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RESEARCH ARTICLE

Set-theoretical solutions of the pentagon equation on Clifford semigroups

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Abstract

Given a set-theoretical solution of the pentagon equation $s : S \times S \to S \times S$ on a set S and writing $s(a, b) = (a \cdot b,\ \theta_a(b))$, with $\cdot$ a binary operation on S and θ_a a map from S into itself, for every $a \in S$, one naturally obtains that $(S,\ \cdot)$ is a semigroup. In this paper, we focus on solutions defined in Clifford semigroups $(S,\ \cdot)$ satisfying special properties on the set of all idempotents $\mathrm{E}(S)$. Into the specific, we provide a complete description of *idempotent-invariant solutions*, namely, those solutions for which θ_a remains invariant in $\mathrm{E}(S)$, for every $a \in S$. Moreover, we construct a family of *idempotent-fixed solutions*, i.e., those solutions for which θ_a fixes every element in $\mathrm{E}(S)$ for every $a \in S$, from solutions given on each maximal subgroup of S.

Keywords Pentagon equation · Set-theoretical solution · Inverse semigroups · Clifford semigroups

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Introduction

If V is a vector space over a field F, a linear map $\mathcal{S} : V \otimes V \to V \otimes V$ is said to be a *solution of the pentagon equation* on V if it satisfies the relation

$$\mathcal{S}_{12}\mathcal{S}_{13}\mathcal{S}_{23} = \mathcal{S}_{23}\mathcal{S}_{12}, \tag{1}$$

where $\mathcal{S}_{12} = \mathcal{S} \otimes \mathrm{id}_V, \mathcal{S}_{23} = \mathrm{id}_V \otimes \mathcal{S}, \mathcal{S}_{13} = (\mathrm{id}_V \otimes \Sigma)\, \mathcal{S}_{12}\, (\mathrm{id}_V \otimes \Sigma)$, with Σ the flip operator on $V \otimes V$, i.e., $\Sigma(u \otimes v) = v \otimes u$, for all $u, v \in V$. The pentagon equation arose at first at the beginning of the '80s in [5] as the Biedenharn–Elliott identity for Wigner $6j$-symbols and Racah coefficients in the representation theory for the rotation group. Maillet [21] showed that solutions of the pentagon equation lead to solutions of the tetrahedron equation [32], a generalization of the well-known quantum Yang–Baxter equation [4, 30]. Moreover, in [25, Theorem 3.2], Militaru showed that bijective solutions on finite vector spaces are equivalent to finite Hopf algebras, so the classification of the latter is reduced to the classification of solutions. In the subsequent years, the pentagon equation appeared in literature in several forms with different terminologies according to the specific research areas. We highlight some interesting works as [2, 3, 11, 13, 15–17, 22, 25, 27–29, 31], just to name a few. For a fuller treatment of some applications in which the pentagon equation appears, we suggest the recent paper by Dimakis and Müller-Hoissen [10] (along with the references therein), where the authors dealt with an infinite family of equations named *polygon equations*.

As well as Drinfel'd in [12] translated the study of solutions of the Yang–Baxter equation into set-theoretical terms, Kashaev and Sergeev in [19] began the study of the pentagon equation with a set-theoretical approach. Namely, if S is a set, a map $s : S \times S \to S \times S$ satisfying the following "reversed" relation

$$s_{23}s_{13}s_{12} = s_{12}s_{23}, \tag{2}$$

where $s_{12} = s \times \mathrm{id}_S, s_{23} = \mathrm{id}_S \times s, s_{13} = (\mathrm{id}_S \times \tau)\, s_{12}\, (\mathrm{id}_S \times \tau)$, and $\tau(a, b) = (b, a)$, for all $a, b \in S$, is said to be a *set-theoretical solution of the pentagon equation*, or briefly *solution*, on S. If, in particular, s is a solution on a finite set S, then the linear map $\mathcal{S} : F^{S\times S} \to F^{S\times S}$ defined by $\mathcal{S}(f)(a, b) = f(s(a, b))$, for all $a, b \in S$, is a solution of (1) on the vector space F^S of all functions from S to F. The problem of classifying all possible solutions to the pentagon equation is a fascinating question that is a long way off from being solved. Nevertheless, some partial results have been obtained.

For their purposes, the authors in [19] investigated only bijective maps. This class of solutions was also studied by Kashaev and Reshetikhin in [18], where it is shown that each symmetrically factorizable Lie group is related to a bijective solution. Among these solutions, a description of all those that are involutive, i.e., $s^2 = \mathrm{id}_{S\times S}$, has been recently given by Colazzo, Jespers, and Kubat in [9].

One of the key steps in the classification problem is the fact that every set S admitting a solution s is inherently endowed with a semigroup structure (see [6, Proposition 8]): if we write $s(a, b) = (a \cdot b,\ \theta_a(b))$, then $(S, \cdot)$ becomes a semigroup (hereinafter, we will use juxtaposition for the product in a semigroup). Therefore, the study of solutions (S, s) undoubtedly passes through an exhaustive analysis of the impact of s on the structural description of S. In this vein, in [6, Theorem 15] the authors provide a description of all solutions on a group by means of its normal subgroups. Moreover, in [7], we can find several constructions of solutions defined in the matched product of two semigroups, that is a semigroup including the classical Zappa–Szép product. Furthermore, in [23], the first author studies idempotent solutions, namely, maps satisfying the property $s^2 = s$, and describes this kind of solutions on monoids having central idempotents. In light of these results, it is becoming more and more clear that idempotents in a semigroup play a prominent role in the study of solutions.

The next natural step is to try to extend the property of idempotents being central to a wider class of semigroups. In this light, this paper aims to begin with the study of solutions on Clifford semigroups, i.e. inverse semigroups whose idempotent elements are central (see the seminal monographs [20] and [26] for a full treatment on inverse semigroups). Bearing in mind that the behaviour of Clifford semigroups is very close to that of groups, the description of solutions on groups in [6] is of great utility in the problem of classifying all solutions on a Clifford semigroup. More concretely, it is easy to check that every solution on a group G satisfies that $\theta_a(1) = 1$, for every $a \in G$. Therefore, it motivates us to consider both classes of solutions on a Clifford semigroup S such that θ_a, respectively, fixes every idempotent or remains invariant on every idempotent, for every $a \in S$. We call them, respectively, *idempotent-fixed* and *idempotent-invariant* solutions.

The main results of this paper are the following. Firstly, we provide a complete description of the first class of solutions on a Clifford semigroup S, which includes that made in the context of groups. To this aim, we introduce the *kernel* of an arbitrary solution on S, which turns out to be a normal subsemigroup, that is a subsemigroup containing the idempotents and closed by conjugation. Secondly, for the second class, considering that any Clifford semigroup is a union of a family of pairwise disjoint groups $\{G_e\}_{e \in \mathrm{E}(S)}$, we give a construction of solutions obtained starting from a solution on each group G_e.

Finally, we raise some questions aimed at continuing the study of the solutions in this class of semigroups.

1 Preliminaries

This section aims to briefly introduce some basics of set-theoretical solutions of the pentagon equation. Initially, we recall some notions related to Clifford semigroups useful for our purposes. For a more detailed treatment of this topic, we refer the reader to [8] and [20].

1.1 Basics on Clifford semigroups

Recall that S is an *inverse semigroup* if for each $a \in S$ there exists a unique $a^{-1} \in S$ such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$. They hold $(ab)^{-1} = b^{-1}a^{-1}$ and

$(a^{-1})^{-1} = a$, for all $a, b \in S$. Moreover, $\mathrm{E}(S) = \{aa^{-1} \mid a \in S\}$ and one can consider the following natural partial-order relation

$$\forall\, e, f \in \mathrm{E}(S) \qquad e \leq f \iff e = ef = fe.$$

An inverse semigroup S is *Clifford* if $aa^{-1} = a^{-1}a$, for any $a \in S$, or, equivalently, the idempotents are central in the sense that commute with every element in S.

Given a Clifford semigroup S, we introduce the following relations and the properties involved themselves. They are an easy consequence of the fact that all Green's relations coincide in S and they characterize the structure of S itself. If $a, b \in S$, we define

1. $a \leq b$ if, and only if, $aa^{-1} \leq bb^{-1}$, which is an extension of the natural partial order in S;
2. $a\,\mathcal{R}\,b$ if, and only if, $a \leq b$ and $b \leq a$.

It follows that $\leq$ is a preorder on S and $\mathcal{R}$ is an equivalence relation on S such that

$$G_{aa^{-1}} := [a]_{\mathcal{R}} = \{b \in S \mid bb^{-1} = aa^{-1}\}$$

is a group with identity aa^{-1}, for every $a \in S$. On the other hand, for all $a, b \in S$,

$$a \leq b \iff \exists\, u \in S \quad a = ub \ \vee \ a = bu. \tag{3}$$

Moreover, $\leq$ induces an order relation on the equivalence classes of $\mathcal{R}$, namely, for all $e, f \in \mathrm{E}(S)$, $G_e \leq G_f$ if, and only if, $e \leq f$. The following theorem describes Clifford semigroups.

Theorem 1 *Let S be a Clifford semigroup. Then,*

1. *S is a union of a family of pairwise disjoint groups $\{G_e\}_{e \in \mathrm{E}(S)}$;*
2. *the map $\varphi_{f,e}\colon G_f \to G_e$ given by $\varphi_{f,e}(b) = eb$, for every $b \in G_f$, is a group homomorphism, for all $e, f \in \mathrm{E}(S)$ such that $e \leq f$;*
3. *for all $e, f, g \in \mathrm{E}(S)$ such that $e \leq f \leq g$, then $\varphi_{g,e} = \varphi_{f,e}\varphi_{g,f}$.*

As a consequence of the previous theorem, the product in Clifford semigroups can be written through the group homomorphisms $\varphi_{e,f}$, namely,

$$ab = (efa)(efb) = \varphi_{e,ef}(a)\,\varphi_{f,ef}(b) \in G_{ef},$$

for all $a \in G_e$, $b \in G_f$. In particular, for all $a \in S$, $e \in \mathrm{E}(S)$ such that $a \leq e$, one has $ae = ea = a$.
For the sake of completeness, the converse of Theorem 1 is also true.

1.2 Basics on solutions

Kashaev and Sergeev [19] first dealt with solutions from an algebraic point of view. Recently, the study of these solutions has been recovered in [6, 7, 9, 23]. Following the notation introduced in these works, given a set S and a map s from $S \times S$ into itself, we will write

$$s(a, b) := (ab,\ \theta_a(b)),$$

for all $a, b \in S$, where θ_a is a map from S into itself, for every $a \in S$. Then, s is briefly a *solution* on S if, and only if, the following conditions hold

$$\begin{aligned}(ab)c &= a(bc) \\ \theta_a(b)\theta_{ab}(c) &= \theta_a(bc)\end{aligned} \tag{P1}$$

$$\theta_{\theta_a(b)}\theta_{ab} = \theta_b \tag{P2}$$

for all $a, b, c \in S$. Thus, the first identity naturally gives rise to a semigroup structure on S, which leads the study of solutions to focus on specific classes of semigroups. When describing solutions, it serves to distinguish those solutions that are not isomorphic.

Definition 2 Let S, T be two semigroups and $s(a, b) = (ab, \theta_a(b))$, $t(u, v) = (uv, \eta_u(v))$ two solutions on S and T, respectively. Then, s and t are *isomorphic* if there exists an isomorphism $\psi : S \to T$ such that

$$\psi\theta_a(b) = \eta_{\psi(a)}\psi(b), \tag{4}$$

for all $a, b \in S$, or, equivalently, $(\psi \times \psi)s = t(\psi \times \psi)$.

The following are easy examples of solutions used throughout this paper.

Examples 1

1. Let S be a set and $f, g : S \to S$ idempotent maps such that $fg = gf$. Then, $s(a, b) = (f(a), g(b))$ is a solution on S (cf. [24]).
2. Let S be a semigroup and $\gamma \in \mathrm{End}(S)$ such that $\gamma^2 = \gamma$. Then, the map s given by $s(a, b) = (ab, \gamma(b))$, for all $a, b \in S$, is a solution on S (see [6, Examples 2-2.]).

Let us observe that every Clifford semigroup S gives rise to the following solutions

$$\mathcal{I}(a, b) = (ab, b), \qquad \mathcal{F}(a, b) = \left(ab, bb^{-1}\right), \qquad \mathcal{E}(a, b) = (ab, e), \tag{5}$$

where $e \in \mathrm{E}(S)$ is a fixed idempotent of S, belonging to the class of solutions in 2. of Examples 1.

In [1], solutions of (1) are defined on Hilbert spaces in terms of commutative and cocommutative multiplicative unitary operators (see [1, Definition 2.1]). These operators motivate the following classes of solutions in the set-theoretical case.

Definition 3 A solution $s : S \times S \to S \times S$ is said to be *commutative* if $s_{12}s_{13} = s_{13}s_{12}$ and *cocommutative* if $s_{13}s_{23} = s_{23}s_{13}$.

Solutions in Examples 1-1. are both commutative and cocommutative. In [9, Corollary 3.4], it is proved that if s is an involutive solution, i.e., $s^2 = \mathrm{id}_{S\times S}$, then s is both commutative and cocommutative.
Convention: In the sequel, we assume that S is a Clifford semigroup and simply write that s is a solution on S instead of $s(a, b) = (ab, \theta_a(b))$, for all $a, b \in S$.

2 Properties of solutions on Clifford semigroups

In this section, we show the existence of a normal subsemigroup associated to any solution s on S. We point out that the properties we proved are consistent with those given in the context of groups [6].

Proposition 4 *Let s be a solution on S. Then, the following statements hold:*
1. $\theta_a\left(a^{-1}\right) = \theta_{aa^{-1}}\left(a\right)^{-1}$,
2. $\theta_a\left(a^{-1}a\right) = \theta_a\left(a^{-1}\right)\theta_a\left(a^{-1}\right)^{-1} \in \mathrm{E}(S)$,
3. $\theta_{aa^{-1}} = \theta_{\theta_{a^{-1}}(aa^{-1})}\theta_{a^{-1}}$,

for every $a \in S$.

Proof Let $a \in S$. Then, by (P1), we have

$$\begin{aligned}\theta_a\left(a^{-1}\right)\theta_{aa^{-1}}\left(a\right)\theta_a\left(a^{-1}\right) &= \theta_a\left(a^{-1}a\right)\theta_a\left(a^{-1}\right) = \theta_a\left(a^{-1}a\right)\theta_{aa^{-1}a}\left(a^{-1}\right)\\ &= \theta_a\left(a^{-1}aa^{-1}\right) = \theta_a\left(a^{-1}\right)\end{aligned}$$

and $\theta_{aa^{-1}}\left(a\right)\theta_a\left(a^{-1}\right)\theta_{aa^{-1}}\left(a\right) = \theta_{aa^{-1}}\left(aa^{-1}\right)\theta_{aa^{-1}}\left(a\right) = \theta_{aa^{-1}}\left(aa^{-1}a\right) = \theta_{aa^{-1}}\left(a\right)$, hence $\theta_a\left(a^{-1}\right) = \theta_{aa^{-1}}\left(a\right)^{-1}$, so 1. is satisfied.

Moreover, by 1., we get $\theta_a\left(a^{-1}a\right) = \theta_a\left(a^{-1}\right)\theta_{aa^{-1}}\left(a\right) = \theta_a\left(a^{-1}\right)\theta_a\left(a^{-1}\right)^{-1}$, thus $\theta_a\left(a^{-1}a\right)$ is an idempotent of S.

Finally, by (P2), $\theta_{aa^{-1}} = \theta_{\theta_{a^{-1}}(aa^{-1})}\theta_{a^{-1}aa^{-1}} = \theta_{\theta_{a^{-1}}(aa^{-1})}\theta_{a^{-1}}$, which is our claim. $\square$

Note that the previous result also holds in any inverse semigroup that is not necessarily Clifford.

Now, let us introduce a crucial object in studying solutions on Clifford semigroups.

Definition 5 If s is a solution on S, the following set

$$K = \{a \in S \mid \forall\, e \in \mathrm{E}(S)\ e \leq a \Rightarrow \theta_e(a) \in \mathrm{E}(S)\}$$

is called the *kernel* of s.

Consistently with [6, Lemma 13], our aim is to show that K is a *normal subsemigroup* of the Clifford S, namely, $\mathrm{E}(S) \subseteq K$ and $a^{-1}Ka \subseteq K$, for every $a \in S$. To this end, we first provide a preliminary result.

Lemma 6 *Let s be a solution on S and K the kernel of s. Then, they hold:*

1. $\theta_a(e) \in \mathrm{E}(S)$, *for all $a \in S$ and $e \in \mathrm{E}(S)$ such that $a \leq e$;*
2. $\theta_{ea}(k) \in \mathrm{E}(S)$, *for all $a \in S$, $k \in K$, and $e \in \mathrm{E}(S)$ such that $e \leq a$, $e \leq k$.*

Proof Let $a \in S$ and $e \in \mathrm{E}(S)$. If $a \leq e$, by (P1), we obtain $\theta_a(e) = \theta_a(e)\theta_{ae}(e) = \theta_a(e)^2$, hence 1. follows. Now, if $k \in K$ and $e \leq a$, $e \leq k$, then $\theta_e(k) \in \mathrm{E}(S)$ and, by (P2),

$$\theta_{ea}(k) = \theta_{\theta_{a^{-1}}(ea)}\theta_{a^{-1}ea}(k) = \theta_{\theta_{a^{-1}}(ea)}\theta_e(k).$$

If we prove that $\theta_{a^{-1}}(ea) \leq \theta_e(k)$, by 1., we obtain that $\theta_{ea}(k) \in \mathrm{E}(S)$. We get

$$\begin{aligned}\theta_{a^{-1}}(ea) &= \theta_{a^{-1}}\left(eakk^{-1}\right) = \theta_{a^{-1}}(ea)\,\theta_{a^{-1}ea}\left(kk^{-1}\right) = \theta_{a^{-1}}(ea)\,\theta_e\left(kk^{-1}\right)\\ &= \theta_{a^{-1}}(ea)\,\theta_e(k)\,\theta_{ek}\left(k^{-1}\right).\end{aligned}$$

Hence, by (3), $\theta_{a^{-1}}(ea) \leq \theta_e(k)$. Therefore, the claim follows. □

Corollary 7 *Let s be a solution on S. If $a, b \in S$ are such that $a \leq b$, then $\theta_a(b) \in G_{\theta_a(bb^{-1})}$. Moreover, they hold $\theta_a(bb^{-1}) = \theta_a(b)\theta_a(b)^{-1}$ and $\theta_a(b)^{-1} = \theta_{ab}(b^{-1})$.*

Proof If $a, b \in S$ are such that $a \leq b$, then $a \leq bb^{-1}$ and by Lemma 6-1., $\theta_a(bb^{-1}) \in \mathrm{E}(S)$. Now,

$$\theta_a(b) = \theta_a\left(bb^{-1}b\right) = \theta_a\left(bb^{-1}\right)\theta_{abb^{-1}}(b) = \theta_a\left(bb^{-1}\right)\theta_a(b)$$

and $\theta_a\left(bb^{-1}\right) = \theta_a(b)\theta_{ab}\left(b^{-1}\right)$. Thus, by (3), $\theta_a(b) \leq \theta_a(bb^{-1})$ and $\theta_a(bb^{-1}) \leq \theta_a(b)$, i.e. $\theta_a(b) \in G_{\theta_a(bb^{-1})}$. In addition, by the equality $\theta_a\left(bb^{-1}\right) = \theta_a\left(b^{-1}b\right) = \theta_a\left(b^{-1}\right)\theta_{ab^{-1}}(b)$ and the previous paragraph, it follows that $\theta_a(b)$, $\theta_a(b^{-1})$, and $\theta_a(bb^{-1})$ are in the same group with identity $\theta_a(bb^{-1})$. Moreover, $\theta_a(b)^{-1} = \theta_{ab}\left(b^{-1}\right)$, which completes the proof. □

Theorem 8 *Let s be a solution on S. Then, the kernel K of s is a normal subsemigroup of S.*

Proof Initially, by Lemma 6-1., $\mathrm{E}(S) \subseteq K$. Now, if $k, h \in K$ and $e \in \mathrm{E}(S)$ are such that $e \leq kh$, then $e \leq k$ and $e \leq h$ and thus, $\theta_e(k), \theta_e(h) \in \mathrm{E}(S)$. By Lemma 6-2., we obtain that $\theta_{ek}(h) \in \mathrm{E}(S)$, and so that $\theta_e(kh) = \theta_e(k)\,\theta_{ek}(h) \in \mathrm{E}(S)$.

Now, if $a \in S$, $k \in K$, and $e \in \mathrm{E}(S)$ are such that $e \leq a^{-1}ka$, then $e \leq a$, $e \leq a^{-1}$, and $e \leq k$. Then, $\theta_e(k) \in \mathrm{E}(S)$. Besides,

$$\theta_e\left(a^{-1}ka\right) = \theta_e\left(a^{-1}\right)\theta_{ea^{-1}}(k)\theta_{ea^{-1}k}(a).$$

By Lemma 6-1., $\theta_e\left(a^{-1}\right) \in \mathrm{E}(S)$ and, by Lemma 6-2., $\theta_{ea^{-1}}(k) \in \mathrm{E}(S)$. Furthermore, also $\theta_{ea^{-1}k}(a) \in \mathrm{E}(S)$. In fact, by (P2),

$$\theta_{ea^{-1}k}\left(a\right) = \theta_{\theta_{k^{-1}a}(ea^{-1}k)}\theta_{k^{-1}aea^{-1}k}\left(a\right) = \theta_{\theta_{k^{-1}a}(ea^{-1}k)}\theta_e\left(a\right)$$

and, since

$$\begin{aligned}\theta_{k^{-1}a}\left(ea^{-1}k\right) &= \theta_{k^{-1}a}\left(ea^{-1}kaa^{-1}\right)\theta_{k^{-1}a}\left(ea^{-1}k\right)\theta_{k^{-1}aea^{-1}k}\left(aa^{-1}\right) = \\ &= \theta_{k^{-1}a}\left(ea^{-1}k\right)\theta_e\left(a\right)\theta_{ea}\left(a^{-1}\right),\end{aligned}$$

we obtain that, by (3), $\theta_{k^{-1}a}\left(ea^{-1}k\right) \leq \theta_e\left(a\right)$. So, as before, by Lemma 6-1., we obtain $\theta_{ea^{-1}k}\left(a\right) \in E\left(S\right)$. Therefore, the claim follows. □

We conclude the section by describing the commutative and cocommutative solutions on Clifford semigroups. It is easy to check that a solution $s(a, b) = (ab, \theta_a(b))$ is commutative if, and only if,

$$acb = abc \tag{C1}$$

$$\theta_a = \theta_{ab} \tag{C2}$$

and s is cocommutative if, and only if,

$$a\theta_b(c) = ac \tag{CC1}$$

$$\theta_a\theta_b = \theta_b\theta_a \tag{CC2}$$

for all $a, b, c \in S$.

Proposition 9 *Let s be a solution on S. Then,*

1. *s is commutative if, and only if, S is a commutative Clifford semigroup and $\theta_a = \gamma$, for every $a \in S$, with $\gamma \in \mathrm{End}(S)$ and $\gamma^2 = \gamma$.*
2. *s is cocommutative if, and only if, $\theta_a(b) = b$, for all $a, b \in S$, i.e., $s = \mathcal{I}$.*

Proof At first, we suppose that $s(a, b) = (ab, \theta_a(b))$ is a commutative solution. Then, by (C1), taking $a = cc^{-1}$, we obtain that S is commutative. Moreover, by (C2), we get $\theta_a = \theta_{ab} = \theta_{ba} = \theta_b$. Hence, $\theta_a = \gamma$, for every $a \in S$, and by the definition of solution we obtain the rest of the claim. The converse trivially follows by 2. in Examples 1.

Now, assume that $s(a, b) = (ab, \theta_a(b))$ is a cocommutative solution. Then, by (CC1), taking $a = cc^{-1}$, we obtain

$$cc^{-1}\theta_b(c) = c, \quad \text{for all } b, c \in S.$$

Set $e_0 := \theta_b(c)\theta_b(c)^{-1}$, it follows that $cc^{-1} \leq e_0$. On the other hand, again by (CC1), $e\theta_b(c) = ec$, for every $e \in \mathrm{E}(S)$. In particular, $\theta_b(c) = e_0\theta_b(c) = e_0c$. Thus, $e_0 \leq cc^{-1}$ and so $e_0 = cc^{-1}$. Therefore, we get $\theta_b(c) = c$, that is our claim. □

3 A description of idempotent-invariant solutions

In this section, we provide a description of a specific class of solutions on a Clifford semigroup, the idempotent-invariant ones, which includes the result contained in [6, Theorem 15].

Definition 10 A solution s on S is said to be *idempotent-invariant* or $E\,(S)$-*invariant* if it holds the identity

$$\theta_a(e) = \theta_a(f), \tag{6}$$

for all $a \in S$ and $e, f \in \mathrm{E}(S)$.

An easy example of $\mathrm{E}(S)$-invariant solution is $\mathcal{E}(a, b) = (ab, e)$ in (5), with $e \in \mathrm{E}(S)$.

Example 2 Let us consider the commutative Clifford monoid $S = \{1, a, b\}$ with identity 1 and such that $a^2 = a$, $b^2 = a$, and $ab = b$. Then, other than the map $\mathcal{E}$ in (5), there exists the idempotent-invariant solution $s(a, b) = (ab, \gamma(b))$ with $\gamma : S \to S$ the map given by $\gamma(1) = \gamma(a) = a$ and $\gamma(b) = b$, which belongs to the class of solutions in 2. of Examples 1.

Next, we show how to construct an idempotent-invariant solution on S starting from a specific congruence on S. Recall that the restriction of a congruence ρ in a Clifford semigroup S to $\mathrm{E}(S)$ is also a congruence on $\mathrm{E}(S)$, called the *trace* of ρ and usually denoted by $\tau = \operatorname{tr}\rho$ (for more details, see [14, Section 5.3]).

Proposition 11 *Let S be a Clifford semigroup, ρ a congruence on S such that S/ρ is a group and $\mathcal{R}$ a system of representatives of S/ρ. If $\mu : S \to \mathcal{R}$ is a map such that*

$$\mu\,(ab) = \mu\,(a)\,\mu\,(a)^{-1}\,\mu\,(ab)\,, \quad \textit{(the product considered is the operation in } S\textit{)} \tag{7}$$

for all $a, b \in S$, and $\mu(a)\,\rho\,a$, for every $a \in S$, then the map $s : S \times S \to S \times S$ given by

$$s(a, b) = \Big(ab, \mu\,(a)^{-1}\,\mu\,(ab)\Big),$$

for all $a, b \in S$, is an $\mathrm{E}(S)$-*invariant solution on S.*

Proof Let $a, b, c \in S$. Set $\theta_a(b) := \mu\,(a)^{-1}\,\mu\,(ab)$, by (7), we obtain

$$\theta_a(b)\theta_{ab}(c) = \mu\,(a)^{-1}\,\mu\,(ab)\,\mu\,(ab)^{-1}\,\mu\,(abc) = \mu\,(a)^{-1}\,\mu\,(abc) = \theta_a(bc).$$

Now, if we compare

$$\begin{aligned}\theta_{\theta_a(b)}\theta_{ab}(c) : &= \mu\left(\mu\left(a\right)^{-1}\mu\left(ab\right)\right)^{-1}\mu\left(\mu\left(a\right)^{-1}\mu\left(ab\right)\mu\left(ab\right)^{-1}\mu\left(abc\right)\right)\\ &= \mu\left(\mu\left(a\right)^{-1}\mu\left(ab\right)\right)^{-1}\mu\left(\mu\left(a\right)^{-1}\mu\left(abc\right)\right) && \text{by (7)}\end{aligned}$$

and $\theta_b(c) := \mu(b)^{-1}\mu(bc)$, to get the claim it is enough to show that

$$\mu(x)^{-1}\mu(xy) = \mu(y),$$

for all $x, y \in S$. Indeed, by [14, Proposition 5.3.1], tr $\rho = \mathrm{E}(S) \times \mathrm{E}(S)$, and so

$$\mu(x)^{-1}\mu(xy)\ \rho\ x^{-1}xy\ \rho\ y^{-1}yy\ \rho\ y\ \rho\ \mu(y).$$

Finally, if $a \in S$ and $e, f \in \mathrm{E}(S)$, we obtain that

$$\mu(ae)\ \rho\ ae\ \rho\ af\ \rho\ \mu(af),$$

hence $\mu\left(ae\right) = \mu\left(af\right)$. Thus, $\theta_a(e) = \mu(a)^{-1}\mu\left(ae\right) = \mu(a)^{-1}\mu\left(af\right) = \theta_a(f)$. Therefore, the claim follows. □

Our aim is to show that all idempotent invariant solutions can be constructed exactly as in Proposition 11. Firstly, let us collect some useful properties of these maps.

Lemma 12 *Let s be an $E\left(S\right)$-invariant solution on S. Then, the following hold:*

1. $\theta_e = \theta_f$,
2. $\theta_{ae} = \theta_a$,
3. $\theta_a\left(e\right) \in \mathrm{E}\left(S\right)$,
4. $\theta_e\theta_a = \theta_e$,
5. $\theta_a\left(b\right) = \theta_a\left(eb\right)$,
6. $\theta_e(a)^{-1} = \theta_{ea}\left(a^{-1}\right)$,

for all $e, f \in \mathrm{E}\left(S\right)$ and $a, b \in S$.

Proof Let $e, f \in \mathrm{E}(S)$ and $a, b \in S$.

1. Since $\theta_e = \theta_{\theta_f(e)}\theta_{fe} = \theta_{\theta_f(fe)}\theta_{ffe} = \theta_{fe}$ and, similarly $\theta_f = \theta_{ef}$, it yields that $\theta_f = \theta_e$.
2. We have that

$$\begin{aligned}\theta_{ae} &= \theta_{\theta_{a^{-1}}(ae)}\theta_{aa^{-1}e}\\ &= \theta_{\theta_{a^{-1}}(a)\theta_{a^{-1}a}(e)}\theta_{aa^{-1}} && aa^{-1}e \in \mathrm{E}\left(S\right)\\ &= \theta_{\theta_{a^{-1}}(a)\theta_{a^{-1}a}(a^{-1}a)}\theta_{aa^{-1}} && \text{by (6)}\\ &= \theta_{\theta_{a^{-1}}(a)}\theta_{aa^{-1}} = \theta_a.\end{aligned}$$

3. According to 2., it follows that $\theta_a(e) = \theta_a(ee) = \theta_a(e)\,\theta_{ae}(e) = \theta_a(e)\,\theta_a(e)$, i.e., $\theta_a(e) \in \mathrm{E}(S)$.
4. According to 2., we obtain that $\theta_e = \theta_{\theta_a(e)}\theta_{ae} = \theta_e\theta_{ae} = \theta_e\theta_a$.
5. Note that, by 2., $\theta_a(b) = \theta_a\left(bb^{-1}b\right) = \theta_a\left(bb^{-1}\right)\theta_{abb^{-1}}(b) = \theta_a(e)\,\theta_{ae}(b) = \theta_a(eb)$.
6. Applying 1., we get $\theta_e(a)\,\theta_{ea}\left(a^{-1}\right)\theta_e(a) = \theta_e\left(aa^{-1}\right)\theta_{eaa^{-1}}(a) = \theta_e(a)$ and, on the other hand,

$$\begin{aligned}\theta_{ea}\left(a^{-1}\right)\theta_e(a)\theta_{ea}\left(a^{-1}\right) &= \theta_{ea}\left(a^{-1}\right)\theta_e\left(aa^{-1}\right) = \theta_{ea}\left(a^{-1}\right)\theta_{eaa^{-1}}\left(aa^{-1}\right)\\ &= \theta_{ea}\left(a^{-1}\right).\end{aligned}$$

Therefore, the claim follows. □

To prove the converse of Proposition 11, we need to recall the notion of the congruence pair of inverse semigroups that are Clifford (see [14, p. 155]). Given a Clifford semigroup S, a congruence τ on $\mathrm{E}(S)$ is said to be *normal* if

$$\forall\, e, f \in \mathrm{E}(S) \quad e\,\tau\, f \implies \forall\, a \in S \quad a^{-1}ea\,\tau\, a^{-1}fa.$$

If K is a normal subsemigroup of S, the pair (K, τ) is named a *congruence pair* of S if

$$\forall\, a \in S,\ e \in \mathrm{E}(S) \quad ae \in K \ \text{ and } \ (e, a^{-1}a) \in \tau \implies a \in K.$$

Given a congruence ρ, denoted by $\operatorname{Ker}\rho$ the union of all the idempotent ρ-classes, its properties can be described entirely in terms of $\operatorname{Ker}\rho$ and $\operatorname{tr}\rho$.

Theorem 13 (cf. Theorem 5.3.3 in [14]) *Let S be an inverse semigroup. If ρ is a congruence on S, then* $(\operatorname{Ker}\rho, \operatorname{tr}\rho)$ *is a congruence pair. Conversely, if (K, τ) is a congruence pair, then*

$$\rho_{(K,\tau)} = \{(a, b) \in S \times S \mid \left(a^{-1}a, b^{-1}b\right) \in \tau,\ ab^{-1} \in K\}$$

is a congruence on S. Moreover, $\operatorname{Ker}\rho_{(K,\tau)} = K$, $\operatorname{tr}\rho_{(K,\tau)} = \tau$, *and* $\rho_{(\operatorname{Ker}\rho, \operatorname{tr}\rho)} = \rho$.

Lemma 14 *Let s be an* $\mathrm{E}(S)$*-invariant solution on S, $\tau = \mathrm{E}(S) \times \mathrm{E}(S)$, and K the kernel of s. Then, (K, τ) is a congruence pair of S.*

Proof At first, let us observe that the kernel K of s can be written as

$$K = \{a \in S \mid \forall\, e \in \mathrm{E}(S) \quad \theta_e(a) \in \mathrm{E}(S)\}.$$

Now, let $a \in S$ and $e \in \mathrm{E}(S)$ such that $ae \in K$. To get the claim it is enough to show that if $f \in \mathrm{E}(S)$, then $\theta_f(a) \in \mathrm{E}(S)$, i.e., $a \in K$. By 1. and 5. in Lemma 12, we obtain that

$$\theta_f(a) = \theta_{ef}(a) = \theta_{ef}(ae) \in \mathrm{E}(S),$$

which is our claim. □

The following result completely describes idempotent-invariant solutions.

Theorem 15 *Let s be an $\mathrm{E}(S)$-invariant solution on S. Then, the map θ_e satisfies (7), for every $e \in \mathrm{E}(S)$, and*

$$\theta_a(b) = \theta_e(a)^{-1}\theta_e(ab),$$

for all $a, b \in S$ and $e \in \mathrm{E}(S)$. Moreover, there exists the congruence pair (K, τ), with K the kernel of S and $\tau = \mathrm{E}(S) \times \mathrm{E}(S)$, such that $\theta_e(S)$ is a system of representatives of the group $S/\rho_{(K,\tau)}$ and $(\theta_e(a), a) \in \rho_{(K,\tau)}$, for all $e \in \mathrm{E}(S)$ and $a \in S$.

Proof Initially, (7) is satisfied since

$$\theta_e(a)^{-1}\theta_e(a)\theta_e(ab) = \theta_e(a)^{-1}\theta_e(a)\theta_e(a)\theta_{ea}(b) = \theta_e(a)\theta_{ea}(b) = \theta_e(ab),$$

for all $a, b \in S$ and $e \in \mathrm{E}(S)$. Besides,

$$\begin{aligned}
\theta_a(b) &= \theta_a\left(a^{-1}ab\right) && \text{by Lemma 12-5.}\\
&= \theta_a\left(a^{-1}\right)\theta_{aa^{-1}}(ab)\\
&= \theta_{aa^{-1}}(a)^{-1}\theta_{aa^{-1}}(ab), && \text{by Proposition 4-1.}\\
&= \theta_e(a)^{-1}\theta_e(ab) && \text{by Lemma 12-1.}
\end{aligned}$$

for all $a, b \in S$ and $e \in \mathrm{E}(S)$. Moreover, by Lemma 14, (K, τ) is a congruence pair and so, by Theorem 13, $\rho_{(K,\tau)}$ is a congruence such that $\tau = \operatorname{tr}\rho_{(K,\tau)}$. Besides, by [14, Proposition 5.3.1], since $\operatorname{tr}\rho_{(K,\tau)} = \mathrm{E}(S) \times \mathrm{E}(S)$, $S/\rho_{(K,\tau)}$ is a group. Now, let $a \in S$ and $e \in \mathrm{E}(S)$ and let us check that $(\theta_e(a), a) \in \rho_{(K,\tau)}$ by proving that $a^{-1}\theta_e(a) \in K$, i.e., $\theta_e\left(a^{-1}\theta_e(a)\right) \in \mathrm{E}(S)$. To this end, note that

$$\begin{aligned}
\theta_e\left(a^{-1}\theta_e(a)\right) &= \theta_e\theta_a\left(a^{-1}\theta_e(a)\right) && \text{by Lemma 12-4.}\\
&= \theta_e\left(\theta_a\left(a^{-1}\right)\theta_{aa^{-1}}\theta_e(a)\right)\\
&= \theta_e\left(\theta_a\left(a^{-1}\right)\theta_{aa^{-1}}(a)\right) && \text{by Lemma 12-4.}\\
&= \theta_e\left(\theta_a\left(a^{-1}\right)\theta_a\left(a^{-1}\right)^{-1}\right), && \text{by Proposition 4-1.}
\end{aligned}$$

hence, by Lemma 12-3., $\theta_e\left(a^{-1}\theta_e(a)\right) \in \mathrm{E}(S)$. Now, let us verify that $\theta_e(S)$ is a system of representatives of $S/\rho_{(K,\tau)}$. Clearly, $\theta_e(S) \neq \emptyset$ since $\theta_e(e) \in \mathrm{E}(S)$. Besides, if $(\theta_e(b), a) \in \rho_{(K,\tau)}$ we have that $a\,\rho_{(K,\tau)}\,b$, since $(\theta_e(a), a) \in \rho_{(K,\tau)}$. Thus, $ab^{-1} \in K$ and so $\theta_e\left(ab^{-1}\right) \in \mathrm{E}(S)$. This implies that

$$
\begin{aligned}
\theta_e(b) &= \theta_e\left(bb^{-1}\right)\theta_{ebb^{-1}}(b) && \\
&= \theta_e\left(bb^{-1}\right)\theta_{\theta_e(ab^{-1})}(b) && \text{by Lemma 12-1.} \\
&= \theta_e\theta_e\left(ab^{-1}\right)\theta_{\theta_e(ab^{-1})}\theta_{eab^{-1}}(b) && \text{by (6) and Lemma 12-4.} \\
&= \theta_e\left(ab^{-1}\right)\theta_{ab^{-1}}(b) && \text{by Lemma 12-4.} \\
&= \theta_e\left(ab^{-1}\right)\theta_{eab^{-1}}(b) && \text{by Lemma 12-2. and (P2)} \\
&= \theta_e\left(ab^{-1}b\right) && \\
&= \theta_e(a). && \text{by Lemma 12-5.}
\end{aligned}
$$

Therefore, the claim follows. □

Proposition 16 *Let $s(a, b) = (ab, \theta_a(b))$ and $t(u, v) = (uv, \eta_u(v))$ be two $\mathrm{E}(S)$-invariant solutions on S. Then, s and t are isomorphic if, and only if, there exists an inverse semigroup isomorphism ψ of S such that $\psi\theta_e = \eta_e\psi$, i.e., ψ sends the system of representatives $\theta_e(S)$ into the other one $\eta_e(\psi(S))$, for every $e \in \mathrm{E}(S)$.*

Proof Indeed, making explicit the condition (4), we obtain

$$
\psi\left(\theta_e(a)^{-1}\theta_e(ab)\right) = \eta_e(\psi(a))^{-1}\,\eta_e(\psi(ab)), \tag{$*$}
$$

for all $a, b \in S$ and $e \in \mathrm{E}(S)$. Taking $b = a^{-1}$, we get

$$
\begin{aligned}
\psi\left(\theta_e(a)^{-1}\right) &= \psi\left(\theta_{ea}\left(a^{-1}\right)\right) && \text{by Lemma 12-6.} \\
&= \psi\left(\theta_{ea}\left(a^{-1}\right)\theta_e\left(aa^{-1}\right)\right) && \text{by Lemma 12-1.} \\
&= \psi\left(\theta_e(a)^{-1}\theta_e\left(aa^{-1}\right)\right) && \text{by Lemma 12-6.} \\
&= \eta_e(\psi(a))^{-1}\,\eta_e\left(\psi\left(aa^{-1}\right)\right) && \text{by Lemma } (*) \\
&= \eta_{e\psi(a)}\left(\psi(a)^{-1}\right)\eta_e\left(\psi\left(aa^{-1}\right)\right) && \text{by Lemma 12-6.} \\
&= \eta_{e\psi(a)}\left(\psi(a)^{-1}\right) && \text{by Lemma 12-1.} \\
&= \eta_e(\psi(a))^{-1} && \text{by Lemma 12-6.}
\end{aligned}
$$

Hence, since ψ is an inverse semigroup homomorphism, $\psi\theta_e = \eta_e\psi$, for every $e \in \mathrm{E}(S)$. Thus, the claim follows. □

4 A construction of idempotent-fixed solutions

In this section, we deal with a class of solutions different from the idempotent-invariant ones, what we call idempotent-fixed solutions. Bearing in mind that a Clifford semigroup can be seen as a union of groups satisfying certain properties, it is natural to contemplate whether it is possible or not to construct a global solution in a Clifford semigroup from solutions obtained in each of its groups. In this regard, in the case of idempotent-fixed solutions, we manage to construct a family of solutions obtained by starting from given solutions in each group.

Definition 17 Let s be a solution on S. Then, s is *idempotent-fixed* or $\mathrm{E}(S)$*-fixed* if

$$\theta_a(e) = e, \tag{8}$$

for all $a \in S$ and $e \in \mathrm{E}(S)$.

The maps $\mathcal{I}(a, b) = (ab, b)$ and $\mathcal{F}(a, b) = \left(ab, bb^{-1}\right)$ in (5) are idempotent-fixed solutions on S. Clearly, if S is a Clifford that is not a group, i.e., $|\,\mathrm{E}(S)| > 1$, then a solution on S can not be both idempotent-fixed and idempotent-invariant.

The next results contained several properties of idempotent-fixed solutions.

Proposition 18 *Let s be an idempotent-fixed solution on S. Then, $\theta_e = \theta_e\theta_{ae}$, for all $a \in S$ and $e \in \mathrm{E}(S)$. In particular, θ_e is an idempotent map.*

Proof It follows by $\theta_e = \theta_{\theta_a(e)}\theta_{ae} = \theta_e\theta_{ae}$, for all $a \in S$ and $e \in \mathrm{E}(S)$. Taking $a = e$, we obtain that the map θ_e is idempotent. □

Proposition 19 *Let s be an idempotent-fixed solution on S. Then, the following hold:*

1. $\theta_a(b) = bb^{-1}\theta_a(b)$,
2. $\theta_a(b)\,\theta_a(b)^{-1} = bb^{-1}$,
3. $\theta_a(b) = \theta_{abb^{-1}}(b)$,

for all $a, b \in S$.

Proof Let $a, b \in S$. Then, $\theta_a(b) = \theta_a(b)\,\theta_{ab}\left(b^{-1}b\right) = \theta_a(b)\,bb^{-1}$. Moreover, we have that $\theta_a(b)^{-1} = \theta_{ab}\left(b^{-1}\right)$ since

$$\theta_a(b)\,\theta_{ab}\left(b^{-1}\right)\theta_a(b) = \theta_a\left(bb^{-1}\right)\theta_a(b) = bb^{-1}\theta_a(b) = \theta_a(b)$$

and

$$\begin{aligned}\theta_{ab}\left(b^{-1}\right)\theta_a(b)\,\theta_{ab}\left(b^{-1}\right) &= \theta_{ab}\left(b^{-1}\right)\theta_a\left(bb^{-1}\right) = b^{-1}b\,\theta_{ab}\left(b^{-1}\right)\\ &= \theta_{ab}\left(bb^{-1}\right)\theta_{abb^{-1}b}\left(b^{-1}\right) = \theta_{ab}\left(b^{-1}\right).\end{aligned}$$

It follows that $\theta_a(b)\,\theta_a(b)^{-1} = \theta_a(b)\,\theta_{ab}\left(b^{-1}\right) = \theta_a\left(bb^{-1}\right) = bb^{-1}$. Finally, by 1., we have that

$$\theta_{abb^{-1}}(b) = bb^{-1}\theta_{abb^{-1}}(b) = \theta_a\left(bb^{-1}\right)\theta_{abb^{-1}}(b) = \theta_a(b)$$

that completes the proof. □

As a consequence of Proposition 19-1., if s is an idempotent-fixed solution on the Clifford S, it follows that every group in S remains invariant by θ_a, for all $a \in S$. Thus, motivated by the fact that solutions on groups are well-described, it makes sense to provide a method to construct this type of solutions from solutions on each group in S. To this end, the inner structure of a Clifford semigroup makes clear that conditions relating to different solutions on the groups of S must be considered. For instance, Proposition 19-3. shows that $\theta_a(b) = \theta_{\varphi_{e,f}(a)}(b)$, for all $e, f \in \mathrm{E}(S)$, with $e \geq f$, and all $a \in G_e$, $b \in G_f$. In light of these observations, we provide the following family of idempotent-fixed solutions.

Theorem 20 *Let $s^{[e]}(a,b) = \left(ab,\ \theta_a^{[e]}(b)\right)$ be a solution on G_e, for every $e \in \mathrm{E}(S)$. Moreover, for all $e, f \in \mathrm{E}(S)$, let $\epsilon_{e,f} : G_e \to G_f$ be maps satisfying*

$$\epsilon_{e,f} = \varphi_{e,f},\ textif\ e \geq f, \tag{9}$$

$$\theta^{[h]}_{\epsilon_{ef,h}(ab)} = \theta^{[h]}_{\epsilon_{e,h}(a)\epsilon_{f,h}(b)}, \tag{10}$$

$$\epsilon_{f,h}\theta^{[f]}_{\epsilon_{e,f}(a)}(b) = \theta^{[h]}_{\epsilon_{e,h}(a)}\epsilon_{f,h}(b), \tag{11}$$

for all $e, f, h \in \mathrm{E}(S)$ and $a \in G_e$ and $b \in G_f$, set

$$\theta_a(b) := \theta^{[f]}_{\epsilon_{e,f}(a)}(b),$$

for all $a \in G_e$ and $b \in G_f$, then the map $s : S \times S \to S \times S$ given by $s(a,b) = (ab, \theta_a(b))$ is an idempotent-fixed solution on S.

Proof Let $e, f, h \in \mathrm{E}(S)$, $a \in G_e$, $b \in G_f$, and $c \in G_h$. Then, since $s^{[fh]}$ is a solution on G_{fh}, we obtain

$$\begin{aligned}\theta_a(bc) &= \theta_a\left(\varphi_{f,fh}(b)\varphi_{h,fh}(c)\right) = \theta^{[fh]}_{\epsilon_{e,fh}(a)}\left(\varphi_{f,fh}(b)\varphi_{h,fh}(c)\right)\\ &= \theta^{[fh]}_{\epsilon_{e,fh}(a)}\varphi_{f,fh}(b)\,\theta^{[fh]}_{\epsilon_{e,fh}(a)\varphi_{f,fh}(b)}\varphi_{f,fh}(c).\end{aligned}$$

Besides, we have that

$$\theta_a(b)\,\theta_{ab}(c) = \theta^{[f]}_{\epsilon_{e,f}(a)}(b)\,\theta^{[h]}_{\epsilon_{ef,h}(ab)}(c) = \varphi_{f,fh}\theta^{[f]}_{\epsilon_{e,f}(a)}(b)\,\varphi_{h,fh}\theta^{[h]}_{\epsilon_{ef,h}(ab)}(c).$$

Hence, noting that, by (10),

$$\theta^{[fh]}_{\epsilon_{e,fh}(a)}\varphi_{f,fh}(b) = \theta^{[fh]}_{\epsilon_{e,fh}(a)}\epsilon_{f,fh}(b) = \epsilon_{f,fh}\theta^{[f]}_{\epsilon_{e,f}(a)}(b) = \varphi_{f,fh}\theta^{[f]}_{\epsilon_{e,f}(a)}(b)$$

and

$$
\begin{aligned}
\theta^{[fh]}_{\epsilon_{e,fh}(a)\varphi_{f,fh}(b)}\varphi_{f,fh}\,(c) &= \theta^{[fh]}_{\epsilon_{e,fh}(a)\epsilon_{f,fh}(b)}\epsilon_{f,fh}\,(c)\\
&= \theta^{[fh]}_{\epsilon_{ef,fh}(ab)}\epsilon_{f,fh}\,(c) && \text{by (10)}\\
&= \epsilon_{h,fh}\theta^{[h]}_{\epsilon_{ef,h}(ab)}\,(c) && \text{by (11)}\\
&= \varphi_{h,fh}\theta^{[h]}_{\epsilon_{ef,h}(ab)}\,(c)\,,
\end{aligned}
$$

it follows that (P1) is satisfied. In addition,

$$
\begin{aligned}
\theta_{\theta_a(b)}\theta_{ab}\,(c) &= \theta_{\theta^{[f]}_{\epsilon_{e,f}(a)}(b)}\theta^{[h]}_{\epsilon_{ef,h}(ab)}\,(c)\\
&= \theta^{[h]}_{\epsilon_{f,h}\theta^{[f]}_{\epsilon_{e,f}(a)}(b)}\theta^{[h]}_{\epsilon_{ef,h}(ab)}\,(c)\\
&= \theta^{[h]}_{\theta^{[h]}_{\epsilon_{e,h}(a)}\epsilon_{f,h}(b)}\theta^{[h]}_{\epsilon_{e,h}(a)\epsilon_{f,h}(b)}\,(c) && \text{by (11) and (10)}\\
&= \theta^{[h]}_{\epsilon_{f,h}(b)}\,(c) && s^{[h]}\text{is a solution on } G_h\\
&= \theta_b\,(c)\,,
\end{aligned}
$$

thus (P2) holds. Finally, by [6, Lemma 11-1.], $\theta_a(f) = \theta^{[f]}_{\epsilon_{e,f}(a)}(f) = f$ and so s is idempotent-fixed. □

The following is a class of idempotent-fixed solutions on S that can be constructed through Theorem 20 and includes the solutions $\mathcal{I}(a,b) = (ab, b)$ and $\mathcal{F}(a,b) = \left(ab, bb^{-1}\right)$ in (5).

Example 3 Let $s^{[e]}\,(a,b) = \left(ab, \gamma^{[e]}\,(b)\right)$ be the solution on G_e as in 2. of Examples 1 with $\gamma^{[e]}$ an idempotent endomorphism of G_e, for every $e \in \mathrm{E}(S)$. Assume that for all $e, f \in \mathrm{E}(S)$, with $e \geq f$, the group homomorphisms $\varphi_{e,f} : G_e \to G_f$ satisfy $\varphi_{e,f}\gamma^{[e]} = \gamma^{[f]}\varphi_{e,f}$. Take $\epsilon_{e,f} = \varphi_{e,f}$ if $e \geq f$ and $\epsilon_{e,f}\,(x) := f$, otherwise. Then, conditions (10) and (11) are satisfied. Hence, the map

$$
s(a,b) = \left(ab, \gamma^{[f]}(b)\right),
$$

for all $a \in G_e$ and $b \in G_f$, is a solution on S.

As a consequence of Theorem 20, the following construction provides a subclass of idempotent-fixed solutions in Clifford semigroups in which each group G_f is an epimorphic image of G_e, whenever $f \leq e$, for all $e, f \in \mathrm{E}(S)$.

Corollary 21 *Let S be a Clifford semigroup such that $\varphi_{e,f}$ is an epimorphism, for all $e, f \in \mathrm{E}(S)$ with $f \leq e$. Let $s^{[e]}(a,b) = \left(ab, \theta^{[e]}_a(b)\right)$ be a solution on G_e and set $N_e := \prod_{f \leq e} \ker \varphi_{e,f}$, for every $e \in \mathrm{E}(S)$. Suppose that*

1. $\theta_a^{[e]} = \theta_b^{[e]}$*, for all* $e \in \mathrm{E}(S)$ *and all* $a, b \in G_e$ *with* $aN_e = bN_e$*,*
2. $\varphi_{e,f}\theta_a^{[e]}(b) = \theta_{\varphi_{e,f}(a)}^{[f]}\varphi_{e,f}(b)$*, for all* $e, f \in \mathrm{E}(S)$ *with* $f \leq e$*, and all* $a, b \in G_e$*.*

Set $\theta_a(b) := \theta_{b'}^{[f]}(b)$*, with* $b' \in G_f$ *such that* $\varphi_{f,ef}(b) = \varphi_{e,ef}(a)$*, for all* $e, f \in \mathrm{E}(S)$*, and all* $a \in G_e$*,* $b \in G_f$*. Then, the map* $s\colon S \times S \to S \times S$ *given by* $s(a, b) = (ab, \theta_a(b))$ *is an idempotent-fixed solution on* S*.*

Proof Initially, by 1., note that θ_a is well-defined, for every $a \in S$. Now, let $e, f \in \mathrm{E}(S)$ and consider $T_{e,f}$ a system of representatives of $\ker \varphi_{f,ef}$ in G_f. Since $\varphi_{f,ef}$ is an epimorphism, for every $a \in G_e$, we can define a map $\epsilon_{e,f}(a) := x \in T_{e,f}$, with $\varphi_{e,ef}(a) = \varphi_{f,ef}(x)$. Specifically, in the case that $f \leq e$, it follows that $\epsilon_{e,f} = \varphi_{e,f}$. Therefore, for all $e, f \in \mathrm{E}(S)$ and all $a \in G_e$, $b \in G_f$, it holds $\theta_a(b) = \theta_{\epsilon_{e,f}(a)}^{[f]}(b)$. Note that, by 1., the last equality is independent of the choice of $T_{e,f}$. Moreover, applying properties in Theorem 1 of homomorphisms $\varphi_{e,f}$, for all $e, f \in \mathrm{E}(S)$ with $f \leq e$, and the assumptions, it is a routine computation to check that conditions (10) and (11) of Theorem 20 are satisfied. □

Let us observe that the kernel of an idempotent-fixed solution s can be rewritten as

$$K = \{a \in S \mid \forall\, e \in \mathrm{E}(S),\ e \leq a,\ \theta_e(a) = aa^{-1}\}.$$

Denoted by K_e the kernel of each solution $s^{[e]}$ on G_e, i.e., the normal subgroup

$$K_e = \{a \in G_e \mid \theta_e^{[e]}(a) = e\}.$$

of G_e, we have the following result that clarifies the previous construction in Theorem 2 20 is not a description.

Proposition 22 *Let* s *be an idempotent-fixed solution on* S *constructed as in Theorem 20 and suppose that* $\epsilon_{e,f}(e) = f$*, for all* $e, f \in \mathrm{E}(S)$ *with* $e \leq f$*. Assume that each* G_e *admits a solution* $s^{[e]}$ *and let* K_e *be the kernel of such a map* $s^{[e]}$*, for every* $e \in \mathrm{E}(S)$*. Then,* $K = \bigcup\limits_{e \in \mathrm{E}(S)} K_e$*.*

Proof Indeed, let $a \in K \cap G_e$. Then, we get $e = aa^{-1} = \theta_e(a) = \theta_e^{[e]}(a)$. Thus, $a \in K_e$. On the other hand, if $a \in K_e$ and $f \in \mathrm{E}(S)$ is such that $f \leq a$, then, since $\epsilon_{e,f}(e) = f$, we obtain $\theta_f(a) = \theta_{\epsilon_{f,e}(f)}^{[e]}(a) = \theta_e^{[e]}(a) = e$, i.e., $a \in K$. □

In light of the previous discussion, the following question arises.

Question 1 Complete a description of all the idempotent-fixed solutions.

To conclude, we observe that not every solution on S lies in the class of idempotent invariant or idempotent-fixed solutions. Indeed, even in Clifford semigroups of low order, it is possible to construct such an example.

Example 4 Let $S = \{1,\ a,\ b\}$ be the Clifford monoid in Example 2. Then, the maps

$$\theta_1(x) = a, \quad \text{for every} \;\; x \in S,$$
$$\theta_a = \theta_b : S \to S, \quad \text{given by} \;\; \theta_a(1) = 1, \;\; \theta_a(a) = \theta_a(b) = a$$

give rise to a solution on S that is neither idempotent invariant nor idempotent fixed.

Question 2 Find and study other classes of solutions on Clifford semigroups, including, for instance, the map in Example 4.

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