Applied Mathematics & Optimization (2021) 83:1523–1545
https://doi.org/10.1007/s00245-019-09595-w

Multiple Solutions with Sign Information for a Class of Parametric Superlinear $(p, 2)$-Equations

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Published online: 18 July 2019

Abstract
We consider a parametric nonlinear, nonhomogeneous Dirichlet problem driven by the sum of a p-Laplacian (with $p > 2$) and a Laplacian (a two phase equation). The reaction consists of a parametric $(p-1)$-superlinear term and a $(p-1)$-sublinear perturbation. We show that for all $\lambda > 0$ big, the problem has at least three nontrivial smooth solutions, all with sign information. Also we determine their asymptotic behaviour as the parameter $\lambda \to \infty$. When we strengthen the regularity of the perturbation term, we produce a second nodal solution, for a total of four solutions, all with sign information.

Keywords Two-phase problem · Constant sign solutions · Extremal solutions · Nodal solutions · Nonlinear regularity · Comparison principle · Asymptotic behaviour · critical groups

Mathematics Subject Classification 35J20 · 35J60 · 58E05

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following parametric $(p, 2)$-equation (two-phase problem):

L. Gasiński: The research was supported by the National Science Center of Poland under Project No. 2015/19/B/ST1/01169.

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$$\begin{cases} -\Delta_p u(z) - \Delta u(z) = \lambda |u(z)|^{r-2} u(z) + f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ 2 < p < r < p^*, \ \lambda > 0, \end{cases} \tag{P_λ}$$

where p^* is the critical Sobolev exponent corresponding to p, namely

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N, \end{cases}$$

and for every $q \in (1, \infty)$ by Δ_q we denote the q-Laplace differential operator defined by

$$\Delta_q u = \operatorname{div}(|Du|^{q-2} Du) \quad \forall u \in W_0^{1,q}(\Omega).$$

when $q = 2$, we have the usual Laplace differential operator and so we write $\Delta_2 = \Delta$. In our problem (P_λ) the differential operator is nonhomogeneous and this is a source of difficulties in its analysis. In the reaction we have two terms. One is parametric and $(p-1)$-superlinear (since $2 < p < r$) with $\lambda > 0$ being the parameter. The perturbation $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \longmapsto f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \longmapsto f(z, x)$ is continuous) which is $(p-1)$-sublinear. Using variational tools from the critical point theory together with suitable truncation and comparison techniques and critical groups (Morse theory), we show that for all $\lambda > 0$ big, problem (P_λ) has at least three nontrivial smooth solutions all with sign information (two of constant sign and the third nodal (sign changing)). If we strengthen the regularity of $f(z, \cdot)$, we prove the existence of a second nodal solution, for a total of four nontrivial smooth solutions, all with sign information.

We mention that $(p, 2)$-equations and more generally two phase problems arise in many mathematical models of physical phenomena. In this direction we mention the works of Zhikov [36,37] on elasticity theory and of Cherfils-Il'yasov [4] on reaction-diffusion systems. Recently there have been some existence and multiplicity results for different classes of parametric $(p, 2)$-equations. We mention works of Chorfi-Rădulescu [5], Gasiński-Papageorgiou [9,10,12,13,16], Papageorgiou-Rădulescu [25], Papageorgiou-Rădulescu-Repovš [27], Papageorgiou-Scapellato [29, 30], Yang-Bai [35].

Finally such sensitivity analysis for parametric equations is also important in the study of optimization and control problems. It provides information about the tolerance of the systems on the variation of the parameter and in which range we expect to find optimal solutions (see Papageorgiou [22,23] and Sokołowski [32]).

2 Mathematical Background

In the analysis of problem (P_λ) we will use the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : \ u|_{\partial\Omega} = 0\}$. By $\|\cdot\|$ we will denote the norm of the

Sobolev space $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality, we have

$$\|u\| = \|Du\|_p \quad \forall u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is ordered with positive (order) cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \quad \text{for all } \ z \in \Omega,\ \frac{\partial u}{\partial n}|_{\partial\Omega} < 0\right\},$$

with n being the outward unit normal vector on $\partial\Omega$. For $q \in (1, \infty)$, let $A_q\colon W_0^{1,q}(\Omega) \longrightarrow W^{-1,q'}(\Omega) = W_0^{1,q}(\Omega)^*$ ($\frac{1}{q} + \frac{1}{q'} = 1$) be the nonlinear map defined by

$$\langle A_q(u), h\rangle = \int_\Omega |Du|^{q-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \forall u, h \in W_0^{1,p}(\Omega).$$

From Gasiński-Papageorgiou [11, Problem 2.192], we have the following properties of A_q.

Proposition 2.1 *The map A_q is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus maximal monotone too) and of type $(S)_+$ (that is, if $u_n \xrightarrow{w} u$ in $W_0^{1,q}(\Omega)$ and $\limsup_{n\to+\infty}\langle A_q(u_n), u_n - u\rangle \leq 0$, then $u_n \longrightarrow u$ in $W_0^{1,q}(\Omega)$).*

Note that for $q = 2$, we have $A_2 = A \in \mathcal{L}(H_0^1(\Omega); H^{-1}(\Omega))$.
Let

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leq p \end{cases}$$

(the critical Sobolev exponent corresponding to p) and let $f_0\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function such that

$$|f_0(z, x)| \leq a_0(z)(1 + |x|^{q-1}) \quad \text{for a.a. } z \in \Omega,\ \text{all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)_+$ and $1 < q \leq p^*$. We set

$$F_0(z, x) = \int_0^x f_0(z, s)\, ds$$

and consider the C^1-functional $\varphi_0 \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_0(z,u)\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

The next proposition is a particular case of a more general result proved by Gasiński-Papageorgiou [8] (subcritical case) and Papageorgiou-Rădulescu [26] (critical case). The result is an outgrowth of the nonlinear regularity theory of Lieberman [19,20]. Related regularity results can be found in the more recent works of Ragusa–Tachikawa [33,34].

Proposition 2.2 *If $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\varrho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \forall h \in C_0^1(\overline{\Omega}), \quad \|h\|_{C_0^1(\overline{\Omega})} < \varrho_0,$$

then $u_0 \in C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and it is a local $W_0^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\varrho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \forall h \in W_0^{1,p}(\Omega), \quad \|h\| < \varrho_1.$$

As we already mentioned in the Introduction our methods involve comparison arguments. In this direction, useful will be the following strong comparison principle, which is a special case of a more general result due to Gasiński-Papageorgiou [14, Proposition 3.2]. First we introduce the following notation. Given $h_1, h_2 \in L^\infty(\Omega)$, we write $h_1 \preceq h_2$ if for every $K \subseteq \Omega$ compact, we can find $\varepsilon = \varepsilon(K) > 0$ such that

$$h_1(z) + \varepsilon \leq h_2(z) \quad \text{for a.a. } z \in K.$$

If $h_1, h_2 \in C(\Omega)$ and $h_1(z) < h_2(z)$ for all $z \in \Omega$, then $h_1 \preceq h_2$.

Proposition 2.3 *If $\widehat{\xi} \geq 0$, $h_1, h_2 \in L^\infty(\Omega)$, $h_1 \preceq h_2$ and $u \in C_0^1(\overline{\Omega})$, $v \in \operatorname{int} C_+$ satisfy*

$$\begin{cases} -\Delta_p u - \Delta u + \widehat{\xi}|u|^{p-2}u = h_1 \text{ in } \Omega, \\ -\Delta_p v - \Delta v + \widehat{\xi} v^{p-1} = h_2 \text{ in } \Omega, \end{cases}$$

then $v - u \in \operatorname{int} C_+$.

Next let us recall some basic facts about the spectrum of $(-\Delta, H_0^1(\Omega))$ which we will need in the sequel. We know that the spectrum $\widehat{\sigma}(2)$ consists of a sequence $\{\widehat{\lambda}_k(2)\}_{k\geq 1}$ of distinct eigenvalues such that $\widehat{\lambda}_k(2) \to +\infty$ as $k \to +\infty$. Also for every $k \in \mathbb{N}$, by $E(\widehat{\lambda}_k(2))$ we denote the corresponding eigenspace. Standard regularity theory implies that

$$E(\widehat{\lambda}_k(2)) \subseteq C_0^1(\overline{\Omega}) \quad \forall k \in \mathbb{N}.$$

We know that $\widehat{\lambda}_1(2) > 0$ and it is simple, that is, $\dim E(\widehat{\lambda}_1(2)) = 1$. Also we have the following variational characterization for $\widehat{\lambda}_1(2) > 0$:

$$\widehat{\lambda}_1(2) = \inf\left\{\frac{\|Du\|_2^2}{\|u\|_2^2} : u \in H_0^1(\Omega),\ u \neq 0\right\}.$$

This infimum is realized on $E(\widehat{\lambda}_1(2))$ and from this expression it is easy to see that the element of $E(\widehat{\lambda}_1(2)) \subseteq C_0^1(\overline{\Omega})$ do not change sign. Indeed note that in the above expression we can replace u by $|u|$ (see also Gasiński-Papageorgiou [7, Theorem 6.1.21, p. 716]). By $\widehat{u}_1(2)$ we denote the positive, L^2-normalized (that is, $\|\widehat{u}_1(2)\|_2 = 1$) eigenfunction corresponding to $\widehat{\lambda}_1(2) > 0$. The strong maximum principle implies that $\widehat{u}_1(2) \in \operatorname{int} C_+$. Note that all the other eigenvalues have nodal eigenfunctions. These properties lead to the following simple lemma (see Gasiński-Papageorgiou [11, Problem 5.67]).

Lemma 2.4 *If $\vartheta_0 \in L^\infty(\Omega)$, $\vartheta_0(z) \leq \widehat{\lambda}_1(2)$ for a.a. $z \in \Omega$, $\vartheta_0 \not\equiv \widehat{\lambda}_1(2)$, then there exists $c_0 > 0$ such that*

$$c_0\|u\|^2 \leq \|Du\|_2^2 - \int_\Omega \vartheta_0(z)u^2\,dz \quad \forall u \in H_0^1(\Omega).$$

We will also consider a weighted eigenvalue problem for $(-\Delta, H_0^1(\Omega))$. So, let $\vartheta \in L^\infty(\Omega)$, $0 \leq \vartheta(z)$ for a.a. $z \in \Omega$, $\vartheta \not\equiv 0$. We consider the following linear eigenvalue problem

$$\begin{cases} -\Delta y(z) = \lambda\vartheta(z)y(z) \text{ in } \Omega, \\ y|_{\partial\Omega} = 0. \end{cases}$$

The spectrum of this problem is a sequence of distinct eigenvalues $\{\widetilde{\lambda}_k(2,\vartheta)\}_{k\geq 1}$ which have the same properties as the sequence $\{\widehat{\lambda}_k(2) = \widetilde{\lambda}_k(2,1)\}_{k\geq 1}$. In particular $\widetilde{\lambda}_1(2,\vartheta) > 0$, it is simple and has eigenfunctions in $C_0^1(\overline{\Omega})$ of constant sign. All other eigenvalues have nodal eigenfunctions. These properties lead to the following monotonicity property for the map $\vartheta \longmapsto \widetilde{\lambda}_1(2,\vartheta)$ (see Motreanu-Motreanu-Papageorgiou [21, Proposition 9.47]).

Lemma 2.5 *If $\vartheta_1, \vartheta_2 \in L^\infty(\Omega)$, $0 \leq \vartheta_1(z) \leq \vartheta_2(z)$ for a.a. $z \in \Omega$, $\vartheta_1 \not\equiv 0$, $\vartheta_1 \not\equiv \vartheta_2$, then $\widetilde{\lambda}_1(2,\vartheta_2) < \widetilde{\lambda}_1(2,\vartheta_1)$.*

Next let us recall some basic definitions and facts concerning critical groups which we will be used in our proofs.

Let X be a Banach space, $\varphi \in C^1(X;\mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets

$$\begin{aligned} K_\varphi &= \{u \in X : \varphi'(u) = 0\} \text{ (the critical set of } \varphi), \\ K_\varphi^c &= \{u \in K_\varphi : \varphi(u) = c\}, \\ \varphi^c &= \{u \in X : \varphi(u) \leq c\}. \end{aligned}$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every $k \in \mathbb{N}_0$ by $H_k(Y_1, Y_2)$ we denote the k-th relative singular homology group with integer coefficients. Suppose that $u \in K_\varphi$ is isolated and $\varphi(u) = c$ (that is, $u \in K_\varphi^c$). The critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \forall k \in \mathbb{N}_0.$$

Here U is a neighbourhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology, implies that the above definition is independent of the particular choice of the neighbourhood U.

Suppose that $\varphi \in C^1(X; \mathbb{R})$ satisfies the Palais-Smale condition (the PS-condition for short; see Gasiński-Papageorgiou [7, Definition 5.1.5]) and that $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. Then the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \forall k \in \mathbb{N}_0.$$

The definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$. Indeed, let $c' < c < \inf \varphi(K_\varphi)$. From Corollary 5.3.13 of Papageorgiou-Rădulescu-Repovš [28], we have that $\varphi^{c'}$ is a strong deformation retract of φ^c. Then Corollary 6.1.24 of [28] implies that

$$H_k(X, \varphi^c) = H_k(X, \varphi^{c'}) \quad \forall k \in \mathbb{N}_0.$$

Suppose that K_φ is finite. We introduce the following quantities:

$$M(t, u) = \sum_{k \geq 0} \operatorname{rank} C_k(\varphi, u) t^k \quad \forall t \in \mathbb{R},\ u \in K_\varphi,$$
$$P(t, \infty) = \sum_{k \geq 0} \operatorname{rank} C_k(\varphi, \infty) t^k \quad \forall t \in \mathbb{R}.$$

The Morse relation says that

$$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t) Q(t), \tag{2.1}$$

where $Q(t) = \sum_{k \geq 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative coefficients.

Finally we fix our notation. For $x \in \mathbb{R}$, we let $x^\pm = \max\{\pm x, 0\}$ and for $u \in W_0^{1,p}(\Omega)$ we define $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. We know that

$$u^\pm \in W_0^{1,p}(\Omega),\ u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Also, given a measurable function $g \colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ (for example a Carathéodory function), we set

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)) \quad \forall u \in W_0^{1,p}(\Omega)$$

(the Nemytski map corresponding to g). By δ_{ki} we denote the Kronecker symbol defined by

$$\delta_{ki} = \begin{cases} 1 & \text{if } k = i, \\ 0 & \text{if } k \neq i. \end{cases}$$

Finally, if $u, v \in W_0^{1,p}(\Omega)$, $v \leq u$, then we define

$$[v, u] = \{y \in W_0^{1,p}(\Omega) : \ v(z) \leq y(z) \leq u(z) \quad \text{for a.a. } z \in \Omega\}.$$

Also by $\operatorname{int}_{C_0^1(\overline{\Omega})}[v, u]$ we define the interior in the $C_0^1(\overline{\Omega})$-norm topology of $[v, u] \cap C_0^1(\overline{\Omega})$.

3 Three Solutions with Sign Information

In this section without assuming any differentiability properties of $f(z, \cdot)$ we show that for all $\lambda > 0$ big, problem (P_λ) has at least three nontrivial smooth solutions all with sign information.

The assumptions on the perturbation term $f(z, x)$ are the following:
$\underline{H(f)_1}$ $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) there exist functions $\widehat{\vartheta}_0, \vartheta_0 \in L^\infty(\Omega)$ such that

$$0 \leq \widehat{\vartheta}_0(z) \leq \vartheta_0(z) \leq \widehat{\lambda}_1(2) \quad \text{for a.a. } z \in \Omega, \ \widehat{\vartheta}_0 \not\equiv 0, \ \vartheta_0 \not\equiv \widehat{\lambda}_1(2),$$
$$\widehat{\vartheta}_0(z) \leq \liminf_{x \to 0} \frac{f(z, x)}{x} \leq \limsup_{x \to 0} \frac{f(z, x)}{x} \leq \vartheta_0(z) \quad \text{uniformly for a.a. } z \in \Omega.$$

(ii) $\lim_{x \to \pm\infty} \dfrac{f(z, x)}{|x|^{p-2}x} = 0$ uniformly for a.a. $z \in \Omega$;

(iii) $f(z, x)x \geq 0$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$.

Evidently the function $f(z, x) = \vartheta(z)x$ with $\vartheta \in L^\infty(\Omega)$, $0 \leq \vartheta(z) \leq \widehat{\lambda}_1(2)$, $\vartheta \not\equiv 0$, $\vartheta \not\equiv \widehat{\lambda}_1(2)$ satisfies hypotheses $H(f)_1$.

We let $F(z, x) = \int_0^x f(z, s)\, ds$.

Proposition 3.1 *If hypotheses $H(f)_1$ hold, then for all $\lambda > 0$ big, problem (P_λ) has at least two constant sign solutions $u_\lambda \in \operatorname{int} C_+$ and $v_\lambda \in -\operatorname{int} C_+$.*

Proof First we produce the positive solution.

Let $\varphi_\lambda^+ : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the C^1-functional defined by

$$\varphi_\lambda^+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \frac{\lambda}{r}\|u^+\|_r^r - \int_\Omega F(z, u^+)\, dz \quad \forall u \in W_0^{1,p}(\Omega).$$

On account of hypotheses $H(f)_1(i)$, (ii), given $\varepsilon > 0$, we can find $c_1 = c_1(\varepsilon) > 0$ such that

$$F(z,x) \le \frac{1}{2}(\vartheta_0(z)+\varepsilon)x^2 + c_1|x|^r. \tag{3.1}$$

Assuming that $\lambda \ge 1$, using (3.1), Lemma 2.4, for all $u \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}\varphi_\lambda^+(u) &\ge \frac{1}{p}\|u\|^p + \frac{1}{2}\left(\|Du\|_2^2 - \int_\Omega \vartheta_0(z)u^2\,dz - \varepsilon\|u\|^2_{H_0^1(\Omega)}\right) - \lambda c_2\|u\|^r \\ &\ge c_3\|u\|^p - \lambda c_2\|u\|^r = (c_3 - \lambda c_2\|u\|^{r-p})\|u\|^p \end{aligned}\tag{3.2}$$

for some $c_2, c_3 > 0$ (by choosing $\varepsilon > 0$ small). So, if $\varrho_\lambda \in (0, \frac{c_3}{\lambda c_2})$, then for $\|u\| = \varrho_\lambda$ we have

$$\varphi_\lambda^+(u) \ge m_\lambda^+ > 0 \quad \forall \|u\| = \varrho_\lambda, \tag{3.3}$$

with $\varrho_\lambda \to 0^+$ as $\lambda \to \infty$. Let $t \in (0,1)$ and $\overline{u}_0 \in \operatorname{int} C_+$. We have

$$\varphi_\lambda^+(t\overline{u}_0) \le \frac{t^p}{p}\|D\overline{u}_0\|_p^p + \frac{t^2}{2}\|D\overline{u}_0\|_2^2 - \lambda\frac{t^r}{r}\|\overline{u}_0\|_r^r \le c_4t^2 - \lambda c_5 t^r \tag{3.4}$$

for some $c_4, c_5 > 0$ (see hypothesis $H(f)_1(iii)$ and recall that $t \in (0,1)$, $2 < p$).

For fixed $t \in (0,1)$, from (3.3) we see that we can find $\widetilde{\lambda} \ge 1$ such that

$$\varphi_\lambda^+(t\overline{u}_0) < 0 \quad \forall \lambda \ge \widetilde{\lambda}, \tag{3.5}$$

and

$$\|t\widetilde{u}_0\| > \varrho_\lambda \tag{3.6}$$

(recall that $\varrho_\lambda \to 0^+$ as $\lambda \to \infty$).

Hypothesis $H(f)_1(ii)$ implies that given $\varepsilon > 0$, we can find $M = M(\varepsilon) \ge 1$ such that

$$F(z,x) \le \frac{\varepsilon}{p}|x|^p \le \frac{\varepsilon}{p}|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \ge M \ge 1. \tag{3.7}$$

We consider the Carathéodory function

$$k_\lambda(z,x) = \lambda|x|^{r-2}x + f(z,x).$$

We set $K_\lambda(z,x) = \int_0^x k_\lambda(z,s)\,ds$ and let $q \in (p,r)$. We have

$$qK_\lambda(z,x) \le \frac{\lambda q}{r}|x|^r + \frac{\varepsilon q}{r}|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \ge M \tag{3.8}$$

(see (3.7)). Also using hypothesis $H(f)_1(iii)$ we have

$$k_\lambda(z,x)x \ge \lambda|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{3.9}$$

From (3.8) and (3.9), we see that by choosing $\varepsilon \in (0, \lambda(r-q))$, we have

$$0 < qK_\lambda(z,x) \leq k_\lambda(z,x)x \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M. \tag{3.10}$$

Using (3.10) (essentially the Ambrosetti–Rabinowitz condition; see Motreanu-Motreanu-Papageorgiou [21]), we can easily check that

$$\varphi_\lambda^+ \text{ satisfies the Palais-Smale condition.} \tag{3.11}$$

Then (3.3), (3.5), (3.6) and (3.11) permit the use of the mountain pass theorem on the functional φ_λ^+ for all $\lambda \geq \widetilde{\lambda}$. So, we can find $u_\lambda \in W_0^{1,p}(\Omega)$ such that

$$u_\lambda \in K_{\varphi_\lambda^+} \quad \text{and} \quad \varphi_\lambda^+(0) = 0 < m_\lambda^+ \leq \varphi_\lambda^+(u_\lambda) \tag{3.12}$$

(see (3.3)). From (3.12) it follows that $u_\lambda \neq 0$ and

$$(\varphi_\lambda^+)'(u_\lambda) = 0,$$

so

$$\langle A_p(u_\lambda), h\rangle + \langle A(u_\lambda), h\rangle = \int_\Omega \left(\lambda (u_\lambda^+)^{p-1} + f(z, u_\lambda^+)\right) h\, dz \quad \forall h \in W_0^{1,p}(\Omega). \tag{3.13}$$

In (3.13) we choose $h = -u_\lambda^- \in W_0^{1,p}(\Omega)$. We have

$$\|Du_\lambda^-\|_p \leq 0,$$

so

$$u_\lambda \geq 0, \quad u_\lambda \neq 0.$$

From (3.13) we have

$$\begin{cases} -\Delta_p u_\lambda(z) - \Delta u_\lambda(z) = \lambda u_\lambda(z)^{r-1} + f(z, u_\lambda(z)) \text{ for a.a. } z \in \Omega, \\ u_\lambda|_{\partial\Omega} = 0. \end{cases} \tag{3.14}$$

From (3.14) and Theorem 7.1 of Ladyzhenskaya-Ural'tseva [18, p. 286], we have that $u_\lambda \in L^\infty(\Omega)$. Then applying Theorem 1 of Lieberman [19], we infer that

$$u_\lambda \in C_+ \setminus \{0\}.$$

From (3.14) and hypothesis $H(f)_1(iii)$, we have

$$\Delta_p u_\lambda(z) + \Delta u_\lambda(z) \leq 0 \quad \text{for a.a. } z \in \Omega,$$

so $u_\lambda \in \operatorname{int} C_+$ (see Pucci-Serrin [31, pp. 111, 120]).

For the negative solution, we consider the C^1-functional $\varphi_\lambda^-: W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\varphi_\lambda^-(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \frac{\lambda}{r}\|u^-\|_p^p - \int_\Omega F(z, -u^-)\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Reasoning as above, using this time the functional φ_λ^-, we produce a negative solution $v_\lambda \in -\text{int}\, C_+$ for all $\lambda \geq \widetilde{\lambda}$ (increasing $\widetilde{\lambda} \geq 1$ if necessary). □

The next result determines the asymptotic behaviour of the two constant sign solutions as $\lambda \to \infty$.

Proposition 3.2 *If hypotheses $H(f)_1$ hold, then $u_\lambda, v_\lambda \to 0$ in $C_0^1(\overline{\Omega})$ as $\lambda \to \infty$.*

Proof Recall that $u_\lambda \in \text{int}\, C_+$ is a critical point of φ_λ^+ of mountain pass type (see the proof of Proposition 3.1). So, we have

$$\begin{aligned}
\varphi_\lambda^+(u_\lambda) &\leq \max_{0\leq s\leq 1} \varphi_\lambda^+(st\overline{u}_0) \\
&\leq \max_{0\leq s\leq 1}\left(\frac{s^p}{p}\|D(t\overline{u}_0)\|_p^p + \frac{s^2}{2}\|D(t\overline{u}_0)\|_2^2 - \frac{\lambda s^r}{r}\|t\overline{u}_0\|_r^r\right) \\
&\leq \max_{0\leq s\leq 1}\left(\frac{s^2}{2}\left(\|D(t\overline{u}_0)\|_p^p + \|D(t\overline{u}_0)\|_2^2\right) - \frac{\lambda s^r}{r}\|t\overline{u}_0\|_r^r\right) \\
&= \max_{0\leq s\leq 1}(c_6 s^2 - \lambda c_7 s^r) = c_6\left(\frac{2c_6}{\lambda c_7 r}\right)^{\frac{2}{r-2}} - \lambda c_7\left(\frac{2c_6}{\lambda c_7 r}\right)^{\frac{r}{r-2}} \\
&= \left(\frac{2c_6}{\lambda c_7 r}\right)^{\frac{2}{r-2}} c_6 \frac{r-2}{r} = \frac{c_8}{\lambda^{\frac{q}{r-2}}},
\end{aligned} \tag{3.15}$$

with $c_6 = \frac{1}{2}(\|D(t\overline{u}_0)\|_p^p + \|D(t\overline{u}_0)\|_2^2) > 0$, $c_7 = \frac{1}{r}\|t\overline{u}_0\|_r^r > 0$ and some $c_8 > 0$ (see hypothesis $H(f)_1(iii)$ and recall that $s \in [0, 1]$, $2 < p$).

We have

$$q\varphi_\lambda^+(u_\lambda) = \frac{q}{p}\|u_\lambda\|^p + \frac{q}{2}\|Du_\lambda\|_2^2 - \int_\Omega qK_\lambda(z, u_\lambda)\,dz \tag{3.16}$$

and

$$0 = -\langle(\varphi_\lambda^+)(u_\lambda), u_\lambda\rangle = -\|u_\lambda\|^p - \|Du_\lambda\|_2^2 + \int_\Omega k_\lambda(z, u_\lambda)u_\lambda\,dz. \tag{3.17}$$

We add (3.16) and (3.17) and use (3.15). Then

$$\left(\frac{q}{p} - 1\right)\|u_\lambda\|^p + \int_\Omega (k_\lambda(z, u_\lambda)u_\lambda - qK_\lambda(z, u_\lambda))\,dz \leq \frac{qc_8}{\lambda^{\frac{q}{r-2}}}$$

(since $2 < q$), so

$$\frac{q-p}{p}\|u_\lambda\|^p \leq \frac{c_8}{\lambda^{\frac{q}{r-2}}} + c_9,$$

for some $c_9 > 0$ (see (3.10)), thus

$$\text{the sequence } \{u_\lambda\}_{\lambda \geq \widetilde{\lambda}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{3.18}$$

Since $u_\lambda \in \operatorname{int} C_+$ is a solution of (P_λ), we have

$$\|u_\lambda\|^p + \|Du_\lambda\|_2^2 = \lambda\|u_\lambda\|_r^r + \int_\Omega f(z, u_\lambda)u_\lambda\, dz,$$

so

$$\lambda\|u_\lambda\|_r^r \leq c_{10} \quad \forall \lambda \geq \widetilde{\lambda}$$

for some $c_{10} > 0$ (see hypothesis $H(f)_1(iii)$ and (3.18)), thus

$$u_\lambda \longrightarrow 0 \quad \text{in } L^r(\Omega) \text{ as } \lambda \to 0^+. \tag{3.19}$$

We know that

$$\begin{cases} -\Delta_p u_\lambda(z) - \Delta u_\lambda(z) = \lambda u_\lambda(z)^{r-1} + f(z, u_\lambda(z)) \text{ for a.a. } z \in \Omega, \\ u_\lambda|_{\partial\Omega} = 0, \ \lambda \geq \widetilde{\lambda}. \end{cases} \tag{3.20}$$

From (3.18), (3.20) and Theorem 7.1 of Ladyzhenskaya-Ural'tseva [18, p. 286], we see that we can find $c_{11} > 0$ such that

$$\|u_\lambda\|_\infty \leq c_{11} \quad \forall \lambda \geq \widetilde{\lambda}.$$

Invoking Theorem 1 Lieberman [19], we infer that there exist $\alpha \in (0, 1)$ and $c_{12} > 0$ such that

$$u_\lambda \in C_0^{1,\alpha}(\overline{\Omega}), \quad \|u_\lambda\|_{C_0^{1,\lambda}(\overline{\Omega})} \leq c_{12} \quad \forall \lambda \geq \widetilde{\lambda}. \tag{3.21}$$

From (3.21), the compactness of the embedding $C_0^{1,\alpha}(\overline{\Omega}) \subseteq C_0^1(\overline{\Omega})$ and (3.19), we conclude that

$$u_\lambda \longrightarrow 0 \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } \lambda \to +\infty.$$

In a similar fashion, working this time with φ_λ^-, we show that

$$v_\lambda \longrightarrow 0 \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } \lambda \to +\infty.$$

□

Next we will show that for all $\lambda \geq \widetilde{\lambda}$ problem (P_λ) has extremal constant sign solutions, that is, there is a smallest positive solution and a biggest negative solution.

To this end, we introduce the following two sets

$$S_\lambda^+ \quad \text{- set of positive solutions for } (P_\lambda),$$

$$S_\lambda^- \quad \text{- set of negative solutions for } (P_\lambda).$$

We know (see Proposition 3.1) that

$$\emptyset \neq S_\lambda^+ \subseteq \operatorname{int} C_+ \quad \text{and} \quad \emptyset \neq S_\lambda^- \subseteq -\operatorname{int} C_+ \quad \forall \lambda \geq \widetilde{\lambda}.$$

Proposition 3.3 *If hypotheses $H(f)_1$ hold, then for all $\lambda > 0$ big, problem (P_λ) has*
- *a smallest positive solution $u_\lambda^* \in \operatorname{int} C_+$;*
- *a biggest negative solution $v_\lambda^* \in -\operatorname{int} C_+$.*

Proof From Filippakis-Papageorgiou [6], we know that the set S_λ^+ is downward directed (that is, if $u_1, u_2 \in S_\lambda^+$, then there exists $u \in S_\lambda^+$ such that $u \leq u_1$, $u \leq u_2$). Then invoking Lemma 3.10 of Hu-Papageorgiou [17, p. 178], we can find a decreasing sequence $\{u_n\}_{n\geq 1} \subseteq S_\lambda^+$ such that

$$\inf_{n\geq 1} u_n = \inf S_\lambda^+, \quad 0 \leq u_n \leq u_1 \quad \forall n \in \mathbb{N}. \tag{3.22}$$

We have

$$\begin{aligned} &\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle = \lambda \int_\Omega u_n^{r-1} h\, dz + \int_\Omega f(z, u_n) h\, dz \\ &\forall h \in W_0^{1,p}(\Omega),\ n \in \mathbb{N}. \end{aligned} \tag{3.23}$$

Choosing $h = u_n \in W_0^{1,p}(\Omega)$ and using (3.22), we infer that the sequence $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, by passing to a suitable subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u_\lambda^* \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \longrightarrow u_\lambda^* \quad \text{in } L^r(\Omega). \tag{3.24}$$

In (3.23) we choose $h = u_n - u_\lambda^* \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.24). We obtain

$$\lim_{n\to+\infty} \left(\left\langle A_p(u_n), u_n - u_\lambda^*\right\rangle + \left\langle A(u_n), u_n - u_\lambda^*\right\rangle\right) = 0,$$

so

$$\limsup_{n\to+\infty} \left(\left\langle A_p(u_n), u_n - u_\lambda^*\right\rangle + \left\langle A(u_\lambda^*), u_n - u_\lambda^*\right\rangle\right) \leq 0$$

(from the monotonicity of A), thus

$$\limsup_{n\to+\infty} \left\langle A_p(u_n), u_n - u_\lambda^*\right\rangle \leq 0$$

(see (3.24)) and hence we get

$$u_n \longrightarrow u_\lambda^* \quad \text{in } W_0^{1,p}(\Omega) \tag{3.25}$$

(see Proposition 2.1). Suppose that $u_\lambda^* = 0$. Then from (3.25) we have

$$\|u_n\| \longrightarrow 0 \quad \text{as } n \to \infty. \tag{3.26}$$

We set $y_n = \frac{u_n}{\|u_n\|}$, for $n \in \mathbb{N}$. We have $\|y_n\| = 1$ for all $n \in \mathbb{N}$. From (3.23), we have

$$\|u_n\|^{p-2}\langle A_n(y_n), h\rangle + \langle A(y_n), h\rangle = \int_\Omega \left(\frac{\lambda u_n^{r-1}}{\|u_n\|} + \frac{N_f(u_n)}{\|u_n\|}\right) h\, dz$$

for all $h \in W_0^{1,p}(\Omega)$, all $n \in \mathbb{N}$, so

$$\begin{cases} -\|u_n\|^{p-2}\Delta_p y_n(z) - \Delta y_n(z) = \frac{\lambda}{\|u_n\|} u_n(z)^{r-1} + \frac{1}{\|u_n\|} f(z, u_n(z)) \\ \hfill \text{for a.a. } z \in \Omega, \\ u_n|_{\partial\Omega} = 0. \end{cases} \tag{3.27}$$

Note that $\{\frac{N_f(u_n)}{\|u_n\|}\}_{n\geq 1} \subseteq L^{p'}(\Omega)$ and $\{\frac{\lambda u_n^{r-1}}{\|u_n\|}\}_{n\geq 1} \subseteq L^{p'}(\Omega)$ (see (3.22)). So, from (3.27) as before using the nonlinear regularity theory (see Ladyzhenskaya-Ural'tseva [18] and Lieberman [19]), at least for a subsequence, we can have

$$y_n \longrightarrow y \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } n \to +\infty. \tag{3.28}$$

We have

$$\frac{\lambda u_n^{r-1}}{\|u_n\|} \xrightarrow{w} 0 \quad \text{in } L^{r'}(\Omega) \tag{3.29}$$

and

$$\frac{N_f(u_n)}{\|u_n\|} \xrightarrow{w} \vartheta y \quad \text{in } L^{p'}(\Omega), \tag{3.30}$$

with $\widehat{\vartheta}_0(z) \leq \vartheta(z) \leq \vartheta_0(z)$ a.e. on Z (see hypothesis $H(f)_1(i)$ and (3.26)). So, if in (3.26) we pass to the limit as $n \to \infty$ and use (3.26), (3.28), (3.29) and (3.30), we have

$$\begin{cases} -\Delta y(z) = \vartheta(z) y(z) \text{ for a.a. } z \in \Omega, \\ y|_{\partial\Omega} = 0. \end{cases} \tag{3.31}$$

Using (3.30) and Lemma 2.5, we have

$$1 = \widetilde{\lambda}_1(2, \widehat{\lambda}_1(2)) < \widetilde{\lambda}_1(2, \vartheta),$$

so $y = 0$ (see (3.31)). This is a contradiction since $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and we have (3.28).

Therefore $u_\lambda^* \neq 0$ and then using (3.25) we see that

$$u_\lambda^* \in S_\lambda^* \quad \text{and} \quad u_\lambda^* = \inf S_\lambda^*.$$

The set S_λ^- is upward directed (that is, if $v_1, v_2 \in S_\lambda^-$ we can find $v \in S_\lambda^-$ such that $v_1 \leq v$, $v_2 \leq v$; see Filippakis-Papageorgiou [6]). Reasoning as above, we produce

$$v_\lambda^* \in S_\lambda^- \quad \text{and} \quad v_\lambda^* = \sup S_\lambda^-.$$

□

Using these extremal constant sign solutions, we can produce a nodal solution.

Proposition 3.4 *If hypotheses $H(f)_1$ hold, then for all $\lambda > 0$ big, problem (P_λ) admits a nodal solution*

$$y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega}).$$

Proof Using the two extremal constant sign solutions $u_\lambda^* \in \operatorname{int} C_+$ and $v_\lambda^* \in -\operatorname{int} C_+$ produced in Proposition 3.3, we introduce the following truncation of the reaction in problem (P_λ):

$$\widehat{k}_\lambda(z,x) = \begin{cases} \lambda|v_\lambda^*(z)|^{r-2}v_\lambda^*(z) + f(z, v_\lambda^*(z)) & \text{if } x < v_\lambda^*, \\ \lambda|x|^{r-2}x + f(z,x) & \text{if } v_\lambda^*(z) \leq x \leq u_\lambda^*(z), \\ \lambda u_\lambda^*(z)^{r-1} + f(z, u_\lambda^*(z)) & \text{if } u_\lambda^*(z) < x. \end{cases} \tag{3.32}$$

We also consider the positive and negative truncations of $\widehat{k}_\lambda(z,\cdot)$, namely the Carathéodory functions

$$\widehat{k}_\lambda^\pm(z,x) = \widehat{k}_\lambda(z, \pm x^\pm). \tag{3.33}$$

We set $\widehat{K}_\lambda(z,x) = \int_0^x \widehat{k}_\lambda(z,s)\,ds$, $\widehat{K}_\lambda^\pm(z,x) = \int_0^x \widehat{k}_\lambda^\pm(z,s)\,ds$ and consider the C^1-functionals $\widehat{\varphi}_\lambda, \widehat{\varphi}_\lambda^\pm : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \widehat{K}_\lambda(z,u)\,dz \quad \forall u \in W_0^{1,p}(\Omega),$$
$$\widehat{\varphi}_\lambda^\pm(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \widehat{K}_\lambda^\pm(z,u)\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Using (3.32) and (3.33) and the nonlinear regularity theory (see Ladyzhenskaya-Ural'tseva [18] and Lieberman [19]), we easily check that

$$K_{\widehat{\varphi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap C_0^1(\Omega), \quad K_{\widehat{\varphi}_\lambda^+} \subseteq [0, u_\lambda^*] \cap C_+, \quad K_{\widehat{\varphi}_\lambda^-} \subseteq [v_\lambda^*, 0] \cap (-C_+).$$

The extremality of u_λ^* and v_λ^* implies that

$$K_{\widehat{\varphi}_\lambda^+} = \{0, u_\lambda^*\}, \quad K_{\widehat{\varphi}_\lambda^-} = \{0, v_\lambda^*\}. \tag{3.34}$$

From (3.32) and (3.33) we see that $\widehat{\varphi}_\lambda^+$ is coercive. Also using the Sobolev embedding theorem, we have that $\widehat{\varphi}_\lambda^+$ is sequentially weakly lower semicontinuous. So, by the

Weierstrass-Tonelli theorem, we can find $\widetilde{u}_\lambda^* \in W^{1,p}$, such that

$$\widetilde{\varphi}_\lambda^+(\widetilde{u}_\lambda^*) = \inf\left\{\widehat{\varphi}_\lambda^+(u) : u \in W_0^{1,p}(\Omega)\right\}. \tag{3.35}$$

Let $\overline{u}_0 \in \operatorname{int} C_+$. Using Proposition 4.1.22 of Papageorgiou-Rădulescu-Repovš [28], we can find $t \in (0,1)$ small such that $0 \leq t\overline{u}_0 \leq u_\lambda^*$. Then using (3.32), (3.33) and hypothesis $H(f)_1(iii)$, we have

$$\begin{aligned}\widehat{\varphi}_\lambda^+(t\overline{u}_0) &\leq \frac{t^p}{p}\|D\overline{u}_0\|_p^p + \frac{t^2}{2}\|D\overline{u}_0\|_2^2 - \lambda\frac{t^r}{r}\|\overline{u}_0\|_r^r \\ &\leq c_{13}t^2 - \lambda c_{14}t^r,\end{aligned}$$

for some $c_{13}, c_{14} > 0$ (recall that $t \in (0,1)$, $2 < p$).

Fixing $t \in (0,1)$, from the above inequality we see that for $\lambda \geq 1$ big, we have

$$\widehat{\varphi}_\lambda^+(t\overline{u}_0) < 0,$$

so

$$\widehat{\varphi}_\lambda^+(\widetilde{u}_\lambda^*) < 0 = \widehat{\varphi}_\lambda^+(0)$$

(see (3.35)) and thus

$$\widetilde{u}_\lambda^* \neq 0. \tag{3.36}$$

Note that $\widetilde{u}_\lambda^* \in K_{\widehat{\varphi}_\lambda^+}$ (see (3.35)). Then from (3.34) and (3.36) we infer that

$$\widetilde{u}_\lambda^* = u_\lambda^* \in \operatorname{int} C_+. \tag{3.37}$$

From (3.32) and (3.33) it is clear that

$$\widehat{\varphi}_\lambda|_{C_+} = \widehat{\varphi}^+|_{C_+},$$

so u_λ^* is a local $C_0^1(\overline{\Omega})$-minimizer of $\widehat{\varphi}_\lambda$ (see (3.37)), and by Proposition 2.2, we get that

$$u_\lambda^* \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \widehat{\varphi}_\lambda. \tag{3.38}$$

Similarly, using this time the functional $\widehat{\varphi}_\lambda^-$, we show that

$$v_\lambda^* \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \widehat{\varphi}_\lambda. \tag{3.39}$$

We may assume that

$$\widehat{\varphi}_\lambda(v_\lambda^*) \leq \widehat{\varphi}_\lambda(u_\lambda^*).$$

The reasoning is the same if the opposite inequality holds, using this time (3.39) instead of (3.38).

On account of (3.34) we see that we may assume that

$$K_{\widehat{\varphi}_\lambda} \text{ is finite.} \tag{3.40}$$

Otherwise on account of the extremality of u_λ^* and v_λ^*, we see that we already have an infinity of smooth nodal solutions (see (3.34)) and we are done.

From (3.38), (3.40) and Theorem 5.7.6 of Papageorgiou-Rădulescu-Repovš [28], we can find $\varrho \in (0, 1)$ small such that

$$\widehat{\varphi}_\lambda(v_\lambda^*) \leq \widehat{\varphi}_\lambda(u_\lambda^*) < \inf\{\widehat{\varphi}_\lambda(u) : \|u - u_\lambda^*\| = \varrho\} = \widehat{m}_\lambda, \quad \|v_\lambda^* - u_\lambda^*\| > \varrho. \tag{3.41}$$

Note that $\widehat{\varphi}_\lambda$ is coercive (see (3.32)). Then Proposition 5.1.15 of Papageorgiou-Rădulescu-Repovš [28] implies that

$$\widehat{\varphi}_\lambda \text{ satisfies the Palais-Smale condition.} \tag{3.42}$$

From (3.41) and (3.42) we see that we can apply the mountain pass theorem. So, there exists $y_\lambda \in W_0^{1,p}(\Omega)$ such that

$$y_\lambda \in K_{\widehat{\varphi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega}) \quad \text{and} \quad \widehat{m}_\lambda \leq \widehat{\varphi}_\lambda(y_\lambda). \tag{3.43}$$

From (3.41) and (3.43) we see that

$$y_\lambda \notin \{u_\lambda^*, v_\lambda^*\} \tag{3.44}$$

So, if we show that $y_\lambda \neq 0$, then y_λ will be the desired nodal solution. Since y_λ is a critical point of $\widehat{\varphi}_\lambda$ of mountain pass type, we have

$$C_1(\widehat{\varphi}_\lambda, y_\lambda) \neq 0 \tag{3.45}$$

(see Papageorgiou-Rădulescu-Repovš [28, Theorem 6.5.8]).

From hypotheses $H(f)_1(i), (ii)$, we see that given $\varepsilon > 0$, we can find $c_{15} = c_{15}(\varepsilon) > 0$ such that

$$F(z, x) \leq \frac{1}{2}(\vartheta_0(z) + \varepsilon)x^2 + c_{15}|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{3.46}$$

Then taking $\lambda \geq 1$ and using (3.46), for $u \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned} \widehat{\varphi}_\lambda(u) &\geq \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\left(\|Du\|_2^2 - \int_\Omega \vartheta_0(z)u^2\,dz - \varepsilon c_{16}\|u\|^2\right) - \lambda c_{17}\|u\|^r \\ &\geq \frac{1}{p}\|u\|^p + \frac{1}{2}(c_0 - \varepsilon c_{16})\|u\|^2 - \lambda c_{17}\|u\|^r \end{aligned}$$

for some $c_{16}, c_{17} > 0$ (see Lemma 2.4).

Choosing $\varepsilon \in (0, \frac{c_0}{c_{16}})$, we see that

$$\widehat{\varphi}_\lambda(u) \geq \frac{1}{p}\|u\|^p - \lambda c_{17}\|u\|^r.$$

Since $r > p$, we can find $\varrho_\lambda \in (0, \delta)$ such that

$$\widehat{\varphi}_\lambda(u) \geq 0 \quad \forall \|u\| \leq \varrho_\lambda,$$

so

$$u = 0 \text{ is a local minimizer of } \widehat{\varphi}_\lambda,$$

thus

$$C_k(\widehat{\varphi}_\lambda, 0) = \delta_{k,0}\mathbb{Z} \quad \forall k \in \mathbb{N}_0. \tag{3.47}$$

From (3.47), (3.45) and (3.44), we infer that

$$y_\lambda \notin \{0, u_\lambda^*, v_\lambda^*\},$$

so $y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega})$ (see (3.43)) is nodal. □

If we strengthen the hypotheses on the perturbation $f(z, \cdot)$ we can improve the conclusion of Proposition 2.2. The new hypotheses on f are the following:
$\underline{H(f)_2}$ $f: \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, hypotheses $H(f)_2(i), (ii), (iii)$ are the same as the corresponding hypotheses hypotheses $H(f)_1(i), (ii), (iii)$ and

(iv) for every $\varrho > 0$, there exists $\widehat{\xi}_\varrho > 0$ such that for a.a. $z \in \Omega$, the function $x \longmapsto f(z, x) + \widehat{\xi}_\varrho |x|^{p-2}x$ is nondecreasing on $[-\varrho, \varrho]$.

Remark 3.5 Evidently hypothesis $H(f)_2(iv)$ implies a lower local Lipschitz condition for $f(z, \cdot)$.

Proposition 3.6 *If hypotheses $H(f)_2$ hold, then for all $\lambda > 0$ big, problem (P_λ) has a nodal solution*

$$y_\lambda \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_\lambda^*, u_\lambda^*].$$

Proof From Proposition 3.4, we know that for all $\lambda > 0$ big, problem (P_λ) has a nodal solution

$$y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega}). \tag{3.48}$$

Let $\varrho = \max\{\|u_\lambda^*\|_\infty, \|v_\lambda^*\|_\infty\}$ and let $\widehat{\xi}_\varrho > 0$ be as postulated by hypotheses $H(f)_2(iv)$. Let $\widetilde{\xi}_\varrho > \widehat{\xi}_\varrho$. We have

$$\begin{aligned}
&-\Delta_p y_\lambda - \Delta y_\lambda + \widetilde{\xi}_\varrho |y_\lambda|^{p-2} y_\lambda \\
&\quad \leq \lambda (u_\lambda^*)^{r-1} + f(z, u_\lambda^*) + \widehat{\xi}_\varrho (u_\lambda^*)^{p-1} + (\widetilde{\xi}_\varrho - \widehat{\xi}_\varrho)(u_\lambda^*)^{p-1}
\end{aligned}$$

$$\leq -\Delta_p u_\lambda^* - \Delta u_\lambda^* + \widetilde{\xi}_\varrho (u_\lambda^*)^{p-1} \quad \text{for a.a. } z \in \Omega \tag{3.49}$$

(see hypothesis $H(f)_2(iv)$ and (3.48)).

Let $a\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be defined by

$$a(y) = |y|^{p-2} y + y \quad \forall y \in \mathbb{R}^N.$$

Evidently $a \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ (recall that $2 < p$) and

$$\nabla a(y) = |y|^{p-2} \left(\mathrm{id} + (p-2) \frac{y \otimes y}{|y|^2} \right) + \mathrm{id} \quad \forall y \in \mathbb{R}^N,$$

so

$$\big(\nabla a(y) \xi, \xi \big)_{\mathbb{R}^N} \geq |\xi|^2 \quad \forall y, \xi \in \mathbb{R}^N.$$

Note that

$$\operatorname{div} a(Du) = \Delta_p u + \Delta u \quad \forall u \in W_0^{1,p}(\Omega).$$

So, invoking the tangency principle of Pucci-Serrin [31, Theorem 2.5.2], we obtain

$$y_\lambda(z) < u_\lambda^*(z) \quad \forall z \in \Omega.$$

Since $y_\lambda, u_\lambda^* \in C_0^1(\overline{\Omega})$, we have

$$(\widetilde{\xi}_\varrho - \widehat{\xi}_\varrho)(y_\lambda)^{p-2} y_\lambda \preceq (\widetilde{\xi}_\varrho - \widehat{\xi}_\varrho)(u_\lambda^*)^{p-1}.$$

Then using Proposition 2.3, we have

$$u_\lambda^* - y_\lambda \in \operatorname{int} C_+.$$

Similarly we show that

$$y_\lambda - v_\lambda^* \in \operatorname{int} C_+.$$

We conclude that

$$y_\lambda \in \operatorname{int}_{C_0^1(\overline{\Omega})} [v_\lambda^*, u_\lambda^*].$$

□

We can now state our first multiplicity theorem.

Theorem 3.7 *(a) If hypotheses $H(f)_1$ hold, then for all $\lambda > 0$ big, problem (P_λ) has at least three nontrivial solutions*

$$u_\lambda \in \operatorname{int} C_+, \quad v_\lambda \in -\operatorname{int} C_+, \quad y_\lambda \in [v_\lambda, u_\lambda] \cap C_0^1(\overline{\Omega}) \text{ nodal}$$

and $u_\lambda, v_\lambda, y_\lambda \longrightarrow 0$ in $C_0^1(\overline{\Omega})$ as $\lambda \to +\infty$.

(b) If hypotheses $H(f)_2$ hold, then for all $\lambda > 0$ big, problem (P_λ) has at least three nontrivial solutions

$$u_\lambda \in \operatorname{int} C_+, \quad v_\lambda \in -\operatorname{int} C_+, \quad y_\lambda \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_\lambda, u_\lambda] \text{ nodal}$$

and $u_\lambda, v_\lambda, y_\lambda \longrightarrow 0$ in $C_0^1(\overline{\Omega})$ as $\lambda \to +\infty$.

4 Four Solutions with Sign Information

In this section by strengthening the regularity of $f(z, \cdot)$, we can improve the above multiplicity theorem and produce a second nodal solution, for a total of four nontrivial smooth solutions, all with sign information.

The new hypotheses on the perturbation $f(z, x)$ are the following:

$\underline{H(f)_3}$ $f: \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function such that $f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$ for a.a. $z \in \Omega$ and

- **(i)** $|f'_x(z, x)| \le a_0(z)(1 + |x|^{q-1})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a_0 \in L^\infty(\Omega)$, $1 < q < p^*$;
- **(ii)** $f'_x(z, 0) = \lim\limits_{x \to 0} \frac{f(z,x)}{x}$ uniformly for a.a. $z \in \Omega$ and

$$0 \le f'_x(z, 0) \le \widehat{\lambda}_1(2) \text{ for a.a. } z \in \Omega, \;\; f'_x(\cdot, 0) \not\equiv 0, \;\; f'_x(\cdot, 0) \not\equiv \widehat{\lambda}_1(2);$$

- **(iii)** $\lim\limits_{x \to \pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = 0$ uniformly for a.a. $z \in \Omega$;
- **(iv)** $f(z, x)x \ge 0$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$;
- **(v)** for every $\varrho > 0$, there exists $\widehat{\xi}_\varrho > 0$ such that for a.a. $z \in \Omega$ the function $x \longmapsto f(z, x) + \widehat{\xi}_\varrho |x|^{p-2}x$ is nondecreasing on $[-\varrho, \varrho]$.

Evidently the function $f(z, x) = \vartheta(z)x + |x|^{q-2}x$ with $0 \le \vartheta(z) \le \widehat{\lambda}_1(2)$ for a.a. $z \in \Omega$, $\vartheta \not\equiv 0$, $\vartheta \not\equiv \widehat{\lambda}_1(2)$ and $2 < q < p$, satisfies hypotheses $H(f)_3$.

Proposition 4.1 *If hypotheses $H(f)_3$ hold, then for all $\lambda > 0$ big, problem (P_λ) has at least two nodal solutions*

$$y_\lambda, \widehat{y}_\lambda \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_\lambda^*, u_\lambda^*].$$

Proof From Theorem 3.7(b), we already have a nodal solution

$$y_\lambda \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_\lambda^*, u_\lambda^*]. \tag{4.1}$$

We consider the energy (Euler) functional $\psi_\lambda : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \frac{\lambda}{r}\|u\|_r^r - \int_\Omega F(z,u)\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Also, we consider the function $\widehat{\varphi}_\lambda : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ from the proof of Proposition 3.4. Hypotheses $H(f)_3$ imply that

$$\varphi_\lambda \in C^2(W_0^{1,p}(\Omega)), \quad \widehat{\varphi}_\lambda \in C^{2-0}(W_0^{1,p}(\Omega)). \tag{4.2}$$

We consider the homotopy

$$h(t,u) = (1-t)\varphi_\lambda(u) + t\widehat{\varphi}_\lambda(u) \quad \forall t \in [0,1], \text{ all } u \in W_0^{1,p}(\Omega).$$

Suppose we could find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \longrightarrow t \text{ in } [0,1], \quad u_n \longrightarrow y_n \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad h'_u(t_n,u_n) = 0 \text{ for all } n \in \mathbb{N}. \tag{4.3}$$

From the equality in (4.3), we have

$$\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle + \int_\Omega \big((1-t_n)k_\lambda(z,u_n) + t_n\widehat{k}_\lambda(z,u_n)\big)h\,dz$$
$$\forall h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}$$

(see the proofs of Propositions 3.1 and 3.4), so

$$\begin{cases} -\Delta_p u_n(z) - \Delta u_n(z) = (1-t)k_\lambda(z,u_n(z)) + t\widehat{k}(z,u_n(z)) & \text{for a.a. } z \in \Omega, \\ u_n|_{\partial\Omega} = 0, \text{ for all } n \in \mathbb{N}. \end{cases} \tag{4.4}$$

As before (see the proof of Proposition 3.2), from (4.4), (4.3) and the nonlinear regularity theory, we have

$$u_n \longrightarrow y_\lambda \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } n \to +\infty,$$

so

$$u_n \in [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega}) \quad \forall n \geq n_0. \tag{4.5}$$

Again without any loss of generality we assume that $K_{\widehat{\varphi}_\lambda}$ is finite (see (3.40)). Then finiteness of $K_{\widehat{\varphi}_\lambda}$ and (4.5), (3.32) lead to a contradiction. So, (4.3) cannot occur and then the homotopy invariance property of the critical groups (see Theorem 6.3.8 of Papageorgiou-Rădulescu-Repovš [28]) implies that

$$C_k(\varphi_\lambda, y_\lambda) = C_k(\widehat{\varphi}_\lambda, y_\lambda) \quad \forall k \in \mathbb{N}_0. \tag{4.6}$$

Recall that $C_1(\widehat{\varphi}_\lambda, y_\lambda) \neq 0$ (see (3.45)). Hence $C_1(\varphi_\lambda, y_\lambda) \neq 0$ (see (4.6)). Then (4.2) and Claim 3 of Papageorgiou-Rădulescu [24, p. 412], imply that

$$C_k(\varphi_\lambda, y_\lambda) = \delta_{k,1}\mathbb{Z} \quad \forall k \in \mathbb{N}_0,$$

so

$$C_k(\widehat{\varphi}_\lambda, y_\lambda) = \delta_{k,1}\mathbb{Z} \quad \forall k \in \mathbb{N}_0 \tag{4.7}$$

(see (4.6)). We know that $u_\lambda^*, v_\lambda^*, 0$ are local minimizers of $\widehat{\varphi}_\lambda$ (see (3.38), (3.39), (3.39)). Hence we have

$$C_k(\widehat{\varphi}_\lambda, u_\lambda^*) = C_k(\widehat{\varphi}_\lambda, v_\lambda^*) = C_k(\widehat{\varphi}_\lambda, 0) = \delta_{k,0}\mathbb{Z} \quad \forall k \in \mathbb{N}_0. \tag{4.8}$$

Since $\widehat{\varphi}_\lambda$ is coercive, we have

$$C_k(\widehat{\varphi}_\lambda, \infty) = 0 \quad \forall k \in \mathbb{N}_0 \tag{4.9}$$

(see Papageorgiou-Rădulescu-Repovš [28, Proposition 6.2.24]).

If $K_{\widehat{\varphi}_\lambda} = \{0, u_\lambda^*, v_\lambda^*, y_\lambda\}$, then from (4.7), (4.8), (4.9) and the Morse reaction (see (2.1)) with $t = -1$, we have

$$3(-1)^0 + (-1)^1 = (-1)^0,$$

so $(-1)^0 = 0$, a contradiction. So, there exists $\widehat{y}_\lambda \in K_{\widehat{\lambda}_\lambda}$, $\widehat{y}_\lambda \notin \{0, u_\lambda^*, v_\lambda^*, y_\lambda\}$. Then $\widehat{y}_\lambda \in C_0^1(\overline{\Omega})$ is a second nodal solution of (P_λ) (see (3.34)) district from y_λ. Moreover, using Proposition 2.3, we have $\widehat{y}_\lambda \in \mathrm{int}_{C_0^1(\overline{\Omega})}[v_\lambda^*, u_\lambda^*]$ (see the proof of Proposition 3.6). □

Now we can state our second multiplicity theorem for problem (P_λ).

Theorem 4.2 *If hypotheses $H(f)_3$ hold, then for all $\lambda > 0$ big, problem (P_λ) has at least four nontrivial solutions*

$$u_\lambda \in \mathrm{int}\, C_+, \quad v_\lambda \in -\mathrm{int}\, C_+, \quad y_\lambda, \widehat{y}_\lambda \in \mathrm{int}_{C_0^1(\overline{\Omega})}[v_\lambda, u_\lambda] \ \textit{nodal}$$

and $u_\lambda, v_\lambda, y_\lambda, \widehat{y}_\lambda \longrightarrow 0$ in $C_0^1(\overline{\Omega})$ as $\lambda \to +\infty$.

Remark 4.3 It will be interesting to extend the results of this work to problems with convection (that is, f depends also on Du). Helpful in that respect can be the recent work of Bai-Gasiński-Papageorgiou [2] (see also Bai-Gasiński-Papageorgiou [1], Candito-Gasiński-Papageorgiou [3] and Gasiński-Papageorgiou [15]).

Acknowledgements The authors wish to thank the referee for his/her corrections and remarks.

References

1. Bai, Y., Gasiński, L., Papageorgiou, N.S.: Nonlinear nonhomogeneous Robin problems with dependence on the gradient. Bound. Value Probl. **2018**(17), 1–24 (2018)
2. Bai, Y., Gasiński, L., Papageorgiou, N.S.: Nonlinear Dirichlet problems with the combined effects of singular and convection terms. Electron. J. Differ. Equ. **2019**(57), 1–13 (2019)
3. Candito, P., Gasiński, L., Papageorgiou, N.S.: Nonlinear nonhomogeneous Robin problems with convection. Ann. Acad. Sci. Fenn. Math. **44**, 755–767 (2019)
4. Cherfils, L., Il'yasov, Y.: On the stationary solutions of generalized reaction diffusion equations with $p\&q$-Laplacian. Commun. Pure Appl. Anal. **4**(1), 9–22 (2005)
5. Chorfi, N., Rădulescu, V.D.: Continuous spectrum for some classes of $(p, 2)$-equations with linear or sublinear growth. Miskolc Math. Notes **17**(2), 817–826 (2016)
6. Filippakis, M.E., Papageorgiou, N.S.: Multiple constant sign and nodal solutions for nonlinear elliptic equations with the p-Laplacian. J. Differ. Equ. **245**(7), 1883–1922 (2008)
7. Gasiński, L., Papageorgiou, N.S.: Nonlinear Analysis. Chapman & Hall/CRC, Boca Raton, FL (2006)
8. Gasiński, L., Papageorgiou, N.S.: Multiple solutions for nonlinear coercive problems with a nonhomogeneous differential operator and a nonsmooth potential. Set-Valued Var. Anal. **20**(3), 417–443 (2012)
9. Gasiński, L., Papageorgiou, N.S.: A pair of positive solutions for (p, q)-equations with combined nonlinearities. Commun. Pure Appl. Anal. **13**(1), 203–215 (2014)
10. Gasiński, L., Papageorgiou, N.S.: Dirichlet (p, q)-equations at resonance. Discret. Contin. Dyn. Syst. **34**(5), 2037–2060 (2014)
11. Gasiński, L., Papageorgiou, N.S.: Exercises in Analysis. Part 2. Nonlinear Analysis. Springer, Cham (2016)
12. Gasiński, L., Papageorgiou, N.S.: Asymmetric $(p, 2)$-equations with double resonance. Calc. Var. Partial Differ. Equ. **56**(3), 1–23 (2017). Art. 88
13. Gasiński, L., Papageorgiou, N.S.: Multiplicity theorems for $(p, 2)$-equations. J. Nonlinear Convex Anal. **18**(7), 1297–1323 (2017)
14. Gasiński, L., Papageorgiou, N.S.: Positive solutions for the Robin p-Laplacian problem with competing nonlinearities. Adv. Calc. Var. **12**(1), 31–56 (2019)
15. Gasiński, L., Papageorgiou, N.S.: Nonlinear Dirichlet problems with sign changing drift coefficient. Appl. Math. Lett. **90**, 209–214 (2019)
16. Gasiński, L., Papageorgiou, N.S.: Multiple solutions for $(p, 2)$-equations with resonance and concave terms. Results Math. **74**(2), 1–34 (2019)
17. Hu, S., Papageorgiou, N.S.: Handbook of Multivalued Analysis. Theory, vol. 1. Kluwer Academic Publishers, Dordrecht (1997)
18. Ladyzhenskaya, A.O., Ural'tseva, N.N.: Linear and Quasilinear Elliptic Equations. Academic Press, New York (1968)
19. Lieberman, G.M.: Boundary regularity for solutions of degenerate elliptic equations. Nonlinear Anal. **12**(11), 1203–1219 (1988)
20. Lieberman, G.M.: The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations. Commun. Partial Differ. Equ. **16**(2–3), 311–361 (1991)
21. Motreanu, D., Motreanu, V.V., Papageorgiou, N.S.: Topological and Variational Methods with Applications to Nonlinear Boundary Value Problems. Springer, New York (2014)
22. Papageorgiou, N.S.: On parametric evolution inclusions of the subdifferential type with applications to optimal control problems. Trans. Am. Math. Soc. **347**(1), 203–231 (1995)
23. Papageorgiou, N.S.: Optimal control and admissible relaxation of uncertain nonlinear elliptic systems. J. Math. Anal. Appl. **197**(1), 27–41 (1996)
24. Papageorgiou, N.S., Rădulescu, V.D.: Qualitative phenomena for some classes of quasilinear elliptic equations with multiple resonance. Appl. Math. Optim. **69**(3), 393–430 (2014)
25. Papageorgiou, N.S., Rădulescu, V.D.: Bifurcation of positive solutions for nonlinear nonhomogeneous Robin and Neumann problems with competing nonlinearities. Discret. Contin. Dyn. Syst. **35**(10), 5003–5036 (2015)
26. Papageorgiou, N.S., Rădulescu, V.D.: Nonlinear nonhomogeneous Robin problems with superlinear reaction term. Adv. Nonlinear Stud. **16**, 737–764 (2016)
27. Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.D.: On a class of parametric $(p, 2)$-equations. Appl. Math. Optim. **75**(2), 193–228 (2017)

28. Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.D.: Nonlinear Analysis—Theory and Methods, vol. 1. Springer, Cham (2019)
29. Papageorgiou, N.S., Scapellato, A.: Nonlinear Robin problems with general potential and crossing reaction. Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **30**(1), 1–29 (2019)
30. Papageorgiou, N.S., Scapellato, A.: Constant sign and nodal solutions for parametric $(p,2)$-equations. Adv. Nonlinear Anal. **9**(1), 449–478 (2019)
31. Pucci, P., Serrin, J.: The Maximum Principle. Birkhäuser, Basel (2007)
32. Sokołowski, J.: Optimal control in coefficients for weak variational problems in Hilbert space. Appl. Math. Optim. **7**(4), 283–293 (1981)
33. Ragusa, M.A., Tachikawa, A.: On continuity of minimizers for certain quadratic growth functionals. J. Math. Soc. Jpn. **57**(3), 691–700 (2005)
34. Ragusa, M.A., Tachikawa, A.: Boundary regularity of minimizers of $p(x)$-energy functionals. Ann. Inst. H. Poincaré Anal. Non Linéaire **33**(2), 451–476 (2016)
35. Yang, D., Bai, C.: Nonlinear elliptic problem of 2-q-Laplacian type with asymmetric nonlinearities. Electron. J. Differ. Equ. **170**, 1–13 (2014)
36. Zhikov, V.V.: Averaging of functionals of the calculus of variations and elasticity theory. Math. USSR-Izv. **29**, 33–66 (1987)
37. Zhikov, V.V.: On variational problems and nonlinear elliptic equations with nonstandard growth conditions. J. Math. Sci. **173**(5), 463–570 (2011)

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