A New Class of History–Dependent Evolutionary Variational–Hemivariational Inequalities with Unilateral Constraints

Stanisław Migórski^{1,2} · Biao Zeng³

Accepted: 6 October 2020 / Published online: 16 October 2020
© The Author(s) 2020

Abstract

In this paper we study a new abstract evolutionary variational–hemivariational inequality which involves unilateral constraints and history–dependent operators. First, we prove the existence and uniqueness of solution by using a mixed equilibrium formulation with suitable selected functions together with a fixed-point principle for history–dependent operators. Then, we apply the abstract result to show the unique weak solvability to a dynamic viscoelastic frictional contact problem. The contact law involves a unilateral Signorini-type condition for the normal velocity combined with the nonmonotone normal damped response condition while the friction condition is a version of the Coulomb law of dry friction in which the friction bound depends on the accumulated slip.

Keywords Variational–hemivariational inequality · History–dependent operator · Unilateral constraint · Existence and uniqueness · Frictional contact

Mathematics Subject Classification 47J20 · 47J22 · 49J40 · 49J45 · 74G25 · 74G30 · 74M15

Project was supported by the European Union’s Horizon 2020 Research and Innovation Programme under the Marie Skłodowska-Curie Grant Agreement No. 823731 CONMECH, the Natural Science Foundation of Guangxi (Nos. 2018GXNSFAA281353, 2019GXNSFBA185005), the Beibu Gulf University Project No. 2018KYQD06, Special Funds of Guangxi Distinguished Experts Construction Engineering, Guangxi, P.R. China, the Start-up Project of Scientific Research on Introducing Talents at School Level in Guangxi University for Nationalities (No. 2019KJQD04), and the Xiangsihu Young Scholars Innovative Research Team of Guangxi University for Nationalities (No. 2019RSCXSHQN02). The first author is also supported by the Ministry of Science and Higher Education of Republic of Poland under Grants Nos. 4004/GGPJII/H2020/2018/0 and 440328/PnH2/2019.

Extended author information available on the last page of the article

1 Introduction

Mathematical models based on variational and hemivariational inequalities are particularly useful to determine and characterize the qualitative properties of systems in mechanics, economics, engineering sciences, etc. The theory of hemivariational and variational–hemivariational inequalities has been initiated in early 1980s with the pioneering works of Panagiotopoulos, see [35–37]. Since then, because of new and efficient approaches which combine convex and nonsmooth analysis, the theory has undergone a remarkable development in pure and applied mathematics, see monographs [12,16,20,21,33,34,37].

Hemivariational inequalities are variational descriptions of physical phenomena that include nonconvex, nondifferentiable and locally Lipschitz functions. They play an important role in a description of diverse mechanical problems arising in solid and fluid mechanics. We refer to [2–4,12,19,20,26,28,34,38,39] and the references therein for the recent results on the mathematical theory of contact mechanics and related issues.

In this paper we study the Cauchy problem for a new class of evolutionary first order variational–hemivariational inequalities in infinite dimensional spaces which involve history–dependent operators and a set of constraints. Given a reflexive Banach space V , and a nonempty, closed and convex set of constraints $K \subset V$, the problem reads as follows: find $w: (0, T) \rightarrow V$ such that $w(t) \in K$ for a.e. $t \in (0, T)$ and

$$\begin{aligned} & \langle w'(t) + A(t, w(t)) + (R_1 w)(t) - f(t), v - w(t) \rangle_{V^* \times V} \\ & + j^0(t, (Sw)(t), Mw(t); Mw - Mw(t)) \\ & + \varphi(t, (R_2 w)(t), Mw) - \varphi(t, (R_2 w)(t), Mw(t)) \geq 0 \end{aligned} \quad (1.1)$$

for all $v \in K$, a.e. $t \in (0, T)$. Here, the nonlinear operators A , R_1 , R_2 , S , and an affine operator M are prescribed, and (1.1) is supplemented by an initial condition. Since the inequality (1.1) contains two potentials, φ which is a convex function and j which is a locally Lipschitz function, it is called a variational–hemivariational inequality. Operators R_1 , R_2 and S are called history–dependent, for their properties and applications, see e.g. [18,25,29,30,32,40].

The main novelty of the paper is to establish a new result on existence, uniqueness and regularity of solution to the inequality (1.1). We extend the main theorem of [18, Theorem 9], where a particular case $K = V$ (no constraints) has been studied. It is worth mentioning that in the study of (1.1), one can not proceed as in [18]. Specifically, since the set of constraints is nonempty, closed and convex, the indicator function I_K of K is proper, convex and lower semicontinuous, so its subdifferential ∂I_K is maximal monotone. After inserting the term $I_K(v) - I_K(w(t))$ in (1.1) makes an equivalent problem with v in the whole space. However, it is clear that I_K is not a locally Lipschitz function, so the existence results of [18, Theorems 6 and 9] are not applicable in this case. Note also that virtually in every mechanical system there is a situation in which unilateral effects occur, see [16]. We study the problem (1.1) in the framework of evolution triple of spaces, exploit recent result on the mixed equilibrium inequality, see [6, Theorem 3.4] and [7, Theorem 2.4], and a fixed-point principle

for history-dependent operators. The second novelty is to obtain the unique weak solvability to a dynamic frictional contact problem for a viscoelastic material with long memory and unilateral constraints in velocity. The model of this contact problem is new and, to the best of our knowledge, it has not been solved in the literature yet. The contact condition involves a unilateral Signorini-type condition for the normal velocity combined with the nonmonotone normal damped response condition. The friction condition is a version of the Coulomb law of dry friction in which the friction bound depends on the accumulated slip. Moreover, a similar unique weak solvability result can be proved for other unilateral boundary condition and a nonmonotone frictional law governed by a nonconvex potential. Results on existence of a weak solution to dynamic viscoelastic unilateral contact problems with constraints in displacement can be found in [1,9,13,15,23] and with constraints in velocity in [17,31].

The outline of this paper is the following. In Sect. 2 we recall the notation and some preliminary materials. The main result of the paper on constrained history-dependent variational-hemivariational inequality is stated and proved in Sect. 3. Finally, in Sect. 4, we examine a dynamic nonsmooth contact problem of viscoelasticity with multivalued contact and friction conditions for which we prove existence and uniqueness of a weak solution.

2 Essential Tools

In this section we introduce the notation and recall some preliminary results.

Let $(X, \|\cdot\|_X)$ be a Banach space, X^* denote its dual space and $\langle \cdot, \cdot \rangle_{X^* \times X}$ be the duality pairing between X^* and X . By $\mathcal{J}: X \rightarrow 2^{X^*}$ we denote the duality mapping defined by

$$\mathcal{J}u = \left\{ u^* \in X^* \mid \langle u^*, u \rangle_{X^* \times X} = \|u\|_X^2 = \|u^*\|_{X^*}^2 \right\} \text{ for all } u \in X.$$

It is known, see e.g. [10,41], that if X is a reflexive Banach space, then an equivalent norm can be introduced so that X is a strictly convex space Banach space and, therefore, the duality map is a single-valued and continuous operator. The symbols “ $\rightarrow$ ” and “ $\rightharpoonup$ ” denote the strong and the weak convergence, respectively. For a set $D \subset X$, $\text{conv}(D)$ is the convex hull of D . The notation $\mathcal{L}(E, F)$ stands for the space of linear bounded operators from a Banach space E to a Banach space F , and it is well known that $\mathcal{L}(E, F)$ is a Banach space endowed with the usual norm $\|\cdot\|_{\mathcal{L}(E, F)}$. For a set $S \subset X$, we write $\|S\|_X = \sup\{\|u\|_X \mid u \in S\}$.

We recall some concepts for single-valued operators and bifunctions which can be found in [2,6,7,11,28].

Definition 1 A single-valued operator $A: X \rightarrow X^*$ is said to be

- (i) demicontinuous, if $u_n \rightarrow u$ in X implies $Au_n \rightarrow Au$ in X^* ,
- (ii) monotone, if $\langle Au - Av, u - v \rangle_{X^* \times X} \geq 0$ for all $u, v \in X$,
- (iii) maximal monotone, if it is monotone and the conditions $(u, u^*) \in X \times X^*$ and $\langle u^* - Av, u - v \rangle_{X^* \times X} \geq 0$ for all $v \in X$ imply $u^* = Au$,

- (iv) quasimonotone, if $\limsup \langle Au_n, u_n - u \rangle_{X^* \times X} \geq 0$ for any sequence $\{u_n\} \subset X$ with $u_n \rightharpoonup u$ in X ,
- (v) pseudomonotone, if for any sequence $\{u_n\} \subset X$ such that $u_n \rightharpoonup u$ in X and $\limsup \langle Au_n, u_n - u \rangle_{X^* \times X} \leq 0$, we have $\liminf \langle Au_n, u_n - v \rangle_{X^* \times X} \geq \langle Au, u - v \rangle_{X^* \times X}$ for all $v \in X$,
- (vi) bounded, if it maps bounded subsets of X into bounded subsets of X^* .

Definition 2 A function $f: X \rightarrow \mathbb{R}$ is said to be

- (i) (resp. weakly) upper semicontinuous (usc) at $x_0 \in X$, if for any sequence $\{x_n\} \subset X$ with (resp. $x_n \rightharpoonup x_0$) $x_n \rightarrow x_0$, we have $\limsup f(x_n) \leq f(x_0)$,
- (ii) (resp. weakly) lower semicontinuous (lsc) at $x_0 \in X$, if for any sequence $\{x_n\} \subset X$ with (resp. $x_n \rightharpoonup x_0$) $x_n \rightarrow x_0$, we have $f(x_0) \leq \liminf f(x_n)$,
- (iii) f is said to be (resp. weakly) usc (lsc) on X , if f is (resp. weakly) usc (lsc) at x , for all $x \in X$.

Definition 3 Let K be a nonempty, closed and convex subset of X . A real-valued function $F: K \times K \rightarrow \mathbb{R}$, called later a bifunction, is said to be

- (i) monotone, if $F(u, v) + F(v, u) \leq 0$ for all $u, v \in K$,
- (ii) quasimonotone, if for all $\{u_n\} \subset K$ with $u_n \rightharpoonup u$ in X , we have $\liminf F(u_n, u) \leq 0$,
- (iii) pseudomonotone, if for all $\{u_n\} \subset K$ with $u_n \rightharpoonup u$ in X and $\liminf F(u_n, u) \geq 0$, we have $\limsup F(u_n, v) \leq F(u, v)$ for all $v \in K$.

Definition 4 Let K be a nonempty, closed and convex subset of X . Let $F: K \times K \rightarrow \mathbb{R}$ be a real-valued bifunction with $F(u, u) = 0$ for all $u \in K$. The bifunction F is said to be maximal monotone if for every $u \in K$ and for every convex function $\psi: K \rightarrow \mathbb{R}$ with $\psi(u) = 0$, we have

$$\psi(v) \geq F(v, u) \text{ for all } v \in K \implies \psi(v) \geq -F(u, v) \text{ for all } v \in K.$$

The following properties collected from [6, 7] will be useful in next sections. Below, K denotes a nonempty, closed and convex subset of X .

- Remark 5**
- (i) If $A: X \rightarrow X^*$ is pseudomonotone (respectively, quasimonotone), then the bifunction $F: K \times K \rightarrow \mathbb{R}$ defined by $F(u, v) = \langle Au, v - u \rangle_{X^* \times X}$ for $u, v \in K$, is pseudomonotone (respectively, quasimonotone).
  - (ii) If the bifunction F is usc with respect to the first argument for the weak topology, then it is pseudomonotone, and in particular, if the condition $F(u, u) \geq 0$ for all $u \in K$ is satisfied, then F is quasimonotone.
  - (iii) If $F, G: K \times K \rightarrow \mathbb{R}$ are pseudomonotone bifunctions such that $F(u, u) \leq 0$ and $G(u, u) \leq 0$ for all $u \in K$, then $F + G$ is also pseudomonotone.
  - (iv) If $A: X \rightarrow X^*$ is maximal monotone, then the bifunction $F: K \times K \rightarrow \mathbb{R}$ defined by $F(u, v) = \langle Au, v - u \rangle_{X^* \times X}$ for $u, v \in K$, is monotone and maximal monotone.
  - (v) If the bifunction $F: K \times K \rightarrow \mathbb{R}$ is such that $F(u, v) + F(v, u) = 0$ for all $u, v \in K$, then F is monotone and maximal monotone.

We recall an abstract result on existence of solution to the mixed equilibrium problem involving bifunctions of the following form. Let U be a subset of a reflexive Banach space X .

Problem 6 Find $u \in U$ such that

$$F(u, v) + G(u, v) + H(u, v) \geq 0 \text{ for all } v \in U.$$

We need the following hypotheses on the data of Problem 6.

$H(U)$: U is a nonempty, closed and convex subset of X .

$H(F)$: $F: U \times U \rightarrow \mathbb{R}$ is such that

- (1) F is monotone and maximal monotone,
- (2) $F(u, \cdot)$ is convex and lsc for all $u \in U$,
- (3) $F(u, u) = 0$ for all $u \in U$.

$H(G)$: $G: U \times U \rightarrow \mathbb{R}$ is such that

- (1) G is pseudomonotone,
- (2) for each finite subset D of U , $G(\cdot, v)$ is usc on $\text{conv}(D)$, for all $v \in U$,
- (3) $G(u, \cdot)$ is convex for all $u \in U$,
- (4) $G(u, u) = 0$ for all $u \in U$.

$H(H)$: $H: U \times U \rightarrow \mathbb{R}$ is such that

- (1) H is quasimonotone,
- (2) $H(\cdot, v)$ is usc for all $v \in U$,
- (3) $H(u, \cdot)$ is convex for all $u \in U$,
- (4) $H(u, u) = 0$ for all $u \in U$.

(H_{coer}) : There exists a nonempty weakly compact subset W such that for each $\lambda > 0$ small enough, there exists a weakly compact and convex subset B_λ of U satisfying the following condition:

$$\forall u \in U \setminus W, \exists v \in B_\lambda \text{ such that } G(u, v) + H(u, v) + \lambda \langle Ju, v - u \rangle_{X^* \times X} < F(v, u).$$

We have the following existence result whose proof can be found in [6, Theorem 3.4] or [7, Theorem 2.4].

Theorem 7 Assume that the hypotheses $H(U)$, $H(F)$, $H(G)$, $H(H)$ and (H_{coer}) hold. Then, Problem 6 has at least one solution $u \in U$.

Remark 8 (i) If, additionally, U is a weakly compact set, then the condition (H_{coer}) can be omitted in Theorem 7.

(ii) If $F(u, \cdot)$ is convex and lsc for all $u \in U$, and $F(u, u) = 0$ for all $u \in U$, then the condition (H_{coer}) is satisfied, if for some $v_0 \in U$, we have

$$\frac{G(u, v_0) + H(u, v_0) + \lambda \langle Ju, v_0 - u \rangle}{\|u - v_0\|_X} \rightarrow -\infty \text{ uniformly in } \lambda, \text{ as } \|u - v_0\|_X \rightarrow +\infty.$$

Next, we recall the generalized directional derivative and the generalized gradient for a locally Lipschitz function, see [8, 10, 28].

Definition 9 Let X be a Banach space and $j: X \rightarrow \mathbb{R}$ be a locally Lipschitz function, that is, for each $x \in X$, there are a neighborhood $\mathcal{N} = \mathcal{N}(x)$ and a constant $k_{\mathcal{N}} > 0$ such that $|j(w) - j(z)| \leq k_{\mathcal{N}}\|w - z\|_X$ for all $w, z \in \mathcal{N}$. The generalized directional derivative of j at $x \in X$ in the direction $v \in X$, denoted by $j^0(x; v)$, is defined by

$$j^0(x; v) = \limsup_{y \rightarrow x, \lambda \downarrow 0} \frac{j(y + \lambda v) - j(y)}{\lambda}.$$

The generalized gradient of j at x , denoted by $\partial j(x)$, is given by

$$\partial j(x) = \{x^* \in X^* \mid \langle x^*, v \rangle \leq j^0(x; v) \text{ for all } v \in X\}.$$

Finally, we state the fixed-point result which is a consequence of the Banach contraction principle, see [22, Lemma 7] or [40, Corollary 27].

Lemma 10 Let $\mathbb{X}$ be a Banach space, $0 < T < \infty$. Let $\Lambda: L^2(0, T; \mathbb{X}) \rightarrow L^2(0, T; \mathbb{X})$ be an operator such that

$$\|(\Lambda\eta_1)(t) - (\Lambda\eta_2)(t)\|_{\mathbb{X}}^2 \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathbb{X}}^2 ds$$

for all $\eta_1, \eta_2 \in L^2(0, T; \mathbb{X})$, a.e. $t \in (0, T)$ with a constant $c > 0$. Then Λ has a unique fixed point in $L^2(0, T; \mathbb{X})$, i.e., there exists a unique $\eta^* \in L^2(0, T; \mathbb{X})$ such that $\Lambda\eta^* = \eta^*$.

3 Existence and Uniqueness Result

The goal of this paper is to study the evolutionary variational–hemivariational inequality with constraints and history–dependent operators.

Let (V, H, V^*) be an evolution triple of spaces, that is, V is a separable, reflexive Banach space, H is a separable Hilbert space, the embedding $V \subset H$ is continuous and V is dense in H , see e.g. [28, Definition 1.52]. We also assume that the embedding $V \subset H$ is compact. Given $0 < T < +\infty$, we introduce the following Bochner spaces

$$\mathcal{V} = L^2(0, T; V), \quad \mathcal{V}^* = L^2(0, T; V^*), \quad \mathcal{W} = \{v \in \mathcal{V} \mid v' \in \mathcal{V}^*\},$$

where v' denotes the distributional derivative of v . From the reflexivity of V , it follows that both $\mathcal{V}$ and $\mathcal{V}^*$ are reflexive Banach spaces, see [41, p. 411]. Endowed with the graph norm, $\mathcal{W}$ is a separable, reflexive Banach space. It is well known that the embeddings $\mathcal{V} \subset L^2(0, T; H) \subset \mathcal{V}^*$ are continuous and $\mathcal{W} \subset L^2(0, T; H)$ is compact, see, e.g., [11, Theorem 3.4.13]. The duality pairing between $\mathcal{V}^*$ and $\mathcal{V}$ is denoted by

$$\langle w, v \rangle_{\mathcal{V}^* \times \mathcal{V}} = \int_0^T \langle w(t), v(t) \rangle_{V^* \times V} dt \quad \text{for } w \in \mathcal{V}^*, v \in \mathcal{V},$$

where $\langle \cdot, \cdot \rangle_{V^* \times V}$ is the duality pairing between V^* and V . We recall that the operator $L: D(L) \subset \mathcal{V} \rightarrow V^*$ defined by

$$Lv = v' \text{ for all } v \in D(L), \quad (3.1)$$

where $D(L) = \{v \in \mathcal{W} \mid v(0) = 0\}$ is linear and maximal monotone, see, e.g., [41, Proposition 32.10, p. 855].

Let X , Y and Z be Banach spaces. The evolutionary history–dependent variational–hemivariational inequality is of the form.

Problem 11 Find $w \in \mathcal{W}$ such that $w(t) \in K$ for a.e. $t \in (0, T)$ and

$$\begin{cases} \langle w'(t) + A(t, w(t)) + (R_1 w)(t) - f(t), v - w(t) \rangle_{V^* \times V} \\ \quad + j^0(t, (Sw)(t), Mw(t); Mv - Mw(t)) \\ \quad + \varphi(t, (R_2 w)(t), Mv) - \varphi(t, (R_2 w)(t), Mw(t)) \geq 0 \\ \quad \text{for all } v \in K, \text{ a.e. } t \in (0, T), \\ w(0) = w_0. \end{cases}$$

The hypotheses for Problem 11 are following.

$H(A)$: $A: (0, T) \times V \rightarrow V^*$ is such that

- (1) $A(\cdot, v)$ is measurable on $(0, T)$ for all $v \in V$,
- (2) $A(t, \cdot)$ is demicontinuous on V for a.e. $t \in (0, T)$,
- (3) $\|A(t, v)\|_{V^*} \leq a_0(t) + a_1 \|v\|_V$ for all $v \in V$, a.e. $t \in (0, T)$ with a function $a_0 \in L^2(0, T)$ satisfying $a_0 \geq 0$ a.e. in $(0, T)$, and a constant $a_1 \geq 0$,
- (4) $A(t, \cdot)$ is strongly monotone for a.e. $t \in (0, T)$, i.e., for a constant $m_A > 0$,

$$\langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle_{V^* \times V} \geq m_A \|v_1 - v_2\|_V^2$$

for all $v_1, v_2 \in V$, a.e. $t \in (0, T)$.

$H(j)$: $j: (0, T) \times Z \times X \rightarrow \mathbb{R}$ is such that

- (1) $j(\cdot, z, v)$ is measurable on $(0, T)$ for all $z \in Z, v \in X$,
- (2) $j(t, \cdot, v)$ is continuous on Z for all $v \in X$, a.e. $t \in (0, T)$,
- (3) $j(t, z, \cdot)$ is locally Lipschitz on X for all $z \in Z$, a.e. $t \in (0, T)$,
- (4) $\|\partial j(t, z, v)\|_{V^*} \leq c_{0j}(t) + c_{1j} \|z\|_Z + c_{2j} \|v\|_X$ for all $z \in Z, v \in X$, a.e. $t \in (0, T)$ with $c_{0j} \in L^2(0, T)$, $c_{0j}, c_{1j}, c_{2j} \geq 0$,
- (5) $j^0(t, z_1, v_1; v_2 - v_1) + j^0(t, z_2, v_2; v_1 - v_2) \leq \bar{m}_{2j} \|z_1 - z_2\|_Z \|v_1 - v_2\|_X + m_{2j} \|v_1 - v_2\|_X^2$ for all $z_i \in Z, v_i \in X, i = 1, 2$, a.e. $t \in (0, T)$ with $\bar{m}_{2j} \geq 0, m_{2j} \geq 0$.

$H(\varphi)$: $\varphi: (0, T) \times Y \times X \rightarrow \mathbb{R}$ is such that

- (1) $\varphi(\cdot, y, v)$ is measurable on $(0, T)$ for all $y \in Y, v \in X$,
- (2) $\varphi(t, \cdot, v)$ is continuous on Y for all $v \in X$, a.e. $t \in (0, T)$,
- (3) $\varphi(t, y, \cdot)$ is convex and lsc on X for all $y \in Y$, a.e. $t \in (0, T)$,

- (4) $\varphi(t, y_1, v_2) - \varphi(t, y_1, v_1) + \varphi(t, y_2, v_1) - \varphi(t, y_2, v_2) \leq \beta_\varphi \|y_1 - y_2\|_Y \|v_1 - v_2\|_X$ for all $y_i \in Y, v_i \in X, i = 1, 2$, a.e. $t \in (0, T)$ with $\beta_\varphi \geq 0$,
- (5) $\|\partial\varphi(t, y, v)\|_{X^*} \leq c_{0\varphi} + c_{1\varphi}\|y\|_Y + c_{2\varphi}\|v\|_V$ for all $y \in Y, v \in V$, a.e. $t \in (0, T)$ with $c_{0\varphi} \in L^2(0, T), c_{0\varphi}, c_{1\varphi}, c_{2\varphi} \geq 0$.

$\underline{H(R, S)}$: $R_1: \mathcal{V} \rightarrow \mathcal{V}^*, R_2: \mathcal{V} \rightarrow L^2(0, T; Y)$, and $S: \mathcal{V} \rightarrow L^2(0, T; Z)$ are such that

- (1) $\|(R_1 v_1)(t) - (R_1 v_2)(t)\|_{V^*} \leq c_{R_1} \int_0^t \|v_1(s) - v_2(s)\|_V ds$
for all $v_1, v_2 \in \mathcal{V}$, a.e. $t \in (0, T)$ with $c_{R_1} > 0$,
- (2) $\|(R_2 v_1)(t) - (R_2 v_2)(t)\|_Y \leq c_{R_2} \int_0^t \|v_1(s) - v_2(s)\|_V ds$
for all $v_1, v_2 \in \mathcal{V}$, a.e. $t \in (0, T)$ with $c_{R_2} > 0$,
- (3) $\|(S v_1)(t) - (S v_2)(t)\|_Z \leq c_S \int_0^t \|v_1(s) - v_2(s)\|_V ds$
for all $v_1, v_2 \in \mathcal{V}$, a.e. $t \in (0, T)$ with $c_S > 0$.

$\underline{H(K)}$: K is a closed and convex subset of V with $0 \in K$.

$\underline{H(M)}$: $M: V \rightarrow X$ is such that

- (1) M is an affine operator.
- (2) the Nemitsky operator $\mathcal{M}: \mathcal{V} \rightarrow L^2(0, T; X)$ corresponding to M is compact.

$\underline{(H_1)}$: $f \in \mathcal{V}^*, w_0 \in V$.

$\underline{(H_2)}$: $m_A > m_{2j} \|A_M\|^2$, where $A_M: V \rightarrow X$ is defined by $A_M v = Mv - M0$ for $v \in V$.

Remark 12 Recall that in the hypothesis $\underline{H(M)}$, the operator M is an affine operator if and only if the operator A_M is linear. The operator A_M is called the linear part of M . In $\underline{(H_2)}$ and in what follows, the constant $\|A_M\|$ denotes the norm in $\mathcal{L}(V, X)$ of the linear part A_M of M . It is obvious that if $M \in \mathcal{L}(V, X)$, then $A_M = M$.

First, we consider Problem 11 in the case of $w_0 = 0$. We have the following existence and uniqueness result.

Theorem 13 Under hypotheses $\underline{H(A)}, \underline{H(j)}, \underline{H(\varphi)}, \underline{H(R, S)}, \underline{H(K)}, \underline{H(M)}, \underline{(H_1)}$ with $w_0 = 0$, and $\underline{(H_2)}$, Problem 11 has a unique solution.

Proof It is performed in several steps.

Step 1 Let $\xi \in \mathcal{V}^*, \eta \in L^2(0, T; Y)$ and $\zeta \in L^2(0, T; Z)$ be fixed and consider the following auxiliary problem.

Problem 14 Find $w = w_{\xi\eta\zeta} \in \mathcal{W}$ with $w(t) \in K$ for a.e. $t \in (0, T)$ such that

$$\left\{ \begin{array}{l} \langle w'(t) + A(t, w(t)) - f(t) + \xi(t), v - w(t) \rangle_{V^* \times V} \\ \quad + j^0(t, \zeta(t), Mw(t); Mw - Mw(t)) \\ \quad + \varphi(t, \eta(t), Mw) - \varphi(t, \eta(t), Mw(t)) \geq 0 \\ \quad \text{for all } v \in K, \text{ a.e. } t \in (0, T), \\ w(0) = 0. \end{array} \right. \quad (3.2)$$

We show that the solution to Problem 14 is unique. For simplicity, we skip the subscripts ξ , η and ζ for this part of the proof. Let $w_i \in \mathcal{W}$, $i = 1, 2$ be solutions to Problem 14, i.e., $w_i(t) \in K$ for a.e. $t \in (0, T)$, $w_i(0) = 0$ and

$$\langle w'_i(t) + A(t, w_i(t)) - f(t) + \xi(t), v - w_i(t) \rangle_{V^* \times V} + j^0(t, \zeta(t), Mw_i(t); Mv - Mw_i(t)) + \varphi(t, \eta(t), Mv) - \varphi(t, \eta(t), Mw_i(t)) \geq 0 \text{ for all } v \in K, \text{ a.e. } t \in (0, T), i = 1, 2.$$

From the above inequalities, we obtain

$$\langle w'_1(t) - w'_2(t), w_1(t) - w_2(t) \rangle_{V^* \times V} + \langle A(t, w_1(t)) - A(t, w_2(t)), w_1(t) - w_2(t) \rangle_{V^* \times V} \leq j^0(t, \zeta(t), Mw_1(t); Mw_2(t) - Mw_1(t)) + j^0(t, \zeta(t), Mw_2(t); Mw_1(t) - Mw_2(t)).$$

Integrating the above inequality on $(0, t)$, using the integration by parts formula, and the hypothesis $H(A)(4)$ on the left hand side, and applying the hypothesis $H(j)(5)$ to the right hand side, we have

$$\begin{aligned} & \frac{1}{2} \|w_1(t) - w_2(t)\|_H^2 + m_A \int_0^t \|w_1(s) - w_2(s)\|_V^2 ds \\ & \leq m_{2j} \int_0^t \|Mw_1(s) - Mw_2(s)\|_X^2 ds \end{aligned}$$

for all $t \in [0, T]$. Then, by $H(M)$, we obtain

$$(m_A - m_{2j} \|A_M\|^2) \|w_1 - w_2\|_{L^2(0,t;V)}^2 \leq 0,$$

for all $t \in [0, T]$. Hence, due to condition (H_2) , we have $w_1 = w_2$. This completes the proof of uniqueness of the solution.

Step 2 We now prove that Problem 14 has a solution. To this end, we introduce

$$\mathcal{K} = L^2(0, T; K) := \{v \in \mathcal{V} \mid v(t) \in K \text{ for a.e. } t \in (0, T)\}, \quad (3.3)$$

and $\mathcal{K}_1 = D(L) \cap \mathcal{K}$, where, recall, $D(L) = \{w \in \mathcal{W} \mid w(0) = 0\}$. Consider the following problem.

Problem 15 Find $w \in \mathcal{K}_1$ such that

$$\begin{aligned} & \int_0^T \langle w'(t) + A(t, w(t)) - f(t) + \xi(t), z(t) - w(t) \rangle_{V^* \times V} dt \\ & + \int_0^T j^0(t, \zeta(t), Mw(t); Mz(t) - Mw(t)) dt \\ & + \int_0^T (\varphi(t, \eta(t), Mz(t)) - \varphi(t, \eta(t), Mw(t))) dt \geq 0 \text{ for all } z \in \mathcal{K}_1. \end{aligned}$$

Lemma 16 Problems 14 and 15 are equivalent.

Proof Let $w \in \mathcal{W}$ be a solution to Problem 14. This means that $w \in \mathcal{K}$ and $w(0) = 0$ which entails $w \in \mathcal{K}_1$. Let $z \in \mathcal{K}_1$. Then, we have

$$\begin{aligned} & \langle w'(t) + A(t, w(t)) - f(t) + \xi(t), z(t) - w(t) \rangle_{V^* \times V} \\ & + j^0(t, \zeta(t), Mw(t); Mz(t) - Mw(t)) \\ & + \varphi(t, \eta(t), Mz(t)) - \varphi(t, \eta(t), Mw(t)) \geq 0 \text{ for a.e. } t \in (0, T). \end{aligned}$$

Integrating the last inequality on $(0, T)$, since $z \in \mathcal{K}_1$ is arbitrary, we deduce that $w \in \mathcal{K}_1$ is a solution to Problem 15.

Vice versa, assume that $w \in \mathcal{K}_1$ solves Problem 15. Since $0 \in K$ and $\mathcal{K}$ is a convex set, by using [24, Theorem 9.1, p.270], we deduce that the condition (9.18) in [24, p.269] holds, which, in particular, implies that

$$D(L) \cap \mathcal{K} \text{ is dense in } \mathcal{K}. \quad (3.4)$$

Exploiting the density (3.4), we obtain that $w \in \mathcal{K}_1$ is a solution to the following problem.

Problem 17 Find $w \in \mathcal{K}_1$ such that

$$\begin{aligned} & \int_0^T \langle w'(t) + A(t, w(t)) - f(t) + \xi(t), \bar{z}(t) - w(t) \rangle_{V^* \times V} dt \\ & + \int_0^T j^0(t, \zeta(t), Mw(t); M\bar{z}(t) - Mw(t)) dt \\ & + \int_0^T (\varphi(t, \eta(t), M\bar{z}(t)) - \varphi(t, \eta(t), Mw(t))) dt \geq 0 \text{ for all } \bar{z} \in \mathcal{K}. \end{aligned}$$

Indeed, let $\bar{z} \in \mathcal{K} = L^2(0, T; K)$. From (3.4), there exists a sequence $z_n \in D(L) \cap \mathcal{K}$ such that $z_n \rightarrow \bar{z}$ in $\mathcal{V}$. By Problem 15, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} & \int_0^T \langle w'(t) + A(t, w(t)) - f(t) + \xi(t), z_n(t) - w(t) \rangle_{V^* \times V} dt \\ & + \int_0^T j^0(t, \zeta(t), Mw(t); Mz_n(t) - Mw(t)) dt \\ & + \int_0^T (\varphi(t, \eta(t), Mz_n(t)) - \varphi(t, \eta(t), Mw(t))) dt. \end{aligned} \quad (3.5)$$

We will pass to the limit in (3.5), as $n \rightarrow \infty$. By passing to a subsequence if necessary, we deduce, see e.g. [28, Theorem 2.39], that $z_n(t) \rightarrow \bar{z}(t)$ in V , for a.e. $t \in (0, T)$ and there is $g \in L^2(0, T)$ such that $\|z_n(t)\|_V \leq g(t)$ for a.e. $t \in (0, T)$. Hence, $Mz_n(t) \rightarrow M\bar{z}(t)$ in X for a.e. $t \in (0, T)$. The upper semicontinuity of j^0 with respect to its last variable entails

$$\limsup j^0(t, \zeta(t), Mw(t); Mz_n(t) - Mw(t)) \leq j^0(t, \zeta(t), Mw(t); M\bar{z} - Mw(t))$$

for a.e. $t \in (0, T)$. The latter together with $H(j)(4)$ and the following estimate

$$\begin{aligned} |j^0(t, \zeta(t), Mw(t); Mz_n(t) - Mw(t))| &\leq \|\partial j(t, \zeta(t), Mw(t))\|_{X^*} \|Mz_n(t) - Mw(t)\|_X \\ &\leq (c_{0j}(t) + c_{1j} \|\zeta\|_Z + c_{2j} \|Mw(t)\|_X) (\|M\|g(t) + \|Mw(t)\|_X) =: h(t) \end{aligned}$$

for a.e. $t \in (0, T)$ with $h \in L^1(0, T)$, allows to apply the Fatou lemma to get

$$\begin{aligned} \limsup \int_0^T j^0(t, \zeta(t), Mw(t); Mz_n(t) - Mw(t)) dt \\ \leq \int_0^T j^0(t, \zeta(t), Mw(t); M\bar{z}(t) - Mw(t)) dt. \end{aligned} \quad (3.6)$$

Next, since the function $\varphi(t, \eta(t), \cdot)$ is continuous on X for a.e. $t \in (0, T)$, see e.g. [14, Corollary 2.5, p.13], we have

$$\varphi(t, \eta(t), Mz_n(t)) - \varphi(t, \eta(t), Mw(t)) \rightarrow \varphi(t, \eta(t), M\bar{z}(t)) - \varphi(t, \eta(t), Mw(t))$$

for a.e. $t \in (0, T)$. It follows from $H(\varphi)(5)$ that

$$\begin{aligned} |\varphi(t, \eta(t), Mz_n(t)) - \varphi(t, \eta(t), Mw(t))| &\leq \|\partial \varphi(t, \eta(t), Mz_n(t))\|_{X^*} \|Mz_n(t) - Mw(t)\|_X \\ &\leq (c_{0\varphi}(t) + c_{1\varphi} \|\eta(t)\|_Y + c_{2\varphi} \|M\|g(t)) (\|M\|g(t) + \|Mw(t)\|_X) =: \tilde{h}(t) \end{aligned}$$

for a.e. $t \in (0, T)$ with $\tilde{h} \in L^1(0, T)$. By the Lebesgue dominated convergence theorem, we deduce

$$\begin{aligned} \lim \int_0^T (\varphi(t, \eta(t), Mz_n(t)) - \varphi(t, \eta(t), Mw(t))) dt \\ = \int_0^T (\varphi(t, \eta(t), M\bar{z}(t)) - \varphi(t, \eta(t), Mw(t))) dt. \end{aligned} \quad (3.7)$$

Using (3.6) and (3.7) and taking the upper limit in (3.5), we conclude that $w \in \mathcal{K}_1$ is a solution to Problem 17.

In the final part of the proof, we show that Problems 14 and 17 are equivalent. It is obvious that Problem 14 implies Problem 17. The converse implication follows from [5, Lemma 2.3] (applied with $\varphi: V \times V \rightarrow \mathbb{R}$, $\varphi(u, v) = \langle Au, v - u \rangle$ which under $H(A)$ is monotone-convex with a suitable growth condition). We conclude that Problems 14 and 15 are equivalent, which completes the proof of the lemma. $\square$

It follows from Step 1 and Lemma 16 that Problems 14 and 15 have the same unique solution. Therefore, to show the existence of a solution to Problem 14, it is enough to prove that there exists a solution to Problem 15.

In what follows, we prove that Problem 15 has at least one solution. This can be done in several ways. For our purpose, it is sufficient to apply Theorem 7. We introduce the following additional notation which allow to rewrite Problem 15 in the form of the

mixed equilibrium inequality as stated in Problem 6. Let the operators $\mathcal{A}: \mathcal{V} \rightarrow \mathcal{V}^*$, $\mathcal{M}: \mathcal{V} \rightarrow L^2(0, T; X)$, and the functions $\Phi: \mathcal{V} \rightarrow \mathbb{R}$, $J: \mathcal{V} \rightarrow \mathbb{R}$ be defined by

$$\begin{aligned}(\mathcal{A}w)(t) &= A(t, w(t)) \text{ for all } w \in \mathcal{V}, \text{ a.e. } t \in (0, T), \\(\mathcal{M}w)(t) &= Mw(t) \text{ for all } w \in \mathcal{V}, \text{ a.e. } t \in (0, T), \\ \Phi(w) &= \int_0^T \varphi(t, \eta(t), Mw(t)) dt \text{ for } w \in \mathcal{V}, \\ J(w) &= \int_0^T j(t, \zeta(t), Mw(t)) dt \text{ for } w \in \mathcal{V}.\end{aligned}$$

Next, we introduce the bifunctions $F, G, H: \mathcal{K}_1 \times \mathcal{K}_1 \rightarrow \mathbb{R}$ defined by

$$F(w, z) = \langle Lw, z - w \rangle_{\mathcal{V}^* \times \mathcal{V}} + \Phi(z) - \Phi(w), \quad (3.8)$$

$$G(w, z) = \langle \mathcal{A}w - f + \xi, z - w \rangle_{\mathcal{V}^* \times \mathcal{V}}, \quad (3.9)$$

$$H(w, z) = J^0(w; z - w) \quad (3.10)$$

for $w, z \in \mathcal{K}_1$. Under these notation, Problem 15 can be equivalently formulated as follows.

Problem 18 Find $w \in \mathcal{K}_1$ such that

$$F(w, z) + G(w, z) + H(w, z) \geq 0 \text{ for all } z \in \mathcal{K}_1. \quad (3.11)$$

The existence of a solution to Problem 18 is demonstrated by using Theorem 7. We will verify the hypotheses of this theorem for $U = \mathcal{K}_1$.

Claim 1 The bifunction F defined by (3.8) satisfies the condition $H(F)$.

First, we prove $H(F)(1)$. Since $F(w, z) + F(z, w) = -\langle Lw - Lz, w - z \rangle_{\mathcal{V}^* \times \mathcal{V}} \leq 0$ for all $w, z \in \mathcal{K}_1$, it is clear that F is monotone. We show that F is maximal monotone. Let $F_1(w, z) = \langle Lw, z - w \rangle_{\mathcal{V}^* \times \mathcal{V}}$ for $w, z \in \mathcal{K}_1$. Assume that for every $w \in \mathcal{K}_1$ and every convex function $\psi: \mathcal{K}_1 \rightarrow \mathbb{R}$ with $\psi(w) = 0$, we have

$$\psi(z) \geq F(z, w) \text{ for all } z \in \mathcal{K}_1,$$

i.e., $\psi(v) \geq F_1(z, w) + \Phi(w) - \Phi(z)$ for all $z \in \mathcal{K}_1$. Hence

$$\psi(z) + \Phi(z) - \Phi(w) \geq F_1(z, w) \text{ for all } z \in \mathcal{K}_1. \quad (3.12)$$

By the maximal monotonicity of the operator $L: \mathcal{V} \rightarrow \mathcal{V}^*$, it follows that F_1 is maximal monotone. Thus, the condition (3.12) implies

$$\psi(z) + \Phi(z) - \Phi(w) \geq -F_1(w, z) \text{ for all } z \in \mathcal{K}_1.$$

Thus

$$\psi(z) \geq -F_1(w, z) - \Phi(w) + \Phi(z) = -F(w, z) \text{ for all } z \in \mathcal{K}_1,$$

which means that bifunction F is maximal monotone.

Condition $H(F)(2)$ follows from hypothesis $H(M)$ and $H(\varphi)(3)$. We will show that $F(w, \cdot)$ is convex and lsc for all $w \in \mathcal{K}_1$. Let $w, z_1, z_2 \in \mathcal{K}_1$ and $\lambda \in (0, 1)$. By $H(\varphi)(3)$, we have

$$\begin{aligned} F(w, \lambda z_1 + (1 - \lambda)z_2) &= \langle Lw, \lambda z_1 + (1 - \lambda)z_2 - w \rangle_{\mathcal{V}^* \times \mathcal{V}} + \Phi(\lambda z_1 + (1 - \lambda)z_2) - \Phi(w) \\ &\leq \lambda \langle Lw, z_1 - w \rangle_{\mathcal{V}^* \times \mathcal{V}} + (1 - \lambda) \langle Lw, z_2 - w \rangle_{\mathcal{V}^* \times \mathcal{V}} \\ &\quad + \lambda(\Phi(z_1) - \Phi(w)) + (1 - \lambda)(\Phi(z_2) - \Phi(w)) \\ &= \lambda F(w, z_1) + (1 - \lambda)F(w, z_2), \end{aligned}$$

which implies that $F(w, \cdot)$ is convex for all $w \in \mathcal{K}_1$. Moreover, for $z_n, z \in \mathcal{K}_1$ with $z_n \rightarrow z$ in $\mathcal{V}$, as $n \rightarrow \infty$, from $H(M)$, we obtain $\mathcal{M}z_n \rightarrow \mathcal{M}z$ in $L^2(0, T; X)$. Hence, by passing to a subsequence, if necessary, we may suppose that $(\mathcal{M}z_n)(t) \rightarrow (\mathcal{M}z)(t)$ in X for a.e. $t \in (0, T)$, i.e., $Mz_n(t) \rightarrow Mz(t)$ in X for a.e. $t \in (0, T)$. Then, from Fatou's lemma and $H(\varphi)(3)$, it follows that

$$\begin{aligned} \liminf \Phi(z_n) &= \liminf \int_0^T \varphi(t, \eta(t), Mz_n(t)) dt \\ &\geq \int_0^T \liminf \varphi(t, \eta(t), Mz_n(t)) dt \geq \int_0^T \varphi(t, \eta(t), Mz(t)) dt = \Phi(z). \end{aligned}$$

Thus for $w \in \mathcal{V}$, we have

$$\begin{aligned} \liminf F(w, z_n) &= \liminf [\langle Lw, z_n - w \rangle_{\mathcal{V}^* \times \mathcal{V}} + \Phi(z_n) - \Phi(w)] \\ &\geq \liminf \langle Lw, z_n - w \rangle_{\mathcal{V}^* \times \mathcal{V}} + \liminf \Phi(z_n) - \Phi(w) \\ &\geq \langle Lw, z - w \rangle_{\mathcal{V}^* \times \mathcal{V}} + \Phi(z) - \Phi(w) = F(w, z), \end{aligned}$$

which implies that $F(w, \cdot)$ is lsc for all $w \in \mathcal{K}_1$. The condition $H(F)(3)$ is obvious. Hence, we deduce that Claim 1 holds.

Claim 2 *The bifunction G defined by (3.9) satisfies the condition $H(G)$.*

We first show that, for each finite subset D of $\mathcal{K}_1$, $G(\cdot, z)$ is usc on $\text{conv}(D)$ for all $z \in \mathcal{K}_1$. Let $\{w_n\} \subset \text{conv}(D)$ be such that $w_n \rightarrow w$ in $\mathcal{V}$. Since $\text{conv}(D)$ is a closed and convex set, we deduce that $w \in \text{conv}(D)$. From hypothesis $H(A)(2)$ and [41, Proposition 27.7(b)], it follows that $\mathcal{A}: \mathcal{V} \rightarrow \mathcal{V}^*$ is demicontinuous. Thus, we have $\mathcal{A}w_n \rightarrow \mathcal{A}w$ in $\mathcal{V}^*$ and

$$\begin{aligned} \limsup G(w_n, z) &= \limsup (\langle \mathcal{A}w_n, z - w_n \rangle_{\mathcal{V}^* \times \mathcal{V}} + \langle \xi - f, z - w_n \rangle_{\mathcal{V}^* \times \mathcal{V}}) \\ &= \limsup \langle \mathcal{A}w_n, z - w_n \rangle_{\mathcal{V}^* \times \mathcal{V}} + \lim \langle \xi - f, z - w_n \rangle_{\mathcal{V}^* \times \mathcal{V}} \\ &= \langle \mathcal{A}w, z - w \rangle_{\mathcal{V}^* \times \mathcal{V}} + \langle \xi - f, z - w \rangle_{\mathcal{V}^* \times \mathcal{V}} = G(w, z). \end{aligned}$$

Hence $G(\cdot, z)$ is usc on $\text{conv}(D)$ for all $z \in \mathcal{K}_1$, i.e., the condition $H(G)(2)$ holds. It follows from hypothesis $H(A)(2),(4)$ that G is pseudomonotone on $\mathcal{K}_1$. The proof is analogous to the one in [7, Lemma 3.4] or in [6, Lemma 4.4], and, for this reason, it is omitted. Thus, $H(G)(1)$ is satisfied. The conditions $H(G)(3),(4)$ can be easily verified. In conclusion, Claim 2 holds.

Claim 3 *The bifunction H defined by (3.10) satisfies the condition $H(H)$.*

For the proof of $H(H)(1)$, let $\{w_n\} \subset \mathcal{K}_1$ with $w_n \rightarrow w$ in $\mathcal{W}$. From $H(M)$, we obtain $\mathcal{M}w_n \rightarrow \mathcal{M}w$ in $L^2(0, T; X)$. From the converse Lebesgue dominated convergence theorem, see, e.g., [28, Theorem 2.39], we can find $\eta \in L^2(0, T)$ such that passing to a subsequence, if necessary, we may suppose that $\|Mw_n(t)\|_X \leq \eta(t)$ for a.e. $t \in (0, T)$ and $Mw_n(t) \rightarrow Mw(t)$ in X for a.e. $t \in (0, T)$. Next, we consider the function $h_n: (0, T) \rightarrow \mathbb{R}$ defined by

$$h_n(t) = j^0(t, \zeta(t), Mw_n(t); Mw(t) - Mw_n(t)) \text{ for a.e. } t \in (0, T).$$

Using $H(j)(4)$ and [28, Proposition 3.23(iii)], we obtain

$$\begin{aligned} |h_n(t)| &= |j^0(t, \zeta(t), Mw_n(t); Mw(t) - Mw_n(t))| \\ &\leq \|\partial j(t, \zeta(t), Mw_n(t))\|_{X^*} \|Mw(t) - Mw_n(t)\|_X \\ &\leq (c_{0j}(t) + c_{1j}\|\zeta(t)\|_Z + c_{2j}\|Mw_n(t)\|_X)(\|Mw(t)\|_X + \|Mw_n(t)\|_X) \end{aligned}$$

for a.e. $t \in (0, T)$. Hence, we have $|h_n(t)| \leq \bar{h}(t)$ a.e. $t \in (0, T)$ with $\bar{h} \in L^1(0, T)$, $\bar{h}(t) = (c_{0j}(t) + c_{1j}\|\zeta(t)\|_Z + c_{2j}\eta(t))(\|Mw(t)\|_X + \eta(t))$ for a.e. $t \in (0, T)$. Moreover, from [28, Proposition 3.23(ii)], we know that $j^0(t, z, \cdot; \cdot)$ is usc on $X \times X$ for $t \in (0, T)$, $z \in Z$. So, by applying Fatou's lemma, we obtain

$$\begin{aligned} \liminf H(w_n, w) &= \liminf J^0(w_n; w - w_n) \\ &\leq \limsup J^0(w_n; w - w_n) \\ &\leq \limsup \int_0^T j^0(t, \zeta(t), Mw_n(t); Mw(t) - Mw_n(t)) dt \\ &\leq \int_0^T \limsup j^0(t, \zeta(t), Mw_n(t); Mw(t) - Mw_n(t)) dt \\ &\leq \int_0^T j^0(t, \zeta(t), Mw(t); 0) dt = 0. \end{aligned}$$

Hence $H(H)(1)$ holds, i.e., H is quasimonotone. From the upper semicontinuity of $J^0(\cdot; z)$ for all $z \in \mathcal{V}$, see [28, Proposition 3.23(ii)], we infer the condition $H(H)(2)$. Next, since $J^0(w; \cdot)$ is positively homogeneous and subadditive for all $w \in \mathcal{V}$, see e.g. [28, Proposition 3.23(i)], it is convex. This implies the condition $H(H)(3)$. Condition $H(H)(4)$ is obvious. We conclude that Claim 3 holds.

Claim 4 *The condition (H_{coer}) holds.*

From Remark 8(ii), the condition (H_{coer}) holds if

$$\frac{G_\lambda(w, v_0)}{\|w - v_0\|_{\mathcal{V}}} \rightarrow -\infty \text{ uniformly in } \lambda, \text{ as } \|w - v_0\|_{\mathcal{V}} \rightarrow +\infty, \quad (3.13)$$

for some $v_0 \in \mathcal{K}_1$, where $G_\lambda(w, v_0) = G(w, v_0) + H(w, v_0) + \lambda \langle \mathcal{J}w, v_0 - w \rangle_{\mathcal{V}^* \times \mathcal{V}}$ for $w \in \mathcal{K}_1$.

By hypotheses $H(A)(3)$, (4) and $H(j)(4)$, (5), we have

$$\begin{aligned} G_\lambda(w, 0) &= \langle \mathcal{A}w - f, -w \rangle_{\mathcal{V}^* \times \mathcal{V}} + J^0(w; -w) + \lambda \langle \mathcal{J}w, -w \rangle_{\mathcal{V}^* \times \mathcal{V}} \\ &= \langle \mathcal{A}w - \mathcal{A}0, 0 - w \rangle_{\mathcal{V}^* \times \mathcal{V}} + \langle \mathcal{A}0, 0 - w \rangle_{\mathcal{V}^* \times \mathcal{V}} + \langle f, w \rangle_{\mathcal{V}^* \times \mathcal{V}} \\ &\quad + J^0(w; 0 - w) + J^0(0; w - 0) - J^0(0; w - 0) + \lambda \langle \mathcal{J}w, -w \rangle_{\mathcal{V}^* \times \mathcal{V}} \\ &\leq -m_A \|w\|_{\mathcal{V}}^2 + \|a_0\|_{L^2} \|w\|_{\mathcal{V}} + m_{2j} (\|A_M\| \|w\|_{\mathcal{V}} + \|M0\|_X)^2 \\ &\quad + (c_{0j} + c_{1j} \|\zeta\|_{L^2(0,T;Z)}) (\|A_M\| \|w\|_{\mathcal{V}} + \|M0\|_X) + \|f\|_{\mathcal{V}^*} \|w\|_{\mathcal{V}} - \lambda \|w\|_{\mathcal{V}}^2 \\ &\leq -(m_A - m_{2j} \|A_M\|^2) \|w\|_{\mathcal{V}}^2 + (\|a_0\|_{L^2} + 2m_{2j} \|A_M\| \|M0\|_X \\ &\quad + (c_{0j} + c_{1j} \|\zeta\|_{L^2(0,T;Z)}) \|A_M\| + \|f\|_{\mathcal{V}^*}) \|w\|_{\mathcal{V}} \\ &\quad + (c_{0j} + c_{1j} \|\zeta\|_{L^2(0,T;Z)}) \|M0\|_X + m_{2j} \|M0\|_X^2, \end{aligned}$$

and hence

$$\begin{aligned} \frac{G_\lambda(w, 0)}{\|w\|_{\mathcal{V}}} &\leq -(m_A - m_{2j} \|A_M\|^2) \|w\|_{\mathcal{V}} + \|a_0\|_{L^2} + 2m_{2j} \|A_M\| \|M0\|_X \\ &\quad + (c_{0j} + c_{1j} \|\zeta\|_{L^2(0,T;Z)}) \|A_M\| + \|f\|_{\mathcal{V}^*} \\ &\quad + \frac{(c_{0j} + c_{1j} \|\zeta\|_{L^2(0,T;Z)}) \|M0\|_X + m_{2j} \|M0\|_X^2}{\|w\|_{\mathcal{V}}}. \end{aligned}$$

We deduce that the condition (3.13) is satisfied with $v_0 = 0$.

Having verified all hypotheses of Theorem 7, we deduce that Problem 18 has a solution $w \in \mathcal{K}_1$. Hence, Problem 15 has a solution. By Step 1 and Lemma 16, we conclude that Problem 14 has a unique solution $w \in \mathcal{W}$. This completes the proof of Step 2.

Step 3 Let $(\xi_i, \eta_i, \zeta_i) \in L^2(0, T; V^* \times Y \times Z)$, $i = 1, 2$ and $w_1 = w_{\xi_1 \eta_1 \zeta_1}$, $w_2 = w_{\xi_2 \eta_2 \zeta_2} \in \mathcal{W}$ with $w_1(t), w_2(t) \in K$ for a.e. $t \in (0, T)$, be the unique solutions to Problem 14 corresponding to (ξ_1, η_1, ζ_1) and (ξ_2, η_2, ζ_2) , respectively. We will show the following estimate

$$\|w_1 - w_2\|_{L^2(0,t;V)} \leq c \left(\|\zeta_1 - \zeta_2\|_{L^2(0,t;Z)} + \|\eta_1 - \eta_2\|_{L^2(0,t;Y)} + \|\xi_1 - \xi_2\|_{L^2(0,t;V^*)} \right) \quad (3.14)$$

for all $t \in [0, T]$, where c is a positive constant.

By the definition of solution to Problem 14, it follows that

$$\begin{aligned} & \langle w_1'(t) + A(t, w_1(t)) - f(t) + \xi_1(t), w_2(t) - w_1(t) \rangle_{V^* \times V} \\ & + j^0(t, \zeta_1(t), Mw_1(t); Mw_2(t) - Mw_1(t)) \\ & + \varphi(t, \eta_1(t), Mw_2(t)) - \varphi(t, \eta_1(t), Mw_1(t)) \geq 0 \end{aligned}$$

for a.e. $t \in (0, T)$ and

$$\begin{aligned} & \langle w_2'(t) + A(t, w_2(t)) - f(t) + \xi_2(t), w_1(t) - w_2(t) \rangle_{V^* \times V} \\ & + j^0(t, \zeta_2(t), Mw_2(t); Mw_1(t) - Mw_2(t)) \\ & + \varphi(t, \eta_2(t), Mw_1(t)) - \varphi(t, \eta_2(t), Mw_2(t)) \geq 0 \end{aligned}$$

for a.e. $t \in (0, T)$, and $w_1(0) = w_2(0) = 0$. We add these two inequalities to obtain

$$\begin{aligned} & \langle w_1'(t) - w_2'(t), w_2(t) - w_1(t) \rangle_{V^* \times V} + \langle A(t, w_1(t)) - A(t, w_2(t)), w_2(t) - w_1(t) \rangle_{V^* \times V} \\ & + j^0(t, \zeta_1(t), Mw_1(t); Mw_2(t) - Mw_1(t)) + j^0(t, \zeta_2(t), Mw_2(t); Mw_1(t) - Mw_2(t)) \\ & + \varphi(t, \eta_1(t), Mw_2(t)) - \varphi(t, \eta_1(t), Mw_1(t)) + \varphi(t, \eta_2(t), Mw_1(t)) - \varphi(t, \eta_2(t), Mw_2(t)) \\ & \geq \langle \xi_1(t), w_1(t) - w_2(t) \rangle_{V^* \times V} - \langle \xi_2(t), w_2(t) - w_1(t) \rangle_{V^* \times V} \end{aligned}$$

for a.e. $t \in (0, T)$. Next, we integrate the above inequality on $(0, t)$, and use hypotheses $H(A)(4)$, $H(j)(5)$ and $H(\varphi)(4)$ to get

$$\begin{aligned} & \frac{1}{2} \|w_1(t) - w_2(t)\|_H^2 - \frac{1}{2} \|w_1(0) - w_2(0)\|_H^2 + m_A \int_0^t \|w_1(s) - w_2(s)\|_V^2 ds \\ & \leq \bar{m}_{2j} \|A_M\| \int_0^t \|\zeta_1(s) - \zeta_2(s)\|_Z \|w_1(s) - w_2(s)\|_V ds \\ & + m_{2j} \|A_M\|^2 \int_0^t \|w_1(s) - w_2(s)\|_V^2 ds \\ & + \beta_\varphi \|A_M\| \int_0^t \|\eta_1(s) - \eta_2(s)\|_Y \|w_1(s) - w_2(s)\|_V ds \\ & + \int_0^t \|\xi_1(s) - \xi_2(s)\|_{V^*} \|w_1(s) - w_2(s)\|_V ds \end{aligned}$$

for all $t \in [0, T]$. Next, using hypothesis (H_2) and the Hölder inequality, we have

$$\begin{aligned} & (m_A - m_{2j} \|A_M\|^2) \|w_1 - w_2\|_{L^2(0,t;V)}^2 \leq \bar{m}_{2j} \|A_M\| \|\zeta_1 - \zeta_2\|_{L^2(0,t;Z)} \|w_1 - w_2\|_{L^2(0,t;V)} \\ & + \beta_\varphi \|A_M\| \|\eta_1 - \eta_2\|_{L^2(0,t;Y)} \|w_1 - w_2\|_{L^2(0,t;V)} + \|\xi_1 - \xi_2\|_{L^2(0,t;V^*)} \|w_1 - w_2\|_{L^2(0,t;V)} \end{aligned}$$

for all $t \in [0, T]$. Hence, by (H_2) , the inequality (3.14) follows.

Step 4 In this part of the proof we use a fixed point argument. We define the operator $\Lambda: L^2(0, T; V^* \times Y \times Z) \rightarrow L^2(0, T; V^* \times Y \times Z)$ by

$$\Lambda(\xi, \eta, \zeta) = (R_1 w_{\xi\eta\zeta}, R_2 w_{\xi\eta\zeta}, S w_{\xi\eta\zeta}) \quad \text{for all } (\xi, \eta, \zeta) \in L^2(0, T; V^* \times Y \times Z),$$

where $w_{\xi\eta\zeta} \in \mathcal{W}$ denotes the unique solution to Problem 14 corresponding to (ξ, η, ζ) .

From hypothesis $H(R, S)$, inequality (3.14) and by the Hölder inequality, we find a constant $c > 0$ such that

$$\begin{aligned} & \|\Lambda(\xi_1, \eta_1, \zeta_1)(t) - \Lambda(\xi_2, \eta_2, \zeta_2)(t)\|_{V^* \times Y \times Z}^2 \\ &= \|(R_1 w_1)(t) - (R_1 w_2)(t)\|_{V^*}^2 + \|(R_2 w_1)(t) - (R_2 w_2)(t)\|_Y^2 + \|(S w_1)(t) - (S w_2)(t)\|_Z^2 \\ &\leq \left(c_{R_1} \int_0^t \|w_1(s) - w_2(s)\|_V ds\right)^2 + \left(c_{R_2} \int_0^t \|w_1(s) - w_2(s)\|_V ds\right)^2 \\ &\quad + \left(c_S \int_0^t \|w_1(s) - w_2(s)\|_V ds\right)^2 \leq c \|w_1 - w_2\|_{L^2(0,t;V)}^2 \\ &\leq c \left(\|\xi_1 - \xi_2\|_{L^2(0,t;Z)}^2 + \|\eta_1 - \eta_2\|_{L^2(0,t;Y)}^2 + \|\xi_1 - \xi_2\|_{L^2(0,t;V^*)}^2\right), \end{aligned}$$

which entails

$$\begin{aligned} & \|\Lambda(\xi_1, \eta_1, \zeta_1)(t) - \Lambda(\xi_2, \eta_2, \zeta_2)(t)\|_{V^* \times Y \times Z}^2 \\ &\leq c \int_0^t \|(\xi_1, \eta_1, \zeta_1)(s) - (\xi_2, \eta_2, \zeta_2)(s)\|_{V^* \times Y \times Z}^2 ds \end{aligned} \quad (3.15)$$

for a.e. $t \in (0, T)$. By Lemma 10, we deduce that there exists a unique fixed point (ξ^*, η^*, ζ^*) of Λ , i.e.,

$$(\xi^*, \eta^*, \zeta^*) \in L^2(0, T; V^* \times Y \times Z) \text{ and } \Lambda(\xi^*, \eta^*, \zeta^*) = (\xi^*, \eta^*, \zeta^*).$$

Step 5 Let $(\xi^*, \eta^*, \zeta^*) \in L^2(0, T; V^* \times Y \times Z)$ be the unique fixed point of the operator Λ . We define $w_{\xi^*\eta^*\zeta^*} \in \mathcal{W}$ to be the unique solution to Problem 14 corresponding to (ξ^*, η^*, ζ^*) . From the definition of the operator Λ , we have

$$\xi^* = R_1(w_{\xi^*\eta^*\zeta^*}), \quad \eta^* = R_2(w_{\xi^*\eta^*\zeta^*}) \text{ and } \zeta^* = S(w_{\xi^*\eta^*\zeta^*}).$$

Finally, we use these relations in Problem 14, and conclude that $w_{\xi^*\eta^*\zeta^*}$ is the unique solution to Problem 11. This completes the proof of the theorem. $\square$

Next, we consider the constrained variational–hemivariational inequality with the nonhomogeneous initial condition $w_0 \in V$, $w_0 \neq 0$.

Problem 19 Find $w \in \mathcal{W}$ such that $w(t) \in K$ for a.e. $t \in (0, T)$ and

$$\begin{cases} \langle w'(t) + A(t, w(t)) + (R_1 w)(t) - f(t), v - w(t) \rangle_{V^* \times V} \\ \quad + j^0(t, (S w)(t), M w(t); M v - M w(t)) \\ \quad + \varphi(t, (R_2 w)(t), M v) - \varphi(t, (R_2 w)(t), M w(t)) \geq 0 \\ \quad \text{for all } v \in K, \text{ a.e. } t \in (0, T), \\ w(0) = w_0. \end{cases}$$

Theorem 20 Under hypotheses $H(A)$, $H(j)$, $H(\varphi)$, $H(R, S)$, $H(K)$, $H(M)$, (H_1) , and (H_2) , Problem 19 has a unique solution.

Proof Let $\tilde{w}(t) = w(t) - w_0$ and $\tilde{K} = \{v - w_0 \mid v \in K\} \subset V$. We define operators $\tilde{A}: (0, T) \times V \rightarrow V^*$ and $\tilde{M}: V \rightarrow X$ by

$$\tilde{A}(t, v) = A(t, v + w_0) \text{ for } v \in V, \text{ a.e. } t \in (0, T), \quad (3.16)$$

$$\tilde{M}v = M(v + w_0) \text{ for } v \in V. \quad (3.17)$$

With this notation, Problem 19 can be reformulated as follows.

Problem 21 Find $\tilde{w} \in \mathcal{W}$ such that $\tilde{w}(t) \in \tilde{K}$ for a.e. $t \in (0, T)$ and

$$\left\{ \begin{array}{l} \langle \tilde{w}'(t) + \tilde{A}(t, \tilde{w}(t)) + (R_1 \tilde{w})(t) - f(t), v - \tilde{w}(t) \rangle_{V^* \times V} \\ \quad + j^0(t, (S(\tilde{w} + w_0))(t), \tilde{M}\tilde{w}(t); \tilde{M}v - \tilde{M}\tilde{w}(t)) \\ \quad + \varphi(t, (R_2(\tilde{w} + w_0))(t), \tilde{M}v) - \varphi(t, (R_2(\tilde{w} + w_0))(t), \tilde{M}\tilde{w}(t)) \geq 0 \\ \quad \text{for all } v \in \tilde{K}, \text{ a.e. } t \in (0, T), \\ \tilde{w}(0) = 0. \end{array} \right.$$

We will apply Theorem 13 to obtain that Problem 21 has a unique solution $\tilde{w} \in \mathcal{W}$. It is sufficient to show that the operators $\tilde{A}$ and $\tilde{M}$ defined by (3.16) and (3.17) satisfy the hypotheses $H(A)$ and $H(M)$.

We first verify that $\tilde{A}$ satisfies $H(A)$. The conditions $H(A)(1), (2)$ are obvious. For $v \in V$, a.e. $t \in (0, T)$, we have

$$\|\tilde{A}(t, v)\|_{V^*} = \|A(t, v + w_0)\|_{V^*} \leq a_0(t) + a_1\|v + w_0\|_V \leq a_0(t) + a_1\|w_0\|_V + a_1\|v\|_V.$$

Hence, $H(A)(3)$ holds with $\tilde{a}_0(t) = a_0(t) + a_1\|w_0\|_V$ and $\tilde{a}_1 = a_1$. Moreover, for $v_1, v_2 \in V$, a.e. $t \in (0, T)$, we have

$$\begin{aligned} & \langle \tilde{A}(t, v_1) - \tilde{A}(t, v_2), v_1 - v_2 \rangle_{V^* \times V} \\ &= \langle A(t, v_1 + w_0) - A(t, v_2 + w_0), v_1 - v_2 \rangle_{V^* \times V} \\ &= \langle A(t, v_1 + w_0) - A(t, v_2 + w_0), (v_1 + w_0) - (v_2 + w_0) \rangle_{V^* \times V} \\ &\geq m_A \|(v_1 + w_0) - (v_2 + w_0)\|_V^2 = m_A \|v_1 - v_2\|_V^2. \end{aligned}$$

Hence, $H(A)(4)$ holds with $m_{\tilde{A}} = m_A$.

Next, we check that $\tilde{M}$ satisfies hypothesis $H(M)$ with $A_{\tilde{M}} = A_M$ and $\tilde{M}0 = A_M w_0 + M0$. From the linearity of the operator $A_M: V \rightarrow X$, $A_M v = Mv - M0$ for $v \in V$ and (3.17), we have

$$A_{\tilde{M}} v = \tilde{M}v - \tilde{M}0 = M(v + w_0) - Mw_0 = A_M(v + w_0) + M0 - (A_M w_0 + M0) = A_M v$$

for $v \in V$, and

$$\tilde{M}0 = Mw_0 = A_M w_0 + M0,$$

which means that $\widetilde{M}$ is an affine operator. Furthermore, for $v \in V$, a.e. $t \in (0, T)$, we get

$$(\widetilde{\mathcal{M}}v)(t) = \widetilde{M}(v(t)) = M(v(t) + w_0) = \mathcal{M}(v + w_0)(t).$$

It follows from the compactness of $\mathcal{M}$ that $\widetilde{\mathcal{M}}$ is compact. Therefore, $\widetilde{M}$ satisfies $H(M)$.

It follows from Theorem 13 that Problem 21 has a unique solution $\widetilde{w} \in \mathcal{W}$. Therefore, $w \in \mathcal{W}$ given by $w(t) = \widetilde{w}(t) + w_0$ is the unique solution to Problem 19. This completes the proof. $\square$

4 Unilateral Frictional Contact Problem

In this section, we study a unilateral viscoelastic frictional contact problem to which our main results of Sect. 3 can be applied. We give the classical formulation of the contact problem, provide its variational formulation, and prove a result on its unique weak solvability.

Consider the following physical setting. A viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$. The boundary of Ω , denoted by Γ , is assumed to be Lipschitz continuous. We denote by $\mathbf{v}$ the outward unit normal at Γ . We suppose that Γ consists of three mutually disjoint and measurable parts $\overline{\Gamma}_D$, $\overline{\Gamma}_N$ and $\overline{\Gamma}_C$ such that $\text{meas}(\Gamma_D) > 0$. The symbol $\mathbb{S}^d$ denotes the space of $d \times d$ symmetric matrices. The standard inner products and norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_i v_i, \quad \|\mathbf{u}\| = (\mathbf{u} \cdot \mathbf{u})^{1/2} \quad \text{for } \mathbf{u} = (u_i), \mathbf{v} = (v_i) \in \mathbb{R}^d, \\ \boldsymbol{\sigma} \cdot \boldsymbol{\tau} &= \sigma_i \tau_i, \quad \|\boldsymbol{\sigma}\| = (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})^{1/2} \quad \text{for } \boldsymbol{\sigma} = (\sigma_{ij}), \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d, \end{aligned}$$

respectively. For a vector field $\mathbf{v}$, v_ν and $\mathbf{v}_\tau$ denote its normal and tangential components on the boundary defined by

$$v_\nu = \mathbf{v} \cdot \boldsymbol{\nu} \quad \text{and} \quad \mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}.$$

Given a tensor $\boldsymbol{\sigma}$, the symbols σ_ν and $\boldsymbol{\sigma}_\tau$ denote its normal and tangential components on the boundary, i.e.,

$$\sigma_\nu = (\boldsymbol{\sigma} \boldsymbol{\nu}) \cdot \boldsymbol{\nu} \quad \text{and} \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}.$$

The classical model for the contact process on the finite time interval is the following.

Problem 22 Find a displacement field $\mathbf{u} : \Omega \times (0, T) \rightarrow \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times (0, T) \rightarrow \mathbb{S}^d$ such that for all $t \in (0, T)$,

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}'(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{C}(t-s)\boldsymbol{\varepsilon}(\mathbf{u}'(s)) ds \quad \text{in } \Omega, \quad (4.1)$$

$$\mathbf{u}''(t) = \text{Div } \boldsymbol{\sigma}(t) + \mathbf{f}_0(t) \quad \text{in } \Omega, \quad (4.2)$$

$$\mathbf{u}(t) = \mathbf{0} \quad \text{on } \Gamma_D, \quad (4.3)$$

$$\boldsymbol{\sigma}(t)\mathbf{v} = \mathbf{f}_N(t) \quad \text{on } \Gamma_N, \quad (4.4)$$

$$\begin{aligned} u'_v(t) &\leq g, \quad \sigma_v(t) + \eta(t) \leq 0, \quad (u'_v(t) - g)(\sigma_v(t) + \eta(t)) = 0, \\ \eta(t) &\in k(u_v(t)) \partial j_v(u'_v(t)) \end{aligned} \quad \text{on } \Gamma_C, \quad (4.5)$$

$$\begin{aligned} \|\boldsymbol{\sigma}_\tau(t)\| &\leq F_b \left(\int_0^t \|\mathbf{u}_\tau(s)\| ds \right), \\ -\boldsymbol{\sigma}_\tau &= F_b \left(\int_0^t \|\mathbf{u}_\tau(s)\| ds \right) \frac{\mathbf{u}'_\tau(t)}{\|\mathbf{u}'_\tau(t)\|} \quad \text{if } \mathbf{u}'_\tau(t) \neq \mathbf{0} \end{aligned} \quad \text{on } \Gamma_C, \quad (4.6)$$

and

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}'(0) = \mathbf{w}_0 \quad \text{in } \Omega. \quad (4.7)$$

In this problem, equation (4.1) represents the constitutive law for viscoelastic materials with long memory in which $\mathcal{A}$ is the viscosity operators, $\mathcal{B}$ represents the elasticity operator and $\mathcal{C}$ is the relaxation tensor, and $\boldsymbol{\varepsilon}(\mathbf{u})$ denotes the linearized strain tensor defined by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } \Omega.$$

Equation (4.2) is the equation of motion in which $\text{Div } \boldsymbol{\sigma} = (\sigma_{ij,j})$ and $\mathbf{f}_0$ denotes the density of the body forces. The displacement homogeneous boundary condition (4.3) means that the body is fixed on Γ_D , while (4.4) is the traction boundary condition with surface tractions of density $\mathbf{f}_N$ acting on Γ_N . Condition (4.5) is the Signorini unilateral contact boundary condition for the normal velocity in which $g > 0$ and ∂j_v denotes the Clarke subgradient of a prescribed function j_v . Condition $\eta(t) \in k(u_v(t)) \partial j_v(u'_v(t))$ on Γ_C represents the normal damped response condition where k is a given damper coefficient depending on the normal displacement. Condition (4.6) is a version of the Coulomb law of dry friction in which F_b denotes the friction bound. The latter may depend on the total (accumulated slip) represented by

$$\int_0^t \|\mathbf{u}_\tau(\mathbf{x}, s)\| ds$$

at the point $\mathbf{x} \in \Gamma_C$ in the time interval $[0, t]$. Examples and details on mechanical interpretation of conditions (4.5) and (4.6) can be found in [18,28] and references therein. Finally, conditions as in (4.7) are the initial conditions in which $\mathbf{u}_0$ and $\mathbf{w}_0$ represent the initial displacement and the initial velocity, respectively.

To provide the weak formulation of Problem 22, we use the following spaces

$$\mathcal{H} = L^2(\Omega; \mathbb{S}^d), \quad V = \{ \mathbf{v} \in H^1(\Omega; \mathbb{R}^d) \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D \}.$$

The space $\mathcal{H}$ is a Hilbert space endowed with the inner product

$$\langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \rangle_{\mathcal{H}} = \int_{\Omega} \sigma_{ij}(\mathbf{x}) \varepsilon_{ij}(\mathbf{x}) \, dx \quad \text{for all } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathcal{H},$$

and the associated norm $\| \cdot \|_{\mathcal{H}}$. The inner product and the corresponding norm on V are given by

$$(\mathbf{u}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}, \quad \|\mathbf{v}\|_V = \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathcal{H}} \quad \text{for all } \mathbf{u}, \mathbf{v} \in V.$$

Recall, see e.g. [28], that, since $\text{meas}(\Gamma_D) > 0$, V is a Hilbert space. Moreover, the continuity of the trace operator $\gamma: V \rightarrow L^2(\Gamma_C; \mathbb{R}^d)$ implies

$$\|\mathbf{v}\|_{L^2(\Gamma; \mathbb{R}^d)} \leq \|\gamma\| \|\mathbf{v}\|_V \quad \text{for all } \mathbf{v} \in V,$$

where $\|\gamma\|$ denotes the norm of the trace operator in $\mathcal{L}(V, L^2(\Gamma_C; \mathbb{R}^d))$. We define a space of fourth order tensor fields

$$\mathcal{Q}_{\infty} = \{ \boldsymbol{\sigma} = (\sigma_{ijkl}) \mid \sigma_{ijkl} = \sigma_{jikl} = \sigma_{klij} \in L^{\infty}(\Omega), 1 \leq i, j, k, l \leq d \}$$

which is a real Banach space with the norm

$$\|\boldsymbol{\sigma}\|_{\mathcal{Q}_{\infty}} = \sum_{1 \leq i, j, k, l \leq d} \|\sigma_{ijkl}\|_{L^{\infty}(\Omega)} \quad \text{for all } \boldsymbol{\sigma} \in \mathcal{Q}_{\infty}.$$

Our hypotheses on Problem 22 read as follows.

$H(\mathcal{A})$: $\mathcal{A}: \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ is such that

- (1) $\mathcal{A}(\cdot, \boldsymbol{\varepsilon})$ is measurable on Ω for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$.
- (2) there exists $L_{\mathcal{A}} > 0$ such that $\|\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|$ for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega$.
- (3) there exists $m_{\mathcal{A}} > 0$ such that $(\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2$ for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega$.
- (4) $\mathcal{A}(\mathbf{x}, \mathbf{0}) = \mathbf{0}$ for a.e. $\mathbf{x} \in \Omega$.

$H(\mathcal{B})$: $\mathcal{B}: \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ is such that

- (1) $\mathcal{B}(\cdot, \boldsymbol{\varepsilon})$ is measurable on Ω for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$.
- (2) there exists $L_{\mathcal{B}} > 0$ such that $\|\mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{B}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|$ for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega$.
- (3) $\mathcal{B}(\mathbf{x}, \mathbf{0}) = \mathbf{0}$ for a.e. $\mathbf{x} \in \Omega$.

$H(\mathcal{C})$: $\mathcal{C} \in C([0, T]; \mathcal{Q}_{\infty})$.

$H(k)$: $k: \Gamma_C \times \mathbb{R} \rightarrow \mathbb{R}$ is such that

- (1) $k(\cdot, r)$ is measurable on Γ_C for all $r \in \mathbb{R}$.
- (2) there exist k_1, k_2 such that $0 < k_1 \leq k(\mathbf{x}, t) \leq k_2$ for all $r \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_C$.
- (3) there exists $L_k > 0$ such that $|k(\mathbf{x}, r_1) - k(\mathbf{x}, r_2)| \leq L_k |r_1 - r_2|$ for all $r_1, r_2 \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_C$.

$H(j_v)$: $j_v: \Gamma_C \times \mathbb{R} \rightarrow \mathbb{R}$ is such that

- (1) $j_v(\cdot, r)$ is measurable on Γ_C for all $r \in \mathbb{R}$ and there exists $e \in L^2(\Gamma_C)$ such that $j_v(\cdot, e(\cdot)) \in L^1(\Gamma_C)$.
- (2) $j_v(\mathbf{x}, \cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $\mathbf{x} \in \Gamma_C$.
- (3) there are $\bar{c}_0, \bar{c}_1 \geq 0$ such that $|\partial j_v(\mathbf{x}, r)| \leq \bar{c}_0 + \bar{c}_1 |r|$ for all $r \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_C$.
- (4) there exists $\alpha_{j_v} \geq 0$ such that $j_v^0(\mathbf{x}, r_1; r_2 - r_1) + j_v^0(\mathbf{x}, r_2; r_1 - r_2) \leq \alpha_{j_v} |r_1 - r_2|^2$ for all $r_1, r_2 \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_C$.

$H(F_b)$: $F_b: \Gamma_C \times \mathbb{R} \rightarrow \mathbb{R}$ is such that

- (1) $F_b(\cdot, r)$ is measurable on Γ_C for all $r \in \mathbb{R}$.
- (2) there exists $L_{F_b} > 0$ such that $|F_b(\mathbf{x}, r_1) - F_b(\mathbf{x}, r_2)| \leq L_{F_b} |r_1 - r_2|$ for all $r_1, r_2 \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_C$.
- (3) $F_b(\mathbf{x}, r) = 0$ for all $r \leq 0$, $F_b(\mathbf{x}, r) \geq 0$ for all $r \geq 0$ for a.e. $\mathbf{x} \in \Gamma_C$.

(H_3) : $f_0 \in L^2(0, T; L^2(\Omega; \mathbb{R}^d))$, $f_N \in L^2(0, T; L^2(\Gamma_N; \mathbb{R}^d))$, $\mathbf{u}_0, \mathbf{w}_0 \in V$.

Further, we introduce the set of admissible velocity fields U defined by

$$U = \{ \mathbf{v} \in V \mid v_v \leq g \text{ on } \Gamma_C \},$$

and an element $f \in V^*$ by

$$\langle f, \mathbf{v} \rangle_{V^* \times V} = \langle f_0, \mathbf{v} \rangle_{L^2(\Omega; \mathbb{R}^d)} + \langle f_N, \mathbf{v} \rangle_{L^2(\Gamma_N; \mathbb{R}^d)} \quad (4.8)$$

for all $\mathbf{v} \in V$.

We now turn to the weak formulation of Problem 22. Let $(\mathbf{u}, \sigma)$ be a smooth solution to this problem which means that the data are smooth functions such that all the derivatives and all the conditions are satisfied in the usual sense at each point. Let $\mathbf{v} \in U$ and $t \in (0, T)$. We multiply (4.2) by $\mathbf{v} - \mathbf{u}'(t)$, use an the integration by parts formula, and apply the boundary conditions (4.3) and (4.4) to obtain

$$\begin{aligned} & \int_{\Omega} \mathbf{u}''(t) \cdot (\mathbf{v} - \mathbf{u}'(t)) dx + \int_{\Omega} \sigma(t) \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}'(t))) dx \\ &= \int_{\Omega} f_0(t) \cdot (\mathbf{v} - \mathbf{u}'(t)) dx + \int_{\Gamma_N} f_N(t) \cdot (\mathbf{v} - \mathbf{u}'(t)) d\Gamma + \int_{\Gamma_C} \sigma(t) \mathbf{v} \cdot (\mathbf{v} - \mathbf{u}'(t)) d\Gamma. \end{aligned}$$

From (4.5) and the definition of the Clarke subgradient, we have

$$\begin{aligned} \sigma_v(t)(v_v - u'_v(t)) &= (\sigma_v(t) + \eta(t))(v_v - g) - (\sigma_v(t) + \eta(t))(u'_v(t) - g) \\ &\quad - \eta(t)(v_v - u'_v(t)) \geq -k(u_v(t)) j_v^0(u'_v(t); v_v - u'_v(t)) \text{ on } \Gamma_C. \end{aligned} \quad (4.9)$$

On the other hand, the friction law (4.6) can be equivalently written as

$$\sigma_\tau(t) \cdot (v_\tau - u'_\tau(t)) \geq -F_b \left(\int_0^t \|u_\tau(s)\| ds \right) (\|v_\tau\| - \|u'_\tau(t)\|) \text{ on } \Gamma_C. \quad (4.10)$$

Combining (4.9), (4.10) and the decomposition formula, see [28, (6.33)], we obtain

$$\begin{aligned} & F_b \left(\int_0^t \|u_\tau(s)\| ds \right) (\|v_\tau\| - \|u'_\tau(t)\|) + k(u_v(t)) j_v^0(u'_v(t); v_v - u'_v(t)) \\ & + \sigma(t) v \cdot (v - u'(t)) \geq 0 \text{ on } \Gamma_C. \end{aligned}$$

The latter and the definition (4.8) imply

$$\begin{aligned} & \int_\Omega u''(t) \cdot (v - u'(t)) dx + \langle \sigma(t), \varepsilon(v) - \varepsilon(u'(t)) \rangle_{\mathcal{H}} \\ & + \int_{\Gamma_C} F_b \left(\int_0^t \|u_\tau(s)\| ds \right) (\|v_\tau\| - \|u'_\tau(t)\|) d\Gamma \\ & + \int_{\Gamma_C} k(u_v(t)) j_v^0(u'_v(t); v_v - u'_v(t)) d\Gamma \geq \langle f, v - u \rangle_{V^* \times V}. \end{aligned}$$

Finally, exploiting (4.1), we are lead to the following variational formulation of Problem 22.

Problem 23 Find $u: (0, T) \rightarrow V$ such that $u(0) = u_0$, $u'(0) = w_0$ and

$$\begin{aligned} & \int_\Omega u''(t) \cdot (v - u'(t)) dx + \langle \mathcal{A}(\varepsilon(u'(t))), \varepsilon(v) - \varepsilon(u'(t)) \rangle_{\mathcal{H}} \\ & + \langle \mathcal{B}(\varepsilon(u(t))) + \int_0^t \mathcal{C}(t-s) \varepsilon(u'(s)) ds, \varepsilon(v) - \varepsilon(u'(t)) \rangle_{\mathcal{H}} \\ & + \int_{\Gamma_C} F_b \left(\int_0^t \|u_\tau(s)\| ds \right) (\|v_\tau\| - \|u'_\tau(t)\|) d\Gamma \\ & + \int_{\Gamma_C} k(u_v(t)) j_v^0(u'_v(t); v_v - u'_v(t)) d\Gamma \geq \langle f, v - u \rangle_{V^* \times V} \end{aligned}$$

for all $v \in U$, a.e. $t \in (0, T)$.

The following result concerns the unique solvability and regularity of solution to Problem 23.

Theorem 24 Assume hypotheses $H(\mathcal{A})$, $H(\mathcal{B})$, $H(\mathcal{C})$, $H(k)$, $H(j_v)$, $H(F_b)$, (H_3) and the following smallness condition

$$m_{\mathcal{A}} > \alpha_{j_v} k_2 \|\gamma\|^2.$$

Then Problem 23 has a unique solution $u \in C([0, T]; V)$, $u' \in \mathcal{W}$ with $u'(t) \in U$ for a.e. $t \in (0, T)$.

Proof We will apply Theorem 20 with the following functional framework: $X = Y = Z = L^2(\Gamma_C)$, $K = U$, and $M = \gamma$. Let the operator $A: V \rightarrow V^*$, and functions $\varphi: Y \times X \rightarrow \mathbb{R}$ and $j: Z \times X \rightarrow \mathbb{R}$ be defined by

$$\begin{aligned}\langle A\mathbf{w}, \mathbf{v} \rangle_{V^* \times V} &= \langle \mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{w})), \boldsymbol{\varepsilon}(\mathbf{v}) \rangle_{\mathcal{H}} \text{ for } \mathbf{w}, \mathbf{v} \in V, \\ \varphi(y, x) &= \int_{\Gamma_C} F_b(y) \|x\| d\Gamma \text{ for } y \in Y, x \in X, \\ j(z, x) &= \int_{\Gamma_C} k(z) j_v(x) d\Gamma \text{ for } z \in Z, x \in X.\end{aligned}$$

Note that A , φ and j are now independent of t . Moreover, we introduce operators $R_1: \mathcal{V} \rightarrow \mathcal{V}^*$, $R_2: \mathcal{V} \rightarrow L^2(0, T; Y)$, and $S: \mathcal{V} \rightarrow L^2(0, T; Z)$ given by

$$\begin{aligned}((R_1\mathbf{w})(t), \mathbf{v})_{V^* \times V} &= \langle \mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_0)) + \int_0^t \boldsymbol{\varepsilon}(\mathbf{w}(s)) ds, \boldsymbol{\varepsilon}(\mathbf{v}) \rangle_{\mathcal{H}} \\ &\quad + \left\langle \int_0^t \mathcal{C}(t-s) \boldsymbol{\varepsilon}(\mathbf{w}(s)) ds, \boldsymbol{\varepsilon}(\mathbf{v}) \right\rangle_{\mathcal{H}} \text{ for } \mathbf{w} \in \mathcal{V}, \mathbf{v} \in V, t \in (0, T), \\ (R_2\mathbf{w})(t) &= \int_0^t \left\| \int_0^s \mathbf{w}_\tau(r) dr + \mathbf{u}_{0\tau} \right\| ds \text{ for } \mathbf{w} \in \mathcal{V}, t \in (0, T), \\ (S\mathbf{w})(t) &= \int_0^t w_v(s) ds + u_{0v} \text{ for } \mathbf{w} \in \mathcal{V}, t \in (0, T).\end{aligned}$$

Let $\mathbf{w}(t) = \mathbf{u}'(t)$ for all $t \in (0, T)$. Then, with the above notation, Problem 23 can be formulated as follows.

Problem 25 Find $\mathbf{w} \in \mathcal{W}$ such that $\mathbf{w}(t) \in U$ for a.e. $t \in (0, T)$, $\mathbf{w}(0) = \mathbf{w}_0$ and

$$\begin{aligned}& \langle \mathbf{w}'(t) + A(\mathbf{w}(t)) + (R_1\mathbf{w})(t) - \mathbf{f}(t), \mathbf{v} - \mathbf{w}(t) \rangle_{V^* \times V} \\ & \quad + j^0((S\mathbf{w})(t), M\mathbf{w}(t); M\mathbf{v} - M\mathbf{w}(t)) \\ & \quad + \varphi((R_2\mathbf{w})(t), M\mathbf{v}) - \varphi((R_2\mathbf{w})(t), M\mathbf{w}(t)) \geq 0\end{aligned}$$

for all $\mathbf{v} \in U$, a.e. $t \in (0, T)$.

It is clear that the set $K = U$ is a closed and convex subset of V with $\mathbf{0} \in K$, i.e., $H(K)$ holds. The hypotheses $H(A)$, $H(j)$, $H(\varphi)$, $H(R, S)$, $H(M)$, (H_1) , and (H_2) can be verified as in the proof in [18, Theorem 13]. We deduce from Theorem 20 that Problem 25 has a solution $\mathbf{w} \in \mathcal{W}$ such that $\mathbf{w}(t) \in U$ for a.e. $t \in (0, T)$. From the relation $\mathbf{u}(t) = \int_0^t \mathbf{w}(s) ds + \mathbf{u}_0$ for all $t \in (0, T)$, we conclude that $\mathbf{u} \in C([0, T]; V)$, $\mathbf{u}' \in \mathcal{W}$ with $\mathbf{u}'(t) \in U$ for a.e. $t \in (0, T)$. This completes the proof. $\square$

A couple of functions $(\mathbf{u}, \boldsymbol{\sigma})$ which satisfies Problem 23 and (4.1) is called a weak solution to Problem 22. Under the assumptions of Theorem 24, Problem 22 has a unique weak solution. Further, this solution has the regularity $\mathbf{u} \in C([0, T]; V)$, $\mathbf{u}' \in \mathcal{W}$ with $\mathbf{u}'(t) \in U$ for a.e. $t \in (0, T)$, $\boldsymbol{\sigma} \in L^2(0, T; \mathcal{H})$, and $\text{Div } \boldsymbol{\sigma} \in \mathcal{V}^*$.

Finally, we note that the main contribution of this paper consists in extending [18, Theorem 9] from the case of no constraints (i.e., $K = V$) to the case of a nonempty,

closed and convex set of constraints $K \subset V$. Moreover, we have used the recent existence result for an equilibrium problem involving bifunctions which is a novelty for variational–hemivariational inequalities. It is reasonable to conjecture that Problem 11 has solutions without the smallness assumption in (H_2) , however, the uniqueness is likely to be lost.

Acknowledgements The authors are gratefully indebted to the anonymous referees for their insightful comments that improved the quality of the paper.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Ahn, J., Stewart, D.E.: Dynamic frictionless contact in linear viscoelasticity. *IMA J. Numer. Anal.* **29**, 43–71 (2009)
2. Carl, S., Le, V.K., Motreanu, D.: *Nonsmooth Variational Problems and Their Inequalities. Comparison Principles and Applications*. Springer Monogr. Math. Springer, New York (2007)
3. Carl, S., Le, V.K., Motreanu, D.: Evolutionary variational-hemivariational inequalities: existence and comparison results. *J. Math. Anal. Appl.* **345**, 545–558 (2008)
4. Carl, S., Motreanu, D.: Extremal solutions of quasilinear parabolic inclusions with generalized Clarke's gradient. *J. Differ. Equ.* **191**, 206–233 (2003)
5. Carstensen, C., Gwinner, J.: A theory of discretization for nonlinear evolution inequalities applied to parabolic Signorini problems. *Ann. Mat. Pura Appl.* **177**, 363–394 (1999)
6. Chadli, O., Ansari, Q.H., Al-Homidan, S.: Existence of solutions for nonlinear implicit differential equations: an equilibrium problem approach. *Numer. Funct. Anal. Optim.* **37**, 1385–1419 (2016)
7. Chadli, O., Ansari, Q.H., Yao, J.-C.: Mixed equilibrium problems and anti-periodic solutions for nonlinear evolution equations. *J. Optim. Theory Appl.* **168**, 410–440 (2016)
8. Clarke, F.H.: *Optimization and Nonsmooth Analysis*. Wiley, New York (1983)
9. Cocou, M.: Existence of solutions of a dynamic Signorini's problem with nonlocal friction in viscoelasticity. *Z. Angew. Math. Phys.* **53**, 1099–1109 (2002)
10. Denkowski, Z., Migórski, S., Papageorgiou, N.S.: *An Introduction to Nonlinear Analysis: Theory*. Kluwer Academic/Plenum Publishers, Boston (2003)
11. Denkowski, Z., Migórski, S., Papageorgiou, N.S.: *An Introduction to Nonlinear Analysis: Applications*. Kluwer Academic/Plenum Publishers, Boston (2003)
12. Eck, C., Jarušek, J., Krbeč, M.: *Unilateral Contact Problems: Variational Methods and Existence Theorems*, Pure and Applied Mathematics, vol. 270. Chapman/CRC Press, New York (2005)
13. Eck, C., Jarušek, J., Sofonea, M.: A dynamic elastic-visco-plastic unilateral contact problem with normal damped response and Coulomb friction. *Eur. J. Appl. Math.* **21**, 229–251 (2010)
14. Ekeland, I., Temam, R.: *Convex Analysis and Variational Problems*. North-Holland, Amsterdam (1976)
15. Essoufi, El-H., Kabbaj, M.: Existence of solutions of a dynamic Signorini's problem with nonlocal friction for viscoelastic piezoelectric materials. *Bull. Math. Soc. Sc. Math. Roumanie* **48**, 181–195 (2005)
16. Goeleven, D., Motreanu, D., Dumont, Y., Rochdi, M.: *Variational and Hemivariational Inequalities, Theory, Methods and Applications, Volume I: Unilateral Analysis and Unilateral Mechanics*. Kluwer Academic Publishers, Boston (2003)
17. Han, J.F., Migórski, S., Zeng, H.D.: Analysis of a dynamic viscoelastic unilateral contact problem with normal damped response. *Nonlinear Anal. RWA* **28**, 229–250 (2016)

18. Han, W., Migórski, S., Sofonea, M.: Analysis of a general dynamic history-dependent variational-hemivariational inequality. *Nonlinear Anal. RWA* **36**, 69–88 (2017)
19. Han, W., Sofonea, M.: Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity. *Studies in Advanced Mathematics*, vol. 30. Americal Mathematical Society, Providence, RI–International Press, Somerville, MA (2002)
20. Haslinger, J., Miettinen, M., Panagiotopoulos, P.D.: Finite Element Method for Hemivariational Inequalities: Theory, Methods and Applications. Kluwer Academic Publishers, Boston (1999)
21. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications *Classics in Applied Mathematics*, vol. 31. SIAM, Philadelphia (2000)
22. Kulig, A., Migórski, S.: Solvability and continuous dependence results for second order nonlinear inclusion with Volterra-type operator. *Nonlinear Anal.* **75**, 4729–4746 (2012)
23. Kuttler, K., Shillor, M.: Dynamic contact with Signorini's condition and slip rate dependent friction. *Electron. J. Differ. Equ.* **83**, 1–21 (2004)
24. Lions, J.-L.: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod, Gauthier-Villars, Paris (1969)
25. Migórski, S.: Optimal control of history-dependent evolution inclusions with applications to frictional contact. *J. Optim. Theory Appl.* **185**, 574–596 (2020)
26. Migórski, S., Dudek, S.: A new class of variational-hemivariational inequalities for steady Oseen flow with unilateral and frictional type boundary conditions. *Zeitschrift fuer Angewandte Mathematik und Mechanik* **100**, e201900112 (2020)
27. Migórski, S., Ochal, A., Sofonea, M.: History-dependent subdifferential inclusions and hemivariational inequalities in contact mechanics. *Nonlinear Anal. RWA* **12**, 3384–3396 (2011)
28. Migórski, S., Ochal, A., Sofonea, M.: Nonlinear Inclusions and Hemivariational Inequalities. *Models and Analysis of Contact Problems. Advances in Mechanics and Mathematics*, vol. 26. Springer, New York (2013)
29. Migórski, S., Ochal, A., Sofonea, M.: History-dependent variational-hemivariational inequalities in contact mechanics. *Nonlinear Anal. RWA* **22**, 604–618 (2015)
30. Migórski, S., Pączka, D.: Variational inequality with almost history-dependent operator for frictionless contact problems. *J. Math. Anal. Appl.* **485**, 123803 (2020)
31. Migórski, S., Sofonea, M., Zeng, S.D.: Well-posedness of history-dependent sweeping processes. *SIAM J. Math. Anal.* **51**, 1082–1107 (2019)
32. Migórski, S., Zeng, S.D.: Rothe method and numerical analysis for history-dependent hemivariational inequalities with applications to contact mechanics. *Numer. Algorithms* **82**, 423–450 (2019)
33. Motreanu, D., Panagiotopoulos, P.D.: Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities and Applications. Kluwer Academic Publishers, Boston (1999)
34. Naniewicz, Z., Panagiotopoulos, P.D.: Mathematical Theory of Hemivariational Inequalities and Applications. Marcel Dekker Inc, New York (1995)
35. Panagiotopoulos, P.D.: Nonconvex energy functions, hemivariational inequalities and substationary principles. *Acta Mech.* **42**, 160–183 (1983)
36. Panagiotopoulos, P.D.: Inequality Problems in Mechanics and Applications. Convex and Nonconvex Energy Functions. Birkhäuser, Basel (1985)
37. Panagiotopoulos, P.D.: Hemivariational Inequalities, Applications in Mechanics and Engineering. Springer, Berlin (1993)
38. Shillor, M., Sofonea, M., Telega, J.J.: Models and Analysis of Quasistatic Contact. *Lect. Notes Phys.*, vol. 655. Springer, New York (2004)
39. Sofonea, M., Matei, A.: Mathematical Models in Contact Mechanics, London Mathematical Society Lecture Notes Series, vol. 398. Cambridge University Press, Cambridge (2012)
40. Sofonea, M., Migórski, S.: Variational–Hemivariational Inequalities with Applications. Chapman & Hall/CRC, Monographs and Research Notes in Mathematics, Boca Raton (2018)
41. Zeidler, E.: Nonlinear Functional Analysis and Applications II A/B. Springer, New York (1990)

Affiliations

Stanisław Migórski^{1,2} · Biao Zeng³

✉ Stanisław Migórski
stanislaw.migorski@uj.edu.pl

Biao Zeng
biao_zeng@163.com

¹ College of Applied Mathematics, Chengdu University of Information Technology, Chengdu 610225, Sichuan Province, People's Republic of China

² Chair of Optimization and Control, Jagiellonian University in Krakow, ul. Łojasiewicza 6, 30348 Kraków, Poland

³ Faculty of Mathematics and Physics, Guangxi University for Nationalities, Nanning 530006, Guangxi Province, People's Republic of China