Upper Semicontinuity of Trajectory Attractors for 3D Incompressible Navier–Stokes Equation

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Abstract

In this paper, we first establish the existence of a trajectory attractor for the Navier–Stokes–Voight (NSV) equation and then prove upper semicontinuity of trajectory attractors of 3D incompressible Navier–Stokes equation when 3D NSV equation is considered as a perturbative equation of 3D incompressible Navier–Stokes equation.

Keywords Navier–Stokes–Voight equation · Navier–Stokes equation · Trajectory attractors

Mathematics Subject Classification 35M13 · 35Q35

1 Introduction

It is very significant and difficult to study the uniqueness and asymptotic behaviour of the evolution equations and their long-time behavior of solutions can be described by attractors. But, we all know that the uniquenesses of weak solutions for the 3D Navier–Stokes equation is still open until now. To this end, Chepyzhov and Vishik in [6] proposed the trajectory attractor theory, which can describe the long-time behavior of solutions whose uniqueness is not known. We study the relationship between the following 3D Navier–Stokes equation:

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$$\begin{cases} u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla p = g(x), & x \in \Omega, \quad t \geq 0, \\ \nabla \cdot u = 0, \end{cases} \quad (1.1)$$

and the 3D Navier–Stokes–Voight equation:

$$\begin{cases} u_t - \alpha^2 \Delta u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla p = g(x), & x \in \Omega, \quad t \geq 0, \\ \nabla \cdot u = 0, \end{cases} \quad (1.2)$$

subjecting to the noslip boundary condition

$$u|_{\partial\Omega} = 0, \quad (1.3)$$

and the initial conditions

$$u(x, 0) = u_0, \quad x \in \Omega, \quad (1.4)$$

where $\Omega \subseteq \mathbb{R}^3$ is a smooth bounded domain, $u(x, t) = (u^1(x, t), u^2(x, t), u^3(x, t))$ is the velocity vector, $p(x, t)$ is the pressure function, and $g = (g^1(x), g^2(x), g^3(x))$ is an external term, ν is a positive viscous constant.

There has been many literature on the trajectory attractors [3, 7, 14–17, 19]. First, Chepyzhov et al. [4, 5] study the following 3D Navier–Stokes- α model,

$$\begin{cases} v_t - \nu \Delta v + (u \cdot \nabla)v + \nabla P = g(x), & x \in \Omega, \quad t \geq 0, \\ v = u - \alpha^2 \Delta u, \quad \nabla \cdot u = 0, \end{cases} \quad (1.5)$$

and showed that the trajectory attractor of the Navier–Stokes- α model converges to the trajectory attractor of the 3D Navier–Stokes system as $\alpha \rightarrow 0^+$. The difference between [4] and [5] is that they chose different trajectory spaces.

There are also a lot of literature on Navier–Stokes- α model [1, 2, 9, 10, 18]. Zelati and Gal [21] proved the existence of global and exponential attractors, then they prove the convergence of the (strong) global attractor of the 3D Navier–Stokes–Voight model to the (weak) global attractor of the 3D Navier–Stokes equation, i.e., they prove the convergence of the (strong) global attractor of the 3D Navier–Stokes–Voight model to the trajectory attractor of the 3D Navier–Stokes equation. But, they did not prove upper semicontinuity of trajectory attractors. In addition, trajectory attractor was also obtained for the fluid dynamics systems such as the MHD system [8], liquid crystal flow [12], binary fluid mixtures [13], and Cahn–Hilliard–Navier–Stokes equations [11]. These 3D systems all have trajectory attractors but not necessarily have global attractors. Zhao and Zhou [22] first proposed the concept of pullback trajectory attractors, and proved the existence of pullback trajectory attractors to 3D incompressible non-Newtonian fluid.

2 Preliminaries

In this section, we will first introduce some Sobolev spaces which will be used, and their dual spaces. It is also necessary to introduce some notations related to trajectory

attractors. At last, we give a result on the trajectory attractors of 3D incompressible Navier–Stokes equation.

2.1 Trajectory Space

We first introduce some functional spaces and operators. Set

$$\begin{aligned}\mathcal{V} &= \{\varphi(x) = (\varphi^1(x), \varphi^2(x), \varphi^3(x)) \in (C_0^\infty(\overline{\Omega}))^3, \quad \nabla \cdot \varphi = 0, \quad \varphi|_{\partial\Omega} = 0\}; \\ H &= \text{closure of } \mathcal{V} \text{ in } (L^2(\Omega))^3 \text{ with norm } |\cdot|, \quad H' \text{ is the dual space of } H; \\ V &= \text{closure of } \mathcal{V} \text{ in } (H^1(\Omega))^3 \text{ with norm } \|\cdot\|, \quad V' \text{ is the dual space of } V,\end{aligned}$$

where $(\cdot, \cdot)$, $\langle \cdot, \cdot \rangle$ denote the inner products in H and in V respectively. For any $v \in V'$, the expression $\langle u, v \rangle$ means the value of the functional v on a vector $u \in V$. In the sequel, we identify H with its dual and we have the following inclusions,

$$D(A) \subset V \subset H = H' \subset V' \subset D(A)'.$$

We set $H^\eta = (-\Delta)^{\frac{\eta}{2}} H$ and use $H^{-\eta}$ denote the dual space of H^η . Clearly, the embedding $H \hookrightarrow H^{-\eta}$ is compact.

The operator $P : [L^2(\Omega)]^3 \rightarrow H$ denotes the orthogonal projector, and A is the Stokes operator with the domain $D(A) = (H^2(\Omega))^3 \cap V$, the operator A is self-adjoint and positive,

$$Au = -P\Delta u.$$

The bilinear operator $B(u, u)$ on $V \times V \rightarrow V'$ is defined as

$$\langle B(u, v), w \rangle = b(u, v, w),$$

and the trilinear form $b : V \times V \times V$,

$$b(u, v, w) = \int_{\Omega} (u \cdot \nabla) v \cdot w dx = \sum_{i,j=1}^3 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx,$$

satisfies

$$b(u, v, w) = -b(u, w, v), \quad b(u, v, v) = 0, \quad u, v, w \in V, \quad (2.1)$$

and

$$|\langle B(u, v), w \rangle| \leq c|u|_3|v||Aw|, \quad \|B(u, v)\|_{D(A)'} \leq c|u|_3|v|. \quad (2.2)$$

We recall the Poincaré inequality,

$$|u| \leq \frac{1}{\sqrt{\lambda_1}} \|u\|, \quad \forall u \in V, \quad (2.3)$$

where λ_k ($k = 1, 2, \dots$) is the eigenvalue of Stokes operator A , and λ_k satisfies

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_k, \quad \lambda_k \rightarrow +\infty, \quad k \rightarrow \infty.$$

We consider the spaces $\mathcal{F}_+^b$ defined by

$$\mathcal{F}_+^b = \{z(\cdot) \mid z(\cdot) \in L_b^2(\mathbb{R}_+; V) \cap L^\infty(\mathbb{R}_+; H), \partial_t z(\cdot) \in L_b^2(\mathbb{R}_+; D(A)')\},$$

with norm

$$\|z\|_{\mathcal{F}_+^b} = \|z\|_{L_b^2(\mathbb{R}_+; V)} + \|z\|_{L^\infty(\mathbb{R}_+; H)} + \|\partial_t z\|_{L_b^2(\mathbb{R}_+; D(A)')},$$

where

$$\begin{aligned} \|z\|_{L_b^2(\mathbb{R}_+; V)}^2 &= \sup_{t \geq 0} \int_t^{t+1} \|z(s)\|^2 ds, \quad \|z\|_{L^\infty(\mathbb{R}_+; H)} = \operatorname{ess\,sup}_{t \geq 0} |z(t)|, \\ \|\partial_t z\|_{L_b^2(\mathbb{R}_+; D(A)')}^2 &= \sup_{t \geq 0} \int_t^{t+1} \|\partial_t z(s)\|_{D(A)'}^2 ds. \end{aligned}$$

We know that $\mathcal{F}_+^b$ with its norm $\|\cdot\|_{\mathcal{F}_+^b}$ is a Banach space. Similarly, we define the space

$$\mathcal{F}_+^{loc} = \{z(\cdot) \mid z(\cdot) \in L_{loc}^2(\mathbb{R}_+; V) \cap L_{loc}^\infty(\mathbb{R}_+; H), \partial_t z(\cdot) \in L_{loc}^2(\mathbb{R}_+; D(A)')\},$$

and we define a topology Θ_+^{loc} on $\mathcal{F}_+^{loc}$, then we consider a topology sequence $\{z_n(\cdot)\} \subset \mathcal{F}_+^{loc}$, $z_n \rightarrow z$ in the topology Θ_+^{loc} , i.e.,

$$\begin{aligned} z_n &\rightharpoonup z, \text{ weakly in } L^2(0, M; V), \text{ and weakly-star in } L^\infty(0, M; H), n \rightarrow \infty, \\ \partial_t z_n &\rightharpoonup \partial_t z, \text{ weakly in } L^2(0, M; D(A)'), n \rightarrow \infty. \end{aligned}$$

Let $\{T(h) \mid h \geq 0\}$ denote the time translation operator acting on the trajectory space,

$$T(h)z(t) = z(t + h).$$

2.2 Some Useful Lemmas

Lemma 2.1 [20] *Let $y(t) \in C^1[t_0, t_1]$, $y \geq 0$ and the following inequality*

$$y'(t) + ky(t) \leq h(t)$$

holds with $k > 0$, $h(t) \in C[t_0, t_1]$, then

$$y(t) \leq y(0)e^{-kt} + \int_0^t e^{-k(t-s)} h(s) ds.$$

In particular, if $h(t) = C$, then

$$y(t) \leq y(0)e^{-kt} + Ck^{-1}.$$

Lemma 2.2 [6] Suppose that $E_1 \hookrightarrow E \subset E_0$, where E is a Banach space, and the embedding $E_1 \hookrightarrow E$ is compact. Set the space

$$\begin{aligned} W_{p_1, p_0}(0, T; E_1, E_0) \\ = \{\psi(s), s \in [0, T] \mid \psi(s) \in L^{p_1}(0, T; E_1), \psi'(s) \in L^{p_0}(0, T; E_0)\}, \end{aligned}$$

with the norm

$$\|\psi\|_{W_{p_1, p_0}(0, T; E_1, E_0)} = \left(\int_0^T \|\psi(s)\|_{E_1}^{p_1} ds \right)^{\frac{1}{p_1}} + \left(\int_0^T \|\psi'(s)\|_{E_0}^{p_0} ds \right)^{\frac{1}{p_0}}.$$

We assume that $p_1 \geq 1$ and $p_0 \geq 1$. Then the following embedding is compact:

$$W_{p_1, p_0}(0, T; E_1, E_0) \hookrightarrow L^{p_1}(0, T; E).$$

Moreover, when $p_1 = \infty$, then the following embedding is compact:

$$W_{\infty, p_0}(0, T; E_1, E_0) \hookrightarrow C(0, T; E).$$

Lemma 2.3 [4] For any $f(t) \in D(A)'$, assume the operator A is self-adjoint and positive, then the following inequality holds:

$$\|f\|_{D(A)'}^2 \leq \|(1 + \alpha^2 A)f\|_{D(A)'}^2.$$

Lemma 2.4 [4] For any $f(t) \in L^2(0, M; D(A)')$, assume the operator A is self-adjoint and positive, then there holds that

$$\lim_{\alpha \rightarrow 0^+} \|(1 + \alpha^2 A)f(t)\|_{L^2(0, M; D(A)')} = \|f(t)\|_{L^2(0, M; D(A)')}.$$

2.3 Trajectory Attractors of 3D Incompressible Navier–Stokes Equation

In this subsection, we will present the theory of trajectory attractors of 3D incompressible Navier–Stokes equation, which can be found in [6]. With the orthogonal projector P , 3D incompressible Navier–Stokes system can be rewritten as

$$\begin{cases} \partial_t u + \nu Au + B(u, u) = g(x), & x \in \Omega, \quad t > 0, \\ \nabla \cdot u = 0, \quad u|_{\partial\Omega} = 0, \quad u(x, 0) = u_0 \in H. \end{cases} \quad (2.4)$$

Lemma 2.5 If $g \in H$, $u_0 \in H$, and $M > 0$, then problem (2.4) admits a weak solution $u(t) \in L^\infty(0, M; H) \cap L^2(0, M; V)$ such that $u(0) = u_0$ and there holds that

$$\frac{1}{2} \frac{d}{dt} |u(t)|^2 + \nu \|u(t)\|^2 = (g, u(t)), \quad \forall t \in [0, M]. \quad (2.5)$$

Moreover, for any $\psi(t) \in C_0^\infty(0, M)$, there holds that

$$-\frac{1}{2} \int_0^M |u(t)|^2 \psi'(t) dt + \nu \int_0^M \|u(t)\|^2 \psi(t) dt \leq \int_0^M (g, u(t)) \psi'(t) dt. \quad (2.6)$$

Definition 2.6 The trajectory space $\mathcal{K}_0$ is the union of weak solutions $u(t)$ of problem (2.4) with an arbitrary $u_0 \in H$, i.e., $u(t)$ satisfies (2.5)–(2.6). $\mathcal{K}_0$ is called the trajectory space of problem (2.4).

Lemma 2.7 If $g \in H$, then the equation (2.4) has a trajectory attractor $\mathcal{O}_0$.

Proof We refer to [6] for its proof. $\square$

3 Trajectory Attractors of 3D Navier–Stokes–Voigt Equation

The 3D Navier–Stokes–Voigt system can be rewritten as

$$\begin{cases} (1 + \alpha^2 A) \partial_t u + \nu A u + B(u, u) = g(x), & x \in \Omega, \\ \nabla \cdot u = 0, & u|_{\partial\Omega} = 0, \quad u(x, 0) = u_0 \in V. \end{cases} \quad (3.1)$$

Theorem 3.1 If $g \in H$, $u_0 \in V$, and $M > 0$, then problem (3.1) admits a weak solution $u(t) \in L^\infty(0, M; H) \cap L^2(0, M; V)$ such that $u(0) = u_0$ and there holds that

$$\frac{1}{2} \frac{d}{dt} \left(|u(t)|^2 + \alpha^2 \|u(t)\|^2 \right) + \nu \|u(t)\|^2 = (g, u(t)), \quad \forall t \in [0, M]. \quad (3.2)$$

Moreover, for any $\psi(t) \in C_0^\infty(0, M)$, there holds that

$$\begin{aligned} & -\frac{1}{2} \int_0^M \left(|u(t)|^2 + \alpha^2 \|u(t)\|^2 \right) \psi'(t) dt \\ & + \nu \int_0^M \|u(t)\|^2 \psi(t) dt = \int_0^M (g, u(t)) \psi'(t) dt. \end{aligned} \quad (3.3)$$

Definition 3.1 The trajectory space $\mathcal{K}_\alpha^+$ is the union of weak solutions $u(t)$ of problem (3.1) with an arbitrary $u_0 \in V$, i.e., $u(t)$ satisfies (3.2)–(3.3). $\mathcal{K}_\alpha^+$ is called the trajectory space of problem (3.1).

We consider the time translation semigroup $\{T(h)\}$ acting on $\mathcal{K}_\alpha^+$.

Proposition 3.2 *The trajectory space $\mathcal{K}_\alpha^+$ is invariant with the time translation semi-group $\{T(h) \mid h \geq 0\}$:*

$$T(h)\mathcal{K}_\alpha^+ \subseteq \mathcal{K}_\alpha^+, \forall h \geq 0.$$

Proof For any $u(t) \in \mathcal{K}_\alpha^+$,

$$T(h)u(t) = u(h+t).$$

Obviously, $u(h+t)$ is also a weak solution of problem (3.1), hence $u(h+t) \in \mathcal{K}_\alpha^+$. $\square$

Lemma 3.2 *If $g \in H$ and $u(t)$ is a weak solution of problem (3.1), then $u(t)$ satisfies (3.2), and the following inequalities hold:*

$$\begin{aligned} |u(t)|^2 + \alpha^2 \|u(t)\|^2 &\leq \left(|u(0)|^2 + \alpha^2 \|u(0)\|^2 \right) e^{-\frac{v\lambda_1}{1+\lambda_1\alpha^2}t} \\ &\quad + \frac{(1+\lambda_1\alpha^2)|g|^2}{v^2\lambda_1^2}, \end{aligned} \quad (3.4)$$

$$\begin{aligned} v \int_t^{t+1} \|u(s)\|^2 ds &\leq \left(|u(0)|^2 + \alpha^2 \|u(0)\|^2 \right) e^{-\frac{v\lambda_1}{1+\lambda_1\alpha^2}t} + \frac{(1+\lambda_1\alpha^2)|g|^2}{v^2\lambda_1^2} \\ &\quad + \frac{|g|^2}{v\lambda_1}. \end{aligned} \quad (3.5)$$

Proof By the Cauchy inequality, it follows that

$$(g, u(t)) \leq \frac{v}{2} \|u(t)\|^2 + \frac{1}{2v\lambda_1} |g|^2. \quad (3.6)$$

Then inserting (3.6) into (3.2) yields

$$\frac{d}{dt} \left(|u(t)|^2 + \alpha^2 \|u(t)\|^2 \right) + v \|u(t)\|^2 \leq \frac{1}{v\lambda_1} |g|^2. \quad (3.7)$$

Using the Poincaré inequality (2.3), we get

$$|u(t)|^2 + \alpha^2 \|u(t)\|^2 \leq \frac{1}{\lambda_1} \|u(t)\|^2 + \alpha^2 \|u(t)\|^2 = \frac{1+\lambda_1\alpha^2}{\lambda_1} \|u(t)\|^2,$$

i.e.,

$$\frac{\lambda_1}{1+\lambda_1\alpha^2} \left(|u(t)|^2 + \alpha^2 \|u(t)\|^2 \right) \leq \|u(t)\|^2. \quad (3.8)$$

Inserting (3.8) into (3.7), we obtain

$$\frac{d}{dt} \left(|u(t)|^2 + \alpha^2 \|u(t)\|^2 \right) + \frac{v\lambda_1}{1+\lambda_1\alpha^2} \left(|u(t)|^2 + \alpha^2 \|u(t)\|^2 \right) \leq \frac{1}{v\lambda_1} |g|^2. \quad (3.9)$$

Thus applying Gronwall Lemma 2.1 to (3.9), (3.4) follows. Integrating (3.7) on $[t, t + 1]$, we find

$$\begin{aligned} & |u(t+1)|^2 + \alpha^2 \|u(t+1)\|^2 + v \int_t^{t+1} \|u(s)\|^2 ds \\ & \leq |u(t)|^2 + \alpha^2 \|u(t)\|^2 + \frac{1}{v\lambda_1} |g|^2, \end{aligned} \quad (3.10)$$

which, together with (3.4), gives (3.5). The proof is thus complete. $\square$

Lemma 3.3 *If $g \in H$ and $u(t)$ is a weak solution of problem (3.1), then the following inequality holds:*

$$\int_t^{t+1} \|\partial_t u(s)\|_{D(A)'}^2 ds \leq C_1 \left(|u(0)|^2 + \alpha^2 \|u(0)\|^2 \right) e^{-\frac{v\lambda_1}{1+\lambda_1\alpha^2}t} + R_1^2, \quad (3.11)$$

where C_1, R_1 only depend on $v, \lambda_1, |g|$, but are independent of α .

Proof First, using the Poincaré inequality (2.3), we have

$$\|g\|_{D(A)'} \leq \frac{1}{\lambda_1} |g|, \quad (3.12)$$

and

$$\|Au\|_{D(A)'} = |u| \leq \frac{1}{\sqrt{\lambda_1}} \|u\|.$$

Then using (3.5), we get

$$\begin{aligned} & v \int_t^{t+1} \|Au(s)\|_{D(A)'}^2 ds \leq \frac{v}{\lambda_1} \int_t^{t+1} \|u(s)\|^2 ds \\ & \leq \frac{1}{\lambda_1} \left((|u(0)|^2 + \alpha^2 \|u(0)\|^2) e^{-\frac{v\lambda_1}{1+\lambda_1\alpha^2}t} + \frac{(1 + \lambda_1\alpha^2)|g|^2}{v^2\lambda_1^2} + \frac{|g|^2}{v\lambda_1} \right). \end{aligned} \quad (3.13)$$

Second, we derive from inequality (2.2) that

$$\|B(u, u)\|_{D(A)'} \leq c|u|\|u\|, \quad (3.14)$$

which, along with (3.4)–(3.5), yields

$$\begin{aligned} & \int_t^{t+1} \|B(u, u)\|_{D(A)'}^2 ds \leq c^2 \sup_{s \in [t, t+1]} |u(s)|^2 \int_t^{t+1} \|u(s)\|^2 ds \\ & \leq \frac{c^2}{v} \left(|u(0)|^2 + \alpha^2 \|u(0)\|^2 + \frac{(1 + \lambda_1\alpha^2)|g|^2}{v^2\lambda_1^2} \right) \end{aligned}$$

$$\times \left((|u(0)|^2 + \alpha^2 \|u(0)\|^2) e^{-\frac{v\lambda}{1+\lambda_1\alpha^2}t} + \frac{(1+\lambda_1\alpha^2)|g|^2}{v^2\lambda_1^2} + \frac{|g|^2}{v\lambda_1} \right). \quad (3.15)$$

Obviously,

$$\begin{aligned} \int_t^{t+1} \|(1 + \alpha^2 A)\partial_t u(s)\|_{D(A)}^2 ds &\leq v \int_t^{t+1} \|Au(s)\|_{D(A)'}^2 ds \\ &+ \int_t^{t+1} \|B(u, u)\|_{D(A)'}^2 ds + \|g\|_{D(A)'}^2. \end{aligned} \quad (3.16)$$

Combining (3.12), (3.13), (3.15), then (3.16) becomes

$$\begin{aligned} \int_t^{t+1} \|(1 + \alpha^2 A)\partial_t u(s)\|_{D(A)'}^2 ds &\leq C_\alpha (|u(0)|^2 + \alpha^2 \|u(0)\|^2) e^{-\frac{v\lambda}{1+\lambda_1\alpha^2}t} \\ &+ R_\alpha^2, \end{aligned} \quad (3.17)$$

where

$$\begin{cases} C_\alpha = \frac{1}{\lambda_1} + \frac{c^2}{v} \left(|u(0)|^2 + \alpha^2 \|u(0)\|^2 + \frac{(1+\lambda_1\alpha^2)|g|^2}{v^2\lambda_1^2} \right), \\ R_\alpha^2 = C_\alpha \left(\frac{(1+\lambda_1\alpha^2)|g|^2}{v^2\lambda_1^2} + \frac{|g|^2}{v\lambda_1} \right) + \frac{1}{\lambda_1^2}. \end{cases}$$

Since $\alpha \in (0, 1)$, it follows

$$\begin{cases} C_\alpha \leq C_1 = \frac{1}{\lambda_1} + \frac{c^2}{v} \left(|u(0)|^2 + \|u(0)\|^2 + \frac{(1+\lambda_1)|g|^2}{v^2} \right), \\ R_\alpha^2 \leq R_1^2 = C_1 \left(\frac{(1+\lambda_1)|g|^2}{v^2\lambda_1^2} + \frac{|g|^2}{v\lambda_1} \right) + \frac{1}{\lambda_1^2}. \end{cases}$$

Now by applying Lemma 2.3 and (3.17), the proof is thus complete. $\square$

Lemma 3.4 *Let $g \in H$, then*

- (i) $\mathcal{K}_\alpha^+ \subset \mathcal{F}_+^b$;
- (ii) *for any function $u(\cdot) \in \mathcal{K}_\alpha^+$, the translation group $\{T(h)\}$ is continuous in $\mathcal{F}_+^b$, and there holds that*

$$\|T(h)u(\cdot)\|_{\mathcal{F}_+^b}^2 \leq C_0 \left(|u(0)|^2 + \|u(0)\|^2 \right) e^{-\frac{v\lambda_1}{1+\lambda_1\alpha^2}h} + R_0^2, \quad \forall h \geq 0, \quad (3.18)$$

where C_0, R_0 only depend on $v, \lambda_1, |g|$, but are independent of α .

Proof The proof of (3.18) is clearly established by Lemmas 3.2–3.3. Let $u_1(s) \rightarrow u_2(s)$ in $\mathcal{F}_+^b$. Then $u_1(s+h) \rightarrow u_2(s+h)$ in $\mathcal{F}_+^b$ and the translation group $\{T(h)\}$ is continuous in $\mathcal{F}_+^b$. $\square$

Proposition 3.3 *The trajectory space $\mathcal{K}_\alpha^+ \subseteq \mathcal{F}_+^b$ is compact in the topology Θ_+^{loc} .*

Proof We consider an arbitrary sequence $\{u_n\} \subset \mathcal{K}_\alpha^+$, such that $u_n \rightarrow u$ in the topology Θ_+^{loc} as $n \rightarrow \infty$, we need to prove $u \in \mathcal{K}_\alpha^+$. Due to $u_n \rightarrow u$ in the topology Θ_+^{loc} , by the definition of Θ_+^{loc} ,

$$\begin{aligned} u_n &\rightharpoonup u, \text{ weakly in } L^2(0, M; V), \text{ and weakly-star in } L^\infty(0, M; H), n \rightarrow \infty, \\ \partial_t u_n &\rightharpoonup \partial_t u, \text{ weakly in } L^2(0, M; D(A)'), n \rightarrow \infty. \end{aligned}$$

Noting $\{u_n\} \subset \mathcal{K}_\alpha^+$ and using Lemma 3.2, u_n is bounded in $L^2(0, M; V) \cap L^\infty(0, M; H)$, and $u_n \rightarrow u$ strongly in $L^2(0, M; H)$, then $\{u_n\}$ contains a subsequence $\{u_{n_k}\}$ such that

$$B(u_{n_k}, u_{n_k}) \rightharpoonup B(u, u) \text{ weakly in } L^{\frac{4}{3}}(0, M; V'), \quad (3.19)$$

$$u_{n_k} \rightharpoonup u \text{ weakly in } L^2(0, M; V), \quad (3.20)$$

$$u_{n_k} \rightharpoonup u \text{ weakly-star in } L^\infty(0, M; H). \quad (3.21)$$

Since $u_{n_k}(t) \in \mathcal{K}_\alpha^+$, we have

$$(1 + \alpha^2 A) \partial_t u_{n_k} + \nu A u_{n_k} + B(u_{n_k}, u_{n_k}) = g(x). \quad (3.22)$$

Taking limit of (3.22) and using (3.19)–(3.21), we have

$$(1 + \alpha^2 A) \partial_t u + \nu A u + B(u, u) = g(x), \quad (3.23)$$

then u is a weak solution of problem (3.1), i.e., $u \in \mathcal{K}_\alpha^+$. $\square$

Proposition 3.4 *The ball $B_{2R_0} = \{u(\cdot) \mid u(\cdot) \in \mathcal{K}_\alpha^+, \|T(h)u\|_{\mathcal{F}_+^b} \leq 2R_0\}$ in $\mathcal{F}_+^b$ is an absorbing set of the semigroup $\{T(h)\}$. B_{2R_0} is compact in the topology Θ_+^{loc} .*

Proof According to Lemma 3.4, for any bounded set $B \subset \mathcal{K}_\alpha^+$, there exists a time $h_1 > 0$, such that as $h \geq h_1$,

$$T(h)B \subset B_{2R_0},$$

which indicates that B_{2R_0} is an absorbing set of semigroup $T(h)$ on $\mathcal{K}_\alpha^+$. Next, since $B_{2R_0} \subset \mathcal{K}_\alpha^+ \subseteq \mathcal{F}_+^b$, and B_{2R_0} is closed, by Proposition 3.3, B_{2R_0} is compact in the topology Θ_+^{loc} . $\square$

Definition 3.5 [6] A set $\mathcal{O}_\alpha \subset \mathcal{K}_\alpha^+$ is said to be the trajectory attractor of equation, if

- (i) $\mathcal{O}_\alpha$ is compact in the topology Θ_+^{loc} ,
- (ii) $\mathcal{O}_\alpha$ is strictly invariant, i.e., $T(h)\mathcal{O}_\alpha = \mathcal{O}_\alpha$ for all $h \geq 0$,
- (iii) $\mathcal{O}_\alpha$ is a uniformly attracting set for $\mathcal{F}_+^b$.

Theorem 3.5 *If $g \in H$, then the translation semigroup $\{T(h)\}$ acting on $\mathcal{K}_\alpha^+$ has a trajectory attractor $\mathcal{O}_\alpha$. The set $\mathcal{O}_\alpha$ is bounded in $\mathcal{F}_+^b$ and compact in the topology Θ_+^{loc} .*

Proof Let

$$\mathcal{O}_\alpha = B_{2R_0}.$$

- (i) By Proposition 3.4, we know that $\mathcal{O}_\alpha$ is compact in the topology of Θ_+^{loc} , and $T(h)\mathcal{O}_\alpha \subseteq \mathcal{O}_\alpha$ for all $h \geq h_1$.
- (ii) On the other hand, let $u(t) \in B_{2R_0} = \mathcal{O}_\alpha$, by the definition of time translation semigroup $\{T(h)\}$,

$$u(t) = T(h)u(t-h), \quad (3.24)$$

let $\tilde{u}(t) = u(t-h)$, obviously, $\tilde{u}$ is a weak solution of problem (3.1), i.e., $\tilde{u} \in \mathcal{O}_\alpha$, then there exists a time $t_2 > 0$, such that as $t \geq t_2$,

$$\|\tilde{u}(t)\|_{\mathcal{F}_+^b} \leq 2R_0, \quad (3.25)$$

hence, $\tilde{u} \in B_{2R_0} = \mathcal{O}_\alpha$. Then from (3.24)–(3.25), we derive $u(t) \in T(h)B_{2R_0}$ as $t \geq t_2$, i.e., $\mathcal{O}_\alpha \subseteq T(h)\mathcal{O}_\alpha$ as $t \geq t_2$. Letting $t_0 = \max\{h_1, t_2\}$, as $t > t_0$, then we have $\mathcal{O}_\alpha = T(h)\mathcal{O}_\alpha$.

- (iii) By Proposition 3.4, we know that $\mathcal{O}_\alpha$ is an absorbing set of $T(h)$. Then we prove that $\mathcal{O}_\alpha$ is a trajectory attractor of problem (3.1).

□

4 Upper Semicontinuity of the Trajectory Attractors of 3D NS Equation

Theorem 4.1 *Let a sequence $\{u_n(t)\} \subset \mathcal{K}_\alpha^+$, $\alpha_n \rightarrow 0^+$ ($n \rightarrow \infty$), and $u_n(t) \rightarrow u(t)$ in the topology Θ_+^{loc} as $n \rightarrow \infty$. Then $u(t)$ is a weak solution of the 3D Navier–Stokes equation such that u satisfies the inequality (2.6), i.e., $u \in \mathcal{K}_0^+$, where $\mathcal{K}_0^+$ is the trajectory space of problem (2.4).*

Proof Since $u_n(t) \in \mathcal{K}_\alpha^+$, i.e., $u_n(t)$ is a weak solution of equation

$$(1 + \alpha_n^2 A) \partial_t u_n + \nu A u_n + B(u_n, u_n) = g(x), \quad (4.1)$$

and noticing (3.18), $u_n(t)$ satisfies

$$\operatorname{ess\,sup}_{t \geq 0} |u_n(t)|^2 \leq C, \quad (4.2)$$

$$\sup_{t \geq 0} \int_t^{t+1} \|u_n(\tau)\|^2 d\tau \leq C, \quad (4.3)$$

$$\sup_{t \geq 0} \int_t^{t+1} \|\partial_t u_n(\tau)\|^2 d\tau \leq C. \quad (4.4)$$

Since $\partial_t u_n \rightharpoonup \partial_t u$ ($n \rightarrow \infty$) weakly in $L^2(0, M; D(A)')$ and using Lemma 2.4, we have

$$(1 + \alpha_n^2 A) \partial_t u_n \rightharpoonup \partial_t u \quad (n \rightarrow \infty) \text{ weakly in } L^2(0, M; D(A)'). \quad (4.5)$$

Since $u_n \rightharpoonup u$ ($n \rightarrow \infty$) weakly in $L^2(0, M; V)$, then $Au_n \rightharpoonup Au$ ($n \rightarrow \infty$) weakly in $L^2(0, M; V')$. Obviously,

$$Au_n \rightharpoonup Au \quad (n \rightarrow \infty) \text{ weakly in } L^2(0, M; D(A)'). \quad (4.6)$$

By (2.2), (4.2) and (4.3), we have

$$\int_0^M \|B(u_n, u_n)\|_{D(A)'}^2 ds \leq c \int_0^M (|u_n| \|u_n\|)^2 ds \leq C,$$

then

$$B(u_n, u_n) \rightharpoonup B(u, u) \quad (n \rightarrow \infty) \text{ weakly in } L^2(0, M; D(A)'). \quad (4.7)$$

Combining with (4.5)–(4.7), Eq. (4.1) converges to

$$\partial_t u + \nu Au + B(u, u) = g(x), \quad (n \rightarrow \infty) \text{ weakly in } L^2(0, M; D(A)'). \quad (4.8)$$

Next, we prove u satisfies the energy inequality (2.6). Since $u_n(t) \rightarrow u(t)$ in the topology Θ_+^{loc} as $n \rightarrow \infty$, by the Aubin Theorem, there exists a subsequence u_n which we still denote by u_n such that

$$\|u_n(t) - u(t)\|_{L^2(0, M; H)}^2 = \int_0^M |u_n(t) - u(t)|^2 dt \rightarrow 0, \quad n \rightarrow \infty. \quad (4.9)$$

Note that $u_n(t)$ satisfies the energy equality

$$\begin{aligned} & -\frac{1}{2} \int_0^M (|u_n(t)|^2 + \alpha_n^2 \|u_n(t)\|^2) \psi'(t) dt + \nu \int_0^M \|u_n(t)\|^2 \psi(t) dt \\ & = \int_0^M (g, u_n(t)) \psi'(t) dt, \quad \forall \psi'(t) \in C_0^\infty(0, M). \end{aligned} \quad (4.10)$$

Applying (4.9) and the Hölder inequality, we have

$$\begin{aligned} & \int_0^M |u_n(t)|^2 \psi'(t) dt - \int_0^M |u(t)|^2 \psi'(t) dt \\ & \leq \sup_{t \in [0, M]} |\psi'(t)| \int_0^M (|u_n(t)| + |u(t)|) (|u_n(t)| - |u(t)|) dt \\ & \leq \sup_{t \in [0, M]} |\psi'(t)| \left(2 \int_0^M (|u_n(t)|^2 + |u(t)|^2) dt \right)^{\frac{1}{2}} \\ & \quad \times \left(\int_0^M |u_n(t) - u(t)|^2 dt \right)^{\frac{1}{2}} \rightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad (4.11)$$

Similarly,

$$\begin{aligned} & \left| \int_0^M (g, u_n(t)) \psi'(t) dt - \int_0^M (g, u(t)) \psi'(t) dt \right| \leq \int_0^M |g| |u_n(t) - u(t)| \psi'(t) dt \\ & \leq \left(\int_0^M (|g| \psi'(t))^2 dt \right)^{\frac{1}{2}} \left(\int_0^M |u_n(t) - u(t)|^2 dt \right)^{\frac{1}{2}} \rightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad (4.12)$$

Since $u_n \sqrt{\psi} \rightharpoonup u \sqrt{\psi}$ weakly in $L^2(0, M; V)$, i.e., for any $w \in V'$,

$$\int_0^M \langle u \sqrt{\psi}, w \rangle dt = \lim_{n \rightarrow \infty} \int_0^M \langle u_n \sqrt{\psi}, w \rangle dt.$$

Letting $w = u \sqrt{\psi}$, then

$$\int_0^M \|u \sqrt{\psi}\|^2 dt \leq \lim_{n \rightarrow \infty} \left(\int_0^M \|u_n \sqrt{\psi}\|^2 dt \right)^{\frac{1}{2}} \left(\int_0^M \|u \sqrt{\psi}\|^2 dt \right)^{\frac{1}{2}},$$

and $\psi(t)$ is independent of x , we obtain

$$\nu \int_0^M \|u(t)\|^2 \psi dt \leq \nu \lim_{n \rightarrow \infty} \int_0^M \|u_n(t)\|^2 \psi dt. \quad (4.13)$$

Similarly, letting ψ' replace ψ , and using (4.3), we have

$$\lim_{n \rightarrow \infty} \int_0^M \|u_n(t)\|^2 \psi'(t) dt \leq C,$$

then

$$\lim_{n \rightarrow \infty} \alpha_n^2 \int_0^M \|u_n(t)\|^2 \psi'(t) dt = 0. \quad (4.14)$$

Taking limit on both sides of equation (4.10), and using (4.11)–(4.14), we have

$$-\frac{1}{2} \int_0^M |u(t)|^2 \psi'(t) dt + \nu \int_0^M \|u(t)\|^2 \psi(t) dt \leq \int_0^M (g, u(t)) \psi'(t) dt.$$

Thus $u \in \mathcal{K}_0^+$, and the proof is now complete. $\square$

Theorem 4.2 Let $\mathbb{B}_\alpha = \{u(t) \mid |u(t)|^2 + \alpha^2 \|u(t)\|^2 \leq \frac{(1+\lambda_1\alpha^2)|g|^2}{\nu^2\lambda_1^2}\}$, $0 < \alpha \leq 1$, be the bounded sets of solutions of 3D Navier–Stokes–Voigt equation (3.1). Then the following convergence holds:

$$T(h)\mathbb{B}_\alpha \rightarrow \mathcal{O}_0 \text{ in the topology } \Theta_+^{loc} \text{ as } h \rightarrow +\infty, \alpha \rightarrow 0^+, \quad (4.15)$$

where $\mathcal{O}_0$ is the trajectory attractor of 3D incompressible Navier–Stokes equation (2.4).

Proof There exists a sequence $\alpha_n \rightarrow 0^+$, as $n \rightarrow +\infty$. Let $u^{\alpha_n}(t) \in \mathbb{B}_{\alpha_n}$, $u^{\alpha_n}(t)$ is the solutions of 3D Navier–Stokes–Voigt equation (3.1), i.e., $u^{\alpha_n}(t) \in \mathcal{K}_\alpha^+$,

$$T(h)u^{\alpha_n}(t) = u^{\alpha_n}(t+h), \text{ since } T(h)\mathcal{K}_\alpha^+ \subseteq \mathcal{K}_\alpha^+, \text{ hence } u^{\alpha_n}(t+h) \in \mathcal{K}_\alpha^+,$$

i.e.,

$$(1 + \alpha_n^2 A) \partial_t u^{\alpha_n}(t+h) + \nu A u^{\alpha_n}(t+h) + B(u^{\alpha_n}(t+h), u^{\alpha_n}(t+h)) = g(x). \quad (4.16)$$

Since $u^{\alpha_n}(t+h) \in \mathcal{K}_\alpha^+$, then

$$\|u^{\alpha_n}(t+h)\|_{\mathcal{F}_+^b} \leq R. \quad (4.17)$$

By Theorem 4.1, there exists $u(t+h)$ in $\mathcal{F}_+^{loc}$, such that $u^{\alpha_n}(t+h) \rightarrow u(t+h)$ in the topology Θ_+^{loc} as $n \rightarrow +\infty$, and $u(t+h)$ is a weak solution of 3D Navier–Stokes equation, then $u(t+h)$ solves

$$\partial_t u(t+h) + \nu A u(t+h) + B(u(t+h), u(t+h)) = g(x). \quad (4.18)$$

There exists a sequence $h_n \rightarrow +\infty$, as $n \rightarrow +\infty$. Next we prove $\lim_{n \rightarrow +\infty} u(t+h_n) \in \mathcal{O}_0$. (4.17) means that

$$\begin{aligned} \|u(t+h_n)\|_{\mathcal{F}_+^b} &= \sup_{t \geq -h_n} |u(t)| + \left(\sup_{t \geq -h_n} \int_t^{t+1} \|u(s)\|^2 ds \right)^{\frac{1}{2}} \\ &\quad + \left(\sup_{t \geq -h_n} \int_t^{t+1} \|\partial_s u(s)\|_{D(A)}^2 ds \right)^{\frac{1}{2}} \leq R, \end{aligned} \quad (4.19)$$

hence

$$\begin{aligned} \lim_{n \rightarrow +\infty} \|u(t + h_n)\|_{\mathcal{F}_+^b} &= \sup_{t \in \mathbb{R}} |u(t)| + \left(\sup_{t \in \mathbb{R}} \int_t^{t+1} \|u(s)\|^2 ds \right)^{\frac{1}{2}} \\ &+ \left(\sup_{t \in \mathbb{R}} \int_t^{t+1} \|\partial_s u(s)\|_{D(A)'}^2 ds \right)^{\frac{1}{2}} \leq R, \end{aligned} \quad (4.20)$$

i.e., for all $t \in \mathbb{R}$,

$$\begin{aligned} \lim_{n \rightarrow +\infty} u(t + h_n) &= u(t) \in \mathcal{F}^b = L^2(\mathbb{R}, V) \\ &\cap L^\infty(\mathbb{R}, H) \cap \{u \mid \partial_t u \in L^2(\mathbb{R}, D(A)')\}, \end{aligned} \quad (4.21)$$

and $u(t)$ is a weak solution of 3D incompressible Navier–Stokes equation, i.e., $u(t) \in \mathcal{O}_0$. $\square$

From the above argument, we have proved the following proposition.

Proposition 4.3 *According to the definition of $\mathcal{O}_\alpha$, the following convergence holds:*

$$\mathcal{O}_\alpha \rightarrow \mathcal{O}_0 \text{ in the topology } \Theta_+^{loc} \text{ as } \alpha \rightarrow 0^+. \quad (4.22)$$

The following theorem concerns the upper semicontinuity of trajectory attractors of 3D Navier–Stokes equation when the regular term $\alpha^2 \Delta u_t$ in NSV equation is considered as its a perturbative term. Denote by $\text{dist}_X(B_1, B_2)$ the Hausdorff semidistance in space X between B_1 and B_2 , i.e.,

$$\text{dist}_X(B_1, B_2) = \sup_{x \in B_1} \inf_{y \in B_2} d_X(x, y), \quad B_1, B_2 \subset X.$$

Theorem 4.4 *Assume that $\mathcal{O}_\alpha, \mathcal{O}_0$ are the trajectory attractors of 3D Navier–Stokes–Voight equation and Navier–Stokes equation respectively. Then the following convergence holds:*

$$\lim_{\alpha \rightarrow 0^+} \text{dist}_{L^2(0, M; H^{-\delta})}(\mathcal{O}_\alpha, \mathcal{O}_0) = 0, \quad (4.23)$$

here $\delta \in (0, 1]$.

Proof Let $u^\alpha \in \mathcal{O}_\alpha$, by Proposition 4.3, there exists $u(t) \in \mathcal{O}_0$ such that $u^\alpha \rightarrow u(t)$ weakly in $L^2(0, M; V)$ as $\alpha \rightarrow 0^+$, i.e., $\forall \varphi \in V'$, $u^\alpha, u \in V$, there holds that

$$\begin{aligned} \lim_{\alpha \rightarrow 0^+} \int_0^M \langle u^\alpha(t) - u(t), \varphi \rangle dt &= 0, \\ \int_0^M \langle u^\alpha(t) - u(t), \varphi \rangle dt &= \int_0^M (A^{\frac{1}{2}}(u^\alpha(t) - u(t)), A^{\frac{1}{2}}\varphi) dt \end{aligned} \quad (4.24)$$

$$= \int_0^M (u^\alpha(t) - u(t), A\varphi) dt, \quad (4.25)$$

Since $\varphi \in V'$ without loss of generality, letting $\varphi = A^{-1}(u^\alpha(t) - u(t))$, it follows from (4.24), (4.25) that

$$\lim_{\alpha \rightarrow 0^+} \int_0^M |u^\alpha(t) - u(t)|_H^2 dt = 0, \quad (4.26)$$

Let $u^\alpha \in \mathcal{O}_\alpha$, $u(t) \in \mathcal{O}_0$, $w(t) = u^\alpha - u$, then $w(t)$ satisfies

$$\partial_t w + \alpha^2 A u_t^\alpha + \nu A w + B(u^\alpha, w) + B(w, u) = 0, \quad (4.27)$$

then

$$\|\partial_t w\|_{L^2(0,M;D(A)')} \leq \|\alpha^2 A u_t^\alpha + \nu A w + B(u^\alpha, w) + B(w, u)\|_{L^2(0,M;D(A)')}. \quad (4.28)$$

According to (2.2) and the interpolation inequality, we have

$$\begin{aligned} \|B(u^\alpha, w)\|_{L^2(0,M;D(A)')}^2 &\leq c \int_0^M |u^\alpha| \|u^\alpha\| |w|^2 ds \leq c \left(\int_0^M |u^\alpha|^2 |w|^2 ds \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^M \|u^\alpha\|^2 |w|^2 ds \right)^{\frac{1}{2}} \\ &\leq c \left(\sup_{s \in [0,M]} |u^\alpha|^2 \int_0^M |w|^2 ds \right)^{\frac{1}{2}} \times \left(\sup_{s \in [0,M]} |w|^2 \int_0^M \|u^\alpha\|^2 ds \right)^{\frac{1}{2}}. \end{aligned} \quad (4.29)$$

Since $|w|^2 = |u^\alpha - u|^2 \leq 2(|u^\alpha|^2 + |u|^2)$, and using (3.4)–(3.5), we get

$$\begin{aligned} &\max \left\{ \sup_{s \in [0,M]} |u^\alpha|^2, \sup_{s \in [0,M]} |w|^2, \int_0^M \|u^\alpha\|^2 ds \right\} \\ &\leq 2 \left((|u(0)|^2 + \|u(0)\|^2) + \frac{(1 + \lambda_1)|g|^2}{\nu^2 \lambda_1^2} + \frac{|g|^2}{\nu \lambda_1} \right). \end{aligned} \quad (4.30)$$

Then (4.29) becomes

$$\|B(u^\alpha, w)\|_{L^2(0,M;D(A)')}^2 \leq C \left(\int_0^M |w|^2 ds \right)^{\frac{1}{2}}. \quad (4.31)$$

Similarly to $B(w, u)$, we obtain

$$\|B(w, u)\|_{L^2(0,M;D(A)')}^2 \leq C \left(\int_0^M |w|^2 ds \right)^{\frac{1}{2}}, \quad (4.32)$$

and

$$v \|Aw\|_{L^2(0,M;D(A)')}^2 = v \int_0^M |w|^2 ds. \quad (4.33)$$

By Lemma 2.4, taking $f(t) = u_t^\alpha$, we have

$$\lim_{\alpha \rightarrow 0^+} \alpha^2 \|Au_t^\alpha\|_{L^2(0,M;H)} = 0.$$

Obviously, $\|Au_t^\alpha\|_{D(A)'} = |u_t^\alpha| \leq |Au_t^\alpha|$, then

$$\lim_{\alpha \rightarrow 0^+} \alpha^2 \|Au_t^\alpha\|_{L^2(0,M;D(A)')} = 0. \quad (4.34)$$

Combining (4.31)–(4.34) with (4.26), we obtain

$$\lim_{\alpha \rightarrow 0^+} \|\partial_t w\|_{L^2(0,M;D(A)')} = 0. \quad (4.35)$$

Letting $E_1 = H$, $E = H^{-\delta}$, $E_0 = D(A)'$, $p_1 = 2$, $p_0 = 2$, and using Lemma 2.2, we can prove (4.23). □

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