Optimal Control of a Nonlinear PDE Governed by Fractional Laplacian

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Abstract

We consider an optimal control problem containing a control system described by a partial nonlinear differential equation with the fractional Dirichlet–Laplacian, associated to an integral cost. We investigate the existence of optimal solutions for such a problem. In our study we use Filippov’s approach combined with a lower closure theorem for orientor fields.

Keywords Lower closure theorem · Implicit measurable function theorem · Fractional Dirichlet–Laplace operator · Existence of optimal solutions

Mathematics Subject Classification 49J20 · 49K20

Introduction

In the last years fractional Laplace operators has been attracted the intersts of many scientists. This is mainly due to the fact that such operators better describe nonlocal models of many phenomena. In particular, they appear in many fields of science such as economics (cf. [6,19]), probability (cf. [6,10,11,18]), mechanics (cf. [9,11]), material science (cf. [8]), fluid mechanics and hydrodynamics (cf. [12,14–16,31–33]). Recently, optimal control problems containing control systems described by fractional Laplacians have received a lot of attention. We refer [1,20–22,29], where linear–quadratic optimal control problems involving fractional partial differential equations are studied. In [21] the numerical aproximation of such a type of problem, where the linear state equation involves a fractional Laplace operator with its spectral definition, is investigated. In [20,22] first order necessary and sufficient optimality conditions as well as a priori error estimates are derived. PDE constraints contain the integral fractional Laplacian. Some numerical schemes are also proposed there. In [1,29], the

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state equation is described by a fractional power of the second order a symmetric and uniformly elliptic operator. In this work, some regularity results, numerical schemes to approximate the optimal solution and a priori error analysis are presented. We mention also [5], where the optimal control of fractional semilinear PDEs with both spectral and integral fractional Laplacians with distributed control is considered and [2]—here linear PDEs and integral fractional Laplacian are studied. In these works, the necessary and sufficient optimality conditions for such problems are obtained. The existence results is also investigated in [13], where an optimal control problem with a spectral fractional Dirichlet Laplacian is considered. A nonlinear and nonlocal state equation, studied there, has a variational structure and a cost depends also on the fractional Laplacian. We also refer to [3,4], where a some optimal control problem with a fractional p -Laplacian is studied.

In our paper we consider the following optimal control problem:

$$\text{minimize } J(z, u) = \int_{\Omega} f_0(x, z(x), u(x)) dx, \quad (1)$$

subject to

$$\begin{cases} [(-\Delta)_{\omega}^{\beta} z](x) = f(x, z(x), u(x)), & x \in \Omega \text{ a.e.}, \\ u(x) \in M, & x \in \Omega \text{ a.e.}, \end{cases} \quad (2)$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is an open and bounded set, $\beta > 0$, $f, f_0 : \Omega \times \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}$, $M \subset \mathbb{R}^m$ is a nonempty set and $[(-\Delta)_{\omega}^{\beta}]$ denotes a weak fractional Laplace operator of order β with zero Dirichlet boundary values on $\partial\Omega$ (the term “weak” is explained in Sect. 2). This operator is defined through the spectral decomposition of the Laplace operator $-\Delta$ in Ω with zero Dirichlet boundary conditions (cf. Sect. 2).

The necessary optimality conditions for one-dimensional problem (1)–(2) have been derived in [23] and [27] by using Dubovitskii–Milyutin approach [23] and a smooth–convex extremum principle [27]. In order to obtain results of such a type in the case of $\Omega \in \mathbb{R}^N$ more advanced investigations are required. This issue will be addressed in a forthcoming paper.

The main goal of this paper is study the existence of optimal solutions of problem (1)–(2). In view of a nonlinearity of f and f_0 , as well as, a general convexity assumption (H_3) a method of the proof of the main result differs from the method presented in [5]. Our study is based on the lower closure theorem for orientor fields ([17, Theorem 10.7.i]) and a measurable selection theorem of Filippov type ([30, Theorem 2J]). To the best of my knowledge, the existence result for such a nonlinear problem was not investigated by other authors. Also, a combination of mentioned Theorems 10.7.i and 2J, used in the proof of the main result, is new. The existence result of such a type for the one-dimensional problem (1)–(2), where

$$f(x, z, u) = g(x, z) + B(x)u,$$

has been obtained in [26]. In order to solve such a problem, a characterization of a weak lower semicontinuity of integral functionals was applied there.

The outline of this paper is as follows. In Sect. 2, we recall some necessary notions and facts concerning Dirichlet–Laplace operator of fractional order and multifunctions. In Sect. 3, we formulate and prove the main result of this paper, namely a theorem on the existence of optimal solutions for problem (1)–(2). We finish with an illustrative, theoretical example.

1 Preliminaries

In this section we provide some necessary notions and results concerning the fractional Dirichlet–Laplace operator in a weak sense (see [25], more details can be found in [24]), as well as, some necessary facts regarding multifunctions are given [17,30].

Let $\Omega \in \mathbb{R}^N$ be an open and bounded set. We shall denote:

- by $L^2 := L^2(\Omega, \mathbb{R})$ the space of square Lebesgue integrable functions endowed with the norm $\|\cdot\|_{L^2}$,
- by $H^2 := H^2(\Omega, \mathbb{R})$, $H_0^1 := H_0^1(\Omega, \mathbb{R})$ the classical Sobolev spaces endowed with norms $\|\cdot\|_{H^2}$ and $\|\cdot\|_{H_0^1}$, respectively.

1.1 Weak Dirichlet–Laplace Operator of Fractional Order

Definition 1 [24] We say that $u : \Omega \rightarrow \mathbb{R}$ has a weak (minus) Dirichlet–Laplacian if $u \in H_0^1$ and there exists a function $g \in L^2$ such that

$$\int_{\Omega} \nabla u(x) \nabla v(x) dx = \int_{\Omega} g(x) v(x) dx$$

for any $v \in H_0^1$. The function g , denoted by $(-\Delta)_{\omega} u$, is called the weak Dirichlet–Laplacian and $(-\Delta)_{\omega}$ - the weak Dirichlet–Laplace operator.

It is well known (cf. [24, Theorem 3.1]) that if $-\Delta : H_0^1 \cap H^2 \subset L^2 \rightarrow L^2$ is the strong Dirichlet–Laplace operator, $(-\Delta)_{\omega} : \text{dom}((-\Delta)_{\omega}) \subset L^2 \rightarrow L^2$ is a weak Dirichlet–Laplace operator then $H_0^1 \cap H^2 \subset \text{dom}((-\Delta)_{\omega})$ and $-\Delta u = (-\Delta)_{\omega} u$ for all $u \in H_0^1 \cap H^2$.

Remark 1 In [7, Sect. 8.2] $(-\Delta)_{\omega}$ is called the Laplace–Dirichlet operator (without "weak") and denoted by $-\Delta$.

It is known [24] that the spectrum $\sigma((-\Delta)_{\omega})$ of $(-\Delta)_{\omega}$ contains only the eigenvalues of $(-\Delta)_{\omega}$ that can be written in a non-decreasing sequence, repeating each eigenvalue according to its multiplicity $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \rightarrow \infty$. Furthermore, a system $\{e_j\}$ of eigenfunctions of the operator $(-\Delta)_{\omega}$, corresponding to λ_j , is a hilbertian basis in L^2 .

Now, for $\beta > 0$ we define the weak fractional Dirichlet–Laplace operator of order β $[(-\Delta)_\omega]^\beta : \text{dom}([(-\Delta)_\omega]^\beta) \subset L^2 \rightarrow L^2$ as follows (cf. [24, Sect. 3]):

$$([(-\Delta)_\omega]^\beta u)(x) = \sum_{j=1}^{\infty} (\lambda_j)^\beta a_j e_j(x),$$

where

$$\text{dom}([(-\Delta)_\omega]^\beta) = \left\{ u \in L^2; \sum_{j=1}^{\infty} ((\lambda_j)^\beta)^2 a_j^2 < \infty, \right. \\ \left. \text{where } a_j\text{'s is such that } u(x) = \sum_{j=1}^{\infty} a_j e_j(x) \right\}.$$

It is well known [24] that the operator $[(-\Delta)_\omega]^\beta$ is self-adjoint, bijective and its spectrum $\sigma([(-\Delta)_\omega]^\beta)$ contains only proper values $(\lambda_j)^\beta$, $j \in \mathbb{N}$. Moreover, eigenspaces, corresponding to $(\lambda_j)^\beta$'s and eigenspaces for $[(-\Delta)_\omega]$, corresponding to λ_j 's are the same.

Let us consider in the space $\text{dom}([(-\Delta)_\omega]^\beta)$ the following scalar product:

$$\langle u, v \rangle_\beta := \langle u, v \rangle_{L^2} + \langle [(-\Delta)_\omega]^\beta u, [(-\Delta)_\omega]^\beta v \rangle_{L^2}$$

which generates the norm:

$$\|u\|_\beta = \left(\|u\|_{L^2}^2 + \|[(-\Delta)_\omega]^\beta u\|_{L^2}^2 \right)^{\frac{1}{2}}. \quad (3)$$

The operator $[(-\Delta)_\omega]^\beta$ is closed (as a self-adjoint operator), so the space $\text{dom}([(-\Delta)_\omega]^\beta)$ with the scalar product $\langle \cdot, \cdot \rangle_\beta$ is a Hilbert space.

In the rest of this paper we shall use the space $\text{dom}([(-\Delta)_\omega]^\beta)$ with the another scalar product $\langle \cdot, \cdot \rangle_{\sim\beta}$ given by

$$\langle u, v \rangle_{\sim\beta} := \langle [(-\Delta)_\omega]^\beta u, [(-\Delta)_\omega]^\beta v \rangle_{L^2}$$

which determines the norm

$$\|u\|_{\sim\beta} = \|[(-\Delta)_\omega]^\beta u\|_{L^2}. \quad (4)$$

Norms (3) and (4) are equivalent due to the following Poincaré inequality in $\text{dom}([(-\Delta)_\omega]^\beta)$ ([24, inequality (3.2)]):

$$\|u\|_{L^2}^2 \leq N_\beta \|u\|_{\sim\beta}^2, \quad (5)$$

where

$$N_\beta = \begin{cases} 1 & \text{if } \lambda_1 \geq 1 \\ \frac{1}{((\lambda_1)^\beta)^2} & \text{if } \lambda_1 < 1 \end{cases} \quad (6)$$

(here $\lambda_1 > 0$ is the first (the smallest) eigenvalue of the operator $(-\Delta)_\omega$).

From [24, Proposition 3.10] follows the following useful result:

Proposition 1 *If $u_n \rightharpoonup u_0$ weakly in $\text{dom}([(-\Delta)_\omega]^\beta)$ then $u_n \rightarrow u_0$ strongly in L^2 and $[(-\Delta)_\omega]^\beta u_n \rightharpoonup [(-\Delta)_\omega]^\beta u_0$ weakly in L^2 .*

1.2 Multifunctions

Let S be an arbitrary nonempty set equipped with a σ -algebra $\mathcal{B}$ and $\Lambda : S \ni s \rightarrow \Lambda(s) \subset \mathbb{R}^r$ be a closed-valued multifunction.

We shall say that Λ is measurable if for each closed set $C \subset \mathbb{R}^r$ the set $\Lambda^{-1}(C)$ given by

$$\Lambda^{-1}(C) := \{s \in S : \Lambda(s) \cap C \neq \emptyset\}$$

is measurable (i.e. $\Lambda^{-1}(C) \in \mathcal{B}$).

Let us define the set:

$$\text{dom } \Lambda := \{s \in S : \Lambda(s) \neq \emptyset\}.$$

A function $\lambda : \text{dom } \Lambda \rightarrow \mathbb{R}^r$ such that $\lambda(s) \in \Lambda(s)$ for all $s \in \text{dom } \Lambda$, is called a selection of the multifunction Λ .

We shall say that a function $f : S \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is a normal integrand on $S \times \mathbb{R}^n$ if f is lower semicontinuous on $\mathbb{R}^n$ for all $s \in S$ and the epigraph

$$E_f(s) := \text{epi } f(s, \cdot) = \{(w, v) \in \mathbb{R}^{n+1} : v \geq f(s, w)\}$$

is a measurable multifunction.

In the proof of the main result of this paper we apply the following version of Filippov's lemma (cf. [30, Theorem 2J]):

Theorem 1 (Measurable selection theorem) *Let $\Lambda : S \ni s \rightarrow \Lambda(s) \subset \mathbb{R}^r$ be a multifunction of the form*

$$\Lambda(s) := \{w \in C(s) : F(s, w) = a(s) \text{ and } f_i(s, w) \leq \kappa_i, \ i \in J\},$$

where $C : S \ni s \rightarrow \Lambda(s) \subset \mathbb{R}^r$ is a measurable (closed-valued) multifunction, $F : S \times \mathbb{R}^r \rightarrow \mathbb{R}^k$ is a Carathéodory mapping, $(f_i : i \in J)$ is a countable collection of normal integrands on $S \times \mathbb{R}^r$ and $a : S \rightarrow \mathbb{R}^k, \kappa_i : S \rightarrow \mathbb{R} \cup \{\pm\infty\}$ are measurable. Then Λ is the measurable (closed-valued) multifunction and hence Λ has a measurable selection $\lambda : \text{dom } \Lambda \rightarrow \mathbb{R}^r$.

Now, let us assume that (S, ρ) is a metric space and $\Lambda : S \ni s \longrightarrow \Lambda(s) \subset \mathbb{R}^r$ is an arbitrary multifunction.

We say that $\Lambda : S \ni s \longrightarrow \Lambda(s) \subset \mathbb{R}^r$ has property (K) at the point $s_0 \in S$ iff

$$\Lambda(s_0) = \bigcap_{\delta > 0} \text{cl} \left(\bigcup \{ \Lambda(s) : \rho(s, s_0) < \delta \} \right),$$

where $\text{cl}Z$ denotes the closure of the set Z .

We say that Λ has property (K) in S if it has property (K) at every point $s \in S$.

We have (cf. [17, Theorem 8.5.iii])

Theorem 2 *Let $\Lambda : S \ni s \longrightarrow \Lambda(s) \subset \mathbb{R}^r$ be a multifunction. Then Λ has property (K) if and only if the graph of Λ given by*

$$\text{Gr}\Lambda := \{(s, w) : s \in S, w \in \Lambda(s)\},$$

is closed in the product space $S \times \mathbb{R}^r$.

Remark 2 From the above theorem it follows that if Λ has property (K) then its values are closed.

In conclusion, we formulate a key result in our study, namely, a lower closure theorem ([17, Theorem 10.7.i]). First, we give the necessary notation.

Let $G \subset \mathbb{R}^v$ be a measurable set of finite measure, for every $x = (x^1, \dots, x^v) \in G$ let $A(x)$ be a given nonempty subset of $\mathbb{R}^n$ and let

$$A = \{(x, z) : x \in G, z \in A(x)\},$$

whereby $z = (z^1, \dots, z^n)$. For every $(x, z) \in A$ let $\tilde{Q}(x, z)$ be a given subset of the space $\mathbb{R}^{r+1}$.

Theorem 3 *Let us assume that for almost all $x \in G$, the set $A(x)$ is closed, the sets $\tilde{Q}(x, z)$ are closed, convex and have property (K) with respect to $z \in A(x)$. Let $\xi, \xi_k : G \rightarrow \mathbb{R}^r, z, z_k : G \rightarrow \mathbb{R}^n, \lambda, \eta, \lambda_k, \eta_k : G \rightarrow \mathbb{R}, k = 1, 2, \dots$, be measurable functions, $\xi, \xi_k \in (L^1(G))^r, \eta_k \in L^1(G)$, with $z_k \rightarrow z$ in measure on $G, \xi_k \rightarrow \xi$ weakly in $(L^1(G))^r$ as $k \rightarrow \infty$,*

$$z_k(x) \in A(x), \quad (\eta_k(x), \xi_k(x)) \in \tilde{Q}(x, z_k(x)), \quad x \in G, \quad k = 1, 2, \dots,$$

$$-\infty < i = \liminf_{k \rightarrow \infty} \int_G \eta_k(x) dx < +\infty, \quad \eta_k(x) \geq \lambda_k(x),$$

$$\lambda, \lambda_k \in L^1(G), \quad \lambda_k \rightarrow \lambda \text{ weakly in } L^1(G).$$

Then there exists a function $\eta \in L^1(G)$ such that

$$x(x) \in A(x), \quad (\eta(x), \xi(x)) \in \tilde{Q}(x, z(x)), \quad x \in G, \quad \int_G \eta(x) dx \leq i.$$

2 Existence of Optimal Solutions

In this section we shall prove the main result of this paper, namely a theorem on the existence of optimal solutions for problem (1)–(2).

Let

$$\mathcal{U}_M := \{u : \Omega \rightarrow \mathbb{R}^m - \text{measurable on } \Omega; \quad u(x) \in M, \quad t \in \Omega \text{ a.e.}\}$$

be a set of controls.

A pair $(z, u) \in \text{dom}([(-\Delta)_\omega]^\beta) \times \mathcal{U}_M$ is called admissible if it satisfies constraints (2). Then, z is called an admissible trajectory, while u is an admissible strategy.

In what follows, we assume that the system (2) is controllable in the sense that at least one admissible pair exists. Furthermore, we impose on functions f and f_0 the following conditions:

(H_1) the function f is measurable on Ω , continuous on $\mathbb{R} \times \mathbb{R}^m$ and satisfies the following growth condition: there exist $A \geq 0$, $a \in L^2(\Omega, \mathbb{R}_0^+)$ such that

$$|f(x, z, u)| \leq A|z| + a(x), \quad (7)$$

for a.e. $x \in \Omega$ and all $z \in \mathbb{R}$, $u \in \mathbb{R}^m$,

(H_2) the function f_0 is measurable on Ω and continuous on $\mathbb{R} \times \mathbb{R}^m$,

(H_3) the sets

$$Q(x, z) := \{(\mu_0, \mu) \in \mathbb{R} \times \mathbb{R}; \quad \exists_{u \in M} \quad \mu_0 \geq f_0(x, z, u), \quad \mu = f(x, z, u)\} \quad (8)$$

for a.e. $x \in \Omega$ and all $z \in \mathbb{R}$, are convex.

Remark 3 From [28, Theorem 1] it follows that if $A < \lambda_1^\beta$ then for any fixed $u \in \mathcal{U}_M$ there exists a solution of the control system (2) (here λ_1 is the first eigenvalue of the operator $(-\Delta)_\omega$).

Later on we use the following results

Proposition 2 *If assumption (H_1) is satisfied, whereby*

$$A < \frac{1}{\sqrt{2N_\beta}} \quad (9)$$

(N_β is given by (6)), then the set of admissible trajectories is bounded in $\text{dom}([(-\Delta)_\omega]^\beta)$, i.e. there exists a constant $C > 0$ (independent on u) such that for any control $u \in \mathcal{U}_M$ the trajectory $z \in \text{dom}([(-\Delta)_\omega]^\beta)$, corresponding to u , satisfies the inequality

$$\|z\|_{\sim\beta} \leq C. \quad (10)$$

Proof Let us fix any control $u \in \mathcal{U}_M$. Assume that $z \in \text{dom}([(-\Delta)_\omega]^\beta)$ is a solution of the control system (2), corresponding to u . Then, using (7) and the Poincaré inequality (5), we obtain

$$\begin{aligned} \|z\|_{\sim\beta}^2 &= \| [(-\Delta)_\omega]^\beta z \|_{L^2}^2 = \int_{\Omega} |f(x, z(x), u(x))|^2 dx \\ &\leq \begin{cases} 2A^2 \int_{\Omega} |z(x)|^2 dx + 2\|a\|_{L^2}^2 \leq 2A^2 N_\beta \|z\|_{\sim\beta}^2 + 2\|a\|_{L^2}^2 & \text{if } A > 0 \\ \|a\|_{L^2}^2 & \text{if } A = 0. \end{cases} \end{aligned}$$

Thus, putting

$$C = \begin{cases} \sqrt{\frac{2}{1-2A^2 N_\beta}} \|a\|_{L^2} & \text{if } A > 0 \\ \|a\|_{L^2} & \text{if } A = 0, \end{cases}$$

we get (10).

The proof is completed. $\square$

Corollary 1 *The set of admissible trajectories is weakly relatively compact in $\text{dom}([(-\Delta)_\omega]^\beta)$.*

Proposition 3 *If assumptions $(H_1) - (H_3)$ are satisfied and the set M is compact then the multifunction $Q(x, \cdot) : \mathbb{R} \ni z \longrightarrow Q(x, z) \in \mathbb{R} \times \mathbb{R}$ given by (8) has property (K).*

Proof In view of Theorem 2 it is sufficient to show that the graph $\text{Gr}(Q(x, \cdot))$ is closed in $\mathbb{R} \times (\mathbb{R} \times \mathbb{R})$. Indeed, let $\{(z^l, (\mu_0^l, \mu^l))\}_{l \in \mathbb{N}} \in \text{Gr}(Q(x, \cdot))$ and $(z^l, (\mu_0^l, \mu^l)) \longrightarrow (\hat{z}, (\hat{\mu}_0, \hat{\mu}))$ in $\mathbb{R} \times (\mathbb{R} \times \mathbb{R})$. Since $(\mu_0^l, \mu^l) \in Q(x, z^l)$ for all $l \in \mathbb{N}$, therefore there exists $u^l \in M$ such that

$$\mu_0^l \geq f_0(x, z^l, u^l) \quad \text{and} \quad \mu^l = f(x, z^l, u^l), \quad l \in \mathbb{N}.$$

From the fact that M is compact it follows that there exist a subsequence $(u^{l_j})_{j \in \mathbb{N}} \subset M$ and $\hat{u} \in M$ such that $u^{l_j} \xrightarrow{j \rightarrow \infty} \hat{u}$ in $\mathbb{R}^m$. Consequently, continuity of f_0 and f with respect to z implies

$$\begin{array}{ccc} \mu_0^{l_j} & \geq & f_0(x, z^{l_j}, u^{l_j}) \\ \downarrow & j \rightarrow \infty & \downarrow \\ \hat{\mu}_0 & \geq & f_0(x, \hat{z}, \hat{u}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mu^{l_j} & = & f(x, z^{l_j}, u^{l_j}) \\ \downarrow & j \rightarrow \infty & \downarrow \\ \hat{\mu} & = & f(x, \hat{z}, \hat{u}). \end{array}$$

This means that $(\hat{\mu}_0, \hat{\mu}) \in Q(x, \hat{z})$. Since $\hat{z} \in \mathbb{R}$, therefore $(\hat{z}, (\hat{\mu}_0, \hat{\mu})) \in \text{Gr}(Q(x, \cdot))$, so the mapping $Q(x, \cdot)$ has property (K).

The proof is completed. $\square$

In what follows, we assume that for any admissible pair (z, u) the integral (1) is finite. The set of such pairs we will denote by $\mathcal{A}$. Since the control system (2) is controllable, therefore $\mathcal{A} \neq \emptyset$. Moreover, the following two additional hypotheses are required:

(H₄) there exists a constant $\gamma \in \mathbb{R}$ such that

$$\mathcal{A}^\gamma := \{(z, u) \in \mathcal{A} : J(z, u) \leq \gamma\} \neq \emptyset,$$

(H₅) there exists a function $\lambda \in L^1(\Omega, \mathbb{R})$ such that for any pair $(z, u) \in \mathcal{A}^\gamma$

$$f_0(x, z(x), u(x)) \geq \lambda(x), \quad x \in \Omega \text{ a.e.}$$

The pair $(z^*, u^*) \in \mathcal{A}$ is called an optimal solution to problem (1)–(2) if

$$J(z^*, u^*) \leq J(z, u)$$

for any pair $(z, u) \in \mathcal{A}$.

Now, we formulate and prove the main result of this paper:

Theorem 4 Assume that M is a compact set. If assumptions (H₁)–(H₅) with condition (9) are satisfied then problem (1)–(2) has an optimal solution $(z^*, u^*) \in \text{dom}([(-\Delta)_\omega]^\beta) \times \mathcal{U}_M$.

Proof Let us denote

$$s := \inf_{(z, u) \in \mathcal{A}} J(z, u) = \inf_{(z, u) \in \mathcal{A}^\gamma} J(z, u).$$

From assumptions (H₄) and (H₅) it follows that $-\infty < s < \infty$. Let $\{(z^l, u^l)\}_{l \in \mathbb{N}} \subset \mathcal{A}^\gamma$ be a minimizing sequence of J , i.e.

$$\lim_{l \rightarrow \infty} J(z^l, u^l) = s.$$

From Corollary 1 it follows that the sequence of trajectories $(z^l)_{l \in \mathbb{N}} \subset \text{dom}([(-\Delta)_\omega]^\beta)$ contains a subsequence (still denoted by $(z^l)_{l \in \mathbb{N}}$) weakly convergent in $\text{dom}([(-\Delta)_\omega]^\beta)$ to a some function $z^* \in \text{dom}([(-\Delta)_\omega]^\beta)$. Proposition 1 implies

$$z^l \longrightarrow z^* \text{ strongly in } L^2 \tag{11}$$

and

$$[(-\Delta)_\omega]^\beta z^l \rightharpoonup [(-\Delta)_\omega]^\beta z^* \text{ weakly in } L^2. \tag{12}$$

Let us denote:

$$\begin{aligned} G &= \Omega, \quad A(x) = \mathbb{R}, \quad A = \Omega \times \mathbb{R}, \quad \tilde{Q}(x, z) = Q(x, z), \\ \xi^l(x) &= ([(-\Delta)_\omega]^\beta z^l)(x), \quad \xi(x) = ([(-\Delta)_\omega]^\beta z^*)(x) \\ \eta^l(x) &= f_0(x, z^l(x), u^l(x)), \quad \lambda^l(x) = \lambda(x), \quad z(x) = z^*(x) \end{aligned}$$

for a.e. $x \in \Omega$ and all $l \in \mathbb{N}$.

It is clear that for almost all $x \in \Omega$, the sets $\tilde{Q}(x, z)$ are convex (assumption (H_3)), have property (K) with respect to $z \in \mathbb{R}$ (Proposition 3), so are also closed (Remark 2). Of course, functions z, z^l are measurable on Ω and $\xi, \xi^l, \lambda, \lambda^l, \eta^l \in L^1$, $l \in \mathbb{N}$. Moreover, convergences (11) and (12) imply $z^l \rightarrow z$ in measure on Ω and $\xi^l \rightharpoonup \xi$ weakly in L^1 as $l \rightarrow \infty$. We also see that

$$\begin{aligned} z^l &\in A(x), \quad (\eta^l, \xi^l) \in \tilde{Q}(x, z^l), \quad x \in \Omega \text{ a.e.}, \quad l \in \mathbb{N}, \\ \liminf_{l \rightarrow \infty} \int_0^T \eta^l(x) dx &= \lim_{l \rightarrow \infty} J(z^l, u^l) = s \in (-\infty, +\infty), \\ \eta^l(x) &= f_0(x, z^l(x), u^l(x)) \geq \lambda(x) = \lambda^l(x), \quad x \in \Omega \text{ a.e.}, \quad l \in \mathbb{N} \end{aligned}$$

and

$$\lambda^l = \lambda \rightharpoonup \lambda \text{ weakly in } L^1.$$

Consequently, using Theorem 3, we assert that there exists a function $\eta \in L^1$ such that

$$(\eta(x), \xi(x)) \in \tilde{Q}(x, z^*(x)), \quad x \in \Omega \text{ a.e.}$$

and

$$\int_{\Omega} \eta(x) dx \leq s. \quad (13)$$

Now, let us consider the multifunction $\Phi : \Omega \ni x \rightarrow \Phi(x) \subset \mathbb{R}^m$ given by

$$\Phi(x) = \{u \in M : \eta(x) \geq f_0(x, z^*(x), u), \xi(x) = f(x, z^*(x), u)\}.$$

Let us denote:

$$\begin{aligned} S &= \Omega, \quad C : S \ni x \rightarrow C(x) = M \subset \mathbb{R}^m, \\ F : S \times \mathbb{R}^m \ni (x, u) &\rightarrow F(x, u) = f(x, z^*(x), u) \in \mathbb{R}, \\ f_1 : S \times \mathbb{R}^m \ni (x, u) &\rightarrow f_1(x, u) = f_0(x, z^*(x), u) \in \mathbb{R}, \\ a : S \ni x &\rightarrow a(x) = \xi(x) \in \mathbb{R}, \quad \kappa_1 : S \ni x \rightarrow \kappa_1(x) = \eta(x) \in \mathbb{R}, \quad x \in S. \end{aligned}$$

Since M is closed and for each closed set $D \subset \mathbb{R}^m$

$$C^{-1}(D) = \begin{cases} \emptyset & \text{if } M \cap D = \emptyset \\ \Omega & \text{if } M \cap D \neq \emptyset \end{cases}$$

is measurable, therefore C is a closed-valued measurable multifunction. Moreover, the function f_1 is a normal integrand as a Carathéodory function (cf. [30, Proposition 2C]). Of course, functions a and κ_1 are measurable on Ω . Consequently, from Theorem 1 it follows that Φ is a closed-valued, measurable multifunction and there exists a measurable function $u^* : \Omega \rightarrow \mathbb{R}^m$ such that $u^*(x) \in \Phi(x)$ for a.e. $x \in \Omega$. This means that

$$\eta(x) \geq f_0(x, z^*(x), u^*(x)), \quad x \in \Omega \text{ a.e.} \quad (14)$$

and

$$\begin{cases} [(-\Delta)_\omega]^\beta z^*(x) = f(x, z^*(x), u^*(x)), & x \in \Omega \text{ a.e.}, \\ u^*(x) \in M \subset \mathbb{R}^m, & x \in \Omega. \end{cases}$$

Consequently, $(z^*, u^*) \in \mathcal{A}$, whereby (see (13) and (14))

$$s \leq \int_{\Omega} f_0(x, z^*(x), u^*(x)) dx \leq \int_{\Omega} \eta(x) dx \leq s.$$

This means that (z^*, u^*) is an optimal solution to problem (1)–(2).

The proof is completed. $\square$

3 Illustrative Example

In this section we present the following theoretical problem:

$$\text{minimize } J(z, u) = \int_0^\pi (\sin tz(t) + u^3(t)) dt, \quad (15)$$

subject to

$$\begin{cases} [(-\Delta)_\omega]^{\frac{1}{2}} z(t) = Az(t) + u^3(t), & t \in (0, \pi) \text{ a.e.}, \\ u(t) \in [-1, 1], & t \in (0, \pi) \text{ a.e.}, \end{cases} \quad (16)$$

where $0 \leq A < \frac{1}{\sqrt{2}}$.

It is easy to check that all assumptions of [23, Theorem 4] are satisfied. Consequently, if the pair $(z_*, u_*) \in \text{dom}((-\Delta_\omega)^{\frac{1}{2}}) \times \mathcal{U}_{[-1,1]}$ is a locally optimal solution to problem (15)–(16) then there exists a function $\lambda \in \text{dom}((-\Delta_\omega)^{\frac{1}{2}})$ such that

$$(-\Delta_\omega)^{\frac{1}{2}} \lambda(t) = A\lambda(t) + \sin(t), \quad t \in (0, \pi) \text{ a.e.} \quad (17)$$

and

$$3u_*^2(t)(\lambda(t) + 1)(u - u_*(t)) \geq 0, \quad u \in [-1, 1] \quad (18)$$

for a.e. $t \in (0, \pi)$. Let $\lambda(t) = \sum_{j=1}^{\infty} b_j \sqrt{\frac{2}{\pi}} \sin jt$. Then (17) can be written in the following way:

$$\sum_{j=1}^{\infty} j b_j \sqrt{\frac{2}{\pi}} \sin jt = A \sum_{j=1}^{\infty} b_j \sqrt{\frac{2}{\pi}} \sin jt + \sum_{j=1}^{\infty} c_j \sqrt{\frac{2}{\pi}} \sin jt, \quad t \in (0, \pi),$$

whereby $c_j = \int_0^{\pi} \sin(t) \sqrt{\frac{2}{\pi}} \sin jt dt$, $j \in \mathbb{N}$.

Consequently,

$$b_j = \frac{c_j}{j - A} = \begin{cases} \frac{1}{1-A} \sqrt{\frac{\pi}{2}}; & j = 1 \\ 0; & j > 1. \end{cases}$$

This means that

$$\lambda(t) = \frac{1}{1 - A} \sin t, \quad t \in (0, \pi) \text{ a.e.}$$

is a solution of (17). Hence and from (18) we conclude that

$$u_*(t) = -1, \quad t \in (0, \pi) \text{ a.e.} \quad (19)$$

and

$$(-\Delta_{\omega})^{\frac{1}{2}} z_*(t) = A z_*(t) - 1, \quad t \in (0, \pi) \text{ a.e.} \quad (20)$$

The above equation can be solved the same as (17). Then, we obtain

$$z_*(t) = -\frac{4}{\pi} \left(\frac{\sin t}{1 - A} + \frac{\sin 3t}{3(3 - A)} + \frac{\sin 5t}{5(5 - A)} + \dots \right), \quad t \in (0, \pi) \text{ a.e.} \quad (21)$$

It means that the pair (z_*, u_*) given by (21) and (19) is the only pair which can be a locally optimal solution of problem (15)–(16). Moreover, the minimal value of the

cost functional J is equal to

$$\begin{aligned} J(z_*, u_*) &= \int_0^\pi \left(\sin t z_*(t) + u_*^3(t) \right) dt \\ &= -\frac{4}{\pi} \int_0^\pi \sin t \left(\frac{\sin t}{1-A} + \frac{\sin 3t}{3(3-A)} + \frac{\sin 5t}{5(5-A)} + \dots \right) dt - \pi \\ &= -\frac{2}{1-A} - \pi. \end{aligned}$$

Now, we show that (z_*, u_*) is an optimal solution to problem (15)–(16). Indeed, it is clear that assumptions (H_1) , (H_2) , condition (9) ($N_\beta = 1$) are satisfied and the sets

$$Q(t, z) = \{(\mu_0, \mu) \in \mathbb{R}^2; \quad \mu_0 \geq \sin t z + u^3, \quad \mu = Az + u^3, \quad u \in [-1, 1]\},$$

for a.e. $t \in (0, \pi)$ and all $u \in [-1, 1]$, are convex. Furthermore, from [23, Lemma 2] and Proposition 2 it follows that

$$\begin{aligned} |f_0(t, z(t), u(t))| &= |\sin t z(t) + u^3(t)| \leq |z(t)| + 1 \\ &\leq \sqrt{\frac{2}{\pi}} \zeta(2) \|z\|_{\sim \frac{1}{2}} + 1 \leq \sqrt{\frac{\pi}{3}} C + 1 \end{aligned}$$

for all admissible pairs $(z, u) \in \text{dom}((-\Delta_\omega)^{\frac{1}{2}}) \times \mathcal{U}_{[-1,1]}$, where $C = \sqrt{\frac{2}{1-2A^2}}\pi$ and

$\zeta(\gamma) = \sum_{j=1}^{\infty} \frac{1}{j^\gamma}$ denotes the Riemann zeta function. Consequently, for all admissible

pairs $(z, u) \in \text{dom}((-\Delta_\omega)^{\frac{1}{2}}) \times \mathcal{U}_{[-1,1]}$ the integral J is finite and hypothesis (H_4) and (H_5) hold, whereby in (H_5)

$$\lambda(t) = \sqrt{\frac{2\pi}{3(1-2A^2)}}\pi + 1, \quad t \in (0, \pi) \text{ a.e.}$$

This means that all assumptions of Theorem 4 are satisfied, so the pair (z_*, u_*) given by (21) and (19) is the optimal solution to problem (15)–(16).

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