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RESEARCH ARTICLE

Finite semigroups and periodic sums systems in $\beta\mathbb{N}$ and their Ramsey theoretic consequences

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Abstract
Let $m, n \geq 2$ and define $\nu : \omega \to \{0, \ldots, m-1\}$ by $\nu(k) \equiv k \pmod m$. We construct some new finite semigroups in $\beta\mathbb{N}$, in particular, a semigroup generated by m elements of order n with cardinality $m^n + m^{n-1} + \cdots + m$. We also show that, for $n \geq m$, there is a sequence $p_0, \ldots, p_{m-1}$ in $\beta\mathbb{N}$ such that all sums $\sum_{j=i}^{i+k} p_{\nu(j)}$, where $i \in \{0, \ldots, m-1\}$ and $k \in \{0, \ldots, n-1\}$, are distinct and $\sum_{j=i}^{i+n} p_{\nu(j)} = \sum_{j=i}^{i+n-m} p_{\nu(j)}$ for each i. As consequences we derive some new Ramsey theoretic results. In particular, we show that, for $n \geq m$, there is a partition $\{A_{i,k} : (i, k) \in \{0, \ldots, m-1\} \times \{0, \ldots, n-1\}\}$ of $\mathbb{N}$ such that, whenever for each (i, k), $\mathscr{B}_{i,k}$ is a finite partition of $A_{i,k}$, there exist $B_{i,k} \in \mathscr{B}_{i,k}$ and a sequence $(x_j)_{j=0}^{\infty}$ such that for every finite sequence $j_0 < \ldots < j_s$ such that $j_{t+1} \equiv j_t + 1 \pmod m$ for each $t < s$, one has $x_{j_0} + \cdots + x_{j_s} \in B_{i_0,k_0}$, where $i_0 = \nu(j_0)$ and k_0 is s if $s \leq n-1$ and $n - m + \nu(s-n)$ otherwise.

Keywords Stone–Čech compactification · Idempotent · Right cancelable ultrafilter · Finite semigroup · Periodic sums system · Ramsey theory

1 Introduction

The addition of the discrete semigroup $\mathbb{N}$ of natural numbers extends to the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ so that for each $a \in \mathbb{N}$, the left translation $\lambda_a : \beta\mathbb{N} \ni x \mapsto a + x \in \beta\mathbb{N}$ is continuous, and for each $q \in \beta\mathbb{N}$, the right translation $\rho_q : \beta\mathbb{N} \ni x \mapsto x + q \in \beta\mathbb{N}$ is continuous.

We take the points of $\beta\mathbb{N}$ to be the ultrafilters on $\mathbb{N}$, identifying the principal ultrafilters with the points of $\mathbb{N}$. For every $A \subseteq \mathbb{N}$, $\overline{A} = \{p \in \beta\mathbb{N} : A \in p\}$ and

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$A^* = \overline{A} \setminus A$. The subsets $\overline{A}$, where $A \subseteq \mathbb{N}$, form a base for the topology of $\beta\mathbb{N}$, and $\overline{A}$ is the closure of A. For $p, q \in \beta\mathbb{N}$, the ultrafilter $p + q$ has a base consisting of subsets of the form $\bigcup_{x\in A}(x + B_x)$, where $A \in p$ and for each $x \in A$, $B_x \in q$.

Being a compact Hausdorff right topological semigroup, $\beta\mathbb{N}$ has a smallest two sided ideal $K(\beta\mathbb{N})$ which is a disjoint union of minimal right ideals and a disjoint union of minimal left ideals. Every right (left) ideal of $\beta\mathbb{N}$ contains a minimal right (left) ideal, the intersection of a minimal right ideal and a minimal left ideal is a group, and the idempotents in a minimal right (left) ideal form a right (left) zero semigroup, that is, $x + y = y$ ($x + y = x$) for all x, y.

The semigroup $\beta\mathbb{N}$ has important applications to Ramsey theory and to topological dynamics. The first application to Ramsey theory was the proof of Hindman's theorem: whenever $\mathbb{N}$ is finitely colored, there is an infinite sequence all of whose sums are monochrome. An elementary introduction to $\beta\mathbb{N}$ can be found in [4].

In 1979, E. van Douwen asked (in [3], published much later) whether there are topological and algebraic copies of $\beta\mathbb{N}$ contained in $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$. This question was answered in the negative by D. Strauss in [7], where it was in fact established that continuous homomorphisms from $\beta\mathbb{N}$ to $\mathbb{N}^*$ have finite images. It follows that if $\varphi : \beta\mathbb{N} \to \mathbb{N}^*$ is a continuous homomorphism, then $\varphi(\beta\mathbb{N})$ is a finite cyclic semigroup generated by $p = \varphi(1)$. That is, there are $n \geq 1$ and $1 \leq m \leq n$ called the order and the period of p (and of the cyclic semigroup) such that all $ip = \underbrace{p + \cdots + p}_{i}$, where $i \in \{1, \ldots, n\}$, are distinct and $(n + 1)p = (n + 1 - m)p$. Conversely, every element $p \in \mathbb{N}^*$ of finite order determines a continuous homomorphism $\varphi : \beta\mathbb{N} \to \mathbb{N}^*$ by $\varphi(1) = p$. In 1996, the author proved that $\beta\mathbb{N}$ contains no nontrivial finite groups (see [4, Theorem 7.17]). Since the periodic part of a cyclic semigroup is a group, it follows that if $p \in \beta\mathbb{N}$ is an element of order n, then $(n + 1)p = np$, that is, p has period 1.

As distinguished from finite groups, $\beta\mathbb{N}$ does contain bands (semigroups of idempotents): for example, left zero semigroups, right zero semigroups, chains of idempotents (with respect to the order $x \leq y$ if and only if $x + y = y + x = x$), and even rectangular bands (direct products of a left zero semigroup and a right zero semigroup). To ask whether $\beta\mathbb{N}$ contains a finite semigroup distinct from bands is the same as asking whether $\beta\mathbb{N}$ contains an element of order 2 which is the same as asking whether there exists a nontrivial continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$ [4, Question 10.19]. If the answer to this question is positive, then there is a subset A of $\mathbb{N}$ with the following Ramsey theoretic property: whenever A is finitely colored, there is an infinite sequence in the complement of A, all of whose sums two or more terms at a time are monochrome [2].

The question whether $\beta\mathbb{N}$ contains an element of order 2 was solved in the affirmative in [8, Theorem 1]. In [9], some further finite semigroups in $\beta\mathbb{N}$ consisting of idempotents and elements of order 2 were constructed, in particular, null semigroups ($x + y = 0$ for all x, y), and a connection of finite semigroups in $\beta\mathbb{N}$ with Ramsey theory was discussed, see also [1]. In [12], it was shown that for every $m \geq 1$, the direct product of the m-element null semigroup and the rectangular band $2^{\mathfrak{c}} \times 2^{\mathfrak{c}}$ has copies in $\beta\mathbb{N}$ (that the rectangular band $2^{\mathfrak{c}} \times 2^{\mathfrak{c}}$ has copies in $\beta\mathbb{N}$ was established in [5]).

The question whether $\beta\mathbb{N}$ contains an element of finite order $n \geq 3$ was solved in the affirmative in [10, Theorem 3]. In fact, it was shown that for any $m \geq 1$ and $n \geq 2$, $\beta\mathbb{N}$ contains copies of the semigroup $C_{m,n}$ generated by the elements $q = q_1, q_2, \ldots, q_m$ with defining relations $(n+1)q = nq$ and $q_s + q_t = 2q$, where $s, t \in \{1, \ldots, m\}$. (If $m = 1$, this is the cyclic semigroup of order n and period 1, and if $n = 2$, this is the m-element null semigroup.) In [13], it was shown that for any $m \geq 1$ and $n \geq 2$, the direct product of the semigroup $C_{m,n}$ and the left zero semigroup $2^{\mathfrak{c}}$ has copies in $\beta\mathbb{N}$.

Let $m, n \geq 2$ and define $\nu : \omega \to \{0, \ldots, m-1\}$ by $\nu(k) \equiv k \pmod m$.

In [6], it was shown that there is a sequence $p_0, \ldots, p_{m-1}$ in $\beta\mathbb{N}$ such that all sums $\sum_{j=i}^{i+k} p_{\nu(j)}$, where $i \in \{0, \ldots, m-1\}$ and $k \in \{0, \ldots, mn-1-i\}$ for each i, are distinct and $\sum_{j=i}^{mn} p_{\nu(j)} = \sum_{j=i}^{mn-m} p_{\nu(j)}$ for each i.

In this paper, we construct some new finite semigroups in $\beta\mathbb{N}$, in particular, a semigroup generated by m elements of order n with cardinality $m^n + m^{n-1} + \cdots + m$. In fact, we construct large locally finite semigroups. The construction is given in Sect. 2.

In Sect. 3, using those semigroups, we show that, for $n \geq m$, there is a sequence $p_0, \ldots, p_{m-1}$ in $\beta\mathbb{N}$ such that all sums $\sum_{j=i}^{i+k} p_{\nu(j)}$, where $i \in \{0, \ldots, m-1\}$ and $k \in \{0, \ldots, n-1\}$, are distinct and $\sum_{j=i}^{i+n} p_{\nu(j)} = \sum_{j=i}^{i+n-m} p_{\nu(j)}$ for each i. We also discuss all possible finite systems of such periodic sums.

And in Sect. 4, we derive some new Ramsey theoretic results. In particular, we show that, for $n \geq m$, there is a partition $\{A_{i,k} : (i,k) \in \{0, \ldots, m-1\} \times \{0, \ldots, n-1\}\}$ of $\mathbb{N}$ such that, whenever for each (i,k), $\mathscr{B}_{i,k}$ is a finite partition of $A_{i,k}$, there exist $B_{i,k} \in \mathscr{B}_{i,k}$ and a sequence $(x_j)_{j=0}^{\infty}$ such that for every finite sequence $j_0 < \ldots < j_s$ such that $j_{t+1} \equiv j_t + 1 \pmod m$ for each $t < s$, one has $x_{j_0} + \cdots + x_{j_s} \in B_{i_0,k_0}$, where $i_0 = \nu(j_0)$ and k_0 is s if $s \leq n-1$ and $n - m + \nu(s-n)$ otherwise.

2 Construction of semigroups

Let $m \geq 1$, $n \geq 2$, and $l = m + n - 1$. For every $x \in \mathbb{N}$, supp x is a unique finite nonempty subset of $\omega = \mathbb{N} \cup \{0\}$ such that

$$x = \sum_{k \in \operatorname{supp} x} 2^k.$$

Pick an increasing sequence $I_0 \subseteq I_1 \subseteq \ldots \subseteq I_l = \omega$ of subsets of ω such that $I_i \setminus I_{i-1}$ is infinite for each $i \in \{0, 1, \ldots, l\}$ (with $I_{-1} = \emptyset$). Define a function h from $\mathbb{N}$ onto the decreasing chain $0 > 1 > \ldots > l$ of idempotents (with the operation $i * j = \max\{i, j\}$) by

$$h(x) = \min\{i \leq l : \operatorname{supp} x \subseteq I_i\} = \max\{i \leq l : (\operatorname{supp} x) \cap (I_i \setminus I_{i-1}) \neq \emptyset\}$$

and let the same letter h denote its continuous extension $\beta\mathbb{N} \to \{0, 1, \ldots, l\}$. If $x, y \in \mathbb{N}$ and $\max \operatorname{supp} x < \min \operatorname{supp} y$, then $h(x+y) = h(x) * h(y)$. It then follows (see [4, Theorem 4.21]) that for any $u \in \beta\mathbb{N}$ and $v \in \mathbb{H}$, where

$$\mathbb{H} = \bigcap_{n=0}^{\infty} \overline{2^n\mathbb{N}},$$

one has $h(u + v) = h(u) * h(v)$, in particular, the restriction of h to $\mathbb{H}$ is a homomorphism. For each $i \in \{0, 1, \ldots, l\}$, let

$$T_i = h^{-1}(\{0, 1, \ldots, i\}) \cap \mathbb{H}.$$

Then $T_0 \subseteq T_1 \subseteq \ldots \subseteq T_l = \mathbb{H}$ is an increasing sequence of closed subsemigroups of $\mathbb{H}$ such that $h(K(T_i)) = \{i\}$ for each $i \leq l$, and so $T_i \cap \overline{K(T_{i+1})} = \emptyset$ for each $i < l$ and $K(T_l) = K(\beta\mathbb{N}) \cap T_l$ [9, Lemma 3.1], in particular, all $\overline{K(T_0)}, \overline{K(T_1)}, \ldots, \overline{K(T_l)}$ are pairwise disjoint. Moreover, $h(K(\beta\mathbb{N})) = \{l\}$, and so $T_{l-1} \cap \overline{K(\beta\mathbb{N})} = \emptyset$.

To see this, let $u \in K(\beta\mathbb{N})$. Then $u + \beta\mathbb{N}$ is the minimal right ideal of $\beta\mathbb{N}$ containing u and $\beta\mathbb{N} + u$ the minimal left ideal containing u. Let v be the identity of the group $(u + \beta\mathbb{N}) \cap (\beta\mathbb{N} + u)$. Then $u = u + v$ and $v \in K(\mathbb{H})$, so $h(u) = h(u + v) = h(u) * h(v) = h(u) * l = l$.

For each $i \in \{0, 1, \ldots, l\}$, let

$$X_i = \{x \in \mathbb{N} : (\operatorname{supp} x) \cap (I_i \setminus I_{i-1}) \neq \emptyset\}.$$

Notice that for any $v \in \overline{X_i} \cap \mathbb{H}$ and $u \in \beta\mathbb{N}$, $u + v \in \overline{X_i}$, and for any $v \in \overline{X_i}$ and $w \in \mathbb{H}$, $v + w \in \overline{X_i}$.

Define $\phi_i : X_i \to \omega$ by

$$\phi_i(x) = \max((\operatorname{supp} x) \cap (I_i \setminus I_{i-1}))$$

and let the same letter ϕ_i denote its continuous extension $\overline{X_i} \to \beta\omega$. Notice that $\{2^k : k \in I_i \setminus I_{i-1}\} \subseteq X_i$ and, since $\phi_i(2^k) = k$, ϕ_i homeomorphically maps $\overline{\{2^k : k \in I_i \setminus I_{i-1}\}}$ onto $\overline{I_i \setminus I_{i-1}}$. If $x \in \mathbb{N}$, $y \in X_i$ and $\max\operatorname{supp} x < \min\operatorname{supp} y$, then $x + y \in X_i$ and $\phi_i(x + y) = \phi_i(y)$. And if $y \in X_i$, $z \in \mathbb{N} \setminus X_i$ and $\max\operatorname{supp} y < \min\operatorname{supp} z$, then $\phi_i(y + z) = \phi_i(y)$. It then follows that for any $v \in \overline{X_i} \cap \mathbb{H}$ and $u \in \beta\mathbb{N}$, $\phi_i(u + v) = \phi_i(v)$, and for any $v \in \overline{X_i}$ and $w \in \mathbb{H} \setminus \overline{X_i}$, $\phi_i(v + w) = \phi_i(v)$.

To see for example the first statement, we first note that for any $x \in \mathbb{N}$ and $v \in \overline{X_i} \cap \mathbb{H}$, $\phi_i(x + v) = \phi_i(v)$ because the continuous functions $\phi_i \circ \lambda_x$ and ϕ_i agree on $X_i \cap 2^n\mathbb{N}$, where $n = (\max\operatorname{supp} x) + 1$. Then for any $v \in \overline{X_i} \cap \mathbb{H}$ and $u \in \beta\mathbb{N}$, $\phi_i(u + v) = \phi_i(v)$ because the continuous function $\phi_i \circ \rho_v$ is constantly equal to $\phi_i(v)$ on $\mathbb{N}$.

Notice that $K(T_i) \subseteq \overline{X_i} \cap \mathbb{H}$ and $T_{i-1} \subseteq \mathbb{H} \setminus \overline{X_i}$ (with $T_{-1} = \emptyset$).

We shall construct

(i) a chain $e_0 > e_1 > \ldots > e_l$ of idempotents with $e_i \in K(T_i)$,
(ii) for each $i \in \{0, 1, \ldots, l\}$, a left zero semigroup $\{e_{i,\alpha} : \alpha < 2^{\mathfrak{c}}\} \subseteq K(T_i)$ such that $e_{i,0} = e_i$ and $e_{i,\alpha} = e_{0,\alpha} + e_i$ for all $\alpha < 2^{\mathfrak{c}}$, and

(iii) for each $i \in \{1, m+1, \ldots, l-1\}$ (for $i = 1$ if $n = 2$), a right zero semigroup $\{e_i(j) : j \in \omega\} \subseteq K(T_i)$ such that $e_i(0) = e_i$, $e_i(j) < e_{i-1}$ for all $j \in \omega$, and $\phi_i(e_i(j)) \neq \phi_i(e_i(k))$ if $j \neq k$.

Notice that (i) and (ii) imply that

$$e_{i,\alpha} + e_{j,\beta} = e_{i*j,\alpha}$$

for all $i, j \in \{0, 1, \ldots, l\}$ and $\alpha, \beta < 2^{\mathfrak{c}}$.

Indeed,

$$\begin{aligned} e_{i,\alpha} + e_{j,\beta} &= e_{0,\alpha} + e_i + e_{0,\beta} + e_j = e_{0,\alpha} + (e_i + e_0) + e_{0,\beta} + e_j \\ &= e_{0,\alpha} + e_i + (e_0 + e_{0,\beta}) + e_j = e_{0,\alpha} + e_i + e_0 + e_j \\ &= e_{0,\alpha} + e_{i*j} = e_{i*j,\alpha}. \end{aligned}$$

The construction goes by induction on $i \in \{0, 1, \ldots, l\}$.

For $i = 0$, pick an injective $2^{\mathfrak{c}}$-sequence $\{r_{0,\alpha} : \alpha < 2^{\mathfrak{c}}\}$ in $\{2^k : k \in I_0\}^*$.

Lemma 2.1 $(r_{0,\alpha} + T_l) \cap (r_{0,\beta} + T_l) = \emptyset$ *if* $\alpha \neq \beta$.

Proof Consider the function $\mathbb{N} \ni x \mapsto \min \operatorname{supp} x \in \omega$ and let θ denote its continuous extension $\beta\mathbb{N} \to \beta\omega$. If $x, y \in \mathbb{N}$ and $\max \operatorname{supp} x < \min \operatorname{supp} y$, then $\theta(x + y) = \theta(x)$. It then follows that for any $u \in \beta\mathbb{N}$ and $v \in \mathbb{H}$, $\theta(u + v) = \theta(u)$. Consequently, $\theta(r_{0,\alpha} + T_l) = \{\theta(r_{0,\alpha})\}$ and $\theta(r_{0,\beta} + T_l) = \{\theta(r_{0,\beta})\}$. Since $\theta(2^k) = k$, $\theta(r_{0,\alpha}) \neq \theta(r_{0,\beta})$, so $(r_{0,\alpha} + T_l) \cap (r_{0,\beta} + T_l) = \emptyset$. □

For every $\alpha < 2^{\mathfrak{c}}$, choose a minimal right ideal $R_{0,\alpha}$ of T_0 contained in $r_{0,\alpha} + T_0$. Pick a minimal left ideal L_0 of T_0, and for every $\alpha < 2^{\mathfrak{c}}$, let $e_{0,\alpha}$ be the identity of the group $R_{0,\alpha} \cap L_0$. By Lemma 2.1, $e_{0,\alpha} \neq e_{0,\beta}$ if $\alpha \neq \beta$. Put $e_0 = e_{0,0}$.

For $i = 1$, choose a minimal right ideal R_1 of T_1 contained in $e_0 + T_1$. Pick an injective sequence $(r_{1,j})_{j=0}^{\infty}$ in $\{2^k : k \in I_1 \setminus I_0\}^*$, and for every $j \in \omega$, choose a minimal left ideal $L_{1,j}$ of T_1 contained in $T_1 + r_{1,j} + e_0$. For every $j \in \omega$, let $e_1(j)$ be the identity of the group $R_1 \cap L_{1,j}$. Then $\phi_1(e_1(j)) = \phi_1(r_{1,j} + e_0) = \phi_1(r_{1,j})$, so ϕ_1 is injective on $\{e_1(j) : j \in \omega\}$. Since $e_1(j) \in e_0 + T_1$, one has $e_0 + e_1(j) = e_1(j)$, and since $e_1(j) \in T_1 + r_{1,j} + e_0$, one has $e_1(j) + e_0 = e_1(j)$, so $e_1(j) < e_0$. Put $e_1 = e_1(0)$. For every $\alpha < 2^{\mathfrak{c}}$, put $e_{1,\alpha} = e_{0,\alpha} + e_1$. Then $e_{1,\alpha} + e_{1,\beta} = e_{0,\alpha} + e_1 + e_{0,\beta} + e_1 = e_{0,\alpha} + (e_1 + e_0) + e_{0,\beta} + e_1 = e_{0,\alpha} + e_1 + (e_0 + e_{0,\beta}) + e_1 = e_{0,\alpha} + e_1 + e_0 + e_1 = e_{0,\alpha} + e_1 = e_{1,\alpha}$, so $\{e_{1,\alpha} : \alpha < 2^{\mathfrak{c}}\}$ is a left zero semigroup (in $K(T_1)$). Since $e_{1,\alpha} = e_{0,\alpha} + e_1 \in r_{0,\alpha} + T_0 + e_1 \in r_{0,\alpha} + T_1$, by Lemma 2.1, $e_{1,\alpha} \neq e_{1,\beta}$ if $\alpha \neq \beta$.

For $i \in \{2, \ldots, m\}$, pick a minimal right ideal R_i of T_i contained in $e_{i-1} + T_i$ and a minimal left ideal L_i of T_i contained in $T_i + e_{i-1}$ and let e_i be the identity of the group $R_i \cap L_i$. For every $\alpha < 2^{\mathfrak{c}}$, let $e_{i,\alpha} = e_{0,\alpha} + e_i$. Then $\{e_{l,\alpha} : \alpha < 2^{\mathfrak{c}}\}$ is a left zero semigroup and $e_{i,\alpha} \neq e_{i,\beta}$ if $\alpha \neq \beta$.

For $i \in \{m+1, \ldots, l-1\}$ (for $n \geq 3$), choose a minimal right ideal R_i of T_i contained in $e_{i-1} + T_i$. Pick an injective sequence $(r_{i,j})_{j=0}^{\infty}$ in $\{2^k : k \in I_i \setminus I_{i-1}\}^*$, and for every $j \in \omega$, choose a minimal left ideal $L_{i,j}$ of T_i contained in $T_i + r_{i,j} + e_{i-1}$, and let $e_i(j)$ be the identity of the group $R_i \cap L_{i,j}$. Then $\phi_i(e_i(j)) = \phi_i(r_{i,j} + e_0) =$

$\phi_i(r_{i,j})$, so ϕ_i is injective on $\{e_i(j) : j \in \omega\}$, and $e_i(j) < e_{i-1}$ for all j. Put $e_i = e_i(0)$. For every $\alpha < 2^{\mathfrak{c}}$, put $e_{i,\alpha} = e_{0,\alpha} + e_i$. Then $\{e_{i,\alpha} : \alpha < 2^{\mathfrak{c}}\}$ a left zero semigroup and $e_{i,\alpha} \neq e_{i,\beta}$ if $\alpha \neq \beta$.

For $i = l$, pick a minimal right ideal R_l of T_l contained in $e_{l-1} + T_l$ and a minimal left ideal L_l of T_l contained in $T_l + e_{l-1}$ and let e_l be the identity of the group $R_l \cap L_l$. For every $\alpha < 2^{\mathfrak{c}}$, put $e_{l,\alpha} = e_{0,\alpha} + e_l$.

Now for each $\alpha < 2^{\mathfrak{c}}$, let

$$D_{l-1,\alpha} = \begin{cases} \{e_{l,\alpha} + e_1(j) : j \in \mathbb{N}\} & \text{if } n = 2 \\ \{e_{l,\alpha} + e_{l-1}(j) : j \in \mathbb{N}\} & \text{if } n \geq 3. \end{cases}$$

Since $\phi_1(e_{l,\alpha} + e_1(j)) = \phi_1(e_1(j))$ and $\phi_{l-1}(e_{l,\alpha} + e_{l-1}(j)) = \phi_{l-1}(e_{l-1}(j))$, we have that if $n = 2$, ϕ_1 is injective on $D_{l-1,\alpha}$ (and so $|\phi_1(\overline{D_{l-1,\alpha}})| = 2^{\mathfrak{c}}$) and if $n \geq 3$, ϕ_{l-1} is injective on $D_{l-1,\alpha}$ (and so $|\phi_{l-1}(\overline{D_{l-1,\alpha}})| = 2^{\mathfrak{c}}$). For every $\alpha < 2^{\mathfrak{c}}$, pick inductively $q_{l-1,\alpha} \in \overline{D_{l-1,\alpha}} \setminus D_{l-1,\alpha}$ such that

if $n = 2$, $\phi_1(q_{l-1,\alpha}) \neq \phi_1(e_1)$ and all $\phi_1(q_{l-1,\alpha})$ are distinct, and

if $n \geq 3$, $\phi_{l-1}(q_{l-1,\alpha}) \neq \phi_{l-1}(e_{l-1})$ and all $\phi_{l-1}(q_{l-1,\alpha})$ are distinct.

Then by downward induction on $i \in \{m + 1, \ldots, l - 2\}$ (for $n \geq 4$), for each $\alpha < 2^{\mathfrak{c}}$, let

$$D_{i,\alpha} = \{e_{i+1,\alpha} + q_{i+1,\alpha} + e_i(j) : j \in \mathbb{N}\}.$$

Since $\phi_i(e_{i+1,\alpha} + q_{i+1,\alpha} + e_i(j)) = \phi_i(e_i(j))$, ϕ_i is injective on $D_{i,\alpha}$. For every $\alpha < 2^{\mathfrak{c}}$, pick inductively $q_{i,\alpha} \in \overline{D_{i,\alpha}} \setminus D_{i,\alpha}$ such that

$\phi_i(q_{i,\alpha}) \neq \phi_i(e_i)$ and all $\phi_i(q_{i,\alpha})$ are distinct.

For $i = m$ (for $n \geq 3$), for each $\alpha < 2^{\mathfrak{c}}$, let

$$D_{m,\alpha} = \{e_{m+1,\alpha} + q_{m+1,\alpha} + e_1(j) : j \in \mathbb{N}\}.$$

Since $\phi_1(e_{m+1,\alpha} + q_{m+1,\alpha} + e_1(j)) = \phi_1(e_1(j))$, ϕ_1 is injective on $D_{m,\alpha}$. For every $\alpha < 2^{\mathfrak{c}}$, pick inductively $q_{m,\alpha} \in \overline{D_{m,\alpha}} \setminus D_{m,\alpha}$ such that

$\phi_1(q_{m,\alpha}) \neq \phi_1(e_m)$ and all $\phi_1(q_{m,\alpha})$ are distinct.

Since $e_{l,\alpha} \in K(\beta\mathbb{N})$ and $\overline{K(\beta\mathbb{N})}$ is an ideal of $\beta\mathbb{N}$ [4, Theorem 4.44], we have by downward induction that for each $i \in \{m, \ldots, l - 1\}$, $D_{i,\alpha} \subseteq \overline{K(\beta\mathbb{N})}$ and $q_{i,\alpha} \in \overline{K(\beta\mathbb{N})}$.

For each $s \in \{0, 1, \ldots, l\}$, $e_{l,\alpha} = e_{s,\alpha} + e_{l,\alpha}$ and $e_{s,\alpha} \in \overline{X_s}$, so $e_{l,\alpha} \in \overline{X_s}$. It then follows by downward induction that for each $i \in \{m, \ldots, l - 1\}$, $D_{i,\alpha} \subseteq \overline{X_s} \cap \mathbb{H}$ and $q_{i,\alpha} \in \overline{X_s} \cap \mathbb{H}$. We also have that ϕ_1 is injective on $D_{m,\alpha}$ and for each $i \in \{m + 1, \ldots, l - 1\}$ (for $n \geq 3$), ϕ_i is injective on $D_{i,\alpha}$.

An ultrafilter $q \in \mathbb{N}^*$ is *right cancelable (in $\beta\mathbb{N}$)* if the right translation of $\beta\mathbb{N}$ by q is injective. An ultrafilter $q \in \mathbb{N}^*$ is right cancelable if and only if $q \notin \mathbb{N}^* + q$ [4, Theorem 8.18]. From the next lemma we obtain that all $q_{i,\alpha}$, where $i \in \{m, \ldots, l-1\}$ and $\alpha < 2^{\mathfrak{c}}$, are right cancelable.

Lemma 2.2 *Let $i \in \{0, 1, \ldots, l\}$, let D be a countable subset of $\overline{X_i} \cap \mathbb{H}$, and suppose that ϕ_i is injective on D. Then every $q \in \overline{D} \setminus D$ is right cancelable.*

Proof This is [10, Lemma 5]. □

The next lemma gives us relations between $q_{i,\alpha}$ and $e_{s,\beta}$.

Lemma 2.3 *For any $\alpha, \beta < 2^{\mathfrak{c}}$,*

(1) $q_{l-1,\alpha} + e_{l-1,\beta} = e_{l,\alpha}$,
(2) if $n = 2$, then for each $s \in \{1, \ldots, l\}$, $q_{l-1,\alpha} + e_{s,\beta} = e_{l,\alpha}$,
(3) if $n \geq 3$, then for each $i \in \{m+1, \ldots, l-1\}$, $q_{i,\alpha} + e_{i-1,\beta} = q_{i,\alpha}$,
(4) if $n \geq 3$, then for each $i \in \{m, \ldots, l-2\}$, $q_{i,\alpha} + e_{i,\beta} = e_{i+1,\alpha} + q_{i+1,\alpha}$, and
(5) if $n \geq 3$, then for each $s \in \{1, \ldots, m\}$, $q_{m,\alpha} + e_{s,\beta} = e_{m+1,\alpha} + q_{m+1,\alpha}$.

Proof (1) For $n \geq 3$, $(e_{l,\alpha} + e_{l-1}(j)) + e_{l-1,\beta} = e_{l,\alpha} + (e_{l-1}(j) + e_{l-1,\beta}) = e_{l,\alpha} + ((e_{l-1}(j) + e_{l-2,0}) + e_{l-1,\beta}) = e_{l,\alpha} + (e_{l-1}(j) + (e_{l-2,0} + e_{l-1,\beta})) = e_{l,\alpha} + e_{l-1}(j) + e_{l-1,0} = e_{l,\alpha} + e_{l-1,0} = e_{l,\alpha}$, and since $\rho_{e_{l-1,\beta}}$ is constantly equal to $e_{l,\alpha}$ on $D_{l-1,\alpha}$, $\rho_{e_{l-1,\beta}}(q_{l-1,\alpha}) = e_{l,\alpha}$, so $q_{l-1,\alpha} + e_{l-1,\beta} = e_{l,\alpha}$. The case $n = 2$ is included in (2).

(2) $(e_{l,\alpha} + e_1(j)) + e_{s,\beta} = e_{l,\alpha} + (e_1(j) + e_{0,0}) + e_{s,\beta} = e_{l,\alpha} + e_1(j) + (e_{0,0} + e_{s,\beta}) = e_{l,\alpha} + e_1(j) + e_{s,0} = e_{l,\alpha} + e_1(j) + (e_{1,0} + e_{s,0}) = e_{l,\alpha} + (e_1(j) + e_{1,0}) + e_{s,0} = e_{l,\alpha} + e_{1,0} + e_{s,0} = e_{l,\alpha} + e_{s,0} = e_{l,\alpha}$.

(3) For $i = l-1$, $(e_{l,\alpha} + e_{l-1}(j)) + e_{l-2,\beta} = e_{l,\alpha} + (e_{l-1}(j) + e_{l-2,0}) + e_{l-2,\beta} = e_{l,\alpha} + e_{l-1}(j) + (e_{l-2,0} + e_{l-2,\beta}) = e_{l,\alpha} + e_{l-1}(j) + e_{l-2,0} = e_{l,\alpha} + e_{l-1}(j)$, and since $\rho_{e_{l-2,\beta}}$ is the identity on $D_{l-1,\alpha}$, $\rho_{e_{l-2,\beta}}(q_{l-1,\alpha}) = q_{l-1,\alpha}$, so $q_{l-1,\alpha} + e_{l-2,\beta} = q_{l-1,\alpha}$. For $i \leq l-2$, $(e_{i+1,\alpha} + q_{i+1,\alpha} + e_i(j)) + e_{i-1,\beta} = e_{i+1,\alpha} + q_{i+1,\alpha} + (e_i(j) + e_{i-1,0}) + e_{i-1,\beta} = e_{i+1,\alpha} + q_{i+1,\alpha} + e_i(j) + (e_{i-1,0} + e_{i-1,\beta}) = e_{i+1,\alpha} + q_{i+1,\alpha} + e_i(j) + e_{i-1,0} = e_{i+1,\alpha} + q_{i+1,\alpha} + e_i(j)$.

(4) For $i \geq m+1$, $(e_{i+1,\alpha} + q_{i+1,\alpha} + e_i(j)) + e_{i,\beta} = e_{i+1,\alpha} + q_{i+1,\alpha} + (e_i(j) + e_{i-1,0}) + e_{i,\beta} = e_{i+1,\alpha} + q_{i+1,\alpha} + e_i(j) + (e_{i-1,0} + e_{i,\beta}) = e_{i+1,\alpha} + q_{i+1,\alpha} + e_i(j) + e_{i,0} = e_{i+1,\alpha} + q_{i+1,\alpha} + e_{i,0} = e_{i+1,\alpha} + q_{i+1,\alpha}$ because $q_{i+1,\alpha} + e_{i,0} = q_{i+1,\alpha}$ by (3). The case $i = m$ is included in (5).

(5) $e_{m+1,\alpha} + q_{m+1,\alpha} + e_1(j) + e_{s,\beta} = e_{m+1,\alpha} + q_{m+1,\alpha} + (e_1(j) + e_{0,0}) + e_{s,\beta} = e_{m+1,\alpha} + q_{m+1,\alpha} + e_1(j) + (e_{0,0} + e_{s,\beta}) = e_{m+1,\alpha} + q_{m+1,\alpha} + e_1(j) + e_{s,0} = e_{m+1,\alpha} + q_{m+1,\alpha} + e_1(j) + (e_{1,0} + e_{s,0}) = e_{m+1,\alpha} + q_{m+1,\alpha} + (e_1(j) + e_{1,0}) + e_{s,0} = e_{m+1,\alpha} + q_{m+1,\alpha} + e_{1,0} + e_{s,0} = e_{m+1,\alpha} + q_{m+1,\alpha} + e_{s,0} = e_{m+1,\alpha} + q_{m+1,\alpha}$ because by (3), $q_{m+1,\alpha} + e_{s,0} = (q_{m+1,\alpha} + e_{m,0}) + e_{s,0} = q_{m+1,\alpha} + (e_{m,0} + e_{s,0}) = q_{m+1,\alpha} + e_{m,0} = q_{m+1,\alpha}$.

□

From Lemma 2.3 we obtain that for each $i \in \{m, \ldots, l-1\}$ and each $s \in \{1, \ldots, l\}$,

$$q_{i,\alpha} + e_{s,\beta} = \begin{cases} e_{l,\alpha} & \text{if } m = l-1 \\ e_{m+1,\alpha} + q_{m+1,\alpha} & \text{if } s \leq i = m \leq l-2 \\ q_{i,\alpha} \text{ if } i \geq m+1 & \text{and } s < i \\ e_{s+1,\alpha} + q_{s+1,\alpha} & \text{if } i \leq s \leq l-2 \\ e_{l,\alpha} & \text{if } l-1 \leq s \leq l. \end{cases}$$

Indeed, the first and the second cases are Lemma 2.3(2) and Lemma 2.3(5) respectively.

In the third case, using Lemma 2.3(3), $q_{i,\alpha} + e_{s,\beta} = (q_{i,\alpha} + e_{i-1,0}) + e_{s,\beta} = q_{i,\alpha} + (e_{i-1,0} + e_{s,\beta}) = q_{i,\alpha} + e_{i-1,0} = q_{i,\alpha}$.

The fourth case for $i = s$ is Lemma 2.3(4). Then by downward induction on $i \in \{m, m+1, s\}$, for $i < s$, $q_{i,\alpha} + e_{s,\beta} = q_{i,\alpha} + (e_{i,\beta} + e_{s,\beta}) = (q_{i,\alpha} + e_{i,\beta}) + e_{s,\beta} = e_{i+1,\alpha} + q_{i+1,\alpha} + e_{s,\beta} = e_{i+1,\alpha} + (q_{i+1,\alpha} + e_{s,\beta}) = e_{i+1,\alpha} + e_{s+1,\alpha} + q_{s+1,\alpha} = e_{s+1,\alpha} + q_{s+1,\alpha}$.

The fifth case for $i = s = l - 1$ is Lemma 2.3(1). For $i \leq l - 2$, using the already established fourth case, $q_{i,\alpha} + e_{l-1,\beta} = q_{i,\alpha} + e_{l-2,\beta} + e_{l-1,\beta} = e_{l-1,\alpha} + q_{l-1,\alpha} + e_{l-1,\beta} = e_{l-1,\alpha} + e_{l,\alpha} = e_{l,\alpha}$. Then for each i, $q_{i,\alpha} + e_{l,\beta} = q_{i,\alpha} + e_{l-1,\beta} + e_{l,\beta} = e_{l,\alpha} + e_{l,\beta} = e_{l,\alpha}$.

Now consider the subsemigroup Q of $\mathbb{H}$ generated algebraically by the elements $e_{i,\alpha}$ and $q_{s,\beta}$, where $i \in \{1, \ldots, l\}$, $s \in \{m, \ldots, l-1\}$, and $\alpha, \beta < 2^{\mathfrak{c}}$ (we have interchanged i and s, and so are α and β). It follows from the formula above that Q consists of elements of the form

$$e_{i,\alpha},\ q_{s_1,\beta_1} + \ldots + q_{s_t,\beta_t},\ \text{and } e_{i,\alpha} + q_{s_1,\beta_1} + \ldots + q_{s_t,\beta_t},$$

where $i \in \{1, \ldots, l\}$, $t \in \mathbb{N}$, $s_1, \ldots, s_t \in \{m, \ldots, l-1\}$, and $\alpha, \beta_1, \ldots, \beta_t < 2^{\mathfrak{c}}$.

Lemma 2.4 *All elements*

$$e_{i,\alpha},\ q_{s_1,\beta_1} + \ldots + q_{s_t,\beta_t},\ \textit{and } e_{i,\alpha} + q_{s_1,\beta_1} + \ldots + q_{s_t,\beta_t},$$

where $i \in \{1, \ldots, l\}$, $t \in \mathbb{N}$, $s_1, \ldots, s_t \in \{m, \ldots, l-1\}$, *and* $\alpha, \beta_1, \ldots, \beta_t < 2^{\mathfrak{c}}$, *are distinct.*

Proof Assume on the contrary that some two distinct expressions represent the same element. Then canceling the equality by q-s we arrive at one of the following cases:

(1) $u + q_{i,\alpha} = v + q_{s,\beta}$ for some $u, v \in \beta\mathbb{N}$ and $(i, \alpha) \neq (s, \beta)$,
(2) $u + q_{i,\alpha} = q_{s,\beta}$ for some $u \in \beta\mathbb{N}$,
(3) $u + q_{i,\alpha} = e_{s,\beta}$ for some $u \in \beta\mathbb{N}$,
(4) $e_{i,\alpha} = e_{s,\beta}$ with $(i, \alpha) \neq (s, \beta)$.

The last one is obviously impossible.

In (1), we have that $\phi_i(q_{i,\alpha}) = \phi_i(u + q_{i,\alpha}) = \phi_i(v + q_{s,\beta}) = \phi_i(q_{s,\beta})$. If $i = s$, then $\alpha \neq \beta$ and $\phi_i(q_{i,\alpha}) = \phi_i(q_{i,\beta})$, a contradiction. If $i \neq s$, say $i < s$, then $\phi_i(q_{s,\beta}) = \phi_i(q_{s,\beta} + e_{i,0}) = \phi_i(e_{i,0})$ and $\phi_i(q_{i,\alpha}) \neq \phi_i(e_{i,0})$, again a contradiction.

In (2), since $q_{s,\beta}$ is right cancelable, one has $s \neq i$. Suppose $i < s$. Then $\phi_i(q_{i,\alpha}) = \phi_i(q_{s,\beta})$. But $\phi_i(q_{s,\beta}) = \phi_i(e_{i,0})$ (as in (1)) and $\phi_i(q_{i,\alpha}) \neq \phi_i(e_{i,0})$, a contradiction. The case $s < i$ is essentially the same, since applying ϕ_s to $q_{s,\beta} = u + q_{i,\alpha}$ gives us $\phi_s(q_{s,\beta}) = \phi_s(q_{i,\alpha})$.

In (3), since $q_{i,\alpha} \in \overline{K(\beta\mathbb{N})}$, $e_1, \ldots, e_{l-1} \in T_{l-1}$ and $T_{l-1} \cap \overline{K(\beta\mathbb{N})} = \emptyset$, one has $s = l$. Then $\phi_i(q_{i,\alpha}) = \phi_i(e_{l,\beta})$. But $\phi_i(e_{l,\beta}) = \phi_i(e_{l,\beta} + e_{i,0}) = \phi_i(e_{i,0})$ and $\phi_i(q_{i,\alpha}) \neq \phi_i(e_{i,0})$, a contradiction. □

From Lemma 2.4 we obtain that

Corollary 2.5 *As an abstract semigroup, Q is generated by the chain of left zero semigroups $\{e_{i,\alpha} : \alpha < 2^{\mathfrak{c}}\}$, where $i \in \{1, \ldots, l\}$ and for each $i \leq l-1$, $e_{i,\alpha} + e_{i+1,\beta} = e_{i+1,\alpha}$ and $e_{i+1,\beta} + e_{\alpha,i} = e_{i+1,\beta}$, and elements $q_{s,\beta}$, where $s \in \{m, \ldots, l-1\}$ and $\beta < 2^{\mathfrak{c}}$, with the defining relations (1)-(5) in Lemma 2.3.*

Now consider the subsemigroup P of Q generated by the elements

$$p_{s,\alpha,\beta} = e_{s,\alpha} + q_{m,\beta},$$

where $s \in \{1, \ldots, m\}$ and $\alpha, \beta < 2^{\mathfrak{c}}$.

Lemma 2.6 *For all $i \geq 2$, $s_1, \ldots, s_i \in \{1, \ldots, m\}$, and $\alpha_1, \beta_1 \ldots, \alpha_i, \beta_i < 2^{\mathfrak{c}}$,*

$$p_{s_i,\alpha_i,\beta_i} + \ldots + p_{s_1,\alpha_1,\beta_1} = \begin{cases} e_{m+i-1,\alpha_i} + q_{m+i-1,\beta_i} + \ldots + q_{m,\beta_1} & \text{if } i \leq n-1 \\ e_{l,\alpha_i} + q_{l-1,\beta_{n-1}} + \ldots + q_{m,\beta_1} & \text{otherwise.} \end{cases}$$

Proof We use Lemma 2.3. If $n = 2$, then

$$\begin{aligned} p_{s_2,\alpha_2,\beta_2} + p_{s_1,\alpha_1,\beta_1} &= e_{s_2,\alpha_2} + q_{m,\beta_2} + e_{s_1,\alpha_1} + q_{m,\beta_1} \\ &= e_{s_2,\alpha_2} + (q_{m,\beta_2} + e_{s_1,\alpha_1}) + q_{m,\beta_1} \\ &= e_{s_2,\alpha_2} + e_{l,\beta_2} + q_{m,\beta_1} = e_{l,\alpha_2} + q_{m,\beta_1} \text{ and} \\ p_{s_3,\alpha_3,\beta_3} + p_{s_2,\alpha_2,\beta_2} + p_{s_1,\alpha_1,\beta_1} &= (p_{s_3,\alpha_3,\beta_3} + p_{s_2,\alpha_2,\beta_2}) + p_{s_1,\alpha_1,\beta_1} \\ &= e_{l,\alpha_3} + q_{m,\beta_2} + e_{s_1,\alpha_1} + q_{m,\beta_1} \\ &= e_{l,\alpha_3} + (q_{m,\beta_2} + e_{s_1,\alpha_1}) + q_{m,\beta_1} \\ &= e_{l,\alpha_3} + e_{l,\beta_2} + q_{m,\beta_1} = e_{l,\alpha_3} + q_{m,\beta_1}. \end{aligned}$$

Let $n \geq 3$. We first notice that for each $j \in \{1, \ldots, n-2\}$,

$$\begin{aligned} q_{m+j-1,\beta_j} + \ldots + q_{m,\beta_1} + e_{s,\alpha} &= e_{m+j,\beta_j} + q_{m+j,\beta_j} + \ldots + q_{m+1,\beta_1} \text{ and} \\ q_{l-1,\beta_{n-1}} + \ldots + q_{m,\beta_1} + e_{s,\alpha} &= e_{l,\beta_{n-1}} + q_{l-1,\beta_{n-2}} + \ldots + q_{m+1,\beta_1}. \end{aligned}$$

Indeed, inductively, $q_{m,\beta_1} + e_{s,\alpha} = e_{m+1,\beta_1} + q_{m+1,\beta_1}$, and for $j \geq 2$,

$$\begin{aligned} q_{m+j-1,\beta_j} + \ldots + q_{m,\beta_1} + e_{s,\alpha} &= q_{m+j-1,\beta_j} + (q_{m+j-2,\beta_{j-1}} + \ldots + q_{m,\beta_1} + e_{s,\alpha}) \\ &= q_{m+j-1,\beta_j} + e_{m+j-1,\beta_{j-1}} + q_{m+j-1,\beta_{j-1}} \\ &\quad + \ldots + q_{m+1,\beta_1} \\ &= e_{m+j,\beta_j} + q_{m+j,\beta_j} + q_{m+j-1,\beta_{j-1}} + \ldots \\ &\quad + q_{m+1,\beta_1}, \end{aligned}$$

and then

$$\begin{aligned} q_{l-1,\beta_{n-1}} + \ldots + q_{m,\beta_1} + e_{s,\alpha} &= q_{l-1,\beta_{n-1}} + (q_{l-2,\beta_{n-2}} + \ldots + q_{m,\beta_1} + e_{s,\alpha}) \\ &= q_{l-1,\beta_{n-1}} + e_{l-1,\beta_{n-2}} + q_{l-1,\beta_{n-2}} + \ldots + q_{m+1,\beta_1} \\ &= e_{l,\beta_{n-1}} + q_{l-1,\beta_{n-2}} + \ldots + q_{m+1,\beta_1}. \end{aligned}$$

Now by induction on $i \in \{2, \ldots, n-1\}$,

$$\begin{aligned} p_{s_2,\alpha_2,\beta_2} + p_{s_1,\alpha_1,\beta_1} &= e_{s_2,\alpha_2} + q_{m,\beta_2} + e_{s_1,\alpha_1} + q_{m,\alpha_1} = e_{s_2,\alpha_2} \\ &\quad + (q_{m,\beta_2} + e_{s_1,\alpha_1}) + q_{m,\beta_1} \\ &= e_{s_2,\alpha_2} + e_{m+1,\beta_2} + q_{m+1,\beta_2} + q_{m,\beta_1} = e_{m+1,\alpha_2} \\ &\quad + q_{m+1,\beta_2} + q_{m,\beta_1}, \end{aligned}$$

and for $i \geq 2$,

$$\begin{aligned} p_{s_i,\alpha_i,\beta_i} + \ldots + p_{s_1,\alpha_1,\beta_1} &= (p_{s_i,\alpha_i,\beta_i} + \ldots + p_{s_2,\alpha_2,\beta_2}) + p_{s_1,\alpha_1,\beta_1} \\ &= e_{m+i-2,\alpha_i} + q_{m+i-2,\beta_i} + \ldots + q_{m,\beta_2} + e_{s_1,\alpha_1} + q_{m,\beta_1} \\ &= e_{m+i-2,\alpha_i} + e_{m+i-1,\beta_i} + q_{m+i-1,\beta_i} + \ldots \\ &\quad + q_{m+1,\beta_2} + q_{m,\beta_1} \\ &= e_{m+i-1,\alpha_i} + q_{m+i-1,\beta_i} + \ldots + q_{m,\beta_1}, \end{aligned}$$

and then

$$\begin{aligned} p_{s_n,\alpha_n,\beta_n} + \ldots + p_{s_1,\alpha_1,\beta_1} &= (p_{s_n,\alpha_n,\beta_n} + \ldots + p_{s_2,\alpha_2,\beta_2}) + p_{s_1,\alpha_1,\beta_1} \\ &= e_{l-1,\alpha_n} + q_{l-1,\beta_n} + \ldots + q_{m,\beta_2} + e_{s_1,\alpha_1} + q_{m,\beta_1} \\ &= e_{l-1,\alpha_n} + e_{l,\beta_n} + q_{l-1,\beta_{n-1}} + \ldots \\ &\quad + q_{m+1,\beta_2} + q_{m,\beta_1} \\ &= e_{l,\alpha_n} + q_{l-1,\beta_{n-1}} + \ldots + q_{m,\beta_1} \end{aligned}$$

and

$$\begin{aligned} p_{s_{n+1},\alpha_{n+1},\beta_{n+1}} + \ldots + p_{s_1,\alpha_1,\beta_1} &= (p_{s_{n+1},\alpha_{n+1},\beta_{n+1}} + \ldots + p_{s_2,\alpha_2,\beta_2}) + p_{s_1,\alpha_1,\beta_1} \\ &= e_{l,\alpha_{n+1}} + q_{l-1,\beta_n} + \ldots + q_{m,\beta_2} + e_{s_1,\alpha_1} + q_{m,\beta_1} \\ &= e_{l,\alpha_{n+1}} + e_{l,\beta_n} + q_{l-1,\beta_{n-1}} + \ldots \\ &\quad + q_{m+1,\beta_2} + q_{m,\beta_1} \\ &= e_{l,\alpha_{n+1}} + q_{l-1,\beta_{n-1}} + \ldots + q_{m,\beta_1}. \end{aligned}$$

□

It follows from Lemma 2.6 that the subsemigroup P consists of the elements

$$p_{s,\alpha,\beta},\ e_{m+i-1,\alpha} + q_{m+i-1,\beta_i} + \ldots + q_{m,\beta_1},\ \text{and}\ e_{l,\alpha} + q_{l-1,\beta_{n-1}} + \ldots + q_{m,\beta_1},$$

where $s \in \{1, \ldots, m\}, 2 \le i \le n-1$, and $\alpha, \beta_1 \ldots, \beta_{n-1} < 2^{\mathfrak{c}}$, and by Lemma 2.4, all these elements are distinct. Notice that the elements $e_{l,\alpha} + q_{l-1,\beta_{n-1}} + \ldots + q_{m,\beta_1}$ form $K(P)$. Since all $q_{j,\beta}$ are in $\overline{K(\beta\mathbb{N})}$, $P \subseteq \overline{K(\beta\mathbb{N})}$, and since $e_{l,\alpha} \in K(\beta\mathbb{N})$, $K(P) \subseteq K(\beta\mathbb{N})$. Also notice that the subsemigroup generated by $p_{s_1,\alpha_1,\beta_1}, \ldots, p_{s_i,\alpha_i,\beta_i}$ is finite. It then follows that P is locally finite, that is, every finitely generated subsemigroup is finite.

Given cardinals $\kappa \ge 1$ and $\lambda \ge 1$ and integers $m \ge 1$ and $n \ge 2$, let $S(\kappa, \lambda, m, n)$ denote the semigroup whose elements are the words $s\alpha\beta, \alpha\beta_i \ldots \beta_1$, and $*\alpha\beta_{n-1} \ldots \beta_1$, where $s \in \{1, \ldots, m\}, 2 \le i \le n-1, \alpha \in \kappa$, and $\beta, \beta_1, \ldots, \beta_{n-1} \in \lambda$, and defining relations are, for $j \ge 2$,

$$s_j\alpha_j\beta_j + \ldots + s_1\alpha_1\beta_1 = \begin{cases} \alpha_j\beta_j \ldots \beta_1 & \text{if } j \le n-1 \\ *\alpha_j\beta_{n-1} \ldots \beta_1 & \text{otherwise,} \end{cases}$$

so $\alpha\beta_i \ldots \beta_1 = 1\alpha\beta_i + \ldots + 1\alpha\beta_1$, and $*\alpha\beta_{n-1} \ldots \beta_1 = 1\alpha\beta_{n-1} + 1\alpha\beta_{n-1} + \ldots + 1\alpha\beta_1$. If $m = 1$, we write $\alpha\beta$ instead of $1\alpha\beta$.

It is easy to see that the mapping $g : P \to S(2^{\mathfrak{c}}, 2^{\mathfrak{c}}, m, n)$ defined by

$$g(p_{s,\alpha,\beta}) = s\alpha\beta, \; g(e_{m+i-1,\alpha} + q_{m+i-1,\beta_k} + \ldots + q_{m,\beta_1}) = \alpha\beta_i \ldots \beta_1, \text{ and}$$
$$g(e_{m+n-1,\alpha} + q_{m+n-2,\beta_{n-1}} + \ldots + q_{m,\beta_1}) = *\alpha\beta_{n-1} \ldots \beta_1$$

is an isomorphism.

We thus have proved the following result.

Theorem 2.7 *Let $m \ge 1$ and $n \ge 2$ and let $S = S(2^{\mathfrak{c}}, 2^{\mathfrak{c}}, m, n)$. There is an isomorphic embedding $\varepsilon : S \to \mathbb{H}$. Furthermore, ε can be chosen so that $\varepsilon(S) \subseteq \overline{K(\beta\mathbb{N})}$ and $\varepsilon(K(S)) \subseteq K(\beta\mathbb{N})$.*

For each $(\alpha, \beta) \in \kappa \times \lambda$, the subsemigroup of $S(\kappa, \lambda, m, n)$ consisting of the elements $s\alpha\beta$, where $s \in \{1, \ldots, m\}$, and $\alpha\beta\beta, \ldots, \alpha\underbrace{\beta \ldots \beta}_{n-1}, *\alpha\underbrace{\beta \ldots \beta}_{n-1}$ is isomorphic to the semigroup $C_{m,n}$. The semigroup $S(\kappa, 1, m, n)$ consists of the elements $s\alpha 0$ and

$$\alpha 00, \ldots, \alpha\underbrace{0 \ldots 0}_{n-1}, *\alpha\underbrace{0 \ldots 0}_{n-1},$$

where $s \in \{1, \ldots, m\}$ and $\alpha \in \kappa$, and is isomorphic to the direct product of $C_{m,n}$ and the left zero semigroup κ. The semigroup $S(\kappa, \lambda, m, 2)$ consists of the elements $s\alpha\beta$ and $*\alpha\beta$, where $s \in \{1, \ldots, m\}$ and $(\alpha, \beta) \in \kappa \times \lambda$, and is isomorphic to the direct product of $C_{m,2}$ (the m-element null semigroup) and the rectangular band $\kappa \times \lambda$.

Now consider the subsemigroup T of $S = S(\kappa, \kappa, 1, n)$ generated by the elements $\beta\beta$, where $\beta \in \kappa$. Since

$$\beta_j\beta_j + \ldots + \beta_1\beta_1 = \begin{cases} \beta_j\beta_j \ldots \beta_1 & \text{if } j \le n-1 \\ *\beta_j\beta_{n-1} \ldots \beta_1 & \text{otherwise,} \end{cases}$$

T consists of the words $\beta_i\beta_i\ldots\beta_1$ and $*\alpha\beta_{n-1}\ldots\beta_1$, where $1\le i\le n-1$ and $\alpha,\beta_1,\ldots,\beta_{n-1}\in\kappa$. Notice that $K(T)=K(S)$.

Given a cardinal $\kappa\ge 1$ and an integer $n\ge 2$, let $F(\kappa,n)$ denote the semigroup whose elements are the words $\beta_i\ldots\beta_1$, where $1\le i\le n$ and $\beta_1,\ldots,\beta_i\in\kappa$, and defining relations are

$$\beta_j+\ldots+\beta_1=\begin{cases}\beta_j\ldots\beta_1 & \text{if } j\le n\\ \beta_j\beta_{n-1}\ldots\beta_1 & \text{otherwise,}\end{cases}$$

so the operation of $F(\kappa,n)$ is defined by

$$\beta_{i+t}\ldots\beta_{i+1}+\beta_i\ldots\beta_1=\begin{cases}\beta_{i+t}\ldots\beta_1 & \text{if } i+t\le n\\ \beta_{i+t}\beta_{n-1}\ldots\beta_1 & \text{otherwise.}\end{cases}$$

It is easy to see that the mapping $f:T\to F(\kappa,n)$ defined by

$$f(\beta_i\beta_i\ldots\beta_1)=\beta_i\ldots\beta_1 \text{ and } f(*\alpha\beta_{n-1}\ldots\beta_1)=\alpha\beta_{n-1}\ldots\beta_1$$

is an isomorphism.

Thus, we obtain from Theorem 2.7 the following result.

Theorem 2.8 *Let $n\ge 2$ and let $F=F(2^{\mathfrak{c}},n)$. There is an isomorphic embedding $\epsilon:F\to\mathbb{H}$. Furthermore, ϵ can be chosen so that $\epsilon(F)\subseteq\overline{K(\beta\mathbb{N})}$ and $\epsilon(K(F))\subseteq K(\beta\mathbb{N})$.*

The semigroup $F(\kappa,n)$ is generated by the 1-letter words β, where $\beta\in\kappa$, each of which is an element of order n and each $m\ge 1$ of which generate a subsemigroup of cardinality $m^n+m^{n-1}+\ldots+m$.

3 Periodic sums systems

Let $m\ge 2$ and define $\nu=\nu_m:\omega\to\{0,\ldots,m-1\}$ by $\nu(k)\equiv k \pmod m$. Given a sequence $p_0,\ldots,p_{m-1}$ in an additive semigroup, the *periodic sums* are sums of the form $\sum_{j=i}^{i+k}p_{\nu(j)}$, where $i\in\{0,\ldots,m-1\}$ and $k\ge 0$, and $(\sum_{j=i}^{i+k}p_{\nu(j)})_{k=0}^{\infty}$ is the *sequence of periodic sums with initial term* p_i. Suppose that $\{\sum_{j=i}^{i+k}p_{\nu(j)}:k\ge 0\}$ is finite. Then $\sum_{j=i}^{i+m-1}p_{\nu(j)}$ is an element of finite order, say of order s_i and period t_i, that is, all elements $k\sum_{j=i}^{i+m-1}p_{\nu(j)}$, where $k\in\{1,\ldots,s_i\}$, are distinct and $(s_i+1)\sum_{j=i}^{i+m-1}p_{\nu(j)}=(s_i+1-t_i)\sum_{j=i}^{i+m-1}p_{\nu(j)}$. Notice that $k\sum_{j=i}^{i+m-1}p_{\nu(j)}=\sum_{j=i}^{i+km-1}p_{\nu(j)}$. It follows that there is a smallest l_i in $\{s_im,\ldots,(s_i+1)m-1\}$ such that $\sum_{j=i}^{i+l_i}p_{\nu(j)}=\sum_{j=i}^{i+l_i-t_im}p_{\nu(j)}$. We call l_i and t_im the *order* and the *period* of the sequence $(\sum_{j=i}^{i+k}p_{\nu(j)})_{k=0}^{\infty}$. If in addition all elements $\sum_{j=i}^{i+k}p_{\nu(j)}$, where $k\in\{0,\ldots,l_i-1\}$, are distinct, then we call the sequence *cyclic of order l_i and period t_im*.

Lemma 3.1 *(i) t_i is the smallest $t \geq 1$ such that $\sum_{j=i}^{i+l} p_{\nu(j)} = \sum_{j=i}^{i+l-tm} p_{\nu(j)}$ for some $l \geq tm$,*
(ii) l_i is the smallest $l \geq m$ such that $\sum_{j=i}^{i+l} p_{\nu(j)} = \sum_{j=i}^{i+l-tm} p_{\nu(j)}$ for some $t \geq 1$ with $tm \leq l$.

Proof (i) Assume on the contrary that there is $t < t_i$ such that $\sum_{j=i}^{i+l'} p_{\nu(j)} = \sum_{j=i}^{i+l'-tm} p_{\nu(j)}$ for some $l' \geq tm$. It then follows that $\sum_{j=i}^{i+l} p_{\nu(j)} = \sum_{j=i}^{i+l-tm} p_{\nu(j)}$ for all $l \geq l'$. Pick $l = km - 1 \geq l'$ with $k \geq s_i + 1$. Then $k \sum_{j=i}^{i+m-1} p_{\nu(j)} = \sum_{j=i}^{i+km-1} p_{\nu(j)} = \sum_{j=i}^{i+km-1-tm} p_{\nu(j)} = (k - t) \sum_{j=i}^{i+m-1} p_{\nu(j)}$. But we also have that $k \sum_{j=i}^{i+m-1} p_{\nu(j)} = (k-t_i) \sum_{j=i}^{i+m-1} p_{\nu(j)}$, because $\sum_{j=i}^{i+m-1} p_{\nu(j)}$ is an element of order s_i and period t_i and $k \geq s_i + 1$. Consequently, $(k-t) \sum_{j=i}^{i+m-1} p_{\nu(j)} = (k-t_i) \sum_{j=i}^{i+m-1} p_{\nu(j)}$ and $(k-t) - (k-t_i) = t_i - t < t_i$, a contradiction.

(ii) Assume on the contrary that there is $l' < l_i$ such that $\sum_{j=i}^{i+l'} p_{\nu(j)} = \sum_{j=i}^{i+l'-tm} p_{\nu(j)}$ for some t, and consequently, $\sum_{j=i}^{i+l} p_{\nu(j)} = \sum_{j=i}^{i+l-tm} p_{\nu(j)}$ for all $l \geq l'$. Then by (i), $t \geq t_i$. If $t > t_i$, then taking $l = (s_i + 1)m - 1$ gives us $(s_i + 1) \sum_{j=i}^{i+m-1} p_{\nu(j)} = (s_i + 1 - t) \sum_{j=i}^{i+m-1} p_{\nu(j)}$, a contradiction. And if $t = t_i$, then $l' < s_i m$, so taking $l = s_i m - 1$ gives us $s_i \sum_{j=i}^{i+m-1} p_{\nu(j)} = (s_i - t_i) \sum_{j=i}^{i+m-1} p_{\nu(j)}$, again a contradiction.

□

The *periodic sums system generated by the sequence* $p_0, \ldots, p_{m-1}$ is the subset S of the semigroup consisting of all periodic sums $\sum_{j=i}^{i+k} p_{\nu(j)}$, where $i < m$ and $k \geq 0$.

Lemma 3.2 *Suppose that for some $i_0 < m$, $\{\sum_{j=i_0}^{i_0+k} p_{\nu(j)} : k \geq 0\}$ is finite. Then*

(1) S is finite,
(2) there are $t \geq 1$ and $l_i \geq tm$ for each $i < m$ such that $(\sum_{j=i}^{i+k} p_{\nu(j)})_{k=0}^{\infty}$ has order l_i and period tm and $l_i \leq l_{\nu(i+1)} + 1$,
(3) for each $i < m$, $\sum_{j=i}^{i+m-1} p_{\nu(j)}$ is an element of order $s_i = [\frac{l_i}{m}]$ and period t.

Proof For (1) and (2), write $i_0 = \nu(i_1 + 1)$ and suppose that $(\sum_{j=i_0}^{i_0+k} p_{\nu(j)})_{k=0}^{\infty}$ has order l_{i_0} and period tm. From $\sum_{j=i_0}^{i_0+l_{i_0}} p_{\nu(j)} = \sum_{j=i_0}^{i_0+l_{i_0}-tm} p_{\nu(j)}$ we obtain that

$$\sum_{j=i_1}^{i_0+l_{i_0}} p_{\nu(j)} = p_{i_1} + \sum_{j=i_0}^{i_0+l_{i_0}} p_{\nu(j)} = p_{i_1} + \sum_{j=i_0}^{i_0+l_{i_0}-tm} p_{\nu(j)} = \sum_{j=i_1}^{i_0+l_{i_0}-tm} p_{\nu(j)}.$$

It follows that $\{\sum_{j=i_1}^{i_1+k} p_{\nu(j)} : k \geq 0\}$ is finite, and by Lemma 3.1, $(\sum_{j=i_1}^{i_1+k} p_{\nu(j)})_{k=0}^{\infty}$ has order $l_{i_1} \leq l_{i_0} + 1$ and period $t'm$ for some $t' \leq t$. From

$$\sum_{j=i_0}^{m-1+i_1+l_{i_1}} p_{\nu(j)} = \sum_{j=i_0}^{i_0+m-1} p_{\nu(j)} + \sum_{j=i_1}^{i_1+l_{i_1}} p_{\nu(j)} = \sum_{j=i_0}^{i_0+m-1} p_{\nu(j)} + \sum_{j=i_1}^{i_1+l_{i_1}-t'm} p_{\nu(j)}$$
$$= \sum_{j=i_0}^{m-1+i_1+l_{i_1}-t'm} p_{\nu(j)},$$

we obtain that $t' \geq t$. Hence $t' = t$. Then write $i_1 = \nu(i_2 + 1)$ and so on.

For (3), if s is the order of $\sum_{j=i}^{i+m-1} p_{\nu(j)}$, then $l_i \in \{sm, \ldots, (s+1)m - 1\}$, and since $s_i m \in \{l_i - m + 1, \ldots, l_i\}$, one has $s = s_i$. □

It follows from Lemma 3.2 that $|l_i - l_r| \leq m - 1$ and $|s_i - s_r| \leq 1$ for all $i, r \in \{0, \ldots, m-1\}$.

We call the m-tuple $(l_0, \ldots, l_{m-1})$ and the number tm the *order* and the *period* of S.

Let S and S' be two periodic sums systems generated by sequences $p_0, \ldots, p_{m-1}$ and $q_0, \ldots, q_{m-1}$ respectively. A mapping $h : S \to S'$ is a *homomorphism* if there is $s < m$ such that for each $i < m$ and each $k \geq 0$, $h(\sum_{j=i}^{i+k} p_{\nu(j)}) = \sum_{i+s}^{i+s+k} q_{\nu(j)}$. An *isomorphism* is a bijective homomorphism. If S is finite of order $(l_0, l_1, \ldots, l_{m-1})$ and period tm and S' is isomorphic to S, then S' is finite of order $(l_s, l_{\nu(s+1)}, \ldots, \ldots, l_{\nu(s+m-1)})$ for some $s < m$ and period tm. If for each $i < m$, $(\sum_{j=i}^{k} p_{\nu(j)})_{k=i}^{\infty}$ is a cyclic sequence of order l_i and period tm, and all these sequences are pairwise disjoint, then S is said to be a *free finite periodic sums system* of order $(l_0, l_1, \ldots, l_{m-1})$ and period tm.

Lemma 3.3 *Let any $m, l_0, \ldots, l_{m-1}, t \geq 1$ be given such that $tm \leq l_i \leq l_{\nu(i+1)} + 1$ for each $i < m$ and consider the semigroup Q generated by elements $p_0, \ldots, p_{m-1}$ with defining relations $\sum_{j=i}^{i+l_i} p_{\nu(j)} = \sum_{j=i}^{i+l_i-tm} p_{\nu(j)}$, where $i < m$. Then the periodic sums system in Q generated by the sequence $p_0, \ldots, p_{m-1}$ is free of order $(l_0, \ldots, l_{m-1})$ and period tm.*

Proof Let F be the free semigroup over the alphabet $\{0, \ldots, m-1\}$ and let W be the subset of F consisting of words $i_0 \ldots i_k$ such that $k \geq 0$ and $i_{s+1} = \nu(i_s + 1)$ for each $s \leq k - 1$. For each $i \in \{0, \ldots, m-1\}$ and $k \geq 0$, let $w(i, k)$ denote the word $i_0 \ldots i_k$ in W with $i_0 = i$. Let V be the subset of W consisting of words $w(i, k)$, where $i \in \{0, \ldots, m-1\}$ and $k \leq l_i - 1$ for each i, and $K(V)$ the subset of V consisting of words $w(i, k)$, where $i \in \{0, \ldots, m-1\}$ and $l_i - tm \leq k \leq l_i - 1$ for each i.

Let δ be the smallest congruence on F generated by the relations $w(i, l_i) = w(i, l_i - tm)$, where $i \leq m - 1$ (that is, for all $v, w \in F$, $v\delta w$ if and only if v is derivable from w under those relations). Then $Q = F/\delta$ with $p_i = \overline{w(i, 0)}$, where $\overline{w}$ denotes the congruence class of w, and $\sum_{j=i}^{i+k} p_{\nu(j)} = \overline{w(i, k)}$. Clearly, for every $w \in W$, $\overline{w} \subseteq W$ and $\overline{w} \cap V \neq \emptyset$. Also for every $v \in \overline{w}$, v and w have the same first and last letters and $|v| \equiv |w| \pmod{tm}$. It then follows that for all distinct $v, w \in K(V)$, $\overline{v} \cap \overline{w} = \emptyset$.

We claim that for each $w \in V \setminus K(V)$, $\overline{w} = \{w\}$, and consequently, for all distinct $v, w \in V$, $\overline{v} \cap \overline{w} = \emptyset$.

To show this notice that if $w = i_0 \dots i_k \in W$ and $\overline{w} \neq \{w\}$, then there is $s \in \{0, \dots, k\}$ such that $k - s \geq l_{i_s} - tm$. Therefore, it suffices to prove the following statement:

For each $w = i_0 \dots i_k \in W$ and each $s \in \{0, \dots, k\}$, if $k - s \geq l_{i_s} - tm$, then $k \geq l_{i_0} - tm$.

We proceed by induction on s. If $s = 0$, it is obviously true. Fix $r \geq 0$ and suppose that the statement holds for $s = r$ and let $s = r + 1$. Then considering the subword $i_1 \dots i_k$ the inductive hypothesis gives us that $k - 1 \geq l_{i_1} - tm$, so $k \geq l_{i_1} + 1 - tm$. And since $l_{i_1} \geq l_{i_0} - 1$, we obtain that $k \geq l_{i_0} - 1 + 1 - tm = l_{i_0} - tm$. □

The subset V of W in the proof of Lemma 3.3 may be considered as a free finite periodic sums system of order $(l_0, \dots, l_{m-1})$ and period tm, and W itself a free m-generated periodic sums system of infinite order. Then the mapping $\pi : W \to V$ defined by $\pi(w) = \overline{w} \cap V$ (that is, $\pi(w) = w$ if $w \in V$ and $\pi(w)$ is the word $v \in K(V)$ such that v and w have the same first and last letters otherwise) is a homomorphism. We call W *the set of periodic words over* $\{0, \dots, m-1\}$, V *(together with* $K(V)$*) the subset of* W *representing a free finite periodic sums system of order* $(l_0, \dots, l_{m-1})$ *and period* tm, *and* $\pi : W \to V$ *the canonical mapping*.

Remark 3.4 One may consider the semigroup Q' generated by idempotents $p'_0, \dots, p'_{m-1}$ with defining relations $\sum_{j=i}^{i+l_i} p'_{\nu(j)} = \sum_{j=i}^{i+l_i-tm} p'_{\nu(j)}$, where $i < m$. Then the periodic sums system in Q' generated by the sequence $p'_0, \dots, p'_{m-1}$ is also free of order $(l_0, \dots, l_{m-1})$ and period tm.

The proof is practically the same. Let δ' be the smallest congruence on F generated by the relations $w(i, l_i) = w(i, l_i - tm)$ and $w(i, 1) = w(i, 0)$, where $i \leq m - 1$. Then $Q = F/\delta'$ with $p'_i = \overline{w(i, 0)}'$, where $\overline{w}'$ denotes the δ' congruence class of w, and for every $w \in W$, $\overline{w}' \cap W = \overline{w}$.

Since every element of finite order in $\beta\mathbb{N}$ has period 1, it follows that

Theorem 3.5 *Every finite* m*-generated periodic sums system in* $\beta\mathbb{N}$ *has period* m.

In [6] it was shown that for any $m \geq 2$ and $n \geq 2$, there is a free finite m-generated periodic sums system in $\mathbb{H}$ of order $(mn, mn - 1, \dots, mn - m + 1)$. Now using Theorem 2.8 we prove the following result.

Theorem 3.6 *For any* $n \geq m \geq 2$, *there is a free finite* m*-generated periodic sums system in* $\mathbb{H}$ *of order* $(n, n, \dots, n)$.

Proof First consider the main case where $n \geq m + 1$. Let $n' = n - m + 1$ and $F = F(m, n')$. By Theorem 2.8, F has copies in $\mathbb{H}$, so it suffices to construct a free m-generated periodic sums system of order $(n, n, \dots, n)$ in F. For each $i \in \{0, \dots, m-1\}$, let p_i be the 1-letter word i in F, and for each $k \in \{0, \dots, n'+m-1\}$, let $v_{i,k}$ be the word in F representing $\sum_{j=i}^{i+k} p_{\nu(j)}$. Then

$$v_{i,k} = \begin{cases} i\nu(i+1)\dots\nu(i+k) & \text{if } k \leq n' - 1 \\ i\nu(i+k-n'+2)\nu(i+k-n'+3)\dots\nu(i+k) & \text{otherwise.} \end{cases}$$

All words $v_{i,k}$, where $i \in \{0, \ldots, m-1\}$ and $k \in \{0, \ldots, n'+m-2\}$, are distinct (if $k \leq n'-1$, the length of $v_{i,k}$ is $k+1$, and if $n'-1 \leq k \leq n'+m-2$, the length of $v_{i,k}$ is n' and the last letter in $v_{i,k}$ is $\nu(i+k)$), and $v_{i,n'+m-1} = i\nu(i+m+1)\nu(i+m+2)\ldots\nu(i+n'+m-1) = i\nu(i+1)\nu(i+2)\ldots\nu(i+n'-1) = v_{i,n'-1}$.

Now let $n = m$. Consider the rectangular band $\{0, \ldots, m-1\} \times \{0, \ldots, m-1\}$, and for each $i \in \{0, \ldots, m-1\}$, let $p_i = (i, i)$. Then for each $k \in \{0, \ldots, m\}$, $\sum_{j=i}^{i+k} p_{\nu(j)} = (i, \nu(i+k))$, so all sums $\sum_{j=i}^{i+k} p_{\nu(j)}$, where $i, k \in \{0, \ldots, m-1\}$, are distinct and $\sum_{j=i}^{i+m} p_{\nu(j)} = (i, i) = p_i$. □

4 Ramsey theoretic consequences

We first prove a general result. It can be deduced from [9, Theorem 4.4], but for convenience of the reader, we give a straight proof. We shall use the fact that every finite subsemigroup S of $\beta\mathbb{N}$ is contained in $\mathbb{H}$ [9, Lemma 4.1], and so for all $p \in S$ and $j \geq 0$, $2^j\mathbb{N} \in p$.

Theorem 4.1 *Let S be a finite semigroup in $\beta\mathbb{N}$ generated by elements $p_0, \ldots, p_{m-1}$, and for each $p \in S$, let $(A_p(j))_{j=0}^{\infty}$ be a sequence of members of the ultrafilter p. There is a sequence $(x_j)_{j=0}^{\infty}$ such that $x_j \in A_{p_{\nu(j)}}(j) \cap 2^j\mathbb{N}$ and for every finite sequence $j_0 < \ldots < j_s$, if $q = p_{\nu(j_0)} + \ldots + p_{\nu(j_s)}$, then $x_{j_0} + \ldots + x_{j_s} \in A_q(j_0)$.*

Proof We construct inductively a sequence $(x_j)_{j=0}^{\infty}$ satisfying for every j the following conditions in addition to $x_j \in 2^j\mathbb{N}$:

for each finite sequence $j_0 < \ldots < j_s = j$,

$$x_{j_0} + \ldots + x_{j_s} \in A_q(j_0),$$

where $q = p_{\nu(j_0)} + \ldots + p_{\nu(j_s)}$, and for each $p \in S$,

$$x_{j_0} + \ldots + x_{j_s} + p \in \overline{A_{q+p}(j_0)}.$$

To define x_0, for each $p \in S$, choose $P(p) \in p_0$ such that $P(p) + p \subseteq \overline{A_{p_0+p}(0)}$. We can do this because the right translation by p is continuous. Pick

$$x_0 \in A_{p_0}(0) \cap \bigcap_{p \in S} P(p).$$

Then $x_0 \in A_{p_0}(0)$ and for each $p \in S$, $x_0 + p \in P(p) + p \subseteq \overline{A_{p_0+p}(0)}$, so x_0 is as required.

Fix $j \geq 0$ and suppose that we have defined $x_0, \ldots, x_j$ as required. To define x_{j+1}, let F be the set of all sequences $j_0 < \ldots < j_s \leq j$ and let $i = \nu(j+1)$. For each $p \in S$, choose $B(p) \in p_i$ such that $B(p) + p \subseteq \overline{A_{p_i+p}(j+1)}$. Then for each $(j_0, \ldots, j_s) \in F$, choose $C(j_0, \ldots, j_s) \in p_i$ such that $x_{j_0} + \ldots + x_{j_s} + C(j_0, \ldots, j_s) \subseteq A_{q+p_i}(j_0)$, where $q = p_{\nu(j_0)} + \ldots + p_{\nu(j_s)}$, and for each $p \in S$, choose $D(j_0, \ldots, j_s, p) \in p_i$

such that $x_{j_0} + \ldots + x_{j_s} + D(j_0, \ldots, j_s, p) + p \subseteq \overline{A_{q+p_i+p}(j_0)}$. We can do the first because by the inductive hypothesis $x_{j_0} + \ldots + x_{j_s} + p_i \in \overline{A_{q+p_i}(j_0)}$ and λ_x, where $x = x_{j_0} + \ldots + x_{j_s}$, is continuous, and the second because $p_i + p \in S$ and by the inductive hypothesis $x_{j_0} + \ldots + x_{j_s} + p_i + p \in \overline{A_{q+p_i+p}(j_0)}$ and λ_x and ρ_p are continuous. Now pick

$$x_{j+1} \in 2^{j+1}\mathbb{N} \cap A_{p_i}(j+1) \cap \bigcap_{p \in S} B(p) \cap \bigcap_{(j_0,\ldots,j_s)\in F} (C(j_0, \ldots, j_s) \cap \bigcap_{p \in S} D(j_0, \ldots, j_s, p))$$

(all those sets are members of p_i).

To see that x_{j+1} is as required, let any $j_0 < \ldots < j_s = j+1$ be given. If $s = 0$, then $x_{j+1} \in A_{p_i}(j+1)$ and for each $p \in S$, $x_{j+1} + p \in B(w) + p \subseteq \overline{A_{p_i+p}(j+1)}$. If $s \geq 1$, then

$$x_{j_0} + \ldots + x_{j_s} \in x_{j_0} + \ldots + x_{j_{s-1}} + C(j_0, \ldots, j_{s-1}) \subseteq A_{q+p_i}(j_0),$$

where $q = p_{\nu(j_0)} + \ldots + p_{\nu(j_{s-1})}$, and for each $p \in S$,

$$x_{j_0} + \ldots + x_{j_s} + p \in x_{j_0} + \ldots + x_{j_{s-1}} + D(x_{j_0}, \ldots x_{j_{s-1}}, p) + p \subseteq \overline{A_{q+p_i+p}(j_0)}.$$

□

Corollary 4.2 *Let S be a finite semigroup generated by elements $p_0, \ldots, p_{m-1}$ and suppose that S has a copy in $\mathbb{H}$. Then there is a partition $\{A_p : p \in S\}$ of $\mathbb{N}$ such that whenever for each p, $\mathscr{B}_p$ is a finite partition of A_p, there exist $B_p \in \mathscr{B}_p$ and a sequence $(x_j)_{j=0}^{\infty}$ such that $x_j \in B_{p_{\nu(j)}} \cap 2^j\mathbb{N}$ and for every finite sequence $j_0 < \ldots < j_s$, if $q = p_{\nu(j_0)} + \ldots + p_{\nu(j_s)}$, then $x_{j_0} + \ldots + x_{j_s} \in B_q$.*

Proof One may suppose that S is in $\beta\mathbb{N}$. Choose a partition $\{A_p : p \in S\}$ of $\mathbb{N}$ such that $A_p \in p$. To see that this partition is as required, for each p, let $\mathscr{B}_p$ be a finite partition of A_p. Pick $B_p \in \mathscr{B}_p$ such that $B_p \in p$, and for every $j \geq 0$, put $A_p(j) = B_p$. Let $(x_j)_{j=0}^{\infty}$ be a sequence guaranteed by Theorem 4.1. For any $j_0 < \ldots < j_s$, if $q = p_{\nu(j_0)} + \ldots + p_{\nu(j_s)}$, then $x_{j_0} + \ldots + x_{j_s} \in A_p(j_0) = B_q$. □

Now from Theorem 2.8 and Corollary 4.2 we obtain the following result.

Corollary 4.3 *Let $m \geq 1$ and $n \geq 2$ and let F be the set of nonempty words over $\{0, \ldots, m-1\}$ of length $\leq n$. There is a partition $\{A_w : w \in F\}$ of $\mathbb{N}$ such that, whenever for each $w \in F$, $\mathscr{B}_w$ is a finite partition of A_w, there exist $B_w \in \mathscr{B}_w$ and a sequence $(x_j)_{j=0}^{\infty}$ such that $x_j \in 2^j\mathbb{N}$ and for every finite sequence $j_0 < \ldots < j_s$, if*

$$v = \begin{cases} \nu(j_0)\ldots\nu(j_s) & \text{if } s \leq n-1 \\ \nu(j_0)\nu(j_{s-n+2})\ldots\nu(j_s) & \text{otherwise,} \end{cases}$$

then $x_{j_0} + \ldots + x_{j_s} \in B_v$.

Proof Consider F as the semigroup $F(m,n)$. □

Remark 4.4 We have extended the addition of natural numbers to an operation $+$ on $\beta\mathbb{N}$ so as to obtain a right topological semigroup. But one can equally well extend the addition to an operation $*$ on $\beta\mathbb{N}$ so as to obtain a left topological semigroup. The semigroup $(\beta\mathbb{N},*)$ is the opposite of the semigroup $(\beta\mathbb{N},+)$: $p*q=q+p$. There are finite semigroups which have copies in $(\beta\mathbb{N},*)$ and not in $(\beta\mathbb{N},+)$. For example, the 3-element band $\{a,b,c\}$, where $\{a,b\}$ is right zero semigroup and c is zero [11]. At the end of the paper [9] it was wrongly remarked that Theorem 4.4 there, an analogue of Theorem 4.1 here, holds for the semigroup $(\beta\mathbb{N},*)$ as well and so the result can be extended to finite semigroups which have copies in $(\beta\mathbb{N},*)$. In fact Theorem 4.1 holds for $(\beta\mathbb{N},*)$ with a correction:

Let S be a finite semigroup in $(\beta\mathbb{N},*)$ generated by elements $p_0,\dots,p_{m-1}$, and for each $p\in S$, let $(A_p(j))_{j=0}^{\infty}$ be a sequence of members of the ultrafilter p. There is a sequence $(x_j)_{j=0}^{\infty}$ in $\mathbb{N}$ such that $x_j\in A_{p_{\nu(j)}}(j)\cap 2^j\mathbb{N}$ and for every finite sequence $j_0<\dots<j_s$, if $q=p_{\nu(j_s)}*\dots*p_{\nu(j_0)}$, then $x_{j_0}+\dots+x_{j_s}\in A_q(j_0)$.

And since $p_{\nu(j_s)}*\dots*p_{\nu(j_0)}=p_{\nu(j_0)}+\dots+p_{\nu(j_s)}$, this is the result for the semigroup $(S,+)$ in $(\beta\mathbb{N},+)$. Hence, using $(\beta\mathbb{N},*)$ in addition to $(\beta\mathbb{N},+)$ gives no new result.

Theorem 4.5 *Let S be a finite periodic sums system in $\mathbb{H}$ generated by a sequence $p_0,\dots,p_{m-1}$, and for each $p\in S$, let $(A_p(j))_{j=0}^{\infty}$ be a sequence of members of p. There is a sequence $(x_j)_{j=0}^{\infty}$ such that $x_j\in A_{p_{\nu(j)}}(j)\cap 2^j\mathbb{N}$ and for every finite sequence $j_0<\dots<j_s$ such that $j_{t+1}\equiv j_t+1 \pmod m$ for each $t<s$, if $q=p_{\nu(j_0)}+\dots+p_{\nu(j_s)}$, then $x_{j_0}+\dots+x_{j_s}\in A_q(j_0)$.*

Proof Let $(l_0,\dots,l_{m-1})$ be the order of S and let W be the set of periodic words over $\{0,\dots,m-1\}$, V the subset of W representing a free finite periodic sums system of order $(l_0,\dots,l_{m-1})$ and period m, and $\pi:W\to V$ the canonical mapping. Also for each $i\in\{0,\dots,m-1\}$, let $V(i)$ denote the subset of V consisting of words with first letter i. Define $f:W\to S$ by $f(i_0\dots i_k)=p_{i_0}+\dots+p_{i_k}$. Then $f(w)=f(\pi(w))$ for all $w\in W$ and $f(wv)=f(w)+f(v)$ for all $w,v\in W$ such that $wv\in W$.

We construct inductively a sequence $(x_j)_{j=0}^{\infty}$ satisfying for every j the following conditions in addition to $x_j\in 2^j\mathbb{N}$:

for each finite sequence $j_0<\dots<j_s=j$ with $w=\nu(j_0)\dots\nu(j_s)\in W$,

$$x_{j_0}+\dots+x_{j_s}\in A_{f(w)}(j_0)$$

and for each $v\in V(\nu(j+1))$,

$$x_{j_0}+\dots+x_{j_s}+f(v)\in\overline{A_{f(wv)}(j_0)}.$$

To define x_0, for each $v\in V(1)$, choose $P(v)\in p_0$ such that $P(v)+f(v)\subseteq\overline{A_{f(0v)}(0)}$. We can do this because $p_0+f(v)=f(0v)$ and $\rho_{f(v)}$ is continuous. Pick

$$x_0\in A_0(0)\cap\bigcap_{v\in V(1)}P(v).$$

Then $x_0 \in A_0(0)$ and for each $v \in V(1)$, $x_0 + f(v) \in P(v) + f(v) \subseteq \overline{A_{f(0v)}(0)}$, so x_0 is as required.

Fix $j \geq 0$ and suppose that we have defined $x_0, \ldots, x_j$ as required. To define x_{j+1}, let F be the set of all sequences $j_0 < \ldots < j_s \leq j$ with $\nu(j_0) \ldots \nu(j_s) \in W$ and $\nu(j_s) = \nu(j)$ and let $i = \nu(j+1)$ and $r = \nu(j+2)$. For each $v \in V(r)$, choose $B(v) \in p_i$ such that $B(v) + f(v) \subseteq \overline{A_{f(iv)}(j+1)}$. Then for each $(j_0, \ldots, j_s) \in F$, choose $C(j_0, \ldots, j_s) \in p_i$ such that $x_{j_0} + \ldots + x_{j_s} + C(j_0, \ldots, j_s) \subseteq A_{f(wi)}(j_0)$, where $w = \nu(j_0) \ldots \nu(j_s)$, and for each $v \in V(r)$, choose $D(j_0, \ldots, j_s, v) \in p_i$ such that $x_{j_0} + \ldots + x_{j_s} + D(j_0, \ldots, j_s, v) + f(v) \subseteq \overline{A_{f(wiv)}(j_0)}$. We can do the first because by the inductive hypothesis $x_{j_0} + \ldots + x_{j_s} + p_i \in \overline{A_{f(wi)}(j_0)}$ and λ_x, where $x = x_{j_0} + \ldots + x_{j_s}$, is continuous, and the second because $p_i + f(v) = f(iv) = f(\pi(iv))$ and by the inductive hypothesis $x_{j_0} + \ldots + x_{j_s} + f(\pi(iv)) \in \overline{A_{f(w\pi(iv))}(j_0)} = \overline{A_{f(wiv)}(j_0)}$ (since $f(wiv) = f(w) + f(iv) = f(w) + f(\pi(iv)) = f(w\pi(iv))$) and λ_x and $\rho_{f(v)}$ are continuous. Now pick

$$x_{j+1} \in 2^{j+1}\mathbb{N} \cap A_i(j+1) \cap \bigcap_{v \in V(r)} B(v) \cap \bigcap_{(j_0, \ldots, j_s) \in F} (C(j_0, \ldots, j_s) \cap \bigcap_{v \in V(r)} D(j_0, \ldots, j_s, v))$$

(all those sets are members of p_i).

To see that x_{j+1} is as required, let any $j_0 < \ldots < j_s = j+1$ with $\nu(j_0) \ldots \nu(j_s) \in W$ be given. If $s = 0$, then $x_{j+1} \in A_i(j+1)$ and for each $v \in V(r)$, $x_{j+1} + f(v) \in B(v) + f(v) \subseteq \overline{A_{f(iv)}(j+1)}$. If $s \geq 1$, then

$$x_{j_0} + \ldots + x_{j_s} \in x_{j_0} + \ldots + x_{j_{s-1}} + C(j_0, \ldots, j_{s-1}) \subseteq A_{f(wi)}(j_0),$$

where $w = \nu(j_0) \ldots \nu(j_{s-1})$, and for each $v \in V(r)$,

$$\begin{aligned} &x_{j_0} + \ldots + x_{j_s} + f(v) \in x_{j_0} + \ldots + x_{j_{s-1}} + D(x_{j_0}, \ldots, x_{j_{s-1},v}) \\ &+ f(v) \subseteq \overline{A_{f(wiv)}(j_0)}. \end{aligned}$$

□

Corollary 4.6 *Let S be a finite periodic sums system generated by a sequence $p_0, \ldots, p_{m-1}$ and suppose that S has a copy in $\mathbb{H}$. Then there is a partition $\{A_p : p \in S\}$ of $\mathbb{N}$ such that whenever for each p, $\mathscr{B}_p$ is a finite partition of A_p, there exist $B_p \in \mathscr{B}_p$ and a sequence $(x_j)_{j=0}^{\infty}$ such that $x_j \in B_{\nu(j)} \cap 2^j\mathbb{N}$ and for every finite sequence $j_0 < \ldots < j_s$ such that $j_{t+1} \equiv j_t + 1 \pmod{m}$ for each $t < s$, if $q = p_{\nu(j_0)} + \ldots + p_{\nu(j_s)}$, then $x_{j_0} + \ldots + x_{j_s} \in B_q$*

Proof Similar to the proof of Corollary 4.2. □

In [6] it was also deduced from the existence of a free finite m-generated periodic sums system in $\mathbb{H}$ of order $(mn, mn-1, \ldots, mn-m+1)$ that:

There is a partition

$$\{A_{i,k} : i \in \{0, \ldots, m-1\} \text{ and } k \in \{i, \ldots, mn-1\} \text{ for each } i\}$$

of $\mathbb{N}$ such that, whenever for each (i, k), $\mathscr{B}_{i,k}$ is a finite partition of $A_{i,k}$, there exist $B_{i,k} \in \mathscr{B}_{i,k}$ and a sequence $(x_j)_{j=0}^{\infty}$ such that $x_j \in 2^j\mathbb{N}$ and for every finite sequence $j_0 < \ldots < j_s$ such that $j_{t+1} \equiv j_t + 1 \pmod m$ for each $t < s$, if $i_0 = \nu(j_0)$ and

$$k_0 = \begin{cases} i_0 + s & \text{if } i_0 + s \le mn - 1 \\ mn - m + \nu(i_0 + s - mn) & \text{otherwise,} \end{cases}$$

then $x_{j_0} + \ldots + x_{j_s} \in B_{i_0,k_0}$.

Now from Theorem 3.6 and Corollary 4.6 we obtain the following result.

Corollary 4.7 *Let $n \ge m \ge 2$. There is a partition*

$$\{A_{i,k} : (i, k) \in \{0, \ldots, m-1\} \times \{0, \ldots, n-1\}\}$$

of $\mathbb{N}$ such that, whenever for each (i, k), $\mathscr{B}_{i,k}$ is a finite partition of $A_{i,k}$, there exist $B_{i,k} \in \mathscr{B}_{i,k}$ and a sequence $(x_j)_{j=0}^{\infty}$ such that $x_j \in 2^j\mathbb{N}$ and for every finite sequence $j_0 < \ldots < j_s$ such that $j_{t+1} \equiv j_t + 1 \pmod m$ for each $t < s$, if $i_0 = \nu(j_0)$ and

$$k_0 = \begin{cases} s & \text{if } s \le n - 1 \\ n - m + \nu(s - n) & \text{otherwise,} \end{cases}$$

then $x_{j_0} + \ldots + x_{j_s} \in B_{i_0,k_0}$.

Proof Consider $\{0, \ldots, m-1\} \times \{0, \ldots, n-1\}$ as a free finite m-generated periodic sums system of order $(n, \ldots, n)$ with $(i, k) = \sum_{j=i}^{i+k} p_{\nu(j)}$. □

In cases $n = m$ and $n = m + 1$, Corollary 4.7 can be strengthened. The free finite m-generated periodic sums systems of orders $(m, \ldots, m)$ and $(m + 1, \ldots, m + 1)$ constructed in Theorem 3.6 are in fact the $m \times m$ rectangular band and the semigroup $F(m, 2)$. Therefore, by Corollary 4.2, the following stronger results hold.

Corollary 4.8 *For every $m \ge 2$, there is a partition*

$$\{A_{i,k} : (i, k) \in \{0, \ldots, m-1\} \times \{0, \ldots, m-1\}\}$$

of $\mathbb{N}$ such that, whenever for each (i, k), $\mathscr{B}_{i,k}$ is a finite partition of $A_{i,k}$, there exist $B_{i,k} \in \mathscr{B}_{i,k}$ and a sequence $(x_j)_{j=0}^{\infty}$ such that $x_j \in 2^j\mathbb{N}$ and for every finite nonempty $J \subseteq \omega$, if $i_0 = \nu(\min J)$ and $k_0 = \nu(\max J)$, then $\sum_{j \in J} x_j \in B_{i_0,k_0}$.

Corollary 4.9 *For every $m \ge 2$, there is a partition*

$$\{A_{i,k} : (i, k) \in \{0, \ldots, m-1\} \times \{0, \ldots, m\}\}$$

of $\mathbb{N}$ *such that, whenever for each* (i,k), $\mathscr{B}_{i,k}$ *is a finite partition of* $A_{i,k}$, *there exist* $B_{i,k} \in \mathscr{B}_{i,k}$ *and a sequence* $(x_j)_{j=0}^{\infty}$ *such that* $x_j \in 2^j\mathbb{N} \cap B_{\nu(j),0}$ *and for every finite* $J \subseteq \omega$ *with* $|J| \geq 2$, *if* $i_0 = \nu(\min J)$ *and* $k_0 = 1+\nu(\max J)$, *then* $\sum_{j\in J} x_j \in B_{i_0,k_0}$.

In Corollary 4.9, (i,k) is identified with the 1-letter word i of $F(m,2)$ if $k=0$ and the word $i(k-1)$ otherwise. It is a restatement of case $m \geq n = 2$ of Corollary 4.3.

We also notice that a finite periodic sums system generated by two idempotents is a semigroup, and so for such systems, if they have copies in $\beta\mathbb{N}$, also stronger results hold.

For every $n \geq 3$ ($n \geq 2$), a free finite 2-idempotent generated periodic sums system of order $(n, n-1)$ ((n,n)) is the semigroup $S_{n,n-1}$ ($S_{n,n}$) generated by idempotents p_0, p_1 with defining relations $\sum_{j=0}^{n} p_{\nu(j)} = \sum_{j=0}^{n-2} p_{\nu(j)}$ and $\sum_{j=1}^{n} p_{\nu(j)} = \sum_{j=1}^{n-2} p_{\nu(j)}$ ($\sum_{j=1}^{n+1} p_{\nu(j)} = \sum_{j=1}^{n-1} p_{\nu(j)}$). Presently $m = 2$, so $\nu = \nu_2$. We know only three of those semigroups that have copies in $\beta\mathbb{N}$: $S_{2,2}$ (2×2 rectangular band), $S_{3,2}$ (the band (10) in [9, Theorem 2.3]), and $S_{4,3}$ (the semigroup (3) in [9, Corollary 3.11]). For all others we do not know whether they have copies in $\beta\mathbb{N}$, in particular, for $S_{3,3}$ which is a free 2-generated band. We also do not know whether a sum of two idempotents in $\beta\mathbb{N}$ can be an element of order $n \geq 3$.

For every finite nonempty subset $J \subseteq \omega$, write the elements of J as $j_0 < \ldots < j_s$ and let $f(J)$ be the number of all $t < s$ such that $j_{t+1} \equiv j_t + 1 \pmod 2$.

Corollary 4.10 *Let* $n \geq 3$ *and suppose that the semigroup* $S_{n,n-1}$ *has a copy in* $\beta\mathbb{N}$. *Then there is a partition*

$$\{A_{i,k} : i \in \{0,1\} \text{ and } k \in \{i,\ldots,n-1\} \text{ for each } i\}$$

of $\mathbb{N}$ *such that, whenever for each* (i,k), $\mathscr{B}_{i,k}$ *is a finite partition of* $A_{i,k}$, *there exist* $B_{i,k} \in \mathscr{B}_{i,k}$ *and a sequence* $(x_j)_{j=0}^{\infty}$ *such that* $x_j \in 2^j\mathbb{N}$ *and for every finite nonempty* $J \subseteq \omega$, *if* $i_0 = \nu(\min J)$ *and*

$$k_0 = \begin{cases} i_0 + f(J) & \text{if } i_0 + f(J) \leq n-1 \\ n-2+\nu(i_0+f(J)-n) & \text{otherwise,} \end{cases}$$

then $\sum_{j\in J} x_j \in B_{i_0,k_0}$.

Proof Consider $\{(i,k) : i \in \{0,1\}$ and $k \in \{i,\ldots,n-1\}$ for each $i\}$ as the semigroup $S_{n,n-1}$ with $(i,k) = \sum_{j=i}^{k} p_{\nu(j)}$. For any finite nonempty $J \subseteq \omega$, if $i_0 = \nu(\min J)$, then $\sum_{j\in J} p_{\nu(j)} = \sum_{j=i_0}^{i_0+f(J)} p_{\nu(j)}$. Apply Corollary 4.2. □

A subset $A \subseteq \mathbb{N}$ is an IP *set* if it contains an infinite sequence all of whose sums belong to A. By Hindman's Theorem, whenever $\mathbb{N}$ is partitioned into finitely many cells, at least one of the cells is an IP set.

Remark 4.11 All results of this section extend to IP sets, that is, in the statement of each corollary the partitioning set $\mathbb{N}$ can be replaced with any IP set $A \subseteq \mathbb{N}$.

Indeed, let $(a_n)_{n=0}^{\infty}$ be a sequence all of whose sums belong to A. Taking a sum subsystem of $(a_n)_{n=0}^{\infty}$ one may suppose that $\max\operatorname{supp} a_n < \min\operatorname{supp} a_{n+1}$

(see [4, Exercise 5.2.2]), and also that A coincides with the set of all sums of the sequence. Define a bijection $f : \mathbb{N} \to A$ by $f(x) = \sum_{n \in \text{supp}\, x} a_n$. Then whenever $\max \text{supp}\, x < \min \text{supp}\, y$, one has $f(x + y) = f(x) + f(y)$.

Now consider say Corollary 4.6. Let $\{A_p^{\mathbb{N}} : p \in S\}$ be a partition of $\mathbb{N}$ guaranteed by the corollary. Define a partition $\{A_p : p \in S\}$ of A by $A_p = f(A_p^{\mathbb{N}})$.

To see that this partition is as required, let for each p, $\mathscr{B}_p$ be a finite partition of A_p and let $\mathscr{B}_p^{\mathbb{N}} = f^{-1}(\mathscr{B}_p)$. Let $B_p^{\mathbb{N}} \in \mathscr{B}_p^{\mathbb{N}}$ and $(x_j^{\mathbb{N}})_{j=0}^{\infty}$ be as guaranteed by the corollary. One may suppose that $\max \text{supp}\, x_j^{\mathbb{N}} < \min \text{supp}\, x_{j+1}^{\mathbb{N}}$. Define $B_p \in \mathscr{B}_p$ and $(x_j)_{j=0}^{\infty}$ by $B_p = f(\mathscr{B}_p^{\mathbb{N}})$ and $x_j = f(x_j^{\mathbb{N}})$.

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