Wong–Zakai Approximation for Landau–Lifshitz–Gilbert Equation Driven by Geometric Rough Paths

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Abstract

We adapt Lyon’s rough path theory to study Landau–Lifshitz–Gilbert equations (LLGEs) driven by geometric rough paths in one dimension, with non-zero exchange energy only. We convert the LLGEs to a fully nonlinear time-dependent partial differential equation without rough paths term by a suitable transformation. Our point of interest is the regular approximation of the geometric rough path. We investigate the limit equation, the form of the correction term, and its convergence rate in controlled rough path spaces. The key ingredients for constructing the solution and its corresponding convergence results are the Doss–Sussmann transformation, maximal regularity property, and the geometric rough path theory.

Keywords Rough paths theory · Partial differential equation · Landau–Lifshitz–Gilbert equations · Wong–Zakai approximation · Ferromagnetism

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1 Introduction

The stochastic Landau–Lifshitz–Gilbert equations (SLLGEs) describe the behaviour of the magnetisation under the influence of the randomly fluctuating effective field. In this work, we consider the SLLGEs with solutions taking values in the two-dimensional sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. Let D be an open bounded interval of the real line. For simplicity, let D be the interval $(0, 1)$. Let $\mathcal{H}_{\text{eff}}$ denote the L^2 -gradient of the energy functional $\mathcal{E}$ and is called the effective field, i.e. $\mathcal{H}_{\text{eff}}(u) = -\nabla_u \mathcal{E}(u)$. Let $\mathbf{M} : [0, T] \times D \rightarrow \mathbb{R}^3$ denote the magnetisation of a ferromagnetic material occupying the domain D . The deterministic Landau–Lifshitz–Gilbert equations take the form

$$\begin{cases} \frac{\partial \mathbf{M}}{\partial t} = \lambda_1 \mathbf{M} \times \mathcal{H}_{\text{eff}}(\mathbf{M}) - \lambda_2 \mathbf{M} \times (\mathbf{M} \times \mathcal{H}_{\text{eff}}(\mathbf{M})), & t \in (0, T), x \in D, \\ \mathbf{M}_x(t, 0) = 0 = \mathbf{M}_x(t, 1), & t \in (0, T), \\ \mathbf{M}(0, \cdot) = \mathbf{M}_0, & x \in D. \end{cases} \quad (1)$$

The parameters $\lambda_1 \neq 0, \lambda_2 > 0$ are real constants. We assume that the material is saturated at the initial time, i.e.

$$|\mathbf{M}_0(x)|_{\mathbb{R}^3} = 1 \quad \text{for a.e. } x \in D. \quad (2)$$

If both anisotropy and the exchange energies are present (see Visintin [30] and Cimrák [11]), the total magnetic energy $\mathcal{E}$ of the LLGEs is given by

$$\mathcal{E}(\mathbf{M}) = \mathcal{E}_{\text{an}}(\mathbf{M}) + \mathcal{E}_{\text{ex}}(\mathbf{M}) = \int_D \left(\psi(\mathbf{M}(x)) + \frac{1}{2} |\nabla \mathbf{M}(x)|^2 \right) dx,$$

where $\mathcal{E}_{\text{an}}(\mathbf{M}) := \int_D \psi(\mathbf{M}(x)) dx$ is the anisotropy energy and $\mathcal{E}_{\text{ex}}(\mathbf{M}) := \int_D |\nabla \mathbf{M}(x)|^2 dx$ is the exchange energy. The effective field $\mathcal{H}_{\text{eff}}(\mathbf{M})$ takes the form $\Delta \mathbf{M} - \nabla \psi(\mathbf{M})$. In such case, the total magnetic energy is given by

$$\mathcal{E} = \frac{1}{2} \int_D |\nabla \mathbf{M}(x)|^2 dx$$

and we obtain $\mathcal{H}_{\text{eff}}(\mathbf{M}) = \Delta \mathbf{M}$. In particular, the LLGEs is given by

$$\begin{cases} \frac{\partial \mathbf{M}}{\partial t} = \lambda_1 \mathbf{M} \times \Delta \mathbf{M} - \lambda_2 \mathbf{M} \times (\mathbf{M} \times \Delta \mathbf{M}), & \text{in } (0, T) \times D, \\ \frac{\partial \mathbf{M}}{\partial n} = 0, & \text{on } (0, T) \times \partial D, \\ \mathbf{M}(0, \cdot) = \mathbf{M}_0, & \text{in } D. \end{cases} \quad (3)$$

where $\mathbf{n}$ is the unit outward normal vector field on the boundary ∂D and $\frac{\partial \mathbf{M}}{\partial n}$ is the directional derivative of $\mathbf{M}$ in the direction $\mathbf{n}$. Here we assume that the material is saturated at the initial time.

It is well-known that the stationary solutions of system (1) correspond to the equilibrium states of the ferromagnet and are not unique in general. An interesting question in the theory of ferromagnetism is to describe phase transitions between different equilibrium states induced by thermal fluctuations of the field $\mathcal{H}_{\text{eff}}$. Thus, conventionally,

randomly fluctuating fields act responsible for magnetization fluctuations, see Neel [29]. It essentially characterizes deviations from the average magnetization trajectory in an ensemble of noninteracting nanoparticles, see Brown in [2]. According to Brown, the magnetization $\mathbf{M}$ evolves randomly, and for the stochastic version of system (1), one needs to modify the system in order to incorporate random fluctuations of the field $\mathcal{H}_{\text{eff}}$ into the dynamics of $\mathbf{M}$ and to describe noise-induced transitions between equilibrium states of the ferromagnet. In this context, we refer to [3,4] for further clarification of adding noise to the effective field $\mathcal{H}_{\text{eff}}$.

The theory of rough paths was initiated by Terry Lyons in his seminal work [26] as an extension of the classical theory of controlled differential equations. Since its introduction, the theory of ordinary and partial differential equations driven by rough paths has developed intensively. We refer the reader to study the papers of Gubinelli et al. [13,16–18], Friz et al. [9,10,14]. We quote the recent preprint [20], where using rough path formulation, existence, uniqueness and regularity for the SLLGE with Stratonovich noise on the one-dimensional torus has been studied.

In this paper, we are interested in the following form of the LLGEs equation:

$$\begin{cases} d\mathbf{M} = (\lambda_1 \mathbf{M} \times \Delta \mathbf{M} - \lambda_2 \mathbf{M} \times (\mathbf{M} \times \Delta \mathbf{M})) dt + (\mathbf{M} \times g) dX, & \text{in } (0, T) \times D \\ \frac{\partial \mathbf{M}}{\partial n}(t, x) = 0, & \text{for } (t, x) \in (0, \infty) \times \partial D, \\ \mathbf{M}(0, x) = \mathbf{M}_0(x), & \text{for } x \in D, \\ |\mathbf{M}_0(x)|_{\mathbb{R}^3} = 1, & \text{for a.e. } x \in D, \end{cases} \quad (4)$$

where X is a geometric rough path. We adapt Lyons' rough paths theory to study system (1) driven by an irregular noise. By proposing a suitable transformation, we convert system (1) to a fully nonlinear time-dependent partial differential equation without rough paths term. The primary interest is the regular approximation of the geometric rough path.

The rough path theory is informally connected to Wong–Zakai results as it necessarily allows to construct solutions as limits of Wong–Zakai type approximations, for references, see, e.g. the seminal paper by Lyons [26], and the recent monograph by Lyons et al. [27]. There are various works on rough PDEs where weak solutions are constructed by exploiting energy arguments. Here in our work, we applied this way of constructing a solution. For a quick survey, we refer to Bailleul et al. [1], Deya et al. [12], Hocquet et al. [21], Hofmanová et al. [22]. We refer the readers to Section 1.5 in [8] for detailed discussion.

In [19], the authors have studied equations similar to SLLGEs in $\mathbb{R}^d$, for any $d > 0$. However, the noise corresponds to a choice of g in equation (4) to be constant across the domain D , i.e., it can be independent of space and time variables. It is not clear how the concept of solution is defined. Brzeźniak et al. in [3] have proved existence of weak martingale solutions of SLLGEs taking values in a sphere $\mathbb{S}^2$. Furthermore, Li et al. in [5] have generalized the results in [3] with non-zero anisotropic energy $\mathcal{E}_{an}$ and multidimensional noise. Finite dimensional analysis of this problem has been discussed in [23], [25]. Brzeźniak et al. have studied in [4] the one-dimensional case and prove the large deviations principle to the SLLGEs for small noise asymptotic. Brzeźniak et al. have proved in [6,7] existence of weak martingale solution for SLLGEs in three dimensions perturbed by jump noise in the Marcus canonical form with non-

zero anisotropic energy $\mathcal{E}_{an}$, see [6] and non-zero exchange energy $\mathcal{E}_{ex}$ only, see [7]. Recently, in [8,28], the authors have employed Wong–Zakai approximation technique to obtain the solvability and convergence of the time dependent transformed PDEs. The open questions framed in [8] have motivated us, as a first step to adapt Lyons' rough paths theory to study LLGEs driven by geometric rough paths in one dimension.

As a particular case, we can take Stratonovich Brownian rough paths as geometric rough paths. Furthermore, we plan to extend our techniques to a more general setting when the stochastic process is no longer one-dimensional (and nor the corresponding vector fields commute). The motivation to use geometric rough path comes from [24], where they discuss the heat equation with a geometric rough path.

1.1 Problem Description

We introduce some notations and summarize the most important definition of rough paths which are taken from [14].

Notations For a domain $D \subset \mathbb{R}^d$ with $d = 1, 2, 3$, we will use the notation $\mathbb{L}^p$ for the space $L^p(D; \mathbb{R}^3)$ and $\mathbb{W}^{m,p}$ for the Sobolev space $W^{m,p}(D; \mathbb{R}^3)$. We will often write $\mathbb{H}^m$ instead of $\mathbb{W}^{m,2}$. We will also denote, for a Banach space V , by $\mathcal{C}^\alpha([0, T], V)$ the space of all α -Hölder continuous functions $f : [0, T] \rightarrow V$ with $\alpha \in [0, 1]$. We define the path increment $f_{s,t} = f_t - f_s$ where $f_t := f(t)$ (we will write $\dot{f}$ for $\frac{df}{dt}$). For $f \in \mathcal{C}^\alpha([0, T], V)$, the usual α -Hölder semi-norm is given by

$$|f|_{\alpha,[0,T],V} := \sup_{s \neq t \in [0,T]} \frac{|f_{s,t}|}{|t-s|^\alpha}.$$

For simplicity we write the semi-norm as $|f|_{\alpha,[0,T]}$ or $|f|_\alpha$, if there is no danger confusion.

A rough path on an interval $[0, T]$ with values in a Banach space V then consists of a continuous function $X : [0, T] \rightarrow V$, as well as a continuous “second order process” $\mathbb{X} : [0, T]^2 \rightarrow V \times V$, subjected to certain conditions which are given by Definition 3. Generically, we write $\mathcal{C}^\alpha([0, T], V)$ for the space of α -Hölder rough paths and $\mathcal{C}_g^\alpha([0, T], V)$ for the space of α -Hölder geometric rough paths over a Banach space V . For a Banach space H and $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, we denote the space of controlled rough paths by $\mathcal{D}_X^{2\alpha}([0, T], H)$. We note that $\mathcal{C}_g^\alpha([0, T], V)$ is a closed subset of $\mathcal{C}^\alpha([0, T], V)$. Hence, we can define $\mathcal{C}_g^{0,\alpha}([0, T], V)$ as the closure of lifts of smooth paths in $\mathcal{C}^\alpha([0, T], V)$. One has the obvious inclusion $\mathcal{C}_g^{0,\alpha}([0, T], V) \subset \mathcal{C}_g^\alpha([0, T], V)$. Furthermore, we recall that for any $\alpha \in (1/3, 1/2)$ with probability one, the corresponding Stratonovich lift is $\mathbf{B}^{\text{strat}} = (B, \mathbb{B}) \in \mathcal{C}_g^{0,\alpha}([0, T], \mathbb{R})$ where $\mathbb{B}_{s,t} := \int_s^t B_{s,r} \circ dB_r$. For more details about rough paths and their integration, we refer to Appendix 2 and the book [14].

In this paper, we consider the case $d = 1$ and $\alpha \in (\frac{1}{3}, \frac{1}{2})$ and assume that D to be a bounded open interval in $\mathbb{R}$. In particular, we take $D = (0, 1)$. The LLGEs in

consideration in this paper is of the form

$$\begin{cases} d\mathbf{M} = \left(\lambda_1 \mathbf{M} \times \mathbf{M}_{xx} - \lambda_2 \mathbf{M} \times (\mathbf{M} \times \mathbf{M}_{xx}) \right) dt \\ \quad + (\mathbf{M} \times \mathbf{g}) dX, \text{ in } (0, T) \times D, \\ \mathbf{M}_x(t, 0) = 0 = \mathbf{M}_x(t, 1), \forall t \in (0, T), \\ \mathbf{M}(0, \cdot) = \mathbf{M}_0 \text{ in } D, \end{cases} \quad (5)$$

where $T > 0$ is fixed, $\mathbf{g} : D \rightarrow \mathbb{R}^3$ is given function such that $\mathbf{g} \in \mathbb{W}^{4,\infty}$, and $X = (X, \mathbb{X}) \in \mathcal{C}_g^\alpha([0, T], \mathbb{R})$. We recall the fact that if $\tilde{X} = (X - X_0, \mathbb{X})$ then $\tilde{X} \in \mathcal{C}_g^\alpha([0, T], \mathbb{R})$ and $\int_0^t f d\tilde{X} = \int_0^t f dX$ for any $f \in \mathcal{D}_X^{2\alpha}([0, T], \mathbb{R}^3)$. Without loss of generality we assume that $X_0 = 0$.

One of the most fundamental questions related to problems similar to the above is whether the solutions depend in a continuous way of the coefficients (the geometric rough path in our case). Let us describe our approach to this question. We first recall that $\mathcal{C}_g^\alpha([0, T], \mathbb{R})$ is a closed subset of $\mathcal{C}^\alpha([0, T], \mathbb{R})$. Thus we can choose $X^{(n)} = (X^{(n)}, \mathbb{X}^{(n)})$, $n \in \mathbb{N}$ be a sequence in $\mathcal{C}_g^\alpha([0, T], \mathbb{R})$ such that

$$X^{(n)} \rightarrow X \text{ in } \mathcal{C}_g^\alpha([0, T], \mathbb{R}). \quad (6)$$

Let us now consider the corresponding system (7) approximating (5)

$$\begin{cases} d\mathbf{M}^{(n)} = \left(\lambda_1 \mathbf{M}^{(n)} \times \mathbf{M}_{xx}^{(n)} - \lambda_2 \mathbf{M}^{(n)} \times (\mathbf{M}^{(n)} \times \mathbf{M}_{xx}^{(n)}) \right) dt \\ \quad + (\mathbf{M}^{(n)} \times \mathbf{g}) dX^{(n)} \text{ in } (0, T) \times D, \\ \mathbf{M}_x^{(n)}(t, 0) = 0 = \mathbf{M}_x^{(n)}(t, 1), \quad \forall t \in (0, T), \\ \mathbf{M}^{(n)}(0, \cdot) = \mathbf{M}_0 \text{ in } D. \end{cases} \quad (7)$$

We note that if $X \in \mathcal{C}_g^{0,\alpha}([0, T], \mathbb{R}) \subset \mathcal{C}_g^\alpha([0, T], \mathbb{R})$ then one can choose $X^{(n)}$ to be a sequence of piecewise smooth path or (piecewise) C^1 paths and $\mathbb{X}_{s,t}^{(n)} = \int_s^t X_{s,r}^{(n)} dX_r^{(n)}$, see page 17 of [14]. Moreover, in the stochastic case one can take Stratonovich Brownian rough paths $\mathbf{B}^{\text{strat}}$ as X and also sequence $\mathbf{B}^{(n)}$ defined in Proposition 3.6 of [14] as $X^{(n)}$. For more details we refer to Sect. 4.1 and the book [14].

Our primary goal is to prove that the solution to the system (5) exists and is a unique strong solution in controlled rough path spaces, which indeed is a limit as $n \rightarrow \infty$ of the solutions of sequence of the corresponding system (7) approximating (5). In particular, we show the convergence with respect to the distance $d_{X, X^{(n)}, 2\alpha, [0, T]}$ (see Definition 4 in Appendix 2).

We now briefly describe the content of the paper. In Sect. 2, we introduce an auxiliary ODE and state some auxiliary facts necessary for the transformation of system (7) to a deterministic PDE without rough paths term. Sections 3 and 4 are devoted to the proof of the main results. In Sect. 3, we show how a unique weak solution to (7) can be obtained from a unique weak solution of the reformulated equation in controlled rough path spaces; see Theorem 1. Section 4 is devoted to the proof of the convergence

of solutions to the weak solution of the reformulated form; see Theorem 4. Finally, in the Appendix, for the reader's ease, we list several facts that are used in the course of the analysis. We split the Appendix into four subsections. First, in Appendix 1, we present several auxiliary lemmata which are essential to prove Theorem 1. Next, in Appendix 2 we handle some technical issues required in this paper, i.e., we introduce few results (see Lemmas 10, 11 and Corollary 5). These results, another contribution of the paper and have their own interest. In the last two subsections, we recall some simple results and algebraic identities used in this paper.

2 The Auxiliary Equations

In this section, we present some basic results on the the spaces and operators involved in the course of analysis. We also introduce new processes $\mathbf{m}$ and $\mathbf{m}^{(n)}$ gained by the Doss-Sussman transformation from the corresponding processes $\mathbf{M}$ and $\mathbf{M}^{(n)}$. We refer to [4,8,15] for further details about the properties of these processes.

2.1 Preliminaries

We define the Laplacian with the Neumann boundary conditions acting on $\mathbb{R}^3$ -valued functions by

$$\begin{aligned} D(A) &:= \{\mathbf{m} \in \mathbb{H}^2 : \mathbf{m}_x(0) = \mathbf{m}_x(1) = 0 \text{ on } \partial D\}, \\ A\mathbf{m} &:= -\Delta \mathbf{m} = -\mathbf{m}_{xx}, \quad \mathbf{m} \in D(A). \end{aligned}$$

We know that the unbounded operator A is self-adjoint in $\mathbb{L}^2$ and A_1^{-1} is compact for $A_1 := I + A$. Therefore, there exists an orthonormal basis $\{e_n\}_{n=1}^\infty$ of $\mathbb{L}^2$ consisting of eigenvectors of A . Furthermore, we know that, if $V := D(A_1^{1/2})$ is endowed with the graph norm, then V coincides with $\mathbb{H}^1$. Later on, we will write $V \hookrightarrow \mathbb{L}^2 \hookrightarrow V'$ which is a Gelfand triple.

We now present the following interpolation inequality which will be used in later sections.

$$|\mathbf{v}|_{\mathbb{L}^\infty}^2 \leq k^2 |\mathbf{v}|_{\mathbb{L}^2} |\mathbf{v}|_{\mathbb{H}^1}, \quad \forall \mathbf{v} \in \mathbb{H}^1. \quad (8)$$

Next, let us we recall an elementary result from [4].

Lemma 1 *Let $\mathbf{v} \in \mathbb{H}^1$ such that $|\mathbf{v}(x)|_{\mathbb{R}^3} = 1$ for all $x \in D$. Then in $(\mathbb{H}^1)'$, we have*

$$\mathbf{v} \times (\mathbf{v} \times \mathbf{v}_{xx}) = -|\mathbf{v}_x|_{\mathbb{R}^3}^2 \mathbf{v} - \mathbf{v}_{xx}. \quad (9)$$

Let us define the map $G : \mathbb{L}^2 \rightarrow \mathbb{L}^2$ by

$$[G\mathbf{u}](x) = \mathbf{u}(x) \times \mathbf{g}(x), \quad \forall \mathbf{u} \in \mathbb{L}^2, \quad x \in D. \quad (10)$$

Lemma 2 *Suppose $\mathbf{g} \in \mathbb{L}^\infty$. Then, G is a bounded linear map and well defined.*

In Appendix 3, Lemma 12, we list the properties of G . Let $\mathbb{H}^2(D; \mathbb{S}^2)$ be the set of all $\mathbb{R}^3$ -valued functions defined on the domain D belonging to the Sobolev space $\mathbb{H}^1 := H^1(D; \mathbb{R}^3)$ and satisfy the saturation condition (2). In particular, let

$$\mathbb{H}^2(D; \mathbb{S}^2) := \left\{ \mathbf{m} \in \mathbb{H}^2 \text{ such that } |\mathbf{m}(x)|_{\mathbb{R}^3} = 1 \text{ for a.a. } x \in D \right\}. \quad (11)$$

In other words, $\mathbb{H}^2(D; \mathbb{S}^2)$ is the set of all functions belonging to the Sobolev space $\mathbb{H}^2$ whose values are in the sphere. Since D is one-dimensional, $\mathbb{H}^2$ is embedded in $C(D; \mathbb{R}^3)$ and the ‘a.e.’ condition in (11) can be substituted by ‘all’.

2.2 The Doss–Sussmann Transformation and the Corresponding New Processes $\mathbf{m}$ and $\mathbf{m}^{(n)}$

Following the discussions in [8,15], we define a new process $\mathbf{m}$ from $\mathbf{M}$ by

$$\mathbf{m}(t, \mathbf{x}) := e^{-X(t)G} \mathbf{M}(t, \mathbf{x}) \quad \forall t \geq 0, \text{ a.e. } \mathbf{x} \in D, \quad (12)$$

where the operator G is introduced in (10). We know, by identity (91) and Lemma 12, that e^G , $G e^G$ and $G^2 e^G$ are bounded functions. Therefore, by Proposition 7.6 of [14], we obtain

$$e^{X(t)G} - e^{X(s)G} = \int_s^t G e^{X(r)G} dX_r, \quad (13a)$$

$$e^{-X(t)G} - e^{-X(s)G} = - \int_s^t G e^{-X(r)G} dX_r, \quad (13b)$$

and $e^{X(\cdot)G} \in \mathcal{D}_X^{2\alpha}([0, T], \mathcal{L}(\mathbb{L}^2))$. In Lemma 5, we will show that if a process $\mathbf{M}$ solves (5), then $\mathbf{m}$ defined by (12) solves the following non-linear time dependent PDE with random coefficients given below. Later on, in Sect. 3, we will show that there exists indeed a process $\mathbf{m}$ solving (14). Let us consider

$$\begin{cases} \dot{\mathbf{m}}(t) = \lambda_1 \mathbf{m}(t) \times \mathbf{m}_{xx}(t) - \lambda_2 \mathbf{m}(t) \times (\mathbf{m}(t) \times \mathbf{m}_{xx}(t)) \\ \quad + F(t, \mathbf{m}), \text{ in } (0, T) \times D, \\ \mathbf{m}_x(t, 0) = 0 = \mathbf{m}_x(t, 1), \quad \forall t \in (0, T), \\ \mathbf{m}(0, \cdot) = \mathbf{m}_0 \text{ in } D, \end{cases} \quad (14)$$

where F is given by

$$F(t, \mathbf{m}) = \lambda_1 \mathbf{m} \times \tilde{C}(X(t), \mathbf{m}(t, \cdot)) - \lambda_2 \mathbf{m} \times (\mathbf{m} \times \tilde{C}(X(t), \mathbf{m}(t, \cdot))). \quad (15)$$

Here, $\tilde{C}$ is an operator coming up in Lemma 15 and is given by

$$\tilde{C}(s, \mathbf{v}) := e^{-sG} ((\sin s)C + (1 - \cos s)(GC + CG))\mathbf{v}, \quad (16)$$

where $s \in \mathbb{R}$, $\mathbf{v} \in \mathbb{W}_0^{1,\infty}$ and C is an operator coming up in Lemma 14 and is given by

$$C\mathbf{v} = \mathbf{v} \times \mathbf{g}_{xx} + 2 \sum_{i=1}^d \frac{\partial \mathbf{v}}{\partial x_i} \times \frac{\partial \mathbf{g}}{\partial x_i}. \quad (17)$$

From the assumption $X(0) = 0$, we note that the initial condition (2) is equivalent to an analogous one for $\mathbf{m}_0$, i.e.

$$|\mathbf{m}_0(x)|_{\mathbb{R}^3} = 1 \quad \text{for a.e. } x \in D. \quad (18)$$

Note that, we could consider the transformation (12) for all $t \geq 0$. Moreover, as one can easily prove $|[e^{tG}(u)](x)|_{\mathbb{R}^3} = |u(x)|_{\mathbb{R}^3}$ for all $t \in [0, T]$, for a.a. $x \in D$, we see that the following saturation conditions for $\mathbf{M}$ and $\mathbf{m}$ are equivalent:

$$|\mathbf{M}(t, x)|_{\mathbb{R}^3} = 1 \quad \text{for a.e. } x \in D, \text{ for all } t \in [0, T], \quad (19)$$

$$|\mathbf{m}(t, x)|_{\mathbb{R}^3} = 1 \quad \text{for a.e. } x \in D, \text{ for all } t \in [0, T]. \quad (20)$$

For proof of the saturation condition (20), we refer to Lemma 3.15 of [8]. Repeating the same algebraic calculations as done in [15], i.e. using the definition of G and the identity (17), we obtain

$$\begin{aligned} GC\mathbf{m} + CG\mathbf{m} &= (\mathbf{m} \times \mathbf{g}_{xx}) \times \mathbf{g} + (\mathbf{m} \times \mathbf{g}) \times \mathbf{g}_{xx} + 2 \left((\mathbf{m}_x \times \mathbf{g}_x) \times \mathbf{g} \right. \\ &\quad \left. + (\mathbf{m}_x \times \mathbf{g}) \times \mathbf{g}_x + (\mathbf{m} \times \mathbf{g}_x) \times \mathbf{g}_x \right). \end{aligned} \quad (21)$$

Substituting (21) in (16), we get

$$F(t, \mathbf{m}) = \lambda_1 \mathbf{m} \times \tilde{C}(X(t), \mathbf{m}) - \lambda_2 \mathbf{m} \times (\mathbf{m} \times \tilde{C}(X(t), \mathbf{m})), \quad (22)$$

where $\tilde{C}(X(t), \mathbf{m})$ has now the following form

$$\begin{aligned} \tilde{C}(X(t), \mathbf{m}) &= e^{-X(t)G} \sin(X(t)) \left(\mathbf{m} \times \mathbf{g}_{xx} + 2\mathbf{m}_x \times \mathbf{g}_x \right) \\ &\quad + e^{-X(t)G} [1 - \cos(X(t))] \left((\mathbf{m} \times \mathbf{g}_{xx}) \times \mathbf{g} + (\mathbf{m} \times \mathbf{g}) \times \mathbf{g}_{xx} \right. \\ &\quad \left. + 2 \left((\mathbf{m}_x \times \mathbf{g}_x) \times \mathbf{g} + (\mathbf{m}_x \times \mathbf{g}) \times \mathbf{g}_x + (\mathbf{m} \times \mathbf{g}_x) \times \mathbf{g}_x \right) \right). \end{aligned}$$

Applying identity (91), we obtain the following representation

$$\tilde{C}(X(t), \mathbf{m}) = S(X)\mathfrak{S}(\mathbf{m}) + \mathcal{C}(X)\mathfrak{C}(\mathbf{m}), \quad (23)$$

where we define the following abbreviations

$$S(X) := \sin(X(t)) - \sin^2(X(t))G + \sin(X(t))G^2 - \frac{1}{2} \sin(2X(t))G^2, \quad (24)$$

$$\mathfrak{S}(\mathbf{m}) := \mathbf{m} \times \mathbf{g}_{xx} + 2\mathbf{m}_x \times \mathbf{g}_x, \quad (25)$$

$$\begin{aligned} \mathcal{C}(X) &:= 1 - \cos(X(t)) + \left[\frac{1}{2} \sin(2X(t)) - \sin(X(t)) \right] G \\ &\quad + \left[1 - 2\cos(X(t)) + \cos^2(X(t)) \right] G^2 \\ \mathfrak{C}(\mathbf{m}) &:= (\mathbf{m} \times \mathbf{g}_{xx}) \times \mathbf{g} + (\mathbf{m} \times \mathbf{g}) \times \mathbf{g}_{xx} + 2 \left((\mathbf{m}_x \times \mathbf{g}_x) \times \mathbf{g} \right. \end{aligned} \quad (26)$$

$$\left. + (\mathbf{m}_x \times \mathbf{g}) \times \mathbf{g}_x + (\mathbf{m} \times \mathbf{g}_x) \times \mathbf{g}_x \right). \quad (27)$$

Remark 1 Using the representation in (23) and the fact that $\mathbf{g} \in \mathbb{W}^{2,\infty}(D)$, we know that there exists a constant $K > 0$ such that

$$\begin{aligned} |\tilde{C}(X(t), \mathbf{m})|_{\mathbb{R}^3} &\leq K \left[|\mathfrak{S}(\mathbf{m})|_{\mathbb{R}^3} + |\mathfrak{C}(\mathbf{m})|_{\mathbb{R}^3} \right], \\ |\tilde{C}_x(X(t), \mathbf{m})|_{\mathbb{R}^3} &\leq K \left[|\mathfrak{S}(\mathbf{m})|_{\mathbb{R}^3} + |\mathfrak{S}_x(\mathbf{m})|_{\mathbb{R}^3} + |\mathfrak{C}(\mathbf{m})|_{\mathbb{R}^3} + |\mathfrak{C}_x(\mathbf{m})|_{\mathbb{L}^2} \right], \\ |\tilde{C}_{xx}(X(t), \mathbf{m})|_{\mathbb{R}^3} &\leq K \left[|\mathfrak{S}(\mathbf{m})|_{\mathbb{R}^3} + |\mathfrak{S}_x(\mathbf{m})|_{\mathbb{R}^3} + |\mathfrak{S}_{xx}(\mathbf{m})|_{\mathbb{R}^3} + |\mathfrak{C}(\mathbf{m})|_{\mathbb{R}^3} \right. \\ &\quad \left. + |\mathfrak{C}_x(\mathbf{m})|_{\mathbb{R}^3} + |\mathfrak{C}_{xx}(\mathbf{m})|_{\mathbb{R}^3} \right]. \end{aligned}$$

Using the following inequalities

$$\begin{cases} |\mathfrak{S}(\mathbf{m})|_{\mathbb{R}^3} \leq K \left[|\mathbf{m}|_{\mathbb{R}^3} + |\mathbf{m}_x|_{\mathbb{R}^3} \right], \\ |\mathfrak{S}_x(\mathbf{m})|_{\mathbb{R}^3} \leq K \left[|\mathbf{m}|_{\mathbb{R}^3} + |\mathbf{m}_x|_{\mathbb{R}^3} + |\mathbf{m}_{xx}|_{\mathbb{R}^3} \right], \\ |\mathfrak{S}_{xx}(\mathbf{m})|_{\mathbb{R}^3} \leq K \left[|\mathbf{m}|_{\mathbb{R}^3} + |\mathbf{m}_x|_{\mathbb{R}^3} + |\mathbf{m}_{xx}|_{\mathbb{R}^3} + |\mathbf{m}_{xxx}|_{\mathbb{R}^3} \right], \\ |\mathfrak{C}(\mathbf{m})|_{\mathbb{R}^3} \leq K \left[|\mathbf{m}|_{\mathbb{R}^3} + |\mathbf{m}_x|_{\mathbb{R}^3} \right], \\ |\mathfrak{C}_x(\mathbf{m})|_{\mathbb{R}^3} \leq K \left[|\mathbf{m}|_{\mathbb{R}^3} + |\mathbf{m}_x|_{\mathbb{R}^3} + |\mathbf{m}_{xx}|_{\mathbb{R}^3} \right], \\ |\mathfrak{C}_{xx}(\mathbf{m})|_{\mathbb{R}^3} \leq K \left[|\mathbf{m}|_{\mathbb{R}^3} + |\mathbf{m}_x|_{\mathbb{R}^3} + |\mathbf{m}_{xx}|_{\mathbb{R}^3} + |\mathbf{m}_{xxx}|_{\mathbb{R}^3} \right], \end{cases} \quad (28)$$

we obtain

$$|\tilde{C}(X(t), \mathbf{m})|_{\mathbb{R}^3} \leq K \left[|\mathbf{m}|_{\mathbb{R}^3} + |\mathbf{m}_x|_{\mathbb{R}^3} \right], \quad (29)$$

$$|\tilde{C}_x(X(t), \mathbf{m})|_{\mathbb{R}^3} \leq K \left[|\mathbf{m}|_{\mathbb{R}^3} + |\mathbf{m}_x|_{\mathbb{R}^3} + |\mathbf{m}_{xx}|_{\mathbb{R}^3} \right], \quad (30)$$

$$|\tilde{C}_{xx}(X(t), \mathbf{m})|_{\mathbb{R}^3} \leq K \left[|\mathbf{m}|_{\mathbb{R}^3} + |\mathbf{m}_x|_{\mathbb{R}^3} + |\mathbf{m}_{xx}|_{\mathbb{R}^3} + |\mathbf{m}_{xxx}|_{\mathbb{R}^3} \right]. \quad (31)$$

Finally, by straightforward calculations we can estimate the nonlinearity F defined in (22). In particular, there exists a constant $K > 0$ such that using (20), we have

$$|F(t, \mathbf{m})|_{\mathbb{L}^2} \leq K \left[|\lambda_1| + |\lambda_2| \right] \left[1 + |\mathbf{m}_x|_{\mathbb{L}^2} \right], \quad (32)$$

$$|F_x(t, \mathbf{m})|_{\mathbb{L}^2} \leq K \left[|\lambda_1| + |\lambda_2| \right] \left[1 + |\mathbf{m}_x|_{\mathbb{L}^\infty} \right] \left[1 + |\mathbf{m}_x|_{\mathbb{L}^2} + |\mathbf{m}_{xx}|_{\mathbb{L}^2} \right], \quad (33)$$

$$\|F_{xx}(t, \mathbf{m})\|_{\mathbb{L}^2} \leq K \left[|\lambda_1| + \lambda_2 \right] \left[1 + \|\mathbf{m}_x\|_{\mathbb{L}^\infty} \right] \left[1 + \|\mathbf{m}_x\|_{\mathbb{L}^2} + \|\mathbf{m}_{xx}\|_{\mathbb{L}^2} + \|\mathbf{m}_{xxx}\|_{\mathbb{L}^2} \right]. \quad (34)$$

Let us now consider the corresponding system (36) approximating (14). Again, let us define a new auxiliary process $\mathbf{m}^{(n)}$ from $\mathbf{M}^{(n)}$ and $X^{(n)}$ by

$$\mathbf{m}^{(n)}(t, \mathbf{x}) := e^{-X^{(n)}(t)G} \mathbf{M}^{(n)}(t, \mathbf{x}) \quad \forall t \geq 0, \text{ a.e. } \mathbf{x} \in D. \quad (35)$$

By the same calculations as above and Lemma 7, one can show that if $\mathbf{M}^{(n)}$ is a solution to (7), then $\mathbf{m}^{(n)}$ is a solution to

$$\begin{cases} \dot{\mathbf{m}}^{(n)} = \lambda_1 \mathbf{m}^{(n)} \times \mathbf{m}_{xx}^{(n)} - \lambda_2 \mathbf{m}^{(n)} \times (\mathbf{m}^{(n)} \times \mathbf{m}_{xx}^{(n)}) \\ \quad + F^{(n)}(\cdot, \mathbf{m}^{(n)}), \text{ in } (0, T) \times D, \\ \mathbf{m}_x^{(n)}(t, 0) = 0 = \mathbf{m}_x^{(n)}(t, 1), \quad \forall t \in (0, T), \\ \mathbf{m}^{(n)}(0, \cdot) = \mathbf{m}_0 \text{ in } D, \end{cases} \quad (36)$$

and vice versa. Here,

$$F^{(n)}(\cdot, \mathbf{m}^{(n)}) = \lambda_1 \mathbf{m}^{(n)} \times \tilde{C}(X^{(n)}, \mathbf{m}^{(n)}) - \lambda_2 \mathbf{m}^{(n)} \times (\mathbf{m}^{(n)} \times \tilde{C}(X^{(n)}, \mathbf{m}^{(n)})). \quad (37)$$

Moreover, in Theorem 3, we will show that system (36) has indeed a unique solution.

3 Existence and Uniqueness of the Solutions to the Systems (5) and (7)

We begin with the definition of (strong) solution to the system (5).

Definition 1 Let $T > 0$ and $X = (X, \mathbb{X}) \in \mathcal{C}_g^\alpha([0, T], \mathbb{R})$ with $\alpha \in (\frac{1}{3}, \frac{1}{2})$. A path $\mathbf{M} \in \mathcal{C}^\alpha([0, T], \mathbb{L}^2)$ is said to be a (strong) solution of the system (5) if the following properties hold:

(i) $\mathbf{M}$ satisfies the following saturation condition

$$\|\mathbf{M}(t, x)\|_{\mathbb{R}^3} = 1 \text{ for a.e. } x \in D, \text{ for all } t \in [0, T];$$

(ii) $\mathbf{M}$ satisfies the second equation of (5) and $\sup_{t \in [0, T]} \|\mathbf{M}(t)\|_{\mathbb{H}^2} < \infty$;
 (iii) $\mathbf{M}$ is controlled by X , i.e. there exists a path $\mathbf{M}' \in \mathcal{C}^\alpha([0, T], \mathbb{L}^2)$ (being called Gubinelli derivative) such that the reminder term $\mathcal{R}^Y$ is given implicitly through the relation $Y_{s,t} = Y'_s X_{s,t} + \mathcal{R}_{s,t}^Y$, and satisfies $\|\mathcal{R}^Y\|_{2\alpha, H} < \infty$. Moreover,

$$(\mathbf{M}, \mathbf{M}') \in \mathcal{D}_X^{2\alpha}([0, T], \mathbb{L}^2);$$

(iv) $\mathbf{M}$ satisfies the rough differential equation given by

$$\mathbf{M}(t, x) = \mathbf{M}_0(x) + \lambda_1 \int_0^t \mathbf{M}(s, x) \times \mathbf{M}_{xx}(s, x) ds$$

$$\begin{aligned}
 & -\lambda_2 \int_0^t \mathbf{M}(s, x) \times (\mathbf{M}(s, x) \times \mathbf{M}_{xx}(s, x)) \, ds \\
 & + \int_0^t (\mathbf{M}(s, x) \times \mathbf{g}(x)) \, dX_s,
 \end{aligned}$$

where the third integral is interpreted in the sense of (77).

In this section, we prove the following theorem which shows that the solution to the system (5) (resp. (7)) exists and is unique.

Theorem 1 *Let $X = (X, \mathbb{X}) \in \mathcal{C}_g^\alpha([0, T], \mathbb{R})$. Let $X^{(n)} = (X^{(n)}, \mathbb{X}^{(n)})$, $n \in \mathbb{N}$ be a sequence in $\mathcal{C}_g^\alpha([0, T], \mathbb{R})$ such that*

$$X^{(n)} \rightarrow X \text{ in } \mathcal{C}_g^\alpha([0, T], \mathbb{R}).$$

If $\mathbf{M}_0, \mathbf{M}_0^{(n)} \in \mathbb{H}^2(D; \mathbb{S}^2)$, then there exist unique (strong) solutions $\mathbf{M}$ and $\mathbf{M}^{(n)}$ to the respective systems (5) and (7). In addition, we have

- (i) $(\mathbf{M}, \mathbf{M} \times \mathbf{g}) \in \mathcal{D}_X^{2\alpha}([0, T], \mathbb{L}^2)$,
- (ii) $(\mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g}) \in \mathcal{D}_{X^{(n)}}^{2\alpha}([0, T], \mathbb{L}^2)$,
- (iii) and $\mathbf{M}, \mathbf{M}^{(n)} \in L^\infty(0, T; \mathbb{H}^2)$.

Before we present the proof of Theorem 1, we introduce Theorem 2 and Theorem 3. In the first theorem, we show the existence, uniqueness and regularity of the solution to the system (14). In the second theorem, we obtain the same for system (36). Then, using the Doss–Sussmann transformation, we show that from the existence of a unique solution to systems (14) and (36), respectively, follows the existence of the solution of systems (5) and (7), respectively. Similarly, the convergence is proven.

Let us start with introducing the definition of (weak) solution $\mathbf{m}$ to the system (14) and the theorem for existence and regularity of $\mathbf{m}$.

Definition 2 Let $T > 0$. A path $\mathbf{m} \in \mathcal{C}^{2\alpha}([0, T], \mathbb{L}^2)$ is said to be a (weak) solution to the system (14) if the following properties hold:

- (i) $\mathbf{m}$ satisfies the following saturation condition

$$|\mathbf{m}(t, x)|_{\mathbb{R}^3} = 1 \text{ for a.e. } x \in D, \text{ for all } t \in [0, T];$$

- (ii) $\mathbf{m}$ satisfies the second equation of (14) and $\sup_{t \in [0, T]} |\mathbf{m}(t)|_{\mathbb{H}^2} < \infty$;
- (iii) $\mathbf{m}$ satisfies the time dependent differential equation for all $\phi \in \mathbb{H}^1$, $t \in [0, T]$

$$\begin{aligned}
 \langle \mathbf{m}(t), \phi \rangle_{\mathbb{L}^2} &= \langle \mathbf{m}(0), \phi \rangle_{\mathbb{L}^2} - \lambda_1 \int_0^t \int_D \langle \mathbf{m}_x(s, x), \phi_x(x) \times \mathbf{m}(s, x) \rangle_{\mathbb{R}^3} dx \, ds \\
 &\quad - \lambda_2 \int_0^t \int_D \langle \mathbf{m}_x(s, x), (\mathbf{m} \times \phi)_x(s, x) \times \mathbf{m}(s, x) \rangle_{\mathbb{R}^3} dx \, ds \\
 &\quad + \int_0^t \int_D \langle F(s, \mathbf{m}(s, x)), \phi(x) \rangle_{\mathbb{R}^3} dx \, ds.
 \end{aligned} \tag{38}$$

Theorem 2 Let $\mathbf{m}_0 \in \mathbb{H}^2(D; \mathbb{S}^2)$. Then there exists a weak solution $\mathbf{m}$ to the system (14) in the sense of Definition 2 satisfying the following:

(i) there exists a positive constant C , depending on $T, \lambda_1, \lambda_2, |\mathbf{m}_0|_{\mathbb{H}^2}$, such that

$$\sup_{t \in [0, T]} |\mathbf{m}(t)|_{\mathbb{H}^2} \leq C;$$

(ii) for almost every $t \in [0, \infty)$, $\mathbf{m}(t) \times \mathbf{m}_{xx}(t) \in \mathbb{L}^2$ and for every $T > 0$, there exists a positive constant C , depending on $T, \lambda_1, \lambda_2, |\mathbf{m}_0|_{\mathbb{H}^2}$, such that we have

$$\int_0^T |\mathbf{m}(t) \times \mathbf{m}_{xx}(t)|_{\mathbb{L}^2}^2 dt \leq C. \quad (39)$$

Furthermore, we obtain that $\mathbf{m} \in H^1(0, T; \mathbb{L}^2) \cap L^\infty(0, T; \mathbb{H}^2)$ and $\mathbf{m}$ is also a strong solution of the system (14).

Proof For proof, we refer to Theorem 3.2 and Lemma 3.13 of [8]. In addition, for the regularity property of $\mathbf{m}$, we refer to Lemma 8 and Lemma 9 in Appendix 2. $\square$

In the same way as above (see Definition 2 and Theorem 2), we can define weak solutions for the system of equations (36) and obtain the following theorem.

Theorem 3 Let $\mathbf{m}_0 \in \mathbb{H}^2(D; \mathbb{S}^2)$. Then there exists a weak solution $\mathbf{m}^{(n)}$ to the system (36), i.e., it satisfies the following:

(i) for every $T > 0$,

$$\sup_{t \in [0, T]} |\mathbf{m}^{(n)}(t)|_{\mathbb{H}^2} \leq C(T, \lambda_2, |\mathbf{m}_0|_{\mathbb{H}^2}); \quad (40)$$

(ii) for almost every $t \in [0, \infty)$, $\mathbf{m}^{(n)}(t) \times \mathbf{m}_{xx}^{(n)}(t) \in \mathbb{L}^2$ and every $T > 0$ we have

$$\int_0^T |\mathbf{m}^{(n)}(t) \times \mathbf{m}_{xx}^{(n)}(t)|_{\mathbb{L}^2}^2 dt \leq C(T, \lambda_2, |\mathbf{m}_0|_{\mathbb{H}^2}); \quad (41)$$

(iii) $|\mathbf{m}^{(n)}(t, x)|_{\mathbb{R}^3} = 1$, a.e. $x \in D$ and for all $t \in [0, T]$;

(iv) for all $\phi \in \mathbb{H}^1$,

$$\begin{aligned} \langle \mathbf{m}^{(n)}(t), \phi \rangle_{\mathbb{L}^2} &= \langle \mathbf{m}(0), \phi \rangle_{\mathbb{L}^2} \\ &\quad - \lambda_1 \int_0^t \int_D \langle \mathbf{m}_x^{(n)}(s, x), \phi_x(x) \times \mathbf{m}^{(n)}(s, x) \rangle_{\mathbb{R}^3} dx ds \\ &\quad - \lambda_2 \int_0^t \int_D \langle \mathbf{m}_x^{(n)}(s, x), (\mathbf{m}^{(n)} \times \phi)_x(s, x) \times \mathbf{m}^{(n)}(s, x) \rangle_{\mathbb{R}^3} dx ds \\ &\quad + \int_0^t \int_D \langle F(s, \mathbf{m}^{(n)}(s, x)), \phi(x) \rangle_{\mathbb{R}^3} dx ds, \end{aligned}$$

holds for all $t \in [0, T]$.

Moreover, we obtain that $\mathbf{m}^{(n)} \in H^1(0, T; \mathbb{L}^2) \cap L^\infty(0, T; \mathbb{H}^2)$ and $\mathbf{m}^{(n)}$ is the strong solution of the system (36).

Proof of Theorem 1 Thanks to Theorem 2, there exists a unique strong solution $\mathbf{m} \in H^1(0, T; \mathbb{L}^2) \cap L^\infty(0, T; \mathbb{H}^2)$ to the system (14). Hence, applying Lemma 5, we conclude that there exists a unique strong solution $\mathbf{M} \in L^\infty(0, T; \mathbb{H}^2)$ of the system (5) and $(\mathbf{M}, \mathbf{M} \times \mathbf{g}) \in \mathcal{D}_X^{2\alpha}([0, T], \mathbb{L}^2)$. Finally, using Lemma 10, we conclude that the integration against rough paths in (5) is well defined. Proceeding in similar lines, one can observe that there exists a unique strong solution $\mathbf{M}^{(n)} \in L^\infty(0, T; \mathbb{H}^2)$ to the system (7) and $(\mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g}) \in \mathcal{D}_X^{2\alpha}([0, T], \mathbb{L}^2)$, thus completing the proof. $\square$

4 Convergence of Solution in Controlled Rough Path Spaces

In this section we prove the convergence result in the space $\mathcal{D}_X^{2\alpha}([0, T], \mathbb{L}^2)$.

Theorem 4 Let $\mathbf{M}$ and $\mathbf{M}^{(n)}$ be the solutions to the systems (5) and (7), respectively. Then we have the following convergence

$$\mathbf{M}^{(n)} \rightarrow \mathbf{M} \text{ in } \mathcal{D}_X^{2\alpha}([0, T], \mathbb{L}^2), \text{ as } n \rightarrow \infty.$$

Before going into the proof of Theorem 4, we introduce and prove some lemmata and corollaries. These lemmata are essential to show Theorem 4.

Lemma 3 Let $\mathbf{m}$ and $\mathbf{m}^{(n)}$ be the solutions to the systems (14) and (36), respectively. Then we have the following convergence

$$\mathbf{m}^{(n)} \rightarrow \mathbf{m} \text{ in } L^\infty(0, T; \mathbb{L}^2), \text{ as } n \rightarrow \infty.$$

Proof By similar arguments as with the proof of Lemma 5.2 in [8], one can show that there exists a constant $C > 0$ and an integrable function φ_C , so that we have the following estimate

$$\begin{aligned} |\mathbf{m}^{(n)}(t) - \mathbf{m}(t)|_{\mathbb{L}^2}^2 &\leq \left(|\mathbf{m}^{(n)}(0) - \mathbf{m}(0)|_{\mathbb{L}^2}^2 + C \int_0^t |X^{(n)}(s) - X(s)| ds \right) \\ &\quad \times e^{2 \int_0^t \varphi_C(s) ds}, \end{aligned}$$

for $t \in [0, T]$. Using (6), we obtain

$$\mathbf{m}^{(n)} \rightarrow \mathbf{m} \text{ in } L^\infty(0, T; \mathbb{L}^2), \text{ as } n \rightarrow \infty.$$

$\square$

Corollary 1 Let $\mathbf{M}$ and $\mathbf{M}^{(n)}$ be the solutions to the systems (5) and (7), respectively. Then we have the following convergence

$$\mathbf{M}^{(n)} \rightarrow \mathbf{M} \text{ in } L^\infty(0, T; \mathbb{L}^2), \text{ as } n \rightarrow \infty.$$

Proof We note that

$$\begin{aligned}\mathbf{M}^{(n)}(t) - \mathbf{M}(t) &= e^{X^{(n)}(t)G} \mathbf{m}^n(t) - e^{X(t)G} \mathbf{m}(t) \\ &= e^{X^{(n)}(t)G} \left(\mathbf{m}^n(t) - \mathbf{m}(t) \right) + \left(e^{X^{(n)}(t)G} - e^{X(t)G} \right) \mathbf{m}(t).\end{aligned}$$

Since $X^{(n)} \rightarrow X$ in $\mathcal{C}_g^\alpha([0, T], \mathbb{R})$ (compare (6)), we can observe that $X^{(n)} \rightarrow X$ in $L^\infty(0, T; \mathbb{R})$. Therefore, using the identity (91) we have $e^{X^{(n)}G} \rightarrow e^{XG}$ in $L^\infty(0, T; \mathcal{L}(\mathbb{L}^2))$, where $\mathcal{L}(\mathbb{L}^2)$ denote space of bounded linear operators on $\mathbb{L}^2$. Finally, from Lemma 3, we obtain

$$\mathbf{M}^{(n)} \rightarrow \mathbf{M} \text{ in } L^\infty(0, T; \mathbb{L}^2), \text{ as } n \rightarrow \infty,$$

which is the assertion. $\square$

In the next lemma, we investigate the convergence of sequence $\mathbf{m}_{xx}^{(n)}$ in $L^\infty(0, T; \mathbb{L}^2)$.

Lemma 4 Let $\mathbf{m}$ and $\mathbf{m}^{(n)}$ be the solutions to the systems (14) and (36), respectively. Then we have the following convergence

$$\mathbf{m}_{xx}^{(n)} \rightarrow \mathbf{m}_{xx} \text{ in } L^\infty(0, T; \mathbb{L}^2), \text{ as } n \rightarrow \infty.$$

Proof Let $u = \mathbf{m}_{xx}$, $u^{(n)} = \mathbf{m}_{xx}^{(n)}$ and $z = u^{(n)} - u$. We proceed in the same way as we advance in the proof of Lemma 9 in Appendix 2. Substituting z in system (14), we get the following identity

$$\begin{aligned}\frac{\partial z}{\partial t} &= \lambda_1 (2\mathbf{m}_x \times z_x + \mathbf{m} \times z_{xx}) - \lambda_2 \left[z \times (\mathbf{m} \times z) \right. \\ &\quad \left. + 2\mathbf{m}_x \times (\mathbf{m}_x \times z) + 2\mathbf{m}_x \times (\mathbf{m} \times z_x) \right. \\ &\quad \left. + 2\mathbf{m} \times (\mathbf{m}_x \times z_x) + \langle \mathbf{m}, z_{xx} \rangle \mathbf{m} - \langle \mathbf{m}, \mathbf{m} \rangle z_{xx} \right] + F_{xx}(t, \mathbf{m}).\end{aligned}$$

Substituting $z = u^{(n)} - u$, we obtain

$$\begin{aligned}\frac{\partial z}{\partial t} &= \lambda_1 \left[2(\mathbf{m}_x^{(n)} \times u_x^{(n)} - \mathbf{m}_x \times u_x) + (\mathbf{m}^{(n)} \times u_{xx}^{(n)} - \mathbf{m} \times u_{xx}) \right] \\ &\quad - \lambda_2 \left[(u^{(n)} \times (\mathbf{m}^{(n)} \times u^{(n)}) - u \times (\mathbf{m} \times u)) + 2(\mathbf{m}_x^{(n)} \times (\mathbf{m}_x^{(n)} \times u^{(n)}) \right. \\ &\quad \left. - \mathbf{m}_x \times (\mathbf{m}_x \times u)) + 2(\mathbf{m}_x^{(n)} \times (\mathbf{m}^{(n)} \times u_x^{(n)}) - \mathbf{m}_x \times (\mathbf{m} \times u_x)) \right. \\ &\quad \left. + 2(\mathbf{m}^{(n)} \times (\mathbf{m}_x^{(n)} \times u_x^{(n)}) - \mathbf{m} \times (\mathbf{m}_x \times u_x)) + (\langle \mathbf{m}^{(n)}, z_{xx}^{(n)} \rangle \mathbf{m}^{(n)} \right. \\ &\quad \left. - \langle \mathbf{m}, z_{xx} \rangle \mathbf{m}) - (z_{xx}^{(n)} - z_{xx}) \right] + [F_{xx}^{(n)}(t, \mathbf{m}^{(n)}) - F_{xx}(t, \mathbf{m})].\end{aligned}$$

Testing with z and integrating over the interval $(0, 1)$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |z|_{\mathbb{L}^2}^2 &= \lambda_1 \int_0^1 \left\langle 2(\mathbf{m}_x^{(n)} \times u_x^{(n)} - \mathbf{m}_x \times u_x) + (\mathbf{m}^{(n)} \times u_{xx}^{(n)} - \mathbf{m} \times u_{xx}), z \right\rangle dx \\ &\quad - \lambda_2 \int_0^1 \left\langle (u^{(n)} \times (\mathbf{m}^{(n)} \times u^{(n)}) - u \times (\mathbf{m} \times u)) \right. \\ &\quad + 2(\mathbf{m}_x^{(n)} \times (\mathbf{m}_x^{(n)} \times u^{(n)}) - \mathbf{m}_x \times (\mathbf{m}_x \times u)) \\ &\quad + 2(\mathbf{m}_x^{(n)} \times (\mathbf{m}^{(n)} \times u_x^{(n)}) - \mathbf{m}_x \times (\mathbf{m} \times u_x)) \\ &\quad + 2(\mathbf{m}^{(n)} \times (\mathbf{m}_x^{(n)} \times u_x^{(n)}) - \mathbf{m} \times (\mathbf{m}_x \times u_x)) \\ &\quad + \left(\langle \mathbf{m}^{(n)}, z_{xx}^{(n)} \rangle \mathbf{m}^{(n)} - \langle \mathbf{m}, z_{xx} \rangle \mathbf{m} \right) - (z_{xx}^{(n)} - z_{xx}), z \left. \right\rangle dx \\ &\quad + \int_0^1 \left\langle F_{xx}^{(n)}(t, \mathbf{m}^{(n)}) - F_{xx}(t, \mathbf{m}), z \right\rangle dx \\ &:= \sum_{j=1}^3 I_j. \end{aligned} \quad (42)$$

Now we estimate each I_j 's for $j = 1, 2, 3$.

Estimate of I_1 : Integrating by parts and using the fact that $z_x = 0$ and $z_x^{(n)} = 0$ at $x = 0$ and $x = 1$, we obtain

$$\begin{aligned} I_1 &\leq |\lambda_1| \left[|\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} \int_0^1 |z_x| |z| dx + |\mathbf{m}_x^{(n)} - \mathbf{m}_x|_{\mathbb{L}^\infty} \int_0^1 |u_x| |z| dx \right. \\ &\quad \left. + |\mathbf{m}^{(n)} - \mathbf{m}|_{\mathbb{L}^\infty} \int_0^1 |u_x| |z_x| dx \right]. \end{aligned}$$

Using Young's inequality, we obtain for any $\varepsilon > 0$,

$$\begin{aligned} I_1 &\leq \varepsilon |\lambda_1| |z_x|_{\mathbb{L}^2}^2 + |\lambda_1| \left[\frac{|\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty}^2}{2\varepsilon} + \frac{1}{2\varepsilon} \right] |z|_{\mathbb{L}^2}^2 \\ &\quad + |\lambda_1| \left[\frac{\varepsilon}{2} |\mathbf{m}_x^{(n)} - \mathbf{m}_x|_{\mathbb{L}^\infty}^2 + \frac{|\mathbf{m}^{(n)} - \mathbf{m}|_{\mathbb{L}^\infty}^2}{2\varepsilon} \right] |u_x|_{\mathbb{L}^2}^2 \\ &=: \psi_1 |z_x|_{\mathbb{L}^2}^2 + \varphi_1 |z|_{\mathbb{L}^2}^2 + \chi_1. \end{aligned} \quad (43)$$

Above, we introduced the abbreviation $\psi_1 := \varepsilon |\lambda_1|$, $\varphi_1 := |\lambda_1| [|\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty}^2 / (2\varepsilon) + 1 / (2\varepsilon)]$ and $\chi_1 := |\lambda_1| [\varepsilon / 2 |\mathbf{m}_x^{(n)} - \mathbf{m}_x|_{\mathbb{L}^\infty}^2 + |\mathbf{m}^{(n)} - \mathbf{m}|_{\mathbb{L}^\infty}^2 / (2\varepsilon)] |u_x|_{\mathbb{L}^2}^2$.

Estimate of I_2 : By a simple vector algebraic identity and equation (20), we obtain

$$I_2 = -\lambda_2 \int_0^1 \left\langle u^{(n)} \times (\mathbf{m}^{(n)} \times u^{(n)}) - u \times (\mathbf{m} \times u), z \right\rangle dx \quad (44)$$

$$\begin{aligned}
& -2\lambda_2 \int_0^1 \left\langle \mathbf{m}_x^{(n)} \times (\mathbf{m}_x^{(n)} \times u^{(n)}) - \mathbf{m}_x \times (\mathbf{m}_x \times u), z \right\rangle dx \\
& -2\lambda_2 \int_0^1 \left\langle \mathbf{m}_x^{(n)} \times (\mathbf{m}^{(n)} \times u_x^{(n)}) - \mathbf{m}_x \times (\mathbf{m} \times u_x), z \right\rangle dx \\
& -2\lambda_2 \int_0^1 \left\langle \mathbf{m}^{(n)} \times (\mathbf{m}_x^{(n)} \times u_x^{(n)}) - \mathbf{m} \times (\mathbf{m}_x \times u_x), z \right\rangle dx \\
& -\lambda_2 \int_0^1 \left[\langle \mathbf{m}^{(n)}, z_{xx}^{(n)} \rangle \langle \mathbf{m}^{(n)}, z \rangle \right] dx - \langle \mathbf{m}, z_{xx} \rangle \langle \mathbf{m}, z \rangle dx - \lambda_2 |z_x|_{\mathbb{L}^2}^2 \\
& := \sum_{j=1}^5 I_{2,j} - \lambda_2 |z_x|_{\mathbb{L}^2}^2.
\end{aligned}$$

We now derive estimates $I_{2,j}$ with $j = 1, 2, \dots, 5$ for a fixed ε . Proceeding in a similar manner as in $I_1(t)$, we get

$$\begin{aligned}
I_{2,1} & \leq \frac{\varepsilon}{2} |z_x|_{\mathbb{L}^2}^2 + \frac{\lambda_2^2 |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty}^2 |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty}^2}{\varepsilon} |z|_{\mathbb{L}^2}^2 + \lambda_2 |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty}^2 |z|_{\mathbb{L}^2}^2 \\
& + \frac{\varepsilon}{2} |z_x|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{2} |z|_{\mathbb{L}^2}^2 + \frac{\lambda_2^2 |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty}^2 |\mathbf{m}^{(n)} - \mathbf{m}|_{\mathbb{L}^\infty}^2}{2\varepsilon} \left[|u|_{\mathbb{L}^2}^2 + |u_x|_{\mathbb{L}^2}^2 \right] \\
& + \frac{\varepsilon}{2} |z|_{\mathbb{L}^2}^2 + \frac{\lambda_2^2 |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty}^2 |\mathbf{m}_x^{(n)} - \mathbf{m}_x|_{\mathbb{L}^\infty}^2}{2\varepsilon} |u|_{\mathbb{L}^2}^2 \\
& = \varepsilon |z_x|_{\mathbb{L}^2}^2 + \left[\frac{\lambda_2^2 |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty}^2 |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty}^2}{\varepsilon} + \lambda_2 |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty}^2 + \varepsilon \right] |z|_{\mathbb{L}^2}^2 \\
& + \frac{\lambda_2^2 |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty}^2}{2\varepsilon} \left(|\mathbf{m}^{(n)} - \mathbf{m}|_{\mathbb{L}^\infty}^2 \left[|u|_{\mathbb{L}^2}^2 + |u_x|_{\mathbb{L}^2}^2 \right] + |\mathbf{m}_x^{(n)} - \mathbf{m}_x|_{\mathbb{L}^\infty}^2 |u|_{\mathbb{L}^2}^2 \right) \\
& =: \psi_{2,1} |z_x|_{\mathbb{L}^2}^2 + \varphi_{2,1} |z|_{\mathbb{L}^2}^2 + \chi_{2,1}.
\end{aligned}$$

Now we estimate $I_{2,2}$. Applying Young's inequality, we have

$$\begin{aligned}
I_{2,2} & \leq \left[2\lambda_2 |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty}^2 + \varepsilon \lambda_2 |\mathbf{m}_x^{(n)} - \mathbf{m}_x|_{\mathbb{L}^\infty} \left(|\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} + |\mathbf{m}_x|_{\mathbb{L}^\infty} \right) \right] |z|_{\mathbb{L}^2}^2 \\
& + \frac{1}{\varepsilon} \lambda_2 |\mathbf{m}_x^{(n)} - \mathbf{m}_x|_{\mathbb{L}^\infty} \left(|\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} + |\mathbf{m}_x|_{\mathbb{L}^\infty} \right) |u|_{\mathbb{L}^2}^2 \\
& := \varphi_{2,2} |z|_{\mathbb{L}^2}^2 + \chi_{2,2}.
\end{aligned}$$

Proceeding in similar manner as in $I_{2,2}$, we get for $I_{2,3}$ and $I_{2,4}$

$$\begin{aligned}
I_{2,3} & \leq 2\lambda_2 \left[\varepsilon \left(|\mathbf{m}^{(n)} - \mathbf{m}|_{\mathbb{L}^\infty} |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} + |\mathbf{m}_x^{(n)} - \mathbf{m}_x|_{\mathbb{L}^\infty} |\mathbf{m}|_{\mathbb{L}^\infty} \right) \right. \\
& \quad \left. + |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} \right] |z|_{\mathbb{L}^2}^2 \\
& + \frac{1}{\varepsilon} \lambda_2 \left(|\mathbf{m}^{(n)} - \mathbf{m}|_{\mathbb{L}^\infty} |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} + |\mathbf{m}_x^{(n)} - \mathbf{m}_x|_{\mathbb{L}^\infty} |\mathbf{m}|_{\mathbb{L}^\infty} \right) |u_x|_{\mathbb{L}^2}^2
\end{aligned}$$

$$\begin{aligned}
 &:= \varphi_{2,3} |z|_{\mathbb{L}^2}^2 + \chi_{2,3}, \\
 I_{2,4} &\leq 2\lambda_2 \left[\varepsilon \left(|\mathbf{m}^{(n)} - \mathbf{m}|_{\mathbb{L}^\infty} |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} + |\mathbf{m}_x^{(n)} - \mathbf{m}_x|_{\mathbb{L}^\infty} |\mathbf{m}|_{\mathbb{L}^\infty} \right) \right. \\
 &\quad \left. + |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} \right] |z|_{\mathbb{L}^2}^2 \\
 &\quad + \frac{1}{\varepsilon} \lambda_2 \left(|\mathbf{m}^{(n)} - \mathbf{m}|_{\mathbb{L}^\infty} |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} + |\mathbf{m}_x^{(n)} - \mathbf{m}_x|_{\mathbb{L}^\infty} |\mathbf{m}|_{\mathbb{L}^\infty} \right) |u_x|_{\mathbb{L}^2}^2 \\
 &=: \varphi_{2,4} |z|_{\mathbb{L}^2}^2 + \chi_{2,4}.
 \end{aligned}$$

Now, by similar argument as applied to $I_{2,1}$, we obtain

$$\begin{aligned}
 I_{2,5} &\leq \frac{2 + \gamma_3^2}{2\varepsilon} |z|_{\mathbb{L}^2}^2 + \frac{3\varepsilon}{2} |z_x|_{\mathbb{L}^2}^2 + \left(\frac{\gamma_1^2 \varepsilon}{2} + \frac{\gamma_2^2}{2\varepsilon} \right) |u_x^{(n)}|_{\mathbb{L}^2}^2 + \left(\frac{\gamma_4^2 \varepsilon}{2} + \frac{\gamma_5^2}{2\varepsilon} \right) |u_x|_{\mathbb{L}^2}^2 \\
 &=: \psi_{2,5} |z_x|_{\mathbb{L}^2}^2 + \varphi_{2,5} |z|_{\mathbb{L}^2}^2 + \chi_{2,5},
 \end{aligned}$$

where

$$\begin{aligned}
 \gamma_1 &= |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} |\mathbf{m}_x^{(n)} - \mathbf{m}_x|_{\mathbb{L}^\infty} + |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} |\mathbf{m}^{(n)} - \mathbf{m}|_{\mathbb{L}^\infty}, \\
 \gamma_2 &= |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} |\mathbf{m}^{(n)} - \mathbf{m}|_{\mathbb{L}^\infty}, \\
 \gamma_3 &= |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} |\mathbf{m}_x|_{\mathbb{L}^\infty} + |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} |\mathbf{m}|_{\mathbb{L}^\infty} + |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} |\mathbf{m}|_{\mathbb{L}^\infty}, \\
 \gamma_4 &= |\mathbf{m}_x|_{\mathbb{L}^\infty} |\mathbf{m}^{(n)} - \mathbf{m}|_{\mathbb{L}^\infty} + |\mathbf{m}|_{\mathbb{L}^\infty} |\mathbf{m}_x^{(n)} - \mathbf{m}_x|_{\mathbb{L}^\infty}, \\
 \gamma_5 &= |\mathbf{m}|_{\mathbb{L}^\infty} |\mathbf{m}^{(n)} - \mathbf{m}|_{\mathbb{L}^\infty}.
 \end{aligned}$$

Combining the estimates of $I_{2,j}$, $j = 1, 2, \dots, 5$ and substituting back in (44), we get

$$\begin{aligned}
 I_2 &\leq (\psi_{2,1} + \psi_{2,5}) |z_x|_{\mathbb{L}^2}^2 + \left[\sum_{j=1}^5 \varphi_{2,j} \right] |z|_{\mathbb{L}^2}^2 + \left[\sum_{j=1}^5 \chi_{2,j} \right] \\
 &=: (\psi_2 - \lambda_2) |z_x|_{\mathbb{L}^2}^2 + \varphi_2 |z|_{\mathbb{L}^2}^2 + \chi_2.
 \end{aligned} \tag{45}$$

Estimate of I_3 : To simplify the notation, we introduce $y = \mathbf{m}^{(n)} - \mathbf{m}$. By the identities (24) and (26), it can be observed that $\mathcal{C}$, $\mathcal{C}_x$, $\mathcal{C}_{xx}$, S , S_x and S_{xx} are Lipschitz continuous functions. In particular, there exists a constant $K = K(|g|_{W^{4,\infty}}) > 0$ such that the following holds:

$$\begin{cases} |S(X^{(n)}(t)) - S(X(t))| &\leq K |X^{(n)}(t) - X(t)|, \\ |S_x(X^{(n)}(t)) - S_x(X(t))| &\leq K |X^{(n)}(t) - X(t)|, \\ |S_{xx}(X^{(n)}(t)) - S_{xx}(X(t))| &\leq K |X^{(n)}(t) - X(t)|, \\ |\mathcal{C}(X^{(n)}(t)) - \mathcal{C}(X(t))| &\leq K |X^{(n)}(t) - X(t)|, \\ |\mathcal{C}_x(X^{(n)}(t)) - \mathcal{C}_x(X(t))| &\leq K |X^{(n)}(t) - X(t)|, \\ |\mathcal{C}_{xx}(X^{(n)}(t)) - \mathcal{C}_{xx}(X(t))| &\leq K |X^{(n)}(t) - X(t)|. \end{cases} \tag{46}$$

Furthermore, an elementary calculation using the identities (25) and (27) yields that

$$\begin{cases} |\mathfrak{S}(\mathbf{m}^{(n)}) - \mathfrak{S}(\mathbf{m})|_{\mathbb{R}^3} & \leq K \left[|y|_{\mathbb{R}^3} + |y_x|_{\mathbb{R}^3} \right], \\ |\mathfrak{S}_x(\mathbf{m}^{(n)}) - \mathfrak{S}_x(\mathbf{m})|_{\mathbb{R}^3} & \leq K \left[|y|_{\mathbb{R}^3} + |y_x|_{\mathbb{R}^3} + |z|_{\mathbb{R}^3} \right], \\ |\mathfrak{S}_{xx}(\mathbf{m}^{(n)}) - \mathfrak{S}_{xx}(\mathbf{m})|_{\mathbb{R}^3} & \leq K \left[|y|_{\mathbb{R}^3} + |y_x|_{\mathbb{R}^3} + |z|_{\mathbb{R}^3} + |z_x|_{\mathbb{R}^3} \right], \\ |\mathfrak{C}(\mathbf{m}^{(n)}) - \mathfrak{C}(\mathbf{m})|_{\mathbb{R}^3} & \leq K \left[|y|_{\mathbb{R}^3} + |y_x|_{\mathbb{R}^3} \right], \\ |\mathfrak{C}_x(\mathbf{m}^{(n)}) - \mathfrak{C}_x(\mathbf{m})|_{\mathbb{R}^3} & \leq K \left[|y|_{\mathbb{R}^3} + |y_x|_{\mathbb{R}^3} + |z|_{\mathbb{R}^3} \right], \\ |\mathfrak{C}_{xx}(\mathbf{m}^{(n)}) - \mathfrak{C}_{xx}(\mathbf{m})|_{\mathbb{R}^3} & \leq K \left[|y|_{\mathbb{R}^3} + |y_x|_{\mathbb{R}^3} + |z|_{\mathbb{R}^3} + |z_x|_{\mathbb{R}^3} \right]. \end{cases} \quad (47)$$

Hence using the estimates (46), (47) and the identity (28) we have

$$\begin{aligned} |\tilde{C}(X^{(n)}(t), \mathbf{m}^{(n)}) - \tilde{C}(X(t), \mathbf{m})|_{\mathbb{R}^3} & \leq K \left[|y|_{\mathbb{R}^3} + |y_x|_{\mathbb{R}^3} \right] \\ & \quad + K \left[|\mathbf{m}|_{\mathbb{R}^3} + |\mathbf{m}_x|_{\mathbb{R}^3} \right] |X^{(n)} - X|, \end{aligned} \quad (48a)$$

$$\begin{aligned} |\tilde{C}_x(X^{(n)}(t), \mathbf{m}^{(n)}) - \tilde{C}_x(X(t), \mathbf{m})|_{\mathbb{R}^3} & \leq K \left[|y|_{\mathbb{R}^3} + |y_x|_{\mathbb{R}^3} + |z|_{\mathbb{R}^3} \right] \\ & \quad + K \left[|\mathbf{m}|_{\mathbb{R}^3} + |\mathbf{m}_x|_{\mathbb{R}^3} \right. \\ & \quad \left. + |u|_{\mathbb{R}^3} \right] |X^{(n)} - X|, \end{aligned} \quad (48b)$$

$$\begin{aligned} |\tilde{C}_{xx}(X^{(n)}(t), \mathbf{m}^{(n)}) - \tilde{C}_{xx}(X(t), \mathbf{m})|_{\mathbb{R}^3} & \leq K \left[|y|_{\mathbb{R}^3} + |y_x|_{\mathbb{R}^3} + |z|_{\mathbb{R}^3} + |u_x|_{\mathbb{R}^3} \right] \\ & \quad + K \left[|\mathbf{m}|_{\mathbb{R}^3} + |\mathbf{m}_x|_{\mathbb{R}^3} + |u|_{\mathbb{R}^3} \right. \\ & \quad \left. + |u_x|_{\mathbb{R}^3} \right] |X^{(n)} - X|. \end{aligned} \quad (48c)$$

Now we are ready to estimate I_3 . Using the identity (22) and a simple vector algebraic identity, we obtain:

$$\begin{aligned} I_3 &= \lambda_1 \int_0^1 \left\langle \mathbf{m}_{xx}^{(n)} \times \tilde{C}(X^{(n)}, \mathbf{m}^{(n)}) - \mathbf{m}_{xx} \times \tilde{C}(X, \mathbf{m}), z \right\rangle dx \\ & \quad + 2\lambda_1 \int_0^1 \left\langle \mathbf{m}_x^{(n)} \times \tilde{C}_x(X^{(n)}, \mathbf{m}^{(n)}) - \mathbf{m}_x \times \tilde{C}_x(X, \mathbf{m}), z \right\rangle dx \\ & \quad + \lambda_1 \int_0^1 \left\langle \mathbf{m}^{(n)} \times \tilde{C}_{xx}(X^{(n)}, \mathbf{m}^{(n)}) - \mathbf{m} \times \tilde{C}_{xx}(X, \mathbf{m}), z \right\rangle dx \\ & \quad - \lambda_2 \int_0^1 \left\langle u^{(n)} \times (\mathbf{m}^{(n)} \times \tilde{C}(X^{(n)}, \mathbf{m}^{(n)})) - u \times (\mathbf{m} \times \tilde{C}(X, \mathbf{m})), z \right\rangle dx \\ & \quad - \lambda_2 \int_0^1 \left\langle \mathbf{m}^{(n)} \times (u^{(n)} \times \tilde{C}(X^{(n)}, \mathbf{m}^{(n)})) - \mathbf{m} \times (u \times \tilde{C}(X, \mathbf{m})), z \right\rangle dx \end{aligned}$$

$$\begin{aligned}
 & -2\lambda_2 \int_0^1 \left\langle \mathbf{m}_x^{(n)} \times (\mathbf{m}_x^{(n)} \times \tilde{C}(X^{(n)}, \mathbf{m}^{(n)})) \right. \\
 & \quad \left. - \mathbf{m}_x \times (\mathbf{m}_x \times \tilde{C}(X, \mathbf{m})), z \right\rangle dx \\
 & -2\lambda_2 \int_0^1 \left\langle \mathbf{m}_x^{(n)} \times (\mathbf{m}^{(n)} \times \tilde{C}_x(X^{(n)}, \mathbf{m}^{(n)})) \right. \\
 & \quad \left. - \mathbf{m}_x \times (\mathbf{m} \times \tilde{C}_x(X, \mathbf{m})), z \right\rangle dx \\
 & -2\lambda_2 \int_0^1 \left\langle \mathbf{m}^{(n)} \times (\mathbf{m}_x^{(n)} \times \tilde{C}_x(X^{(n)}, \mathbf{m}^{(n)})) \right. \\
 & \quad \left. - \mathbf{m} \times (\mathbf{m}_x \times \tilde{C}_x(X, \mathbf{m})), z \right\rangle dx \\
 & -\lambda_2 \int_0^1 \left\langle \mathbf{m}^{(n)} \times (\mathbf{m}^{(n)} \times \tilde{C}_{xx}(X^{(n)}, \mathbf{m}^{(n)})) \right. \\
 & \quad \left. - \mathbf{m} \times (\mathbf{m} \times \tilde{C}_{xx}(X, \mathbf{m})), z \right\rangle dx \\
 & =: \sum_{j=1}^9 I_{3,j}. \tag{49}
 \end{aligned}$$

We now derive estimates $I_{3,j}$ with $j = 1, 2, \dots, 9$ for fixed ε . Using (48a), (29) and Young's inequality, we have

$$\begin{aligned}
 I_{3,1} & \leq \frac{K}{2\varepsilon} |\lambda_1| \left[\left(|y|_{\mathbb{L}^\infty} + |y_x|_{\mathbb{L}^\infty} \right) + |X^{(n)} - X| \left[|\mathbf{m}|_{\mathbb{L}^\infty} + |\mathbf{m}_x|_{\mathbb{L}^\infty} \right] \right] |z|_{\mathbb{L}^2}^2 \\
 & \quad + \frac{K\varepsilon}{2} |\lambda_1| \left[\left(|y|_{\mathbb{L}^\infty} + |y_x|_{\mathbb{L}^\infty} \right) + |X^{(n)} - X| \left[|\mathbf{m}|_{\mathbb{L}^\infty} + |\mathbf{m}_x|_{\mathbb{L}^\infty} \right] \right] |u^{(n)}|_{\mathbb{L}^2}^2 \\
 & =: \varphi_{3,1} |z|_{\mathbb{L}^2}^2 + \chi_{3,1}.
 \end{aligned}$$

Again using (48a), (29) and Young's inequality, we get

$$\begin{aligned}
 I_{3,4} & \leq \frac{K}{2\varepsilon} \lambda_2 \left[\left(|\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} |X^{(n)} - X| + |y|_{\mathbb{L}^\infty} \right) \left(|\mathbf{m}|_{\mathbb{L}^\infty} + |\mathbf{m}_x|_{\mathbb{L}^\infty} \right) \right. \\
 & \quad \left. + |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} \left(|y|_{\mathbb{L}^\infty} + |y_x|_{\mathbb{L}^\infty} \right) \right] |z|_{\mathbb{L}^2}^2 + \frac{K\varepsilon}{2} \lambda_2 \left[|\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} \left(|y|_{\mathbb{L}^\infty} + |y_x|_{\mathbb{L}^\infty} \right) \right. \\
 & \quad \left. + \left(|\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} |X^{(n)} - X| + |y|_{\mathbb{L}^\infty} \right) \left(|\mathbf{m}|_{\mathbb{L}^\infty} + |\mathbf{m}_x|_{\mathbb{L}^\infty} \right) \right] |u^{(n)}|_{\mathbb{L}^2}^2 \\
 & =: \varphi_{3,4} |z|_{\mathbb{L}^2}^2 + \chi_{3,3}.
 \end{aligned}$$

Proceeding in similar manner as in $I_{3,4}$, we have

$$\begin{aligned}
 I_{3,5} & \leq \frac{K}{2\varepsilon} \lambda_2 \left[\left(|\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} |X^{(n)} - X| + |y|_{\mathbb{L}^\infty} \right) \left(|\mathbf{m}|_{\mathbb{L}^\infty} + |\mathbf{m}_x|_{\mathbb{L}^\infty} \right) \right. \\
 & \quad \left. + |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} \left(|y|_{\mathbb{L}^\infty} + |y_x|_{\mathbb{L}^\infty} \right) \right] |z|_{\mathbb{L}^2}^2 + \frac{K\varepsilon}{2} \lambda_2 \left[|\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} \left(|y|_{\mathbb{L}^\infty} + |y_x|_{\mathbb{L}^\infty} \right) \right.
 \end{aligned}$$

$$\begin{aligned}
& + \left(|\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} [|X^{(n)} - X| + 1] + |y|_{\mathbb{L}^\infty} \right) \left(|\mathbf{m}|_{\mathbb{L}^\infty} + |\mathbf{m}_x|_{\mathbb{L}^\infty} \right) \Big] |u^{(n)}|_{\mathbb{L}^2}^2 \\
& =: \varphi_{3,5} |z|_{\mathbb{L}^2}^2 + \chi_{3,5}.
\end{aligned}$$

Again, proceeding as in $I_{3,4}$, we obtain

$$\begin{aligned}
I_{3,6} & \leq 2K\lambda_2 |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty}^2 \left[\left(|y|_{\mathbb{L}^\infty} + |y_x|_{\mathbb{L}^\infty} \right) \right. \\
& \quad \left. + |X^{(n)} - X| \left[|\mathbf{m}|_{\mathbb{L}^\infty} + |\mathbf{m}_x|_{\mathbb{L}^\infty} \right] \right] |z|_{\mathbb{L}^2}^2 \\
& \quad + 2K\lambda_2 |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} |y_x|_{\mathbb{L}^\infty} \left[|\mathbf{m}|_{\mathbb{L}^\infty} + |\mathbf{m}_x|_{\mathbb{L}^\infty} \right] |z|_{\mathbb{L}^2}^2 \\
& \quad + 2K\lambda_2 |\mathbf{m}_x|_{\mathbb{L}^\infty} |y_x|_{\mathbb{L}^\infty} \left[|\mathbf{m}|_{\mathbb{L}^\infty} + |\mathbf{m}_x|_{\mathbb{L}^\infty} \right] |z|_{\mathbb{L}^2}^2 \\
& =: \varphi_{3,6} |z|_{\mathbb{L}^2}^2.
\end{aligned}$$

Now applying (48b), (30) and Young's inequality, we obtain

$$\begin{aligned}
I_{3,2} & \leq 2|\lambda_1| \left[K |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} \left(\frac{1}{\varepsilon} + 1 \right) + \frac{3}{2\varepsilon} K \left(|X^{(n)} - X| + |y_x|_{\mathbb{L}^\infty} \right) \right] |z|_{\mathbb{L}^2}^2 \\
& \quad + K \left[|X^{(n)} - X| + |y_x|_{\mathbb{L}^\infty} \right] \left(\frac{\varepsilon}{2} |\mathbf{m}|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{2} |\mathbf{m}_x|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{2} |u|_{\mathbb{L}^2}^2 \right) \\
& \quad + 2K|\lambda_1| |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} \left(\frac{\varepsilon}{2} |y|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{2} |y_x|_{\mathbb{L}^2}^2 \right) \\
& =: \varphi_{3,2} |z|_{\mathbb{L}^2}^2 + \chi_{3,2}.
\end{aligned}$$

Again using (48b), (30) and Young's inequality, we get

$$\begin{aligned}
I_{3,7} & \leq 2K\lambda_2 \left[\frac{3}{2\varepsilon} \left[|\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} |X^{(n)} - X| \right. \right. \\
& \quad \left. \left. + |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} |y|_{\mathbb{L}^\infty} + |\mathbf{m}|_{\mathbb{L}^\infty} |y_x|_{\mathbb{L}^\infty} \right] \right. \\
& \quad \left. + \left(\frac{1}{\varepsilon} + 1 \right) |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} \right] |z|_{\mathbb{L}^2}^2 \\
& \quad + 2K|\lambda_2| |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} \frac{\varepsilon}{2} |y|_{\mathbb{L}^2}^2 \\
& \quad + 2K\lambda_2 |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} \frac{\varepsilon}{2} |y_x|_{\mathbb{L}^2}^2 \\
& \quad + 2K|\lambda_2| \left(|\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} |X^{(n)} - X| \right. \\
& \quad \left. + |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} |y|_{\mathbb{L}^\infty} + |\mathbf{m}|_{\mathbb{L}^\infty} |y_x|_{\mathbb{L}^\infty} \right) \left(\frac{\varepsilon}{2} |\mathbf{m}|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{2} |\mathbf{m}_x|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{2} |u|_{\mathbb{L}^2}^2 \right) \\
& =: \varphi_{3,7} |z|_{\mathbb{L}^2}^2 + \chi_{3,7}.
\end{aligned}$$

Proceeding in similar manner as in $I_{3,7}$, we have

$$\begin{aligned}
 I_{3,8} &\leq 2K\lambda_2 \left[\frac{3}{2\varepsilon} \left[|\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} |X^{(n)} - X| \right. \right. \\
 &\quad \left. \left. + |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} |y_x|_{\mathbb{L}^\infty} + |\mathbf{m}_x|_{\mathbb{L}^\infty} |y|_{\mathbb{L}^\infty} \right] \right. \\
 &\quad \left. + \left(\frac{1}{\varepsilon} + 1 \right) |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} \right] |z|_{\mathbb{L}^2}^2 + 2K|\lambda_2| |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} \frac{\varepsilon}{2} |y|_{\mathbb{L}^2}^2 \\
 &\quad + 2K\lambda_2 |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} \frac{\varepsilon}{2} |y_x|_{\mathbb{L}^2}^2 \\
 &\quad + 2K|\lambda_2| \left(|\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} |X^{(n)} - X| \right. \\
 &\quad \left. + |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} |y_x|_{\mathbb{L}^\infty} + |\mathbf{m}_x|_{\mathbb{L}^\infty} |y|_{\mathbb{L}^\infty} \right) \left(\frac{\varepsilon}{2} |\mathbf{m}|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{2} |\mathbf{m}_x|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{2} |u|_{\mathbb{L}^2}^2 \right) \\
 &=: \varphi_{3,8} |z|_{\mathbb{L}^2}^2 + \chi_{3,8}.
 \end{aligned}$$

Now using Young's inequality, (48b) and (30), we get

$$\begin{aligned}
 I_{3,3} &\leq |\lambda_1| \left[K |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} \left(\frac{3}{2\varepsilon} + 1 \right) + \frac{2}{\varepsilon} K \left(|X^{(n)} - X| + |y|_{\mathbb{L}^\infty} \right) \right] |z|_{\mathbb{L}^2}^2 \\
 &\quad + \frac{K\varepsilon}{2} |\lambda_1| |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} |z_x|_{\mathbb{L}^2}^2 + K|\lambda_1| |\mathbf{m}_x^{(n)}|_{\mathbb{L}^\infty} \left(\frac{\varepsilon}{2} |y|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{2} |y_x|_{\mathbb{L}^2}^2 \right) \\
 &\quad + K|\lambda_1| \left[|X^{(n)} - X| + |y|_{\mathbb{L}^\infty} \right] \left(\frac{\varepsilon}{2} |\mathbf{m}|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{2} |\mathbf{m}_x|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{2} |u|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{2} |u_x|_{\mathbb{L}^2}^2 \right) \\
 &=: \psi_{3,3} |z_x|_{\mathbb{L}^2}^2 + \varphi_{3,3} |z|_{\mathbb{L}^2}^2 + \chi_{3,3}.
 \end{aligned}$$

Again using (48c), (31) and Young's inequality, we get

$$\begin{aligned}
 I_{3,9} &\leq K\lambda_2 \left(|\mathbf{m}^{(n)}|_{\mathbb{L}^\infty}^2 + |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} |X^{(n)} - X| \right. \\
 &\quad \left. + |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} |y|_{\mathbb{L}^\infty} + |\mathbf{m}|_{\mathbb{L}^\infty} |y|_{\mathbb{L}^\infty} \right) |z|_{\mathbb{L}^2}^2 \\
 &\quad + K\lambda_2 |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty}^2 \frac{\varepsilon}{2} |z_x|_{\mathbb{L}^2}^2 + K\lambda_2 |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty}^2 \left(\frac{\varepsilon}{2} |y|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{2} |y_x|_{\mathbb{L}^2}^2 \right) \\
 &\quad + K\lambda_2 \left[|\mathbf{m}^{(n)}|_{\mathbb{L}^\infty}^2 |X^{(n)} - X| + |\mathbf{m}^{(n)}|_{\mathbb{L}^\infty} |y|_{\mathbb{L}^\infty} + |\mathbf{m}|_{\mathbb{L}^\infty} |y|_{\mathbb{L}^\infty} \right] \left(\frac{\varepsilon}{2} |\mathbf{m}|_{\mathbb{L}^2}^2 \right. \\
 &\quad \left. + \frac{\varepsilon}{2} |\mathbf{m}_x|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{2} |u|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{2} |u_x|_{\mathbb{L}^2}^2 \right) \\
 &=: \psi_{3,9} |z_x|_{\mathbb{L}^2}^2 + \varphi_{3,9} |z|_{\mathbb{L}^2}^2 + \chi_{3,9}.
 \end{aligned}$$

Thus combining the estimates of $I_{3,j}$, $j = 1, 2, \dots, 9$ and substituting back in (49), we have

$$\begin{aligned} I_3 &\leq (\psi_{3,3} + \psi_{3,9}) |z_x|_{\mathbb{L}^2} + \left[\sum_{j=1}^9 \varphi_{3,j} \right] |z|_{\mathbb{L}^2} + \left[\sum_{j \neq 6} \chi_{3,j} \right] \\ &=: \psi_3 |z_x|_{\mathbb{L}^2}^2 + \varphi_3 |z|_{\mathbb{L}^2}^2 + \chi_3. \end{aligned} \quad (50)$$

Substituting (43), (45) and (50) in (42) we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |z|_{\mathbb{L}^2}^2 &\leq \left[\sum_{j=1}^3 \psi_j \right] |z_x|_{\mathbb{L}^2} + \left[\sum_{j=1}^3 \varphi_j \right] |z|_{\mathbb{L}^2} + \left[\sum_{j=1}^3 \chi_j \right] \\ &= \left[\left(\frac{5}{2} + |\lambda_1| \left[1 + \frac{K |m_x^{(n)}|_{\mathbb{L}^\infty}}{2} \right] + \frac{K |m^{(n)}|_{\mathbb{L}^\infty}^2}{2} \lambda_2 \right) \varepsilon - \lambda_2 \right] |z_x|_{\mathbb{L}^2}^2 \end{aligned} \quad (51)$$

$$+ \left[\sum_{j=1}^3 \varphi_j \right] |z|_{\mathbb{L}^2} + \left[\sum_{j=1}^3 \chi_j \right]. \quad (52)$$

We note by (20) and Corollary 4, we have $m^{(n)}, m_x^{(n)} \in L^\infty(0, T; \mathbb{L}^\infty)$ for all $n \in \mathbb{N}$. Therefore we can choose ε such that

$$\frac{1}{2} \frac{d}{dt} |z(t)|_{\mathbb{L}^2}^2 \leq \left[\sum_{j=1}^3 \varphi_j \right] |z|_{\mathbb{L}^2} + \left[\sum_{j=1}^3 \chi_j \right] =: \varphi(t) |z(t)|_{\mathbb{L}^2} + \chi(t). \quad (53)$$

By Lemma 8, we note that $m, m^n \in L^\infty(0, T; \mathbb{H}^1) \cap L^2(0, T; D(A))$. From Lemma 9 and Corollary 4, we obtain $m^{(n)}, m, m_x^{(n)}, m_x \in L^\infty(0, T; \mathbb{L}^\infty)$, $m_{xx}, m_{xx}^{(n)} \in L^\infty(0, T; \mathbb{L}^2)$ and $u_x, u_x^{(n)} \in L^2(0, T; \mathbb{L}^2)$. Thus φ and χ are integrable on $[0, T]$. Using Gronwall's inequality we get

$$|z(t)|_{\mathbb{L}^2}^2 \leq \left(|z_0|_{\mathbb{L}^2}^2 + \int_0^t \chi(s) ds \right) e^{2 \int_0^t \varphi(s) ds}.$$

Now by Lemma 3 and (8), we note that

$$|m_x^{(n)} - m_x|_{L^\infty(0, T; \mathbb{L}^\infty)}^2 \leq k^2 |m_x^{(n)} - m_x|_{L^\infty(0, T; \mathbb{L}^2)}^2 |m_{xx}^{(n)} - m_{xx}|_{L^\infty(0, T; \mathbb{L}^2)}^2.$$

As $m_{xx}, m_{xx}^{(n)} \in L^\infty(0, T; \mathbb{L}^2)$, we obtain $m_x^{(n)} \rightarrow m_x$ in $L^\infty(0, T; \mathbb{L}^\infty)$. Moreover, we also have $X^{(n)} \rightarrow X$ in $\mathcal{C}_g^\alpha([0, T], \mathbb{R})$. This implies that $z \rightarrow 0$ in $L^\infty(0, T; \mathbb{L}^2)$. Hence

$$m_{xx}^{(n)} \rightarrow m_{xx} \text{ in } L^\infty(0, T; \mathbb{L}^2), \text{ as } n \rightarrow \infty.$$

This completes the proof of Lemma 4. $\square$

Corollary 2 Let $\mathbf{M}$ and $\mathbf{M}^{(n)}$ be the solutions to the systems (5) and (7), respectively. Then we have the following convergence

$$\mathbf{M}_{xx}^{(n)} \rightarrow \mathbf{M}_{xx} \text{ in } L^\infty(0, T; \mathbb{L}^2), \text{ as } n \rightarrow \infty.$$

Proof Using the identity (91), we obtain

$$\begin{aligned} \mathbf{M}_{xx}(t) &= \mathbf{m}_{xx}(t) + \sin(X(t))(\mathbf{m} \times \mathbf{g})_{xx} + (1 - \cos(X(t)))((\mathbf{m} \times \mathbf{g}) \times \mathbf{g})_{xx}, \\ \mathbf{M}_{xx}^{(n)}(t) &= \mathbf{m}_{xx}^{(n)}(t) + \sin(X^{(n)}(t))(\mathbf{m}^{(n)} \times \mathbf{g})_{xx} \\ &\quad + (1 - \cos(X^{(n)}(t)))((\mathbf{m}^{(n)} \times \mathbf{g}) \times \mathbf{g})_{xx}. \end{aligned}$$

Thus, we have

$$\begin{aligned} (\mathbf{M}^{(n)}(t) - \mathbf{M}(t))_{xx} &= (\mathbf{m}^{(n)}(t) - \mathbf{m}(t))_{xx} \\ &\quad + (\sin(X^{(n)}(t)) - \sin(X(t))) (\mathbf{m}^{(n)}(t) \times \mathbf{g})_{xx} \\ &\quad + \sin(X(t)) ((\mathbf{m}^{(n)}(t) - \mathbf{m}(t)) \times \mathbf{g})_{xx} \\ &\quad - (\cos(X^{(n)}(t)) - \cos(X(t))) ((\mathbf{m}^{(n)}(t) \times \mathbf{g}) \times \mathbf{g})_{xx} \\ &\quad + (1 - \cos(X(t))) ((\mathbf{m}^{(n)}(t) - \mathbf{m}(t)) \times \mathbf{g}) \times \mathbf{g}_{xx}. \end{aligned}$$

Using $|\cos(x_1) - \cos(x_2)|, |\sin(x_1) - \sin(x_2)| \leq c|x_1 - x_2|$, and the boundedness of $\sin$ and $\cos$, we get

$$\begin{aligned} |(\mathbf{M}^{(n)} - \mathbf{M})_{xx}|_{L^\infty(0, T; \mathbb{L}^2)} &\leq |(\mathbf{m}^{(n)} - \mathbf{m})_{xx}|_{L^\infty(0, T; \mathbb{L}^2)} \\ &\quad + |X^{(n)} - X|_{L^\infty(0, T; \mathbb{R})} |(\mathbf{m}^{(n)}(t) \times \mathbf{g})_{xx}|_{L^\infty(0, T; \mathbb{L}^2)} \\ &\quad + |((\mathbf{m}^{(n)}(t) - \mathbf{m}(t)) \times \mathbf{g})_{xx}|_{L^\infty(0, T; \mathbb{L}^2)} \\ &\quad + |X^{(n)} - X|_{L^\infty(0, T; \mathbb{R})} \\ &\quad \quad \times |((\mathbf{m}^{(n)}(t) \times \mathbf{g}) \times \mathbf{g})_{xx}|_{L^\infty(0, T; \mathbb{L}^2)} \\ &\quad + |((\mathbf{m}^{(n)}(t) - \mathbf{m}(t)) \times \mathbf{g}) \times \mathbf{g}|_{xx}|_{L^\infty(0, T; \mathbb{L}^2)}. \end{aligned}$$

Using Lemma 4, Lemma 3 and that $X^{(n)} \rightarrow X$ in $\mathcal{C}_g^\alpha([0, T], \mathbb{R})$, we obtain

$$\mathbf{M}_{xx}^{(n)} \rightarrow \mathbf{M}_{xx} \text{ in } L^\infty(0, T; \mathbb{L}^2), \text{ as } n \rightarrow \infty.$$

□

Corollary 3 Let $\mathbf{M}$ and $\mathbf{M}^{(n)}$ be the solutions to the systems (5) and (7), respectively. Then, we have

$$\begin{aligned}\mathbf{M}^{(n)} &\rightarrow \mathbf{M}, \\ \mathbf{M}^{(n)} \times \mathbf{M}_{xx}^{(n)} &\rightarrow \mathbf{M} \times \mathbf{M}_{xx}, \\ \mathbf{M}^{(n)} \times (\mathbf{M}^{(n)} \times \mathbf{M}_{xx}^{(n)}) &\rightarrow \mathbf{M} \times (\mathbf{M} \times \mathbf{M}_{xx})\end{aligned}$$

in $L^\infty(0, T; \mathbb{L}^2)$ as $n \rightarrow \infty$.

Proof This is a direct consequence of Corollaries 1 and 2. $\square$

Now we are ready to prove Theorem 4.

Proof of Theorem 4 Since $(\mathbf{M}, \mathbf{M} \times \mathbf{g}) \in \mathcal{D}_X^{2\alpha}([0, T], \mathbb{L}^2)$ and $(\mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g}) \in \mathcal{D}_{X^{(n)}}^{2\alpha}([0, T], \mathbb{L}^2)$, there exists a number $N \in \mathbb{R}^+$ such that

$$\sup_{s \in [0, T]} |\mathbf{M}_s|_{\mathbb{L}^2} + \sup_{s \in [0, T]} |\mathbf{M}_s \times \mathbf{g}|_{\mathbb{L}^2} + |\mathbf{M}, \mathbf{M} \times \mathbf{g}|_{X, 2\alpha, [0, T], \mathbb{L}^2} \leq N, \quad (54)$$

$$\sup_{s \in [0, T]} |\mathbf{M}_s^{(n)}|_{\mathbb{L}^2} + \sup_{s \in [0, T]} |\mathbf{M}_s^{(n)} \times \mathbf{g}|_{\mathbb{L}^2} + |\mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g}|_{X, 2\alpha, [0, T], \mathbb{L}^2} \leq N, \quad (55)$$

$$|X|_{\alpha, [0, T], \mathbb{R}} + |\mathbb{X}|_{2\alpha, [0, T], \mathbb{R}} \leq N, \quad (56)$$

$$|X^{(n)}|_{\alpha, [0, T], \mathbb{R}} + |\mathbb{X}^{(n)}|_{2\alpha, [0, T], \mathbb{R}} \leq N. \quad (57)$$

Let $0 < T_1 \leq \min\{1, T\}$. In particular, we assume that $C_{\alpha, N, g} T_1^\alpha \leq \frac{1}{2}$, where $C_{\alpha, N, g} > 0$ is a generic constant depending only on α , N , and g and popping up in estimate (59). In the next lines we proof the following estimate

$$\begin{aligned}d_{X, X^{(n)}, 2\alpha, [0, T_1], \mathbb{L}^2}(\mathbf{M}, \mathbf{M} \times \mathbf{g}; \mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g}) \\ = |\mathbf{M} \times \mathbf{g} - \mathbf{M}^{(n)} \times \mathbf{g}|_{\alpha, [0, T_1], \mathbb{L}^2} + |\mathcal{R}^{\mathbf{M}} - \mathcal{R}^{\mathbf{M}^{(n)}}|_{2\alpha, [0, T_1], \mathbb{L}^2}.\end{aligned}$$

Let us compute $|\mathbf{M} \times \mathbf{g} - \mathbf{M}^{(n)} \times \mathbf{g}|_{\alpha, [0, T_1], \mathbb{L}^2}$. Due to the fact that $\mathbf{g} \in \mathbb{W}^{2, \infty}$, we can write for $0 \leq s < t \leq T_1$

$$|\mathbf{M}_{s,t} \times \mathbf{g} - \mathbf{M}_{s,t}^{(n)} \times \mathbf{g}|_{\mathbb{L}^2} \leq |\mathbf{M}_{s,t} - \mathbf{M}_{s,t}^{(n)}|_{\mathbb{L}^2} |\mathbf{g}|_{\mathbb{L}^\infty}.$$

Next, using the identities $\mathbf{M}_{s,t} = \mathbf{M}'_s X_{s,t} + \mathcal{R}_{s,t}^{\mathbf{M}}$ and $\mathbf{M}_{s,t}^{(n)} = (\mathbf{M}^{(n)})'_s X_{s,t}^{(n)} + \mathcal{R}_{s,t}^{\mathbf{M}^{(n)}}$, where $\mathbf{M}' = \mathbf{M} \times \mathbf{g}$ and $(\mathbf{M}^{(n)})' = \mathbf{M}^{(n)} \times \mathbf{g}$, we obtain

$$\begin{aligned}|\mathbf{M}_{s,t} \times \mathbf{g} - \mathbf{M}_{s,t}^{(n)} \times \mathbf{g}|_{\mathbb{L}^2} \\ \leq \left| \mathbf{M}'_s X_{s,t} + \mathcal{R}_{s,t}^{\mathbf{M}} - \left((\mathbf{M}^{(n)})'_s X_{s,t}^{(n)} + \mathcal{R}_{s,t}^{\mathbf{M}^{(n)}} \right) \right|_{\mathbb{L}^2} |\mathbf{g}|_{\mathbb{L}^\infty} \\ \leq \left[\left(|\mathbf{M}'|_{\alpha, [0, T], \mathbb{L}^2} \left| X - X^{(n)} \right|_{\alpha, [0, T], \mathbb{R}} \right. \right.\end{aligned}$$

$$\begin{aligned}
 & + \left| \mathbf{M}' - (\mathbf{M}^{(n)})' \right|_{\alpha, [0, T_1], \mathbb{L}^2} \left| X^{(n)} \right|_{\alpha, [0, T], \mathbb{R}} T_1^\alpha \\
 & + \left(\left| \mathbf{M}'_0 \right|_{\mathbb{L}^2} \left| X - X^{(n)} \right|_{\alpha, [0, T], \mathbb{R}} + \left| \mathbf{M}'_0 - (\mathbf{M}^{(n)})'_0 \right|_{\mathbb{L}^2} \left| X^{(n)} \right|_{\alpha, [0, T], \mathbb{R}} \right) \\
 & + \left| \mathcal{R}^{\mathbf{M}} - \mathcal{R}^{\mathbf{M}^{(n)}} \right|_{2\alpha, [0, T_1], \mathbb{L}^2} T_1^\alpha \left] |\mathbf{g}|_{\mathbb{L}^\infty} |t - s|^\alpha.
 \end{aligned}$$

Using (54)–(57), we have

$$\begin{aligned}
 & |\mathbf{M}_{s,t} \times \mathbf{g} - \mathbf{M}_{s,t}^{(n)} \times \mathbf{g}|_{\mathbb{L}^2} \\
 & \leq C_{N,g} \left[(1 + T_1^\alpha) \left| X - X^{(n)} \right|_{\alpha, [0, T], \mathbb{R}} + \left| \mathbf{M}' - (\mathbf{M}^{(n)})' \right|_{\alpha, [0, T_1], \mathbb{L}^2} T_1^\alpha \right. \\
 & \quad \left. + \left| \mathbf{M} - \mathbf{M}^{(n)} \right|_{L^\infty([0, T], \mathbb{L}^2)} + \left| \mathcal{R}^{\mathbf{M}} - \mathcal{R}^{\mathbf{M}^{(n)}} \right|_{2\alpha, [0, T_1], \mathbb{L}^2} T_1^\alpha \right] |t - s|^\alpha \\
 & = C_{N,g} \left[(1 + T_1^\alpha) \varrho_{\alpha, [0, T], \mathbb{R}}(X, X^{(n)}) \right. \\
 & \quad + T_1^\alpha d_{X, X^{(n)}, 2\alpha, [0, T_1], \mathbb{L}^2}(\mathbf{M}, \mathbf{M} \times \mathbf{g}; \mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g}) \\
 & \quad \left. + \left| \mathbf{M} - \mathbf{M}^{(n)} \right|_{L^\infty([0, T], \mathbb{L}^2)} \right] |t - s|^\alpha.
 \end{aligned}$$

This implies

$$\begin{aligned}
 & |\mathbf{M} \times \mathbf{g} - \mathbf{M}^{(n)} \times \mathbf{g}|_{\alpha, [0, T_1], \mathbb{L}^2} \\
 & \leq C_{N,g} \left[T_1^\alpha \varrho_{\alpha, [0, T], \mathbb{R}}(X, X^{(n)}) \right. \\
 & \quad + T_1^\alpha d_{X, X^{(n)}, 2\alpha, [0, T_1], \mathbb{L}^2}(\mathbf{M}, \mathbf{M} \times \mathbf{g}; \mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g}) \\
 & \quad \left. + \left| \mathbf{M} - \mathbf{M}^{(n)} \right|_{L^\infty([0, T], \mathbb{L}^2)} \right]. \tag{58}
 \end{aligned}$$

We now start to estimate $|\mathcal{R}^{\mathbf{M}} - \mathcal{R}^{\mathbf{M}^{(n)}}|_{2\alpha, [0, T_1], \mathbb{L}^2}$. Using system (5), we obtain

$$\begin{aligned}
 \mathcal{R}_{s,t}^{\mathbf{M}} &= \int_s^t \lambda_1 \mathbf{M}(r) \times \mathbf{M}_{xx}(r) \, dr - \int_s^t \lambda_2 \mathbf{M}(r) \times (\mathbf{M}(r) \times \mathbf{M}_{xx}(r)) \, dr \\
 & \quad + (\mathcal{I}\mathcal{E})_{s,t} - \mathcal{E}_{s,t} + (\mathbf{M}'_s \times \mathbf{g}) \mathbb{X}_{s,t},
 \end{aligned}$$

where $\mathcal{E}_{s,t} := (\mathbf{M}_s \times \mathbf{g})X_{s,t} + (\mathbf{M}'_s \times \mathbf{g})\mathbb{X}_{s,t}$ and

$$(\mathcal{I}\mathcal{E})_{s,t} := \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} \mathcal{E}_{u,v}.$$

In similar way, we also obtain the same identity as above for $\mathcal{R}_{s,t}^{M^{(n)}}$, i.e.

$$\begin{aligned}\mathcal{R}_{s,t}^{M^{(n)}} &= \int_s^t \lambda_1 \mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r) dr - \int_s^t \lambda_2 \mathbf{M}^{(n)}(r) \times (\mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r)) dr \\ &\quad + (\mathcal{I}\mathcal{E}^{(n)})_{s,t} - \mathcal{E}_{s,t}^{(n)} + ((\mathbf{M}_s^{(n)})' \times \mathbf{g}) \mathbb{X}_{s,t}^{(n)},\end{aligned}$$

where $\mathcal{E}_{s,t}^{(n)} := (\mathbf{M}_s^{(n)} \times \mathbf{g})X_{s,t}^{(n)} + ((\mathbf{M}_s^{(n)})' \times \mathbf{g})\mathbb{X}_{s,t}^{(n)}$ and

$$(\mathcal{I}\mathcal{E})_{s,t} := \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} \mathcal{E}_{u,v}^{(n)}.$$

Setting $\Psi = \mathcal{E} - \mathcal{E}^{(n)}$, we use equation (4.11) of [14] with $\beta = 3\alpha$ and replaced $\mathcal{E}$ by Ψ , so that we can write for $0 \leq s < t \leq T_1$

$$\begin{aligned}&|\mathcal{R}_{s,t}^M - \mathcal{R}_{s,t}^{M^{(n)}}|_{\mathbb{L}^2} \\ &\leq |(\mathcal{I}\Psi)_{s,t} - \Psi_{s,t}|_{\mathbb{L}^2} + |(\mathbf{M}_s' \times \mathbf{g})\mathbb{X}_{s,t} - ((\mathbf{M}_s^{(n)})' \times \mathbf{g})\mathbb{X}_{s,t}^{(n)}| \\ &\quad + |\lambda_1| \int_s^t |\mathbf{M}(r) \times \mathbf{M}_{xx}(r) - \mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r)|_{\mathbb{L}^2} dr \\ &\quad + |\lambda_2| \int_s^t |\mathbf{M}(r) \times (\mathbf{M}(r) \times \mathbf{M}_{xx}(r)) \\ &\quad - \mathbf{M}^{(n)}(r) \times (\mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r))|_{\mathbb{L}^2} dr \\ &\leq C_\alpha |\delta\Psi|_{3\alpha, [0, T_1], \mathbb{L}^2} |t - s|^{3\alpha} \\ &\quad + |\mathbf{M}_s' \times \mathbf{g}|_{\mathbb{L}^2} |\mathbb{X}_{s,t} - \mathbb{X}_{s,t}^{(n)}|_{\mathbb{R}} + |\mathbf{M}_s' \times \mathbf{g} - (\mathbf{M}_s^{(n)})' \times \mathbf{g}|_{\mathbb{L}^2} |\mathbb{X}_{s,t}^{(n)}|_{\mathbb{R}} \\ &\quad + |\lambda_1| |\mathbf{M}(r) \times \mathbf{M}_{xx}(r) - \mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r)|_{L^\infty(0, T; \mathbb{L}^2)} |t - s| \\ &\quad + |\lambda_2| |\mathbf{M}(r) \times (\mathbf{M}(r) \times \mathbf{M}_{xx}(r)) \\ &\quad - \mathbf{M}^{(n)}(r) \times (\mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r))|_{L^\infty(0, T; \mathbb{L}^2)} |t - s|,\end{aligned}$$

where

$$\begin{aligned}\delta\Psi_{s,u,t} &:= \Psi_{s,t} - \Psi_{s,u} - \Psi_{u,t} \\ &= \mathcal{R}_{s,u}^{M^{(n)} \times \mathbf{g}} X_{u,t}^{(n)} - \mathcal{R}_{s,u}^{M \times \mathbf{g}} X_{u,t} \\ &\quad + ((\mathbf{M}^{(n)})'_{s,u} \times \mathbf{g}) \mathbb{X}_{u,t}^{(n)} - (\mathbf{M}'_{s,u} \times \mathbf{g}) \mathbb{X}_{u,t}.\end{aligned}$$

For $0 \leq s < u < t \leq T_1$, we get now

$$\begin{aligned}&|\delta\Psi_{s,u,t}| \\ &\leq |\mathbf{g}|_{\mathbb{L}^\infty} N |X^{(n)} - X|_{\alpha, [0, T], \mathbb{R}} |t - s|^{3\alpha} \\ &\quad + |\mathbf{g}|_{\mathbb{L}^\infty} |\mathcal{R}^{M^{(n)}} - \mathcal{R}^M|_{2\alpha, [0, T_1], \mathbb{L}^2} N |t - s|^{3\alpha}\end{aligned}$$

$$\begin{aligned}
 & + |\mathbf{g}|_{\mathbb{L}^\infty} N |\mathbb{X}^{(n)} - \mathbb{X}|_{2\alpha, [0, T], \mathbb{R}} |t - s|^{3\alpha} \\
 & + |\mathbf{g}|_{\mathbb{L}^\infty} |(\mathbf{M}^{(n)})' - \mathbf{M}'|_{\alpha, [0, T_1], \mathbb{L}^2} N |t - s|^{3\alpha} \\
 & \leq C_{N, \mathbf{g}} |t - s|^{3\alpha} \left[\varrho_{\alpha, [0, T], \mathbb{R}}(\mathbf{X}, \mathbf{X}^{(n)}) \right. \\
 & \quad \left. + d_{X, X^{(n)}, 2\alpha, [0, T_1], \mathbb{L}^2}(\mathbf{M}, \mathbf{M} \times \mathbf{g}; \mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g}) \right].
 \end{aligned}$$

Continuing, we obtain

$$\begin{aligned}
 & |\mathcal{R}_{s,t}^{\mathbf{M}} - \mathcal{R}_{s,t}^{\mathbf{M}^{(n)}}|_{\mathbb{L}^2} \\
 & \leq C_{\alpha, N, \mathbf{g}} |t - s|^{2\alpha} |\mathbb{X} - \mathbb{X}^{(n)}|_{2\alpha, [0, T_1], \mathbb{R}} + C_{\alpha, N, \mathbf{g}} |t - s|^{2\alpha} |\mathbf{M} - \mathbf{M}^{(n)}|_{L^\infty(0, T; \mathbb{L}^2)} \\
 & \quad + C_{\alpha, N, \mathbf{g}} |t - s|^{3\alpha} \left[\varrho_{\alpha, [0, T], \mathbb{R}}(\mathbf{X}, \mathbf{X}^{(n)}) \right. \\
 & \quad \left. + d_{X, X^{(n)}, 2\alpha, [0, T_1], \mathbb{L}^2}(\mathbf{M}, \mathbf{M} \times \mathbf{g}; \mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g}) \right] \\
 & \quad + |\lambda_1| |\mathbf{M}(r) \times \mathbf{M}_{xx}(r) - \mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r)|_{L^\infty(0, T; \mathbb{L}^2)} |t - s| \\
 & \quad + |\lambda_2| |\mathbf{M}(r) \times (\mathbf{M}(r) \times \mathbf{M}_{xx}(r)) \\
 & \quad - \mathbf{M}^{(n)}(r) \times (\mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r))|_{L^\infty(0, T; \mathbb{L}^2)} |t - s| \\
 & \leq C_{\alpha, N, \mathbf{g}} |t - s|^{2\alpha} \left[(1 + T_1^\alpha) \varrho_{\alpha, [0, T], \mathbb{R}}(\mathbf{X}, \mathbf{X}^{(n)}) + |\mathbf{M} - \mathbf{M}^{(n)}|_{L^\infty(0, T; \mathbb{L}^2)} \right. \\
 & \quad \left. + T_1^\alpha d_{X, X^{(n)}, 2\alpha, [0, T_1], \mathbb{L}^2}(\mathbf{M}, \mathbf{M} \times \mathbf{g}; \mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g}) \right] \\
 & \quad + |\lambda_1| |\mathbf{M}(r) \times \mathbf{M}_{xx}(r) - \mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r)|_{L^\infty(0, T; \mathbb{L}^2)} |t - s|^{2\alpha} T^{1-2\alpha} \\
 & \quad + |\lambda_2| |\mathbf{M}(r) \times (\mathbf{M}(r) \times \mathbf{M}_{xx}(r)) \\
 & \quad - \mathbf{M}^{(n)}(r) \times (\mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r))|_{L^\infty(0, T; \mathbb{L}^2)} |t - s|^{2\alpha} T^{1-2\alpha}.
 \end{aligned}$$

This implies

$$\begin{aligned}
 & |\mathcal{R}^{\mathbf{M}} - \mathcal{R}^{\mathbf{M}^{(n)}}|_{2\alpha, [0, T_1], \mathbb{L}^2} \\
 & \leq C_{\alpha, N, \mathbf{g}} \left[(1 + T_1^\alpha) \varrho_{\alpha, [0, T], \mathbb{R}}(\mathbf{X}, \mathbf{X}^{(n)}) + |\mathbf{M} - \mathbf{M}^{(n)}|_{L^\infty(0, T; \mathbb{L}^2)} \right. \\
 & \quad \left. + T_1^\alpha d_{X, X^{(n)}, 2\alpha, [0, T_1], \mathbb{L}^2}(\mathbf{M}, \mathbf{M} \times \mathbf{g}; \mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g}) \right] \\
 & \quad + |\lambda_1| |\mathbf{M}(r) \times \mathbf{M}_{xx}(r) - \mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r)|_{L^\infty(0, T; \mathbb{L}^2)} T^{1-2\alpha} \\
 & \quad + |\lambda_2| |\mathbf{M}(r) \times (\mathbf{M}(r) \times \mathbf{M}_{xx}(r)) \\
 & \quad - \mathbf{M}^{(n)}(r) \times (\mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r))|_{L^\infty(0, T; \mathbb{L}^2)} T^{1-2\alpha}. \tag{59}
 \end{aligned}$$

Finally, using (58) and (59), we obtain the estimate

$$\begin{aligned}
 & d_{X, X^{(n)}, 2\alpha, [0, T_1], \mathbb{L}^2}(\mathbf{M}, \mathbf{M} \times \mathbf{g}; \mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g}) \\
 & \leq C_{\alpha, N, \mathbf{g}} \left[(1 + T_1^\alpha) \varrho_{\alpha, [0, T], \mathbb{R}}(\mathbf{X}, \mathbf{X}^{(n)}) + |\mathbf{M} - \mathbf{M}^{(n)}|_{L^\infty(0, T; \mathbb{L}^2)} \right]
 \end{aligned}$$

$$\begin{aligned}
& + T_1^\alpha d_{X, X^{(n)}, 2\alpha, [0, T_1], \mathbb{L}^2}(\mathbf{M}, \mathbf{M} \times \mathbf{g}; \mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g}) \Big] \\
& + |\lambda_1| |\mathbf{M}(r) \times \mathbf{M}_{xx}(r) - \mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r)|_{L^\infty(0, T; \mathbb{L}^2)} T^{1-2\alpha} \\
& + |\lambda_2| |\mathbf{M}(r) \times (\mathbf{M}(r) \times \mathbf{M}_{xx}(r)) \\
& - \mathbf{M}^{(n)}(r) \times (\mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r))|_{L^\infty(0, T; \mathbb{L}^2)} T^{1-2\alpha}. \tag{60}
\end{aligned}$$

Observe, we have taken $C_{\alpha, N, g} T_1^\alpha \leq \frac{1}{2}$, from which it follows that

$$\begin{aligned}
& d_{X, X^{(n)}, 2\alpha, [0, T_1], \mathbb{L}^2}(\mathbf{M}, \mathbf{M} \times \mathbf{g}; \mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g}) \\
& \leq 2C_{\alpha, N, g} \Big[(1 + T_1^\alpha) \varrho_{\alpha, [0, T], \mathbb{R}}(X, X^{(n)}) + |\mathbf{M} - \mathbf{M}^{(n)}|_{L^\infty(0, T; \mathbb{L}^2)} \Big] \\
& + 2|\lambda_1| |\mathbf{M}(r) \times \mathbf{M}_{xx}(r) - \mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r)|_{L^\infty(0, T; \mathbb{L}^2)} T^{1-2\alpha} \\
& + 2|\lambda_2| |\mathbf{M}(r) \times (\mathbf{M}(r) \times \mathbf{M}_{xx}(r)) \\
& - \mathbf{M}^{(n)}(r) \times (\mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r))|_{L^\infty(0, T; \mathbb{L}^2)} T^{1-2\alpha}.
\end{aligned}$$

Noting that the choice T_1 does not depend on the initial condition, we can extend the estimate by gluing techniques to the whole time interval $[0, T]$ and get

$$\begin{aligned}
& d_{X, X^{(n)}, 2\alpha, [(j-1)T_1, jT_1], \mathbb{L}^2}(\mathbf{M}, \mathbf{M} \times \mathbf{g}; \mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g}) \\
& \leq 2C_{\alpha, N, g} \Big[(1 + T_1^\alpha) \varrho_{\alpha, [0, T], \mathbb{R}}(X, X^{(n)}) + |\mathbf{M} - \mathbf{M}^{(n)}|_{L^\infty(0, T; \mathbb{L}^2)} \Big] \\
& + 2|\lambda_1| |\mathbf{M}(r) \times \mathbf{M}_{xx}(r) - \mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r)|_{L^\infty(0, T; \mathbb{L}^2)} T^{1-2\alpha} \\
& + 2|\lambda_2| |\mathbf{M}(r) \times (\mathbf{M}(r) \times \mathbf{M}_{xx}(r)) \\
& - \mathbf{M}^{(n)}(r) \times (\mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r))|_{L^\infty(0, T; \mathbb{L}^2)} T^{1-2\alpha}, \tag{61}
\end{aligned}$$

for $j = 1, \dots, K$ where $K \in \mathbb{N}$ and $K < \infty$ such that $KT_1 < T$ and $(K+1)T_1 \geq T$. Also, we obtain

$$\begin{aligned}
& d_{X, X^{(n)}, 2\alpha, [KT_1, T], \mathbb{L}^2}(\mathbf{M}, \mathbf{M} \times \mathbf{g}; \mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g}) \\
& \leq 2C_{\alpha, N, g} \Big[(1 + T_1^\alpha) \varrho_{\alpha, [0, T], \mathbb{R}}(X, X^{(n)}) + |\mathbf{M} - \mathbf{M}^{(n)}|_{L^\infty(0, T; \mathbb{L}^2)} \Big] \\
& + 2|\lambda_1| |\mathbf{M}(r) \times \mathbf{M}_{xx}(r) - \mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r)|_{L^\infty(0, T; \mathbb{L}^2)} T^{1-2\alpha} \\
& + 2|\lambda_2| |\mathbf{M}(r) \times (\mathbf{M}(r) \times \mathbf{M}_{xx}(r)) \\
& - \mathbf{M}^{(n)}(r) \times (\mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r))|_{L^\infty(0, T; \mathbb{L}^2)} T^{1-2\alpha}. \tag{62}
\end{aligned}$$

Therefore, combining (61) and (62), we achieve

$$\begin{aligned}
& d_{X, X^{(n)}, 2\alpha, [0, T], \mathbb{L}^2}(\mathbf{M}, \mathbf{M} \times \mathbf{g}; \mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g}) \\
& \leq \sum_{j=1}^K d_{X, X^{(n)}, 2\alpha, [(j-1)T_1, jT_1], \mathbb{L}^2}(\mathbf{M}, \mathbf{M} \times \mathbf{g}; \mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g})
\end{aligned}$$

$$\begin{aligned}
 & + d_{X, X^{(n)}, 2\alpha, [KT1, T], \mathbb{L}^2}(\mathbf{M}, \mathbf{M} \times \mathbf{g}; \mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g}) \\
 & \leq 2(K+1)C_{\alpha, N, g} \left[(1 + T_1^\alpha) \varrho_{\alpha, [0, T], \mathbb{R}}(X, X^{(n)}) + |\mathbf{M} - \mathbf{M}^{(n)}|_{L^\infty(0, T; \mathbb{L}^2)} \right] \\
 & \quad + 2(K+1)|\lambda_1| |\mathbf{M}(r) \times \mathbf{M}_{xx}(r) - \mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r)|_{L^\infty(0, T; \mathbb{L}^2)} T^{1-2\alpha} \\
 & \quad + 2(K+1)|\lambda_2| |\mathbf{M}(r) \times (\mathbf{M}(r) \times \mathbf{M}_{xx}(r)) \\
 & \quad - \mathbf{M}^{(n)}(r) \times (\mathbf{M}^{(n)}(r) \times \mathbf{M}_{xx}^{(n)}(r))|_{L^\infty(0, T; \mathbb{L}^2)} T^{1-2\alpha}.
 \end{aligned}$$

Using Corollary 3 and (6), we obtain

$$\mathbf{M}^{(n)} \rightarrow \mathbf{M} \quad \text{in } \mathcal{D}_X^{2\alpha}([0, T], \mathbb{L}^2), \quad \text{as } n \rightarrow \infty.$$

□

4.1 Application to Stochastic Landau–Lifshitz–Gilbert Equations (SLLGEs)

In [8, 28], the authors used the Wong–Zakai approximation to obtain a solution and the convergence of (63). In this section, we show the same results using the results from the previous section and by taking as geometric rough path the rough path generated by the Brownian motion where the stochastic integral is interpreted in the Stratonovich sense. To be more precise, given a real-valued Brownian motion $B = \{B_t, t \in [0, T]\}$ defined on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where $\mathcal{F}_t = \sigma\{B_s : s \in [0, t]\}$, the corresponding geometric rough path is defined by $\mathbf{B}^{\text{strat}} = (B, \mathbb{B})$, where $\mathbb{B}_{s,t} := \int_s^t B_{s,u} \otimes \circ dB_u$, $B_{s,u} := B_u - B_s$, $0 \leq s \leq t \leq T$ (compare [14, p. 69]). In this way, we arrive at the SLLGEs of the following form:

$$\begin{cases} d\mathbf{M} = \left(\lambda_1 \mathbf{M} \times \mathbf{M}_{xx} - \lambda_2 \mathbf{M} \times (\mathbf{M} \times \mathbf{M}_{xx}) \right) dt \\ \quad + (\mathbf{M} \times \mathbf{g}) \circ d\mathbf{B}, \text{ in } (0, T) \times D, \\ \mathbf{M}_x(t, 0) = 0 = \mathbf{M}_x(t, 1), \forall t \in (0, T), \\ \mathbf{M}(0, \cdot) = \mathbf{M}_0 \text{ in } D, \end{cases} \quad (63)$$

Since we know that for any $\alpha \in (1/3, 1/2)$, we have with probability one $\mathbf{B}^{\text{strat}} = (B, \mathbb{B}) \in \mathcal{C}_g^\alpha([0, T], \mathbb{R})$ (compare [14, Proposition 3.4]), the rough path version of (63) becomes

$$\begin{cases} d\mathbf{M} = \left(\lambda_1 \mathbf{M} \times \mathbf{M}_{xx} - \lambda_2 \mathbf{M} \times (\mathbf{M} \times \mathbf{M}_{xx}) \right) dt \\ \quad + (\mathbf{M} \times \mathbf{g}) d\mathbf{B}^{\text{strat}}, \text{ in } (0, T) \times D, \\ \mathbf{M}_x(t, 0) = 0 = \mathbf{M}_x(t, 1), \forall t \in (0, T), \\ \mathbf{M}(0, \cdot) = \mathbf{M}_0 \text{ in } D. \end{cases} \quad (64)$$

Let $B^{(n)}$ be the dyadic piecewise-linear approximations of B , let

$$\mathbb{B}_{s,t}^{(n)} := \int_s^t B_{s,r}^{(n)} dB_r^{(n)},$$

and $\mathbf{B}^{(n)} = (B^{(n)}, \mathbb{B}^{(n)})$ be the corresponding geometric rough path. Here, the integral $\int_0^t B^{(n)} dB^{(n)}$ is understood as classical Riemann-Stieltjes. Since $B^{(n)}$ is a piecewise linear function and continuous function, the Riemann-Stieltjes is well defined. From Proposition 3.6 of [14], we know that $\mathbf{B}^{(n)}$ is a sequence in $\mathcal{C}_g^\alpha([0, T], \mathbb{R})$ and we have with probability one

$$\mathbf{B}^{(n)} \rightarrow \mathbf{B}^{\text{strat}} \text{ in } \mathcal{C}_g^\alpha([0, T], \mathbb{R}). \quad (65)$$

Let us now consider the following approximation of system (64)

$$\begin{cases} d\mathbf{M}^{(n)} = \left(\lambda_1 \mathbf{M}^{(n)} \times \mathbf{M}_{xx}^{(n)} - \lambda_2 \mathbf{M}^{(n)} \times (\mathbf{M}^{(n)} \times \mathbf{M}_{xx}^{(n)}) \right) dt \\ \quad + (\mathbf{M}^{(n)} \times \mathbf{g}) d\mathbf{B}^{(n)}, \\ \mathbf{M}_x^{(n)}(t, 0) = 0 = \mathbf{M}_x^{(n)}(t, 1), \quad \forall t \in (0, T), \\ \mathbf{M}^{(n)}(0, \cdot) = \mathbf{M}_0 \text{ in } D. \end{cases} \quad (66)$$

From Theorem 1, one can conclude that there exist unique solutions $\mathbf{M}$ and $\mathbf{M}^{(n)}$ to the systems (64) and (66), respectively. Furthermore, we have

$$(\mathbf{M}(\cdot, \omega), \mathbf{M}(\cdot, \omega) \times \mathbf{g}) \in \mathcal{D}_{B(\omega)}^{2\alpha}([0, T], \mathbb{L}^2)$$

and

$$(\mathbf{M}^{(n)}(\cdot, \omega), \mathbf{M}^{(n)}(\cdot, \omega) \times \mathbf{g}) \in \mathcal{D}_{B^{(n)}(\omega)}^{2\alpha}([0, T], \mathbb{L}^2)$$

for $\mathbb{P}$ -a.s. $\omega \in \Omega$. Moreover, from Theorem 4, we know

$$\mathbf{M}^{(n)} \rightarrow \mathbf{M} \text{ in } L^\infty(0, T; \mathbb{L}^2) \text{ as } n \rightarrow \infty.$$

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Equivalence of Systems (5) and (14) and Systems (7) and (36)

In this section, we present several auxiliary lemmas to deal with the equivalence and the regularity properties between $\mathbf{m}$ ($\mathbf{m}^n$ resp.) and $\mathbf{M}$ ($\mathbf{M}^n$ resp.). These results are essential to prove Theorem 1.

Technical Lemmas

Now we present the following lemma, which states that by using the transformation (12) we obtain that if $\mathbf{M}$ is a solution to the system (5), then $\mathbf{m}$ is a solution to the system (14) and vice versa.

Lemma 5 Suppose $\mathbf{M}(t) = e^{X(t)G}\mathbf{m}(t)$, $\mathbf{m}_0 \in \mathbb{H}^2(D; \mathbb{S}^2)$ and assume that $g \in \mathbb{W}^{2,\infty}(D)$. Then the following statements are equivalent

- (i) $\mathbf{m} \in H^1(0, T; \mathbb{L}^2) \cap L^\infty(0, T; \mathbb{H}^2)$ satisfies system (14);
- (ii) $\mathbf{M} \in L^\infty(0, T; \mathbb{H}^2)$ satisfies system (5) and $(\mathbf{M}, \mathbf{M} \times \mathbf{g}) \in \mathcal{D}_X^{2\alpha}([0, T], \mathbb{L}^2)$.

Before going to the proof of Lemma 5, we give the following lemma which is essential to the proof of Lemma 5.

Lemma 6 Let X be any continuous function on $[0, T]$ and

$$\mathbf{M}(t, \mathbf{x}) = e^{X(t)G}\mathbf{m}(t, \mathbf{x}) \quad \forall \quad t \geq 0, a.e. \quad \mathbf{x} \in D.$$

Assume that $\mathbf{g} \in \mathbb{W}^{2,\infty}(D)$. If $\mathbf{m}$ belongs to $L^\infty(0, T; \mathbb{H}^2)$, then $\mathbf{M}$ belongs to $L^\infty(0, T; \mathbb{H}^2)$ and vice versa.

Proof Assume that $\mathbf{m}$ belongs to $L^\infty(0, T; \mathbb{H}^2)$. It suffices to prove that $\mathbf{M}_{xx}$ belongs to $L^\infty(0, T; \mathbb{L}^2)$. Using the identity (91) we obtain

$$\begin{aligned} \mathbf{M}_{xx}(t) &= \left(e^{X(t)G}\mathbf{m}(t) \right)_{xx} \\ &= \mathbf{m}_{xx}(t) + \sin(X(t))(\mathbf{m}(t) \times \mathbf{g})_{xx} + \left(1 - \cos(X(t)) \right) ((\mathbf{m}(t) \times \mathbf{g}) \times \mathbf{g})_{xx}. \end{aligned}$$

We know that

$$(\mathbf{m}(t) \times \mathbf{g})_{xx} = \mathbf{m}_{xx}(t) \times \mathbf{g} + 2\mathbf{m}_x(t) \times \mathbf{g}_x + \mathbf{m}(t) \times \mathbf{g}_{xx}.$$

So we get

$$|(\mathbf{m}(t) \times \mathbf{g})_{xx}|_{\mathbb{L}^2} \leq \left| \mathbf{m}_{xx}(t) \right|_{\mathbb{L}^2} |\mathbf{g}|_{\mathbb{L}^\infty} + 2|\mathbf{m}_x(t)|_{\mathbb{L}^2} |\mathbf{g}_x|_{\mathbb{L}^\infty} + |\mathbf{m}(t)|_{\mathbb{L}^2} |\mathbf{g}_{xx}|_{\mathbb{L}^\infty}.$$

Since $\mathbf{m} \in L^\infty(0, T; \mathbb{H}^2)$ and $\mathbf{g} \in \mathbb{W}^{2,\infty}$, we have $(\mathbf{m}(t) \times \mathbf{g})_{xx} \in L^\infty(0, T; \mathbb{L}^2)$. Furthermore, using

$$\frac{\partial}{\partial x} (\mathbf{m}(t) \times \mathbf{g}) = \mathbf{m}_x(t) \times \mathbf{g} + \mathbf{m}(t) \times \mathbf{g}_x,$$

we obtain

$$\left| \frac{\partial}{\partial x} (\mathbf{m}(t) \times \mathbf{g}) \right|_{\mathbb{L}^2} \leq |\mathbf{m}_x(t)|_{\mathbb{L}^2} |\mathbf{g}|_{\mathbb{L}^\infty} + |\mathbf{m}(t)|_{\mathbb{L}^2} |\mathbf{g}_x|_{\mathbb{L}^\infty}.$$

Since $\mathbf{m} \in L^\infty(0, T; \mathbb{H}^2)$ and $\mathbf{g} \in \mathbb{W}^{2,\infty}$, then $\frac{\partial}{\partial x} (\mathbf{m}(t) \times \mathbf{g}) \in L^\infty(0, T; \mathbb{L}^2)$. Therefore, we obtain $((\mathbf{m}(t) \times \mathbf{g}) \times \mathbf{g})_{xx} \in L^\infty(0, T; \mathbb{L}^2)$. Finally, $\mathbf{M}_{xx}$ belongs to $L^\infty(0, T; \mathbb{L}^2)$.

By similar arguments we also have the converse. $\square$

We are now ready to prove Lemma 5.

Proof of Lemma 5 We first assume that the statement (i) is correct. Using Lemma 6, we know $\mathbf{M} \in L^\infty(0, T; \mathbb{H}^2)$. Using (8) and the fact that $\mathbf{m} \in L^\infty(0, T; \mathbb{H}^2)$ and $\frac{d\mathbf{m}}{dt} \in L^2(0, T; \mathbb{L}^2)$, we obtain

$$\mathbf{m} \in C([0, T], \mathbb{L}^2), \quad \mathbf{m} \times \mathbf{m}_{xx}, \quad \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}) \in L^\infty(0, T; \mathbb{L}^2).$$

Thus by (32), we have $F(\cdot, \mathbf{m}(\cdot)) \in L^\infty(0, T; \mathbb{L}^2)$. Therefore, it follows $\mathbf{m} \in C^{2\alpha}([0, T], \mathbb{L}^2)$ and $(\mathbf{M}, \mathbf{M} \times \mathbf{g}) \in \mathcal{D}_X^{2\alpha}([0, T], \mathbb{L}^2)$. Indeed,

$$\begin{aligned} |\mathbf{M}|_{\alpha, [0, T], \mathbb{L}^2} &\leq \left(\sup_{s \in \mathbb{R}} |e^{sG}|_{\mathcal{L}(\mathbb{L}^2)} \right) |\mathbf{m}|_{\alpha, [0, T], \mathbb{L}^2} + |e^{X(\cdot)G}|_{\alpha, [0, T], \mathcal{L}(\mathbb{L}^2)} \left(\sup_{s \in [0, T]} |\mathbf{m}(s)|_{\mathbb{L}^2} \right), \\ |\mathbf{M}'|_{\alpha, [0, T], \mathbb{L}^2} &\leq |g|_{\mathbb{L}^\infty} |\mathbf{M}|_{\alpha, [0, T], \mathbb{L}^2}. \end{aligned}$$

Since $\mathbf{m} \in C^{2\alpha}([0, T], \mathbb{L}^2)$ and $e^{XG} \in \mathcal{D}_X^{2\alpha}([0, T], \mathcal{L}(\mathbb{L}^2))$ we have

$$\mathbf{M}, \mathbf{M}' \in C^\alpha([0, T], \mathbb{L}^2)$$

Furthermore, if $Z(t) := e^{X(t)G}$ then $\mathcal{R}^M$ is given by

$$\begin{aligned} \mathcal{R}_{s,t}^M &= e^{X(t)G} \mathbf{m}(t) - e^{X(s)G} \mathbf{m}(s) - (e^{X(t)G} - e^{X(s)G} - \mathcal{R}_{s,t}^Z) \mathbf{m}(s) \\ &= e^{X(t)G} (\mathbf{m}(t) - \mathbf{m}(s)) + \mathcal{R}_{s,t}^Z \mathbf{m}(s). \end{aligned}$$

Therefore, we conclude

$$|\mathcal{R}_{s,t}^M|_{2\alpha, [0, T], \mathbb{L}^2} \leq \left(\sup_{s \in \mathbb{R}} |e^{sG}|_{\mathcal{L}(\mathbb{L}^2)} \right) |\mathbf{m}|_{2\alpha, [0, T], \mathbb{L}^2} + |\mathcal{R}^Z|_{2\alpha, [0, T], \mathcal{L}(\mathbb{L}^2)} \left(\sup_{s \in [0, T]} |\mathbf{m}(s)|_{\mathbb{L}^2} \right).$$

It follows that $\mathcal{R}^M \in C^{2\alpha}([0, T], \mathbb{L}^2)$ and we have

$$(\mathbf{M}, \mathbf{M} \times \mathbf{g}) \in \mathcal{D}_X^{2\alpha}([0, T], \mathbb{L}^2).$$

We now prove that $\mathbf{M}$ satisfies system (5). It can be clearly observed that $\mathbf{M}$ satisfies the last two equations of (5). Next, using (13a), Corollary 5 and the fact that $\mathbf{m} \in$

$C^{2\alpha}([0, T], \mathbb{L}^2) \cap L^2(0, T; \mathbb{H}^1)$, we deduce

$$\mathbf{M}(t) - \mathbf{M}(0) = \int_0^t G \mathbf{M}(r) d\mathbf{X}_r + \int_0^t e^{X(r)G} \dot{\mathbf{m}}_r dr.$$

Multiplying both sides by a test function $\boldsymbol{\psi} \in C_c^\infty(D)$ and integrating over D we obtain

$$\langle \mathbf{M}(t), \boldsymbol{\psi} \rangle_{\mathbb{L}^2} - \langle \mathbf{M}(0), \boldsymbol{\psi} \rangle_{\mathbb{L}^2} = \int_0^t \langle e^{B(s)G} \dot{\mathbf{m}}_s, \boldsymbol{\psi} \rangle_{\mathbb{L}^2} ds + \int_0^t \langle \mathbf{M} \times \mathbf{g}, \boldsymbol{\psi} \rangle_{\mathbb{L}^2} d\mathbf{X}. \quad (67)$$

Following the similar calculations in Lemma 4.1 of [15], we infer the following

$$\begin{aligned} \langle \mathbf{M}(t), \boldsymbol{\psi} \rangle_{\mathbb{L}^2} - \langle \mathbf{M}(0), \boldsymbol{\psi} \rangle_{\mathbb{L}^2} &= - \int_0^t \lambda_1 \langle \mathbf{M} \times \mathbf{M}_x, \boldsymbol{\psi}_x \rangle_{\mathbb{L}^2} ds \\ &\quad - \lambda_2 \int_0^t \langle \mathbf{m} \times \mathbf{M}_x, (\mathbf{M} \times \boldsymbol{\psi})_x \rangle_{\mathbb{L}^2} ds \\ &\quad + \int_0^t \langle \mathbf{M} \times \mathbf{g}, \boldsymbol{\psi} \rangle_{\mathbb{L}^2} d\mathbf{X}. \end{aligned} \quad (68)$$

Since (68) is satisfied for all $\boldsymbol{\psi} \in C_c^\infty(D)$, using integration by parts, we have for all $t \in [0, T]$

$$d\mathbf{M} = \left(\lambda_1 \mathbf{M} \times \mathbf{M}_{xx} - \lambda_2 \mathbf{M} \times (\mathbf{M} \times \mathbf{M}_{xx}) \right) dt + (\mathbf{M} \times \mathbf{g}) d\mathbf{X}$$

which completes the statement (ii). Conversely, assume statement (ii) is correct. Using Lemma 6, we have $\mathbf{m} \in L^\infty(0, T; \mathbb{H}^2)$. Now, we will prove that $\mathbf{m}$ satisfies system (14). It can be clearly observed that $\mathbf{m}$ satisfies the last two equations of (14). Next, using (13b), Lemma 11, (5) and the fact that $(\mathbf{M}, \mathbf{M} \times \mathbf{g}) \in \mathcal{D}_X^{2\alpha}([0, T], \mathbb{L}^2)$, we deduce

$$\begin{aligned} \mathbf{m}(t) - \mathbf{m}(0) &= \int_0^t e^{-B(r)G} \left[\lambda_1 \mathbf{M}(r) \times \mathbf{M}_{xx}(r) - \lambda_2 \mathbf{M}(r) \times (\mathbf{M}(r) \times \mathbf{M}_{xx}(r)) \right] dr \\ &\quad + 2 \int_0^t G^2 e^{-X(r)G} \mathbf{M}(r) d\mathbb{X}_r - \int_0^t G^2 e^{-X(r)G} \mathbf{M}(r) d[X, X]_r. \end{aligned}$$

We note that $\mathbb{X} : [0, T] \rightarrow \mathbb{R} \otimes \mathbb{R} \cong \mathbb{R}$, thus we obtain

$$\begin{aligned} &2 \int_0^t G^2 e^{-X(r)G} \mathbf{M}(r) d\mathbb{X}_r - \int_0^t G^2 e^{-X(r)G} \mathbf{M}(r) d[X, X]_r \\ &= \sum_{[u,v] \in \mathcal{P}} G^2 e^{-X(u)G} \mathbf{M}(u) \left[2 \text{Sym}(\mathbb{X})_{u,v} - X_{u,v} \otimes X_{u,v} \right]. \end{aligned}$$

Using the property of geometric rough paths $Sym(\mathbb{X})_{u,v} = \frac{1}{2}X_{u,v} \otimes X_{u,v}$ we have

$$2 \int_0^t G^2 e^{-X(r)G} \mathbf{M}(r) d\mathbb{X}_r - \int_0^t G^2 e^{-X(r)G} \mathbf{M}(r) d[X, X]_r = 0.$$

This implies

$$\mathbf{m}(t) - \mathbf{m}(0) = \int_0^t e^{-X(r)G} \left[\lambda_1 \mathbf{M}(r) \times \mathbf{M}_{xx}(r) - \lambda_2 \mathbf{M}(r) \times \left(\mathbf{M}(r) \times \mathbf{M}_{xx}(r) \right) \right] dr.$$

Multiplying both sides by a test function $\psi \in C_c^\infty(D)$ and integrating over D we obtain

$$\begin{aligned} \langle \mathbf{m}(t), \psi \rangle_{\mathbb{L}^2} - \langle \mathbf{m}(0), \psi \rangle_{\mathbb{L}^2} &= \lambda_1 \int_0^t \langle e^{-X(r)G} \mathbf{M}(r) \times \mathbf{M}_{xx}(r), \psi \rangle_{\mathbb{L}^2} dr \\ &\quad - \lambda_2 \int_0^t \left\langle e^{-X(r)G} \mathbf{M}(r) \times \left(\mathbf{M}(r) \times \mathbf{M}_{xx}(r) \right), \psi \right\rangle_{\mathbb{L}^2} dr. \end{aligned}$$

Recalling $\mathbf{m} = e^{-X(r)G} \mathbf{M}$ and using (96), we obtain

$$\begin{aligned} \langle \mathbf{m}(t), \psi \rangle_{\mathbb{L}^2} - \langle \mathbf{m}(0), \psi \rangle_{\mathbb{L}^2} &= \lambda_1 \int_0^t \langle \mathbf{m}(r) \times e^{-X(r)G} \mathbf{M}_{xx}(r), \psi \rangle_{\mathbb{L}^2} dr \\ &\quad - \lambda_2 \int_0^t \left\langle \mathbf{m}(r) \times \left(\mathbf{m}(r) \times e^{-X(r)G} \mathbf{M}_{xx}(r) \right), \psi \right\rangle_{\mathbb{L}^2} dr. \end{aligned}$$

Thus by (104) and (93), we get

$$\begin{aligned} \langle \mathbf{m}(t), \psi \rangle_{\mathbb{L}^2} - \langle \mathbf{m}(0), \psi \rangle_{\mathbb{L}^2} &= \lambda_1 \int_0^t \left\langle \mathbf{M}_{xx}(r), e^{X(r)G} (\psi \times \mathbf{m}(r)) \right\rangle_{\mathbb{L}^2} dr \\ &\quad - \lambda_2 \int_0^t \left\langle \mathbf{M}_{xx}(r), e^{X(r)G} \left((\psi \times \mathbf{m}(r)) \times \mathbf{m}(r) \right) \right\rangle_{\mathbb{L}^2} dr. \end{aligned}$$

Integrating by part and applying (100), we have

$$\begin{aligned} \langle \mathbf{m}(t), \psi \rangle_{\mathbb{L}^2} - \langle \mathbf{m}(0), \psi \rangle_{\mathbb{L}^2} &= -\lambda_1 \int_0^t \left[\left\langle \left(e^{-X(r)G} \mathbf{M}(r) \right)_x, (\psi \times \mathbf{m}(r))_x \right\rangle_{\mathbb{L}^2} \right. \\ &\quad \left. - \left\langle \tilde{C}(X(r), e^{-X(r)G} \mathbf{M}), \psi \times \mathbf{m}(r) \right\rangle_{\mathbb{L}^2} \right] dr \\ &\quad + \lambda_2 \int_0^t \left[\left\langle \left(e^{-X(r)G} \mathbf{M}(r) \right)_x, \left((\psi \times \mathbf{m}(r)) \times \mathbf{m}(r) \right)_x \right\rangle_{\mathbb{L}^2} \right. \\ &\quad \left. - \left\langle \tilde{C}(X(r), e^{-X(r)G} \mathbf{M}), \left((\psi \times \mathbf{m}(r)) \times \mathbf{m}(r) \right) \right\rangle_{\mathbb{L}^2} \right] dr. \end{aligned}$$

Again using $\mathbf{m} = e^{-X(r)G} \mathbf{M}$, (104) and integrating by part, we obtain

$$\begin{aligned} & \langle \mathbf{m}(t), \boldsymbol{\psi} \rangle_{\mathbb{L}^2} - \langle \mathbf{m}(0), \boldsymbol{\psi} \rangle_{\mathbb{L}^2} \\ &= -\lambda_1 \int_0^t \left[-\left\langle \mathbf{m}(r) \times \mathbf{m}_{xx}(r), \boldsymbol{\psi} \right\rangle_{\mathbb{L}^2} - \left\langle \mathbf{m}(r) \times \tilde{C}(X(r), \mathbf{m}(r)), \boldsymbol{\psi} \right\rangle_{\mathbb{L}^2} \right] dr \\ & \quad + \lambda_2 \int_0^t \left[-\left\langle \mathbf{m}(r) \times (\mathbf{m}(r) \times \mathbf{m}_{xx}(r)), \boldsymbol{\psi} \right\rangle_{\mathbb{L}^2} \right. \\ & \quad \left. - \left\langle \mathbf{m}(r) \times (\mathbf{m}(r) \times \tilde{C}(X(r), \mathbf{m}(r))), \boldsymbol{\psi} \right\rangle_{\mathbb{L}^2} \right] dr. \end{aligned}$$

Since it is satisfied for all $\boldsymbol{\psi} \in C_c^\infty(D)$, using integration by parts, we have for all $t \in [0, T]$

$$\dot{\mathbf{m}} = \lambda_1 \mathbf{m} \times \mathbf{m}_{xx} - \lambda_2 \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}) + F(\cdot, \mathbf{m})$$

where $F(t, \mathbf{m})$ is given by (15). By standard argument, one can conclude that $\mathbf{m} \in H^1(0, T; \mathbb{L}^2)$, which completes the proof. $\square$

We consider the systems (7) and (36) and provide a lemma similar to Lemma 5 which establishes the equivalence of these two systems.

Lemma 7 Suppose $\mathbf{M}^{(n)}(t) = e^{X(t)G} \mathbf{m}^{(n)}(t)$, $\mathbf{m}_0 \in \mathbb{H}^2(D; \mathbb{S}^2)$ and assume that $g \in \mathbb{W}^{2,\infty}(D)$. Then the following statements are equivalent:

- (i) $\mathbf{m}^{(n)} \in H^1(0, T; \mathbb{L}^2) \cap L^\infty(0, T; \mathbb{H}^2)$ satisfies system (36),
- (ii) $\mathbf{M}^{(n)} \in L^\infty(0, T; \mathbb{H}^2)$ satisfies system (7) and $(\mathbf{M}^{(n)}, \mathbf{M}^{(n)} \times \mathbf{g}) \in \mathcal{D}_{X^{(n)}}^{2\alpha}([0, T], \mathbb{L}^2)$.

We note that the proof of Lemma 7 is completely analogous to the proof of Lemma 5. We use the fact that $X^{(n)} = (X^{(n)}, \mathbb{X}^{(n)})$ is a geometric rough path, i.e. $\text{Sym}(\mathbb{X}_{s,t}^{(n)}) = \frac{1}{2} X_{s,t}^{(n)} \otimes X_{s,t}^{(n)}$.

Regularity Properties

In this subsection, we first precisely give the following lemma. For proof we refer to Lemma 3.10, Lemma 3.13 and Corollary 6.2 of [8].

Lemma 8 Let $\mathbf{m}_0 \in \mathbb{H}^2(D; \mathbb{S}^2)$ and suppose that $\mathbf{m}$ and $\mathbf{m}^{(n)}$ satisfy the systems (14) and (36), respectively. Then, $\mathbf{m}$ and $\mathbf{m}^{(n)}$ belong to $C([0, T]; \mathbb{L}^2) \cap H^1(0, T; \mathbb{L}^2) \cap L^\infty(0, T; \mathbb{H}^1) \cap L^2(0, T; D(A))$.

We show $L^\infty(0, T; \mathbb{L}^2)$ -regularity of $\mathbf{m}_{xx}$ and $\mathbf{m}_{xx}^{(n)}$.

Lemma 9 Let $\mathbf{m}_0 \in \mathbb{H}^2(D; \mathbb{S}^2)$ and suppose that $\mathbf{m}$ and $\mathbf{m}^{(n)}$ satisfy the systems (14) and (36), respectively. Then, $\mathbf{m}_{xx}, \mathbf{m}_{xx}^{(n)} \in L^\infty(0, T; \mathbb{L}^2)$.

Proof We show that $\mathbf{m}_{xx} \in L^\infty(0, T; \mathbb{L}^2)$. Denoting $z := \mathbf{m}_{xx}$, system (14) can be written as

$$\frac{\partial \mathbf{m}}{\partial t} = \lambda_1 \mathbf{m} \times z - \lambda_2 \mathbf{m} \times (\mathbf{m} \times z) + F(t, \mathbf{m}). \quad (69)$$

Thus one can obtain in the sense of distribution

$$\begin{aligned} \left(\frac{\partial \mathbf{m}}{\partial t} \right)_{xx} &= \lambda_1 (z \times z + 2\mathbf{m}_x \times z_x + \mathbf{m} \times z_{xx}) - \lambda_2 \left[z \times (\mathbf{m} \times z) + 2\mathbf{m}_x \times (\mathbf{m}_x \times z) \right. \\ &\quad \left. + 2\mathbf{m}_x \times (\mathbf{m} \times z_x) + 2\mathbf{m} \times (\mathbf{m}_x \times z_x) + \mathbf{m} \times (z \times z) + \mathbf{m} \times (\mathbf{m} \times z_{xx}) \right] \\ &\quad + F_{xx}(t, \mathbf{m}). \end{aligned}$$

Using the identity $z \times z = 0$ we have

$$\begin{aligned} \frac{\partial z}{\partial t} &= \lambda_1 (2\mathbf{m}_x \times z_x + \mathbf{m} \times z_{xx}) - \lambda_2 \left[z \times (\mathbf{m} \times z) + 2\mathbf{m}_x \times (\mathbf{m}_x \times z) \right. \\ &\quad \left. + 2\mathbf{m}_x \times (\mathbf{m} \times z_x) + 2\mathbf{m} \times (\mathbf{m}_x \times z_x) + \mathbf{m} \times (\mathbf{m} \times z_{xx}) \right] + F_{xx}(t, \mathbf{m}). \end{aligned}$$

By (107) we have

$$\begin{aligned} \frac{\partial z}{\partial t} &= \lambda_1 (2\mathbf{m}_x \times z_x + \mathbf{m} \times z_{xx}) - \lambda_2 \left[z \times (\mathbf{m} \times z) + 2\mathbf{m}_x \times (\mathbf{m}_x \times z) \right. \\ &\quad \left. + 2\mathbf{m}_x \times (\mathbf{m} \times z_x) + 2\mathbf{m} \times (\mathbf{m}_x \times z_x) + \langle \mathbf{m}, z_{xx} \rangle \mathbf{m} \right. \\ &\quad \left. - \langle \mathbf{m}, \mathbf{m} \rangle z_{xx} \right] + F_{xx}(t, \mathbf{m}). \end{aligned} \quad (70)$$

Our goal is to obtain the existence of the fourth derivative of $\mathbf{m}$ in a.e. sense. Let us approximate z by a Faedo-Galerkin approximation. For $k \in \mathbb{N}$, let z_k be the solution of

$$\begin{aligned} \frac{\partial z_k}{\partial t} &= \lambda_1 (2\mathbf{m}_x \times (z_k)_x + \mathbf{m} \times (z_k)_{xx}) - \lambda_2 \left[z_k \times (\mathbf{m} \times z_k) + 2\mathbf{m}_x \times (\mathbf{m}_x \times z_k) \right. \\ &\quad \left. + 2\mathbf{m}_x \times (\mathbf{m} \times (z_k)_x) + 2\mathbf{m} \times (\mathbf{m}_x \times (z_k)_x) + \langle \mathbf{m}, (z_k)_{xx} \rangle \mathbf{m} \right. \\ &\quad \left. - \langle \mathbf{m}, \mathbf{m} \rangle (z_k)_{xx} \right] + F_{xx}(t, \mathbf{m}). \end{aligned} \quad (71)$$

Note, since the system is finite dimensional, a unique solution exists by standard arguments. Taking the dot product with z_k in $\mathbb{R}^3$, we get

$$\begin{aligned} \left\langle \frac{\partial z_k}{\partial t}, z_k \right\rangle &= \lambda_1 \left[2 \langle \mathbf{m}_x \times (z_k)_x, z_k \rangle + \langle \mathbf{m} \times (z_k)_{xx}, z_k \rangle \right] - \lambda_2 \left[\langle z_k \times (\mathbf{m} \times z_k), z_k \rangle \right. \\ &\quad \left. + 2 \langle \mathbf{m}_x \times (\mathbf{m}_x \times z_k), z_k \rangle + 2 \langle \mathbf{m}_x \times (\mathbf{m} \times (z_k)_x), z_k \rangle \right. \\ &\quad \left. + 2 \langle \mathbf{m} \times (\mathbf{m}_x \times (z_k)_x), z_k \rangle \right] \end{aligned}$$

$$+\langle \mathbf{m}, (z_k)_{xx} \rangle \langle \mathbf{m}, z_k \rangle - \langle \mathbf{m}, \mathbf{m} \rangle \langle (z_k)_{xx}, z_k \rangle \Big] + \langle F_{xx}(t, \mathbf{m}), z_k \rangle.$$

Using the saturation condition, i.e. (20), identity (105) and integrating over $(0, 1)$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |z_k|_{\mathbb{L}^2}^2 &= \lambda_1 \left[2 \int_0^1 \langle \mathbf{m}_x \times (z_k)_x, z_k \rangle dx + \int_0^1 \langle \mathbf{m} \times (z_k)_{xx}, z_k \rangle dx \right] \\ &\quad - \lambda_2 \left[2 \int_0^1 \langle \mathbf{m}_x \times (\mathbf{m}_x \times z_k), z_k \rangle dx + 2 \int_0^1 \langle \mathbf{m}_x \times (\mathbf{m} \times (z_k)_x), z_k \rangle dx \right. \\ &\quad + 2 \int_0^1 \langle \mathbf{m} \times (\mathbf{m}_x \times (z_k)_x), z_k \rangle dx + \int_0^1 \langle \mathbf{m}, (z_k)_{xx} \rangle \langle \mathbf{m}, z_k \rangle dx \\ &\quad \left. - \int_0^1 \langle (z_k)_{xx}, z_k \rangle dx \right] + \int_0^1 \langle F_{xx}(t, \mathbf{m}), z_k \rangle dx. \end{aligned}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^3$. Using standard calculation, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |z_k|_{\mathbb{L}^2}^2 &= \lambda_1 \int_0^1 \langle \mathbf{m}_x \times (z_k)_x, z_k \rangle dx - \lambda_2 \left[2 \int_0^1 \langle \mathbf{m}_x \times (\mathbf{m}_x \times z_k), z_k \rangle dx \right. \\ &\quad + 2 \int_0^1 \langle \mathbf{m}_x \times (\mathbf{m} \times (z_k)_x), z_k \rangle dx + 2 \int_0^1 \langle \mathbf{m} \times (\mathbf{m}_x \times (z_k)_x), z_k \rangle dx \\ &\quad - \int_0^1 \langle \mathbf{m}, (z_k)_x \rangle^2 dx - \int_0^1 \langle \mathbf{m}, (z_k)_x \rangle \langle \mathbf{m}_x, z_k \rangle dx \\ &\quad \left. - \int_0^1 \langle \mathbf{m}_x, (z_k)_x \rangle \langle \mathbf{m}, z_k \rangle dx + |(z_k)_x|_{\mathbb{L}^2}^2 \right] + \int_0^1 \langle F_{xx}(t, \mathbf{m}), z_k \rangle dx. \end{aligned} \quad (72)$$

Integration by parts and the fact that that $z_x = 0$ at $x = 0$ and $x = 1$ gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |z_k|_{\mathbb{L}^2}^2 &\leq \left[\frac{5\varepsilon}{2} - \lambda_2 \right] |(z_k)_x|_{\mathbb{L}^2}^2 + \left[\frac{(|\lambda_1| + 6\lambda_2)^2 |\mathbf{m}_x|_{\mathbb{L}^\infty}^2}{2\varepsilon} + 2\varepsilon + 5\lambda_2 |\mathbf{m}_x|_{\mathbb{L}^\infty}^2 \right. \\ &\quad \left. + \frac{1}{2\varepsilon} K [|\lambda_1| + \lambda_2]^2 [1 + |\mathbf{m}_x|_{\mathbb{L}^\infty}^2] \right] |z_k|_{\mathbb{L}^2}^2 + \varepsilon [(1 + |\mathbf{m}_x|_{\mathbb{L}^2})^2] \\ &= \left[\frac{5\varepsilon}{2} - \lambda_2 \right] |(z_k)_x|_{\mathbb{L}^2}^2 + \varphi_1 |z_k|_{\mathbb{L}^2}^2 + \varphi_2 \end{aligned}$$

where

$$\begin{aligned} \varphi_1(t) &= \frac{(|\lambda_1| + 6\lambda_2)^2 |\mathbf{m}_x(t)|_{\mathbb{L}^\infty}^2}{2\varepsilon} + \frac{1}{2\varepsilon} K [|\lambda_1| + \lambda_2]^2 [1 + |\mathbf{m}_x(t)|_{\mathbb{L}^\infty}^2] \\ &\quad + 2\varepsilon + 5\lambda_2 |\mathbf{m}_x(t)|_{\mathbb{L}^\infty}^2, \\ \varphi_2(t) &= \varepsilon [(1 + |\mathbf{m}_x(t)|_{\mathbb{L}^2})^2]. \end{aligned}$$

Choosing $\varepsilon = \frac{\lambda_2}{5}$, we get

$$\frac{1}{2} \frac{d}{dt} \|z_k(t)\|_{\mathbb{L}^2}^2 + \frac{\lambda_2}{2} |(z_k)_x(t)|_{\mathbb{L}^2}^2 \leq \varphi_1(t) \|z_k(t)\|_{\mathbb{L}^2}^2 + \varphi_2(t).$$

This yields

$$\|z_k(t)\|_{\mathbb{L}^2}^2 + \frac{\lambda_2}{2} \int_0^t |(z_k)_x(s)|_{\mathbb{L}^2}^2 ds \leq \left(\|z_0\|_{\mathbb{L}^2}^2 + \varphi_2(t) \|z_k(t)\|_{\mathbb{L}^2}^2 \right) e^{\int_0^t \varphi_1(s) ds}. \quad (73)$$

Lemma 8 yields $|\varphi_2|_{L^\infty(0,T;\mathbb{R})} < \infty$, and by the interpolation inequality (8), we get $\int_0^t \varphi_1(s) ds < \infty$, for $t \in [0, T]$. Using that $z_0 \in \mathbb{L}^2$ and the estimate (73), we can show that (71) has a solution in $L^2(0, T; \mathbb{H}^1) \cap L^2(0, T; \mathbb{H}^{-1})$. Therefore, the weak limit of z_k , denoted by z , exists and satisfies (70). It follows the existence of z_{xx} , i.e., that the fourth derivative of $\mathbf{m}$ exists. Passing to the limit as $k \rightarrow \infty$ in (73), it can be clearly observed that z satisfies

$$\|z(t)\|_{\mathbb{L}^2}^2 + \frac{\lambda_2}{2} \int_0^t \|z_x(s)\|_{\mathbb{L}^2}^2 ds \leq \left(\|z_0\|_{\mathbb{L}^2}^2 + \varphi_2(t) \|z(t)\|_{\mathbb{L}^2}^2 \right) e^{\int_0^t \varphi_1(s) ds}. \quad (74)$$

This completes the proof. $\square$

Corollary 4 Let $\mathbf{m}_0 \in \mathbb{H}^2(D; \mathbb{S}^2)$ and suppose that $\mathbf{m}$ and $\mathbf{m}^{(n)}$ satisfy systems (14) and (36), respectively. Then $\mathbf{m} \times \mathbf{m}_{xx}$, $\mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx})$, $\mathbf{m}^{(n)} \times \mathbf{m}_{xx}^{(n)}$ and $\mathbf{m}^{(n)} \times (\mathbf{m}^{(n)} \times \mathbf{m}_{xx}^{(n)})$ belong to $L^\infty(0, T; \mathbb{L}^2)$. Furthermore, we also have $\mathbf{m}_x$ and $\mathbf{m}_x^{(n)}$ belong to $L^\infty(0, T; \mathbb{L}^\infty)$.

Proof This is a direct consequence from Lemmas 8, 9 and inequality (8). $\square$

Rough Paths

In this section we summarize the most important definitions, lemma and corollary which are necessary for the existence theory and the convergence results. In order to prove the equivalence between $\mathbf{m}$ and $\mathbf{M}$ which is given in Lemma 5, some technical issues require us to introduce Lemmas 10, 11 and Corollary 5. These results, another contribution of the paper, have its own interest.

The following definitions (Definitions 3, 4), and Remark 2 are taken from [14].

Definition 3 For $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, let $\mathcal{C}^\alpha([0, T], V)$ be the space of α -Hölder continuous rough paths over V consisting of those pairs $\mathbf{X} := (X, \mathbb{X})$ where $X : [0, T] \rightarrow V$ and $\mathbb{X} : [0, T]^2 \rightarrow V \otimes V$ such that

$$\begin{aligned} |X|_{\alpha, [0, T], V} &:= \sup_{s \neq t \in [0, T]} \frac{|X_{s,t}|_V}{|t-s|^\alpha} < \infty \text{ and } |\mathbb{X}|_{2\alpha, [0, T], V \otimes V} \\ &:= \sup_{s \neq t \in [0, T]} \frac{|\mathbb{X}_{s,t} - \mathbb{X}_{t,s}|_{V \otimes V}}{|t-s|^{2\alpha}} < \infty \end{aligned}$$

and $\mathbb{X}_{s,t} = \mathbb{X}_{s,u} + \mathbb{X}_{u,t} + X_{s,u} \otimes X_{u,t}$ for all $s, t, u \in [0, T]$ (Chen identity). Moreover, if $\text{Sym}(\mathbb{X}_{s,t}) = \frac{1}{2} X_{s,t} \otimes X_{s,t}$ then we say that X is α -Hölder geometric rough paths and we denote the space by $\mathcal{C}_g^\alpha([0, T], V)$. Also, we define $\mathcal{C}_g^{0,\alpha}([0, T], V)$ as the closure of lifts of smooth paths in $\mathcal{C}^\alpha([0, T], V)$. In addition, given two rough paths $\mathbf{Y}^1, \mathbf{Y}^2 \in \mathcal{C}^\alpha([0, T], V)$, we define the (inhomogenous) α -Hölder rough path metric

$$\varrho_{\alpha,[0,T],V}(\mathbf{Y}^1, \mathbf{Y}^2) := \sup_{s \neq t \in [0,T]} \frac{|Y_{s,t}^1 - Y_{s,t}^2|_V}{|t - s|^\alpha} + \sup_{s \neq t \in [0,T]} \frac{|\mathbb{Y}_{s,t}^1 - \mathbb{Y}_{s,t}^2|_{V^{\otimes 2}}}{|t - s|^{2\alpha}}.$$

In order to present an integration against rough paths, we also need to give the following definition.

Definition 4 Given a path $X \in \mathcal{C}^\alpha([0, T], V)$, we say that $Y \in \mathcal{C}^\alpha([0, T], H)$ is controlled by X , if there exists a path $Y' \in \mathcal{C}^\alpha([0, T], L(V, H))$ (we call Gubinelli derivative) so that the remainder term $\mathcal{R}^Y$ given implicitly through the relation $Y_{s,t} = Y'_s X_{s,t} + \mathcal{R}_{s,t}^Y$, satisfies $|\mathcal{R}^Y|_{2\alpha,H} < \infty$. This defines the space of controlled rough paths denoted by $\mathcal{D}_X^{2\alpha}([0, T], H)$ which is a Banach space under the norm $(Y, Y') \mapsto |Y_0| + |Y'_0| + |Y, Y'|_{2\alpha,H}$ where

$$|Y, Y'|_{X,2\alpha,[0,T],H} := |Y'|_{\alpha,[0,T],L(V,H)} + |\mathcal{R}^Y|_{2\alpha,[0,T],H}. \quad (75)$$

In addition, consider $\mathbf{X} := (X, \mathbb{X}), \tilde{\mathbf{X}} := (\tilde{X}, \tilde{\mathbb{X}}) \in \mathcal{C}^\alpha([0, T], V)$ with $(Y, Y') \in \mathcal{D}_X^{2\alpha}([0, T], H)$ and $(\tilde{Y}, \tilde{Y}') \in \mathcal{D}_{\tilde{X}}^{2\alpha}([0, T], H)$, we define the distance

$$d_{X,\tilde{X},2\alpha,[0,T],H}(Y, Y'; \tilde{Y}, \tilde{Y}') := |Y' - \tilde{Y}'|_{\alpha,[0,T],L(V,H)} + |\mathcal{R}^Y - \mathcal{R}^{\tilde{Y}}|_{2\alpha,[0,T],H}. \quad (76)$$

Remark 2 Let us recall that if $(Y, Y') \in \mathcal{D}_X^{2\alpha}([0, T], H)$ then

$$\sup_{s \in [0,T]} |Y_s|_H \leq T^\alpha |Y'|_{\alpha,[0,T],H} + |Y_0|_V$$

and

$$\sup_{s \in [0,T]} |Y'_s|_{L(V,H)} \leq T^\alpha |Y'|_{\alpha,L(V,H),[0,T]} + |Y'_0|_{L(V,H)},$$

i.e. $\sup_{s \in [0,T]} |Y_s|_H < \infty$ and $\sup_{s \in [0,T]} |Y'_s|_{L(V,H)} < \infty$.

We are now ready to extend Young's integral to that of a path controlled by X against $X = (X, \mathbb{X})$. The definition of the rough integral $\int Y dX$ in terms of compensated Riemann sums then immediately suggests to define the integral of Y against X by

$$\int_0^T Y dX := \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[s,t] \in \mathcal{P}} (Y_s X_{s,t} + Y'_s \mathbb{X}_{s,t}). \quad (77)$$

Example 1 Let $B(t)$, $t \in [0, T]$ is the standard real-valued Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}_t = \sigma\{B(s) : s \in [0, t]\}$. For any $\alpha \in (1/3, 1/2)$ with probability one, $\mathbf{B}^{\text{strat}} = (B, \mathbb{B}) \in \mathcal{C}_g^\alpha([0, T], \mathbb{R})$ where $\mathbb{B}_{s,t} := \int_s^t B_{s,r} \circ dB_r$ (cf. Proposition 3.4 of [14]). Moreover, if $(Y, Y') \in \mathcal{D}_{B(\omega)}^{2\alpha}([0, T], \mathbb{L}^2)$, for $\mathbb{P}$ -a.s. $\omega \in \Omega$, then the rough integral $\int_0^T Y d\mathbf{B}^{\text{strat}}$ exists (cf. Corollary 5.2 of [14]).

Now we introduce the following lemma which gives us the required conditions for the integration against rough paths in (5) and (7) to be well defined.

Lemma 10 *Let $\mathbf{g}$ be as before and for any $\alpha \in (1/3, 1/2)$ we consider $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C}^\alpha([0, T], \mathbb{R})$. If $(Y, Y') \in \mathcal{D}_X^{2\alpha}([0, T], \mathbb{L}^2)$, then $(Y \times \mathbf{g}, Y' \times \mathbf{g}) \in \mathcal{D}_X^{2\alpha}([0, T], \mathbb{L}^2)$ and*

$$|Y \times \mathbf{g}, Y' \times \mathbf{g}|_{X, 2\alpha, [0, T]} \leq |\mathbf{g}|_{\mathbb{L}^\infty} |Y, Y'|_{X, 2\alpha, [0, T]}.$$

Proof We will prove that $Y \times \mathbf{g}, Y' \times \mathbf{g} \in \mathcal{C}^\alpha([0, T], \mathbb{L}^2)$ and $\mathcal{R}_{s,t}^{Y \times \mathbf{g}} = Y_{s,t} \times \mathbf{g} - (Y_s \times \mathbf{g})X_{s,t} \in \mathcal{C}^{2\alpha}$. In order to prove first term we use (108), i.e.

$$\begin{aligned} |Y_{s,t} \times \mathbf{g}|_{\mathbb{L}^2} &= \left(\int_D |Y_{s,t}(x) \times \mathbf{g}(x)|_{\mathbb{R}^3}^2 \right)^{1/2} \\ &\leq |\mathbf{g}|_{\mathbb{L}^\infty} |Y_{s,t}|_{\mathbb{L}^2}. \end{aligned}$$

Since $Y \in \mathcal{C}^\alpha([0, T], \mathbb{L}^2)$, we have $Y \times \mathbf{g} \in \mathcal{C}^\alpha([0, T], \mathbb{L}^2)$. The similar way, we can prove that $Y' \times \mathbf{g} \in \mathcal{C}^\alpha([0, T], \mathbb{L}^2)$ and $\mathcal{R}_{s,t}^{Y' \times \mathbf{g}} \in \mathcal{C}^{2\alpha}$. Furthermore, we have

$$\begin{aligned} |Y \times \mathbf{g}|_{\alpha, [0, T], \mathbb{L}^2} &\leq |\mathbf{g}|_{\mathbb{L}^\infty} |Y|_{\alpha, [0, T], \mathbb{L}^2}, \\ |Y' \times \mathbf{g}|_{\alpha, [0, T], \mathbb{L}^2} &\leq |\mathbf{g}|_{\mathbb{L}^\infty} |Y'|_{\alpha, [0, T], \mathbb{L}^2}, \\ |\mathcal{R}_{s,t}^{Y \times \mathbf{g}}|_{2\alpha, [0, T], \mathbb{L}^2} &\leq |\mathbf{g}|_{\mathbb{L}^\infty} |\mathcal{R}_{s,t}^Y|_{2\alpha, [0, T], \mathbb{L}^2}. \end{aligned}$$

Therefore we have

$$|Y \times \mathbf{g}, Y' \times \mathbf{g}|_{X, 2\alpha, [0, T], \mathbb{L}^2} \leq |\mathbf{g}|_{\mathbb{L}^\infty} |Y, Y'|_{X, 2\alpha, [0, T], \mathbb{L}^2}.$$

□

To prove the equivalence of weak solution between $\mathbf{m}$ and $\mathbf{M}$, we have to apply Itô formula, see [15]. Therefore, we present the following lemma to handle our case.

Lemma 11 *Let $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C}^\alpha([0, T], V)$ and let $(Y, Y'), (Z, Z') \in \mathcal{D}_X^{2\alpha}$ are controlled rough paths of the form*

$$Y_t = \int_0^t Y'_r dX_r + \Gamma_t, \quad (78)$$

$$Z_t = \int_0^t Z'_r dX_r + \Lambda_t \quad (79)$$

for some controlled rough paths $(Y', Y''), (Z', Z'') \in \mathcal{D}_X^{2\alpha}$ and some paths $\Gamma, \Lambda \in \mathcal{C}^{2\alpha}$. Then,

$$\begin{aligned} Y_T Z_T - Y_0 Z_0 &= \int_0^T Y'_r Z_r dX_r + \int_0^T Y_r Z'_r dX_r + \int_0^T (d\Gamma_r Z_r) + \int_0^T Y_r d\Lambda_r \\ &\quad - 2 \int_0^T Y'_r Z'_r d\mathbb{X}_r + \int_0^T Y'_r Z'_r d[X, X]_r, \end{aligned} \quad (80)$$

which we interpret the integrals $\int_0^T (d\Gamma_r Z_r)$, $\int_0^T Y_r d\Lambda_r$ and $\int_0^T Y'_r Z'_r d\mathbb{X}_r$ as Young integrals. Furthermore, the last integral is given by

$$\int_0^T Y'_r Z'_r d[X, X]_r := \lim_{|\mathcal{P}| \rightarrow \infty} \sum_{[u,v] \in \mathcal{P}} Y'_u Z'_u [X, X]_{u,v}$$

where the bracket $[\cdot, \cdot]$ is defined in [14, Definition 5.8, p. 73].

Proof To show (80), note first that a consequence of (78), (79) and Theorem 4.10 of [14], the increments of Y and Z are of the form

$$Y_{s,t} = Y'_s X_{s,t} + Y''_s \mathbb{X}_{s,t} + \Gamma_{s,t} + o(|t-s|), \quad (81)$$

$$Z_{s,t} = Z'_s X_{s,t} + Z''_s \mathbb{X}_{s,t} + \Lambda_{s,t} + o(|t-s|). \quad (82)$$

Thanks to (81) and (82), we have

$$\begin{aligned} Y_T Z_T - Y_0 Z_0 &= \sum_{[u,v] \in \mathcal{P}} -Y_v Z_{v,u} + Y_{u,v} Z_u \\ &= \sum_{[u,v] \in \mathcal{P}} -Y_v (Z'_v X_{v,u} + Z''_v \mathbb{X}_{v,u} + \Lambda_{v,u} + o(|v-u|)) \\ &\quad + (Y'_u X_{u,v} + Y''_u \mathbb{X}_{u,v} + \Gamma_{u,v} + o(|u-v|)) Z_u. \end{aligned}$$

By taking the limit $|\mathcal{P}| \rightarrow 0$, also noting that $\sum_{[u,v] \in \mathcal{P}} o(|v-u|) \rightarrow 0$, we have

$$\begin{aligned} Y_T Z_T - Y_0 Z_0 &= \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} -(Y_v Z'_v X_{v,u} + Y_v Z''_v \mathbb{X}_{v,u} + Y'_v Z'_v \mathbb{X}_{v,u}) \\ &\quad + Y'_v Z'_v \mathbb{X}_{v,u} - Y_v \Lambda_{v,u} + (Y'_u Z_u X_{u,v} + Y''_u Z_u \mathbb{X}_{u,v} + Y'_u Z'_u \mathbb{X}_{u,v}) \\ &\quad - Y'_u Z'_u \mathbb{X}_{u,v} + \Gamma_{u,v} Z_u. \end{aligned}$$

Thanks to Remark 5.11 and Corollary 7.4 of [14], we obtain

$$Y_T Z_T - Y_0 Z_0 = \int_0^T Y_r Z'_r dX_r + \int_0^T Y'_r Z_r dX_r$$

$$\begin{aligned}
& + \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} Y'_v Z'_v \mathbb{X}_{v,u} - \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} Y_v \Lambda_{v,u} \\
& - \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} Y'_u Z'_u \mathbb{X}_{u,v} + \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} \Gamma_{u,v} Z_u.
\end{aligned}$$

By assumption $(Y, Y'), (Z, Z') \in \mathcal{D}_X^{2\alpha}$ and $\Gamma, \Lambda \in \mathcal{C}^{2\alpha}$, we have

$$\begin{aligned}
Y_T Z_T - Y_0 Z_0 &= \int_0^T Y_r Z'_r d\mathbf{X}_r + \int_0^T Y'_r Z_r d\mathbf{X}_r \\
&+ \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} \left[(Y'_v Z'_v - Y'_u Z'_u) \mathbb{X}_{v,u} + Y'_u Z'_u \mathbb{X}_{v,u} \right] \\
&- \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} \left[(Y_v - Y_u) \Lambda_{v,u} + Y_u \Lambda_{v,u} \right] \\
&- \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} Y'_u Z'_u \mathbb{X}_{u,v} + \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} \Gamma_{u,v} Z_u \\
&= \int_0^T Y_r Z'_r d\mathbf{X}_r + \int_0^T Y'_r Z_r d\mathbf{X}_r \\
&+ \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} Y'_u Z'_u \mathbb{X}_{v,u} - \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} Y_u \Lambda_{v,u} \\
&- \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} Y'_u Z'_u \mathbb{X}_{u,v} + \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} \Gamma_{u,v} Z_u.
\end{aligned}$$

Now using the Chen identity, we get

$$\begin{aligned}
Y_T Z_T - Y_0 Z_0 &= \int_0^T Y_r Z'_r d\mathbf{X}_r + \int_0^T Y'_r Z_r d\mathbf{X}_r \\
&+ \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} Y'_u Z'_u X_{u,v} \otimes X_{u,v} - 2 \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} Y'_u Z'_u \mathbb{X}_{u,v} \\
&+ \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} \Gamma_{u,v} Z_u + \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} Y_u \Lambda_{u,v}.
\end{aligned}$$

Applying Lemma 5.9 of [14], we obtain

$$\begin{aligned}
Y_T Z_T - Y_0 Z_0 &= \int_0^T Y'_r Z_r d\mathbf{X}_r + \int_0^T Y_r Z'_r d\mathbf{X}_r + \int_0^T Y'_u Z'_u d[X, X]_u \\
&- 2 \int_0^T Y'_r Z'_r d\mathbb{X}_r + \int_0^T (d\Gamma_r Z_r) + \int_0^T Y_r d\Lambda_r.
\end{aligned}$$

The convergence to the Young integral in the last three integrals follows from $(Y, Y'), (Z, Z') \in \mathcal{D}_X^{2\alpha}$ and $\Gamma, \Lambda \in \mathcal{C}^{2\alpha}$. The proof is complete. $\square$

In the following corollary we write the specific assumption for Y and Z which satisfies the conditions in Lemma 5.

Corollary 5 *Let $X = (X, \mathbb{X}) \in \mathcal{C}^\alpha([0, T], V)$ and let $(Y, Y') \in \mathcal{D}_X^{2\alpha}$ be controlled rough paths of the form*

$$Y_t = \int_0^t Y'_r dX_r + Y_0,$$

for some controlled rough paths $(Y', Y'') \in \mathcal{D}_X^{2\alpha}$ and some paths $Z \in \mathcal{C}^{2\alpha}$. Then,

$$Y_T Z_T - Y_0 Z_0 = \int_0^T Y'_r Z_r dX_r + \int_0^T Y_r dZ_r,$$

which we interpret the last integrals as Young integrals.

Proof The assertion of theorem is shown by Lemma 11. We omit the complete proof of the above Lemma. $\square$

A Few Properties of Operator G and e^{sG}

We recall some simple results from [4, 15].

Lemma 12 *Assume that $g \in \mathbb{L}^\infty$. Let $G : \mathbb{L}^2 \rightarrow \mathbb{L}^2$ be defined by*

$$[Gu](x) = u(x) \times g(x) \quad \forall u \in \mathbb{L}^2, x \in D.$$

Then the operator G is well defined and bounded linear map. Moreover, for any $u, v \in \mathbb{L}^2$,

$$G^* = -G, \tag{83}$$

$$u \times Gv = (u \cdot g)v - (u \cdot v)g, \tag{84}$$

$$u \times G^2v = (u \cdot g)Gv - Gu \times Gv, \tag{85}$$

$$Gu \times Gv = (g \cdot (u \times v))g = G^2u \times G^2v, \tag{86}$$

$$Gu \times G^2v = ((g \cdot u)(g \cdot v) - (u \cdot v))g = -G^2u \times Gv, \tag{87}$$

$$(Gu) \cdot v = -u \cdot (Gv), \tag{88}$$

$$G^{2n+1}u = (-1)^n Gu, \quad n \geq 0, \tag{89}$$

$$G^{2n+2}u = (-1)^n G^2u, \quad n \geq 0. \tag{90}$$

Lemma 13 *For any $s \in \mathbb{R}$ and $u, v \in \mathbb{L}^2$,*

$$e^{sG}u = u + (\sin s)Gu + (1 - \cos s)G^2u, \tag{91}$$

$$e^{-sG}e^{sG}u = u, \tag{92}$$

$$\left(e^{sG}\right)^* = e^{-sG}, \quad (93)$$

$$e^{sG}Gu = Ge^{sG}u, \quad (94)$$

$$e^{sG}G^2u = G^2e^{sG}u, \quad (95)$$

$$e^{sG}(u \times v) = e^{sG}u \times e^{sG}v. \quad (96)$$

Lemma 14 Assume that $g \in \mathbb{H}^2$. For any $u \in \mathbb{H}^1$ and $v \in \mathbb{W}_0^{1,\infty}$,

$$\langle (Gu)_x, v_x \rangle_{\mathbb{L}^2} + \langle u_x, (Gv)_x \rangle_{\mathbb{L}^2} = -\langle Cu, v \rangle_{\mathbb{L}^2} \quad (97)$$

and

$$\langle u_x, (G^2v)_x \rangle_{\mathbb{L}^2} - \langle (G^2u)_x, v_x \rangle_{\mathbb{L}^2} = \langle GCu, v \rangle_{\mathbb{L}^2} + \langle CGu, v \rangle_{\mathbb{L}^2}, \quad (98)$$

where

$$Cu = u \times g_{xx} + 2 \sum_{i=1}^d \frac{\partial u}{\partial x_i} \times \frac{\partial g}{\partial x_i}. \quad (99)$$

Lemma 15 Assume that $g \in \mathbb{H}^2$. For any $s \in \mathbb{R}$, $u \in \mathbb{H}^1$ and $v \in \mathbb{W}_0^{1,\infty}$,

$$\langle (e^{-sG}u)_x, v_x \rangle_{\mathbb{L}^2} - \langle u_x, (e^{sG}v)_x \rangle_{\mathbb{L}^2} = \langle \tilde{C}(s, e^{-sG}u), v \rangle_{\mathbb{L}^2} \quad (100)$$

where

$$\tilde{C}(s, v) = e^{-sG}((\sin s)C + (1 - \cos s)(GC + CG))v, \quad (101)$$

and C is given by (99).

Some Algebraic Identities

Here, we present list all algebraic identities will be used in this paper. Suppose that $a, b, c, d \in \mathbb{R}^3$. Then

$$a \times b = -b \times a, \quad (102)$$

$$\langle a \times (b \times c), d \rangle = \langle c, (d \times a) \times b \rangle, \quad (103)$$

$$\langle a \times b, c \rangle = \langle b, c \times a \rangle, \quad (104)$$

$$\langle a \times b, b \rangle = 0, \quad (105)$$

$$-\langle a \times b, c \rangle = \langle b, a \times c \rangle, \quad (106)$$

$$a \times (b \times c) = \langle a, c \rangle b - \langle a, b \rangle c, \quad (107)$$

$$|a \times b| \leq |a| |b|. \quad (108)$$

In particular, if $\langle a, b \rangle = 0$, then

$$(a \times b) \times b = b \times (b \times a) = \langle b, a \rangle b - \langle b, b \rangle a = -|b|^2 a \quad (109)$$

and

$$a \times (a \times b) = \langle a, b \rangle a - \langle a, a \rangle b = -|a|^2 b. \quad (110)$$

Corollary 6

$$\langle a \times (a \times b), b \rangle = -|a \times b|^2. \quad (111)$$

References

- Bailleul, I., Gubinelli, M.: Unbounded rough drivers. *Annales de la Faculté des sciences de Toulouse : Mathématiques Ser. 6* **26**(4), 795–830 (2017)
- Brown, W.F.: Thermal fluctuations of a single-domain particle. *Phys. Rev.* **130**(5), 1677–1686 (1963)
- Brzeźniak, Z., Goldys, B., Jegaraj, T.: Weak solutions of a stochastic Landau-Lifshitz-Gilbert Equation. *Appl. Math. Res. eXpress* **2013**(1), 1–33 (2012)
- Brzeźniak, Z., Goldys, B., Jegaraj, T.: Large deviations and transitions between equilibria for stochastic Landau-Lifshitz-Gilbert equation. *Arch. Ration. Mech. Anal.* **226**(2), 497–558 (2017)
- Brzeźniak, Z., Liang, L.: Weak solutions of the stochastic Landau-Lifshitz-Gilbert equations with nonzero anisotropy energy. *Appl. Math. Res. Express. AMRX* **2016**(2), 334–375 (2016)
- Brzeźniak, Z., Manna, U.: Stochastic Landau-Lifshitz-Gilbert equation with anisotropy energy driven by pure jump noise. *Comput. Math. Appl.* **77**(2), 1503–1512 (2019)
- Brzeźniak, Z., Manna, U.: Weak solutions of a stochastic Landau-Lifshitz-Gilbert equation driven by pure jump noise. *Commun. Math. Phys.* **371**, 1071–1129 (2019)
- Brzeźniak, Z., Manna, U., Mukherjee, D.: Wong-Zakai approximation for the stochastic Landau-Lifshitz-Gilbert equations. *J. Differ. Equ.* **267**(2), 776–825 (2019)
- Caruana, M., Friz, P.: Partial differential equations driven by rough paths. *J. Differ. Equ.* **247**(1), 140–173 (2009). <https://doi.org/10.1016/j.jde.2009.01.026>
- Caruana, M., Friz, P.K., Oberhauser, H.: A (rough) pathwise approach to a class of non-linear stochastic partial differential equations. *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis* **28**, 27–46 (2011). <https://doi.org/10.1016/j.anihpc.2010.11.002>
- Cimrák, I.: A survey on the numerics and computations for the Landau-Lifshitz equation of micro-magnetism. *Arch. Comput. Methods Eng.* **15**, 277–309 (2008)
- Deya, A., Gubinelli, M., Hofmanová, M., Tindel, S.: A priori estimates for rough PDEs with application to rough conservation laws. *J. Funct. Anal.* **276**(12), 3577–3645 (2019)
- Deya, A., Gubinelli, M., Tindel, S.: Non-linear rough heat equations. *Probab. Theory Relat. Fields* **153**(1), 97–147 (2012). <https://doi.org/10.1007/s00440-011-0341-z>
- Friz, P., Hairer, M.: *A Course on Rough Paths: With an Introduction to Regularity Structures*. Springer (2014). <https://doi.org/10.1007/978-3-319-08332-2>
- Goldys, B., Le, K.N., Tran, T.: A finite element approximation for the stochastic Landau-Lifshitz-Gilbert equation. *J. Differ. Equ.* **260**(2), 937–970 (2016)
- Gubinelli, M., Imkeller, P., Perkowski, N.: Paracontrolled distributions and singular PDEs. *Forum Math. Pi* **3**, e6 (2015). <https://doi.org/10.1017/fmp.2015.2>
- Gubinelli, M., Lejay, A., Tindell, S.: Young integrals and SPDEs. *Potential Anal.* **35**(4), 307–326 (2006). <https://doi.org/10.1007/s11118-006-9013-5>
- Gubinelli, M., Tindel, S.: Rough evolution equations. *Ann. Probab.* **38**(1), 1–75 (2010)
- Guo, B., Pu, X.: Stochastic Landau-Lifshitz equation. *Differ. Integr. Equ.* **22**, 251–274 (2009)
- Gussetti, E., Hocquet, A.: A pathwise stochastic Landau-Lifshitz-Gilbert equation with application to large deviations (2021)
- Hocquet, A., Hofmanová, M.: An energy method for rough partial differential equations. *J. Differ. Equ.* **265**(4), 1407–1466 (2018)
- Hofmanová, M., Leahy, J., Nilssen, T.: On the Navier-Stokes equation perturbed by rough transport noise. *J. Evol. Equ.* **19**, 203–247 (2019)
- Kampeter, T., Mertens, F.G., Moro, E., Sánchez, A., Bishop, A.R.: Magnetic elements at finite temperature and large deviation theory. Kohn, R. V. and Reznikoff, M. G. and Vanden-Eijnden, E. **15**(4), 223–253 (2005)

24. Kim, H.J., Lototsky, S.V.: Heat equation with geometric rough path potential in one space dimension: existence and regularity of solution. *Commun. Korean Math. Soc.* **34**(3), 757–769 (2019)
25. Lelièvre, T., Le Bris, C., Vanden-Eijnden, E.: Analyse de certains schémas de discrétisation pour des équations différentielles stochastiques contraintes. *C. R. Math. Acad. Sci. Paris* **346**(7–8), 471–476 (2008)
26. Lyons, T.J.: Differential equations driven by rough signals. *Revista Matematica Iberoamericana* **14**(2), 215–310 (1998)
27. Lyons, T.J., Caruana, M., Lévy, T.: *Differential Equations Driven by Rough Paths*, Lecture Notes in Mathematics edn. Ecole d'été de Probabilités de Saint-Flour XXXIV - 2004. Springer (2007)
28. Manna, U., Mukherjee, D., Panda, A.A.: Wong-Zakai approximation for the stochastic Landau-Lifshitz-Gilbert equations with anisotropy energy. *J. Math. Anal. Appl.* **480**(1), 123384 (2019)
29. Neel, L.: Influence of thermal fluctuations on the magnetization of ferromagnetic small particles. *Phys. Status Solidi (a)* **122**, 377–389 (1990)
30. Visintin, A.: On Landau-Lifshitz equations for ferromagnetism. *Jpn. J. Appl. Math.* **2**(1), 69–84 (1985)

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