\right).$$

Since $U_{\epsilon,x_{j,\epsilon}}(x)\leq Ce^{-\frac{\sqrt{\alpha}|x-x_{j,\epsilon}|}{\epsilon}}$, $V_{\epsilon,x_{j,\epsilon}}(x)\leq Ce^{-\frac{\sqrt{\alpha}|x-x_{j,\epsilon}|}{\epsilon}}$, by the similar argument as (2.43)–(2.45) we have there exists a $\sigma>0$ such that

$$\frac{\mu}{p+1}\int_{\mathbb{R}^N}\left((\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}})^{p+1}-\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}}^{p+1}-\frac{p+1}{2}\sum_{j\neq i}^{k}U_{\epsilon,x_{j,\epsilon}}^{p}U_{\epsilon,x_{i,\epsilon}}\right)dx$$
$$=C\epsilon^N\sum_{i\neq j}^{k}e^{-\frac{\sqrt{\alpha}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}+O(\epsilon^N\sum_{i\neq j}^{k}e^{-\frac{2(\sqrt{\alpha}-\sigma)|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}),$$

$$\frac{\mu}{p+1}\int_{\mathbb{R}^N}\left((\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}})^{p+1}-\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}}^{p+1}-\frac{p+1}{2}\sum_{j\neq i}^{k}U_{\epsilon,x_{j,\epsilon}}^{p}U_{\epsilon,x_{i,\epsilon}}\right)dx$$
$$=C\epsilon^N\sum_{i\neq j}^{k}e^{-\frac{\sqrt{\alpha}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}+O(\epsilon^N\sum_{i\neq j}^{k}e^{-\frac{2(\sqrt{\alpha}-\sigma)|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}),$$

$$-\frac{2\beta}{p+1}\int_{\mathbb{R}^N}\left((\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}})^{\frac{p+1}{2}}(\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}})^{\frac{p+1}{2}}-\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}}^{\frac{p+1}{2}}V_{\epsilon,x_{j,\epsilon}}^{\frac{p+1}{2}}\right)dx$$
$$+\frac{2\beta}{p+1}\int_{\mathbb{R}^N}\frac{p+1}{4}\left(\sum_{j\neq i}^{k}U_{\epsilon,x_{j,\epsilon}}^{\frac{p-1}{2}}V_{\epsilon,x_{i,\epsilon}}^{\frac{p+1}{2}}U_{\epsilon,x_{\epsilon,i}}+\sum_{j\neq i}^{k}U_{\epsilon,x_{j,\epsilon}}^{\frac{p+1}{2}}V_{\epsilon,x_{i,\epsilon}}^{\frac{p-1}{2}}V_{\epsilon,x_{\epsilon,i}}\right)dx$$
$$=C\epsilon^N\sum_{i\neq j}^{k}e^{-\frac{\sqrt{\alpha}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}+O(\epsilon^N\sum_{i\neq j}^{k}e^{-\frac{2(\sqrt{\alpha}-\sigma)|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}).$$

Thus,

$$I\left(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}},\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}}\right)=(\frac{1}{2}-\frac{1}{p+1})k\epsilon^N\lambda^{1-\frac{N}{2}}A$$
$$+C\epsilon^N\left(\sum_{j=1}^{k}(P(x_{j,\epsilon})-\lambda)+\sum_{j=1}^{k}(Q(x_{j,\epsilon})-\lambda)+\epsilon\right)$$
$$-C\epsilon^N\sum_{i\neq j}^{k}e^{-\frac{\sqrt{\alpha}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}+O(\epsilon^N\sum_{i\neq j}^{k}e^{-\frac{2(\sqrt{\alpha}-\sigma)|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}).$$

□

Proposition 5.2 *There exist positive constants C_1, C_2, C_3 and C_4, such that*

$$I\left(\sum_{j=1}^{k}U_{\epsilon,x_{j,\epsilon}},\sum_{j=1}^{k}V_{\epsilon,x_{j,\epsilon}}\right)$$
$$=\left(\frac{1}{2}-\frac{1}{p+1}\right)A\left(\sum_{j=1}^{k}\epsilon^N\lambda_j^{\frac{p+1}{p-1}-\frac{N}{2}}\mu^{-\frac{2}{p-1}}+\sum_{j=1}^{m}\epsilon^N\overline{\lambda}_j^{\frac{p+1}{p-1}-\frac{N}{2}}\nu^{-\frac{2}{p-1}}\right)$$
$$-C\epsilon^N\sum_{i\neq j}^{k}e^{-\frac{\min\{\sqrt{\lambda_i},\sqrt{\lambda_j}\}|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}+O\left(\epsilon^N\sum_{i\neq j}^{k}e^{-\frac{(2\min\{\sqrt{\lambda_i},\sqrt{\lambda_j}\}-\sigma)|x_{i,\epsilon}-x_{j,\epsilon}|}{\epsilon}}\right)$$
$$-C\epsilon^N\sum_{i\neq j}^{m}e^{-\frac{\min\{\sqrt{\overline{\lambda}_i},\sqrt{\overline{\lambda}_j}\}|z_{i,\epsilon}-z_{j,\epsilon}|}{\epsilon}}+O\left(\epsilon^N\sum_{i\neq j}^{m}e^{-\frac{(2\min\{\sqrt{\overline{\lambda}_i},\sqrt{\overline{\lambda}_j}\}-\sigma)|z_{i,\epsilon}-z_{j,\epsilon}|}{\epsilon}}\right)$$
$$-C\epsilon^N\sum_{i=1}^{k}\sum_{j=1}^{m}e^{-\frac{\min\{\sqrt{\lambda_i},\sqrt{\overline{\lambda}_j}\}|x_{i,\epsilon}-z_{j,\epsilon}|}{\epsilon}}+O\left(\sum_{i=1}^{k}\sum_{j=1}^{m}e^{-\frac{(2\min\{\sqrt{\lambda_i},\sqrt{\overline{\lambda}_j}\}-\sigma)|x_{i,\epsilon}-z_{j,\epsilon}|}{\epsilon}}\right)$$
$$+O(\epsilon^{N+1}),$$

where $A=\int_{\mathbb{R}^N}W^{p+1}dx$ and $\sigma>0$ is a small constant.

Proof By direct computations, we have

$$I\left(\sum_{j=1}^{k}U_{\epsilon,x_j,\mu},\sum_{j=1}^{k}V_{\epsilon,z_j,\nu}\right)$$
$$=\left(\frac{1}{2}-\frac{1}{p+1}\right)\sum_{j=1}^{k}\int_{\mathbb{R}^N}\mu U_{\epsilon,x_j,\mu}^{p+1}+\left(\frac{1}{2}-\frac{1}{p+1}\right)\sum_{j=1}^{m}\int_{\mathbb{R}^N}\nu U_{\epsilon,z_j,\nu}^{p+1}$$
$$+\frac{1}{2}\int_{\mathbb{R}^N}\sum_{j=1}^{k}\left((P(x)-P(x_j))U_{\epsilon,x_j,\mu}^2\right)+\sum_{j=1}^{m}\left((Q(x)-Q(z_j))U_{\epsilon,z_j,\nu}^2\right)dx$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{\mathbb{R}^N}\sum_{j\neq i}^{k}\Big((P(x)-P(x_j))U_{\epsilon,x_j,\mu}U_{\epsilon,x_i,\mu}\Big)+\sum_{j\neq i}^{m}(Q(x)-Q(z_j))U_{\epsilon,z_j,\nu}U_{\epsilon,z_i,\nu}dx\\
&-\frac{\mu}{p+1}\int_{\mathbb{R}^N}\left((\sum_{j=1}^{k}U_{\epsilon,x_j,\mu})^{p+1}-\sum_{j=1}^{k}U_{\epsilon,x_j,\mu}^{p+1}-\frac{p+1}{2}\sum_{j\neq i}^{k}U_{\epsilon,x_j,\mu}^{p}U_{\epsilon,x_i,\mu}\right)dx\\
&-\frac{\nu}{p+1}\int_{\mathbb{R}^N}\left((\sum_{j=1}^{k}U_{\epsilon,z_j,\nu})^{p+1}-\sum_{j=1}^{k}U_{\epsilon,z_j,\nu}^{p+1}-\frac{p+1}{2}\sum_{j\neq i}^{k}U_{\epsilon,z_j,\nu}^{p}U_{\epsilon,z_i,\nu}\right)dx\\
&-\frac{2\beta}{p+1}\int_{\mathbb{R}^N}\left((\sum_{j=1}^{k}U_{\epsilon,x_j,\mu})^{\frac{p+1}{2}}(\sum_{j=1}^{m}U_{\epsilon,z_j,\nu})^{\frac{p+1}{2}}\right)dx.
\end{aligned}
$$

By the definition of $U_{\epsilon,x_j,\mu}$ and $U_{\epsilon,z_j,\nu}$, we get

$$
\begin{aligned}
&\left(\frac{1}{2}-\frac{1}{p+1}\right)\sum_{j=1}^{k}\int_{\mathbb{R}^N}\mu U_{\epsilon,x_j,\mu}^{p+1}+(\frac{1}{2}-\frac{1}{p+1})\sum_{j=1}^{m}\int_{\mathbb{R}^N}\nu U_{\epsilon,z_j,\nu}^{p+1}dx\\
&=\left(\frac{1}{2}-\frac{1}{p+1}\right)\sum_{j=1}^{k}\epsilon^N\lambda_j^{\frac{p+1}{p-1}-\frac{N}{2}}\mu^{-\frac{2}{p-1}}\int_{\mathbb{R}^N}W^{p+1}dx\\
&\quad+\left(\frac{1}{2}-\frac{1}{p+1}\right)\sum_{j=1}^{m}\epsilon^N\overline{\lambda}_j^{\frac{p+1}{p-1}-\frac{N}{2}}\nu^{-\frac{2}{p-1}}\int_{\mathbb{R}^N}W^{p+1}dx.
\end{aligned}
$$

By the similar arguments as (2.42), we can get

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N}\sum_{j=1}^{k}\Big((P(x)-P(x_j))U_{\epsilon,x_j,\mu}^2\Big)+\sum_{j=1}^{m}\Big((Q(x)-Q(z_j))U_{\epsilon,z_j,\nu}^2\Big)dx\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^N}\sum_{j\neq i}^{k}\Big((P(x)-P(x_j))U_{\epsilon,x_j,\mu}U_{\epsilon,x_i,\mu}\Big)+\sum_{j\neq i}^{m}(Q(x)-Q(z_j))U_{\epsilon,z_j,\nu}U_{\epsilon,z_i,\nu}dx\\
&=C\epsilon^N\left(\sum_{i\neq j}^{k}(P(x_i)-P(x_j))+\sum_{i\neq j}^{m}(Q(x_i)-Q(x_j))+\epsilon\right)=O(\epsilon^{N+1}).
\end{aligned}
$$

Since $U_{\epsilon,x_j,\mu}\le Ce^{-\frac{\sqrt{\lambda_i}|x-x_j|}{\epsilon}}, U_{\epsilon,z_j,\nu}\le Ce^{-\frac{\sqrt{\overline{\lambda}_i}|x-z_j|}{\epsilon}}$, where $\lambda_i=P(x_i)$, $\overline{\lambda}_i=Q(z_i)$, by the similar arguments as (2.43)–(2.45), we have

$$
\begin{aligned}
&\frac{\mu}{p+1}\int_{\mathbb{R}^N}\left((\sum_{j=1}^{k}U_{\epsilon,x_j,\mu})^{p+1}-\sum_{j=1}^{k}U_{\epsilon,x_j,\mu}^{p+1}-\frac{p+1}{2}\sum_{j\neq i}^{k}U_{\epsilon,x_j,\mu}^{p}U_{\epsilon,x_i,\mu}\right)dx\\
&\quad=C\epsilon^N\sum_{i\neq j}^{k}e^{-\frac{\min\{\sqrt{\lambda_i},\sqrt{\lambda_j}\}|x_i-x_j|}{\epsilon}}+O\left(\epsilon^N\sum_{i\neq j}^{k}e^{-\frac{(2\min\{\sqrt{\lambda_i},\sqrt{\lambda_j}\}-\sigma)|x_i-x_j|}{\epsilon}}\right),
\end{aligned}
$$

$$\frac{\nu}{p+1}\int_{\mathbb{R}^N}\left((\sum_{j=1}^{k}U_{\epsilon,z_j,\nu})^{p+1}-\sum_{j=1}^{k}U_{\epsilon,z_j,\nu}^{p+1}-\frac{p+1}{2}\sum_{j\neq i}^{k}U_{\epsilon,z_j,\nu}^{p}U_{\epsilon,z_i,\nu}\right)dx$$
$$=C\epsilon^N\sum_{i\neq j}^{m}e^{-\frac{\min\{\sqrt{\lambda_i},\sqrt{\lambda_j}\}|z_i-z_j|}{\epsilon}}+O\left(\epsilon^N\sum_{i\neq j}^{m}e^{-\frac{(2\min\{\sqrt{\lambda_i},\sqrt{\lambda_j}\}-\sigma)|z_i-z_j|}{\epsilon}}\right),$$

$$\frac{2\beta}{p+1}\int_{\mathbb{R}^N}\left((\sum_{j=1}^{k}U_{\epsilon,x_j,\mu})^{\frac{p+1}{2}}(\sum_{j=1}^{m}U_{\epsilon,z_j,\nu})^{\frac{p+1}{2}}\right)dx$$
$$=C\epsilon^N\sum_{i=1}^{k}\sum_{j=1}^{m}e^{-\frac{\min\{\sqrt{\lambda_i},\sqrt{\lambda_j}\}|x_i-z_j|}{\epsilon}}+O\left(\sum_{i=1}^{k}\sum_{j=1}^{m}e^{-\frac{(2\min\{\sqrt{\lambda_i},\sqrt{\lambda_j}\}-\sigma)|x_i-z_j|}{\epsilon}}\right).$$

Thus, by the above computations, we obtain the desired conclusions. □

References

1. Alotaibi, T., Jleli, M., Samet, B., Vetro, C.: First and second critical exponents for an inhomogeneous Schrödinger equation with combined nonlinearities. Z. Angew. Math. Phys. **73**, 17 (2022). https://doi.org/10.1007/s00033-022-01784-y
2. Bartsch, T., Soave, N.: A natural constraint approach to normalized solutions of nonlinear Schrödinger equations and systems. J. Funct. Anal. **272**, 4304–4333 (2017)
3. Bartsch, T., Soave, N.: Multiple normalized solutions for a competing system of Schrödinger equations. Calc. Var. Partial Differ. Equ. **58**, 22–24 (2019)
4. Bartsch, T., Wang, Z., Wei, J.: Bound states for a coupled Schrödinger system. J. Fixed Point Theory Appl. **2**, 353–367 (2007)
5. Bartsch, T., Dancer, N., Wang, Z.: A Liouville theorem, a-priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system. Calc. Var. Partial Differ. Equ. **2**, 345–361 (2010)
6. Benboubke, M.B., et al.: Entropy solutions for elliptic Schrödinger type equations under Fourier boundary conditions. Rend. Circ. Mat. Palermo II Ser. (2022). https://doi.org/10.1007/s12215-022-00822-y
7. Berestycki, H., Terracini, S., Wang, K., Wei, J.: On entire solutions of an elliptic system modeling phase seperations. Adv. Math. **243**, 102–126 (2013)
8. Cao, D., Noussair, E.S.: Multiplicity of positive and nodal solutions for nonlinear elliptic problems in $\mathbb{R}^N$. Ann. Inst. Henri Poincaré Analyse Non Linéaire **13**, 567–588 (1996)
9. Cao, D., Noussair, E.S., Yan, S.: Existence and uniqueness results on single-peaked solutions of a semilinear problem. Ann. Inst. Henri Poincaré, Analyse Non Linéaire **15**, 73–111 (1998)
10. Cao, D., Noussair, E.S., Yan, S.: Solutions with multiple "peaks" for nonlinear elliptic equations. Proc. R. Soc. Edinb. Sect. A **129**, 235–264 (1999)
11. Floer, A., Weinstein, A.: Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential. J. Funct. Anal. **69**, 397–408 (1986)
12. Goubet, O., Manoubi, I.: Standing waves for semilinear Schrödinger equations with discontinuous dispersion. Rend. Circ. Mat. Palermo II Ser **71**, 11591171 (2022). https://doi.org/10.1007/s12215-022-00782-3
13. Guo, Y., Li, B., Wei, J.: Entire nonradial solutions for non-cooperative coupled elliptic system with critical exponent in $\mathbb{R}^3$. J. Differ. Equ. **256**, 3463–3495 (2014)
14. Guo, Y., Luo, S., Zou, W.: The existence, uniqueness and nonexistence of the ground state to the N-coupled Schrödinger systems in $\mathbb{R}^4(n \leq 4)$. Nonlinearity **31**, 314–339 (2018)

15. He, Q., Peng, S., Peng, Y.: Existence, non-degeneracy of proportional positive solutions and least energy solutions for a fractional elliptic system. Adv. Differ. Equ. **22**, 867–892 (2017)
16. Kwong, M.: Uniqueness of positive solutions of $\Delta u - u + u^p = 0 \text{in} \mathbb{R}^n$. Arch. Ration. Mech. Anal. **105**, 243–266 (1989)
17. Lin, T., Wei, J.: Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \leq 3$. Commun. Math. Phys. **255**, 629–653 (2005)
18. Lin, T., Wei, J.: Spikes in two coupled nonlinear Schrödinger equations. Ann. Inst. H. Poincaré Analyse Non Linéaire **22**, 403–439 (2005)
19. Lin, T., Wei, J.: Spikes in two-component systems of nonlinear Schrödinger equations with trapping potential. J. Differ. Equ. **229**, 538–569 (2006)
20. Long, W., Tang, Z., Yang, S.: Many synchronized vector solutions for a Bose–Einstein system. Proc. R. Soc. Edinb. Sect. A **150**, 3293–3320 (2020)
21. Noussair, E., Yan, S.: On positive multipeak solutions of nonlinear elliptic problem. J. Lond. Math. Soc. **62**, 213–227 (2000)
22. Peng, S., Pi, H.: Spike vector solutions for some coupled nonlinear Schrödinger equations. Discrete Contin. Dyn. Syst. **36**, 2205–2227 (2016)
23. Peng, S., Wang, Z.: Segregated and synchronized vector solutions for nonlinear Schrödinger systems. Arch. Rational. Mech. Anal. **208**, 305–339 (2013)
24. Peng, S., Peng, Y., Wang, Z.: On critical systems with Sobolev critical growth. Calc. Var. Partial Differ. Equ. **55**, 142 (2016)
25. Peng, S., Wang, Q., Wang, Z.: On coupled nonlinear Schrödinger systems with mixed couplings. Trans. Am. Math. Soc. **371**, 7559–7583 (2019)
26. Sirakov, B.: Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$. Commun. Math. Phys. **271**, 199–221 (2007)
27. Tang, Z., Xie, H.: Multi-scale spike solutions for nonlinear coupled elliptic systems with critical frequency. Nonlinear Differ. Equ. Appl. **28**, 25 (2021)
28. Wei, J., Weth, T.: Nonradial symmetric bound states for a system of two coupled Schrödinger equations. Rend. Lincei Mat. Appl. **18**, 279–293 (2007)
29. Wei, J., Weth, T.: Radial solutions and phase separation in a system of two Schrödinger equations. Arch. Ration. Mech. Anal. **190**, 83–106 (2008)
30. Wei, J., Wu, Y.: Ground state of nonlinear Schrödinger systems with mixed couplings. J. Math. Pures Appl. **141**, 50–88 (2020)
31. Wei, J., Yan, S.: Infinitely many solutions for the prescribed scalar curvature problem on S^n. J. Funct. Anal. **258**, 3048–3081 (2010)
32. Wei, J., Yan, S.: Infinitely many solutions for the nonlinear Schrödinger equations in $\mathbb{R}^N$. Calc. Var. Partial Differ. Equ. **37**, 423–439 (2010)
33. Wei, J., Yao, W.: Uniqueness of positive solutions to some coupled nonlinear Schrödinger equations. Commun. Pure Appl. Anal. **11**, 1003–1011 (2012)
34. Zhen, M., Zhang, B., Rădulescu, V.: Normalized solutions for nonlinear coupled fractional systems: low and high perturbations in the attractive case. Discrete Contin. Dyn. Syst. **41**, 2653–2676 (2021)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.