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RESEARCH ARTICLE

Ternary rings of operators arising from inverse semigroups

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Abstract
We are interested in properties, especially *injectivity* (in the sense of category theory), of the ternary rings of operators generated by certain subsets of an inverse semigroup via the regular representation. We determine all subsets of the extended bicyclic semigroup which are closed under the triple product xy^*z (called semiheaps) and show that the weakly closed ternary rings of operators generated by them are injective operator spaces.

Keywords Ternary ring of operators · Inverse semigroup · Bicyclic semigroup · Semiheap · Injective operator space

1 Introduction

Ternary rings of operators (TROs) originated in the work of M. R. Hestenes in 1962 [4]. These are linear spaces of operators from one Hilbert space to another which are stable under the triple product XY^*Z, which he called *ternary algebras*. By their nature, these spaces satisfied an associativity condition involving five elements, namely,

$$(XY^*Z)U^*W = XY^*(ZU^*W) = X(UZ^*Y)^*W. \qquad (1)$$

Dedicated to the memory of H. Garth Dales.

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These were subsequently axiomatized and named *associative triple systems* [12]. A milestone in their development in the realm of functional analysis was a Gelfand-Naimark type representation theorem for associative triple systems equipped with an operator type norm [16].

At around the same time as Hestenes' work, unbeknownst to the researchers in the West due partially to the Cold War [5], the concept of *semiheap* was introduced in the Soviet Union [8]. A semiheap is a set together with a single three-variable operation satisfying an abstract version of (1), and akin to the known concepts of *ternary group* and *inverse semigroup*.

Since the concept of semiheap is central to this paper, we provide the formal definition, as stated in [8, p. 56]. By a *semiheap*, we mean a set K together with a singled-valued, everywhere defined ternary operation $[\cdot\cdot\cdot]$, satisfying the condition

$$[[k_1k_2k_3]k_4k_5] = [k_1[k_4k_3k_2]k_5] = [k_1k_2[k_3k_4k_5]].$$

An *inverse semigroup* is a semigroup S in which for every element x there exists a unique element x^*, called the *inverse* or *generalized inverse* of x, such that $x = xx^*x$ and $x^* = x^*xx^*$. For the basic facts on inverse semigroups, see [11, Chapter 1] or [7, Chapter 5].

Semiheaps and their associated structures are closely related to inverse semigroups. In turn, inverse semigroups, together with groupoids, give rise to operator algebras [13]. A ubiquitous example of an inverse semigroup is the *bicyclic semigroup*, given abstractly ([11, Section 3.4]) by the presentation $\langle p, q : pq = 1\rangle$, and concretely ([7, p. 144]) as $\mathbb{N} \times \mathbb{N}$ with the multiplication

$$(m, n)(p, q) = (m - n + \max(n, p), q - p + \max(n, p)) \tag{2}$$

We shall use the following notation: $\mathbb{N} = \{1, 2, \ldots\}$; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $\mathbb{Z} = \mathbb{N}_0 \cup -\mathbb{N}$.

In this paper, we analyze the *extended bicyclic semigroup*, which we call E throughout this paper, in such a way that exhibits its semiheap structure. This inverse semigroup E, which is the set $\mathbb{Z} \times \mathbb{Z}$ together with the multiplication (2), was defined originally in [15, p. 367]; however, in that and most other papers, only binary structures are considered.

Unlike the bicyclic semigroup, the extended bicyclic semigroup is not finitely generated, nor does it have an identity element. Nevertheless, they share the same semigroup identities ([1, Corollary 4.3]).

We shall use the representation of the extended bicyclic semigroup which is based on the realization of the bicyclic semigroup by the unilateral shift ([13, p. 188]), as follows.

Let E_{22} be the bicyclic semigroup, as realized by the unilateral shift; that is,

$$E_{22} = \{a_{ij} = \sum_{k\geq 0} e_{i+k,j+k} : i, j \in \mathbb{N}_0\},$$

where for any $i, j \in \mathbb{Z}$, e_{ij} is the matrix over $\mathbb{Z}$ with 1 in the i, j position and zeros elsewhere, and the $\aleph_0$ by $\aleph_0$ matrix a_{ij} acts as a linear operator on column vectors of complex numbers. (a_{ij} is a bounded operator on the Hilbert space $\ell^2(\mathbb{Z})$.)

Set

$$E = E_{11} \cup E_{12} \cup E_{21} \cup E_{22}$$

where $E_{21} = \{a_{ij} : i \in \mathbb{N}_0, j \in -\mathbb{N}\}$, $E_{11} = \{a_{ij} : i, j \in -\mathbb{N}\}$ and $E_{12} = \{a_{ij} : i \in -\mathbb{N}, j \in \mathbb{N}_0\}$.

We note that for $i, j, p, q \in \mathbb{Z}$, $a_{ij}^* = a_{ji}$, a_{ij} is a partial isometry on $\ell^2(\mathbb{Z})$ and

$$a_{ij}a_{pq} = \begin{cases} a_{i,q+j-p}, & p \leq j \\ a_{i+p-j,q}, & p \geq j \end{cases}, \tag{3}$$

equivalently

$$a_{ij}a_{pq} = a_{i+p-\min(j,p),j+q-\min(j,p)}.$$

In particular, $a_{ij}a_{pq} \neq 0$, $a_{ij}a_{jq} = a_{iq}$, and $a_{ii}a_{pp} = a_{mm}$ with $m = \max(i, p)$.

Remark 1.1 Thus E is an inverse semigroup consisting of partial isometries with inverse a_{ij}^* equal to the adjoint of a_{ij}. E is isomorphic to the extended bicyclic semigroup, and when convenient notationally, we represent a_{ij} in formulas and diagrams simply by $(i, j) \in \mathbb{Z} \times \mathbb{Z}$.

We shall analyze the extended bicyclic semigroup E toward the aims of finding all of the subsemiheaps of E, and showing that the associated W*-TROs, that is, weakly closed TROs, are injective operator spaces.

In our main and only theorem, Theorem 1.2, we classify all of the subsemiheaps of this extended bicyclic semigroup. We then show in Corollary 4.3, via a general result applying to all inverse semigroups [13, Theorem 4.5.2], that each of the examples resulting from this classification has the property that the weakly closed ternary ring of operators it generates is an injective operator space. It is worth pointing out that, although the injectivity of the W*-TROs generated by the classification of subsemiheaps uses deep results in functional analysis ([3, Theorem 2.5], [13, Theorem 4.5.2]), the classification itself is self-contained using only elementary arguments.

Our results are summarized in the following theorem, listing all of the subsemiheaps of the extended bicyclic semigroup. The proof is contained in the references in each statement to later results of this paper.

Theorem 1.2 *The subsemiheaps of the extended bicyclic semigroup are the inductive limits (see Remark 3.1) of sequences of the following semiheaps K:*

- *K is a single point $\{a_{pq}\}$ (Lemma 3.2 and Example 2.4)*
- *$K = \{a_{\alpha_0,\beta_0}\} \cup \{a_{\alpha_0+k_j,\beta_0+k_j} : 1 \leq j \leq n_0\}$, where $1 \leq k_1 < k_2 < \cdots < k_{n_0}$ and $n_0 \in \mathbb{N}$*
  (Proposition 3.3 and Examples 2.4)

- $K = \{a_{\alpha_0\beta_0}\} \cup \{a_{\alpha_0+k_j,\beta_0+k_j} : j \in \mathbb{N}\}$ *where* $1 \leq k_1 < k_2 < \cdots < k_j < \cdots < \infty$ *(Proposition 3.4 and Example 2.4)*
- *There exist* $\sigma, \ell_0 \in \mathbb{N}$ *such that*

$$K = \{a_{\alpha_0\beta_0}\} \cup \{a_{\alpha_0+k_j,\beta_0+k_j} : j = 1, \ldots \ell_0 - 1\} \cup K^{\sigma}_{\alpha_0+k_{\ell_0},\beta_0+k_{\ell_0}}$$

or

$$K = \{a_{\alpha_0\beta_0}\} \cup \{a_{\alpha_0+k_i,\beta_0+k_i} : 1 \leq i < \ell_0\} \cup \left(\bigcup_{i=\ell_0}^{j} K^{\sigma}_{\alpha_0+k_i,\beta_0+k_i} \right)$$

where $1 \leq k_1 < k_2 < \cdots < k_{\ell_0}$. *(Proposition 3.4 and Examples 2.5* and *2.6)*
- $K = K^{p}_{\alpha_0,\beta_0}$ *for some* $p > 0$ *(Proposition 3.18 and Example 2.7)*
- *There exist* $p > 0$ *and* $q > 0$ *such that*

$$K = \bigcup_{i=0}^{n} K^{p}_{\alpha_0+q_i,\beta_0+q_i}.$$

where $a_{\alpha_0+q_i,\beta_0+q_i}$, $0 \leq i < \infty$, *are the points of* K *lying on the diagonal, such that*

$$q = q_0 < q_1 < q_2 < \cdots < q_n < p \quad \textit{and} \quad p < q_{n+1} < q_{n+2} < \cdots.$$

(Proposition 3.18 and Example 2.7)

All of the subsemigroups of the bicyclic semigroup have been determined in [2]. Those subsemigroups which are inverse subsemigroups, which were determined earlier in [14] and later in [6], were also identified in [2, Theorem 7.1]. Since inverse subsemigroups are semiheaps, our results give a new approach to the description of the inverse subsemigroups of the (extended) bicyclic semigroup.

2 Diagrams 1–10 and Examples

In order to analyze the subsemiheaps of the extended bicyclic semigroup E, we prepare some material.

The idempotents of E are the elements a_{ii} with $i \in \mathbb{Z}$ and $a_{ii} \leq a_{jj}$, that is, $a_{ii}a_{jj} = a_{ii}$, if and only if $j \leq i$. From (3), we calculate and find that for $p, q \in \mathbb{Z}$,

$$a_{ii}a_{pq} = \begin{cases} a_{i,q+i-p} & p \leq i \\ a_{pq} & p \geq i \end{cases}$$

$$a_{pq}a_{jj} = \begin{cases} a_{pq} & j \leq q \\ a_{p+j-q,j} & j \geq q \end{cases}$$

and

$$a_{ii}a_{pq}a_{jj} = \begin{cases} a_{i,q+i-p} & p \le i \text{ and } j \le q+i-p \\ a_{j-q+p,q} & p \le i \text{ and } j \ge q+i-p \\ a_{pq} & p \ge i \text{ and } j \le q \\ a_{p+j-q,j} & p \ge i \text{ and } j \ge q \end{cases}$$

In particular, $a_{00}E = E_{21} \cup E_{22}$ and $Ea_{00} = E_{12} \cup E_{22}$. Also, $a_{ii}E$, Ea_{jj} and $a_{ii}Ea_{jj}$ are subsemigroups and semiheaps, and $a_{ii}Ea_{jj}$ is an inverse semigroup if $i = j$. Also,

$$Ea_{jj} = \{a_{pq} : j \le q\}, \quad a_{ii}E = \{a_{pq} : p \ge i\},$$

and

$$a_{ii}E \cap Ea_{jj} = a_{ii}Ea_{jj} = \{a_{pq} : p \ge i, q \ge j\}.$$

From (3), we have the following lemma.

Lemma 2.1 *For any a_{ij}, a_{pq}, a_{rs} in E, we have*

$$a_{ij}a^*_{pq}a_{rs} = \begin{cases} (i)\ a_{i,s+p+j-q-r} & r \le p+j-q,\ q \le j \\ (ii)\ a_{i+r-p-j+q,s} & r \ge p+j-q,\ q \le j \\ (iii)\ a_{i+q-j+r-p,s} & r \ge p,\ q \ge j \\ (iv)\ a_{i+q-j,s+p-r} & r \le p,\ q \ge j \end{cases}$$

It is worth noting, as will be evident in the ten diagrams that follow, all triple products in E which involve only two elements, produce new elements which do not propagate to the left of, or up from the diagram.

Lemma 2.2 and Diagrams 1–5 describe the case in which the slope of the line connecting the two points is negative (or zero or infinite). Lemma 2.3 and Diagrams 6-10 describe the case in which the slope of the line connecting the two points is positive (or zero or infinite).

Lemma 2.2 *If K is a subsemiheap of E, and if $a_{\alpha\beta}, a_{\gamma\delta} \in K$ with $\gamma \ge \alpha$ and $\delta \ge \beta$, then the following elements belong to K:*

- $x_1 = a_{\alpha+\delta-\beta,\beta+\gamma-\alpha}$
- $x_2 = a_{\alpha+\delta-\beta,\delta}$
- $x_3 = a_{\gamma,\beta+\gamma-\alpha}$
- $x_4 = a_{\gamma,\delta+(\delta-\beta)-(\gamma-\alpha)}$ *if* $\gamma - \alpha \le \delta - \beta$
- $x_5 = a_{\gamma+(\gamma-\alpha)-(\delta-\beta),\delta}$ *if* $\gamma - \alpha \ge \delta - \beta$

Proof The following are the eight possible triple products containing two distinct elements, and thus belong to K. They are calculated using Lemma 2.1.

- $a_{\alpha\beta}a^*_{\alpha\beta}a_{\alpha\beta} = a_{\alpha\beta}$
- $a_{\alpha\beta}a^*_{\gamma\delta}a_{\alpha\beta} = a_{\alpha+\delta-\beta,\beta+\gamma-\alpha} = x_1$
- $a_{\alpha\beta}a^*_{\alpha\beta}a_{\gamma\delta} = a_{\gamma\delta}$

- $a_{\alpha\beta}a^*_{\gamma\delta}a_{\gamma\delta} = a_{\alpha+\delta-\beta,\delta} = x_2$
- $a_{\gamma\delta}a^*_{\alpha\beta}a_{\alpha\beta} = a_{\gamma\delta}$
- $a_{\gamma\delta}a^*_{\gamma\delta}a_{\alpha\beta} = a_{\gamma,\beta+\gamma-\alpha} = x_3$
- $a_{\gamma\delta}a^*_{\alpha\beta}a_{\gamma\delta} = \begin{cases} a_{\gamma,\delta+\alpha+\delta-\beta-\gamma} = x_4, & \gamma-\alpha \leq \delta-\beta \\ a_{\gamma+\gamma-\alpha-\delta+\beta,\delta} = x_5, & \gamma-\alpha \geq \delta-\beta \end{cases}$
- $a_{\gamma\delta}a^*_{\gamma\delta}a_{\gamma\delta} = a_{\gamma\delta}$

□

Diagram 1 $\delta-\beta > \gamma-\alpha > 0 \quad (b=\delta-\beta, h=\gamma-\alpha)$

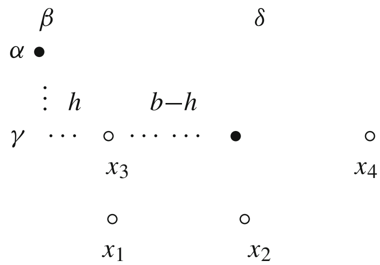


Diagram 2 $\gamma=\alpha, \delta-\beta > 0$

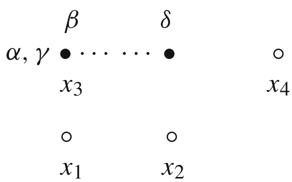


Diagram 3 $\gamma-\alpha > \delta-\beta > 0 \quad (b=\delta-\beta, h=\gamma-\alpha)$

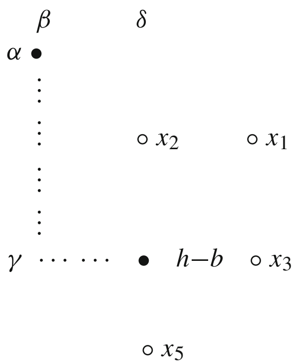


Diagram 4 $\delta = \beta, \gamma - \alpha > 0$

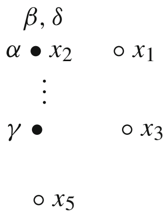
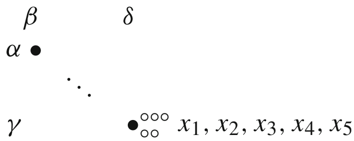


Diagram 5 $\delta - \beta = \gamma - \alpha > 0$



Lemma 2.3 *If K is a subsemiheap of E, and if $a_{\alpha\beta}, a_{\gamma\delta} \in K$ with $\gamma \geq \alpha$, $\delta \leq \beta$, then the following elements belong to K:*

- $x_1 = a_{\alpha,\beta+(\gamma-\alpha)+(\beta-\delta)}$
- $x_2 = a_{\gamma+\beta-\delta,\beta}$
- $x_3 = a_{\gamma,\beta+\gamma-\alpha}$
- $x_4 = a_{\gamma+(\beta-\delta)+(\gamma-\alpha),\delta}$

Proof The following eight products belong to K and can be calculated using Lemma 2.1.

- $a_{\alpha\beta}a^*_{\alpha\beta}a_{\alpha\beta} = a_{\alpha\beta}$
- $a_{\alpha\beta}a^*_{\gamma\delta}a_{\alpha\beta} = a_{\alpha,\beta+(\gamma-\alpha)+(\beta-\delta)} = x_1$
- $a_{\alpha\beta}a^*_{\alpha\beta}a_{\gamma\delta} = a_{\gamma\delta}$
- $a_{\alpha\beta}a^*_{\gamma\delta}a_{\gamma\delta} = a_{\alpha\beta}$
- $a_{\gamma\delta}a^*_{\alpha\beta}a_{\alpha\beta} = a_{\gamma+\beta-\delta,\beta} = x_2$
- $a_{\gamma\delta}a^*_{\gamma\delta}a_{\alpha\beta} = a_{\gamma,\beta+\gamma-\alpha} = x_3$
- $a_{\gamma\delta}a^*_{\alpha\beta}a_{\gamma\delta} = a_{\gamma+\beta-\delta+\gamma-\alpha,\delta} = x_4$
- $a_{\gamma\delta}a^*_{\gamma\delta}a_{\gamma\delta} = a_{\gamma\delta}$

□

Example 2.4 For $\alpha, \beta \in \mathbb{Z}$, and $J \subset \mathbb{N}_0$, $D_{\alpha,\beta}(J) := \{a_{\alpha+j,\beta+j} : j \in J\}$ is a subsemiheap of E.

Example 2.5 For $\alpha, \beta \in \mathbb{Z}$, and $\sigma \in \mathbb{N}$, $K_{\alpha,\beta} = \{a_{\alpha+\ell,\beta+m} : \ell, m \in \mathbb{N}_0\}$, and more generally, $K^\sigma_{\alpha,\beta} := \{a_{\alpha+\ell\sigma,\beta+m\sigma} : \ell, m \in \mathbb{N}_0\}$ are subsemiheaps of E.

Special cases of Lemma 2.3 and their diagrams are as follows:

Diagram 6 $\beta - \delta > \gamma - \alpha > 0 \ \ (b = \beta - \delta, h = \gamma - \alpha)$

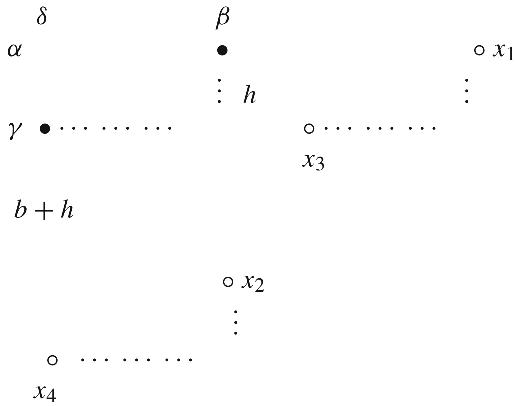


Diagram 7 $\gamma = \alpha, \beta - \delta > 0$

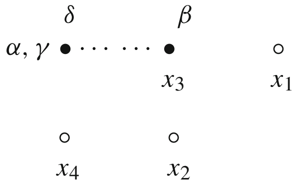


Diagram 8 $\gamma - \alpha > \beta - \delta > 0 \ \ (b = \beta - \delta, h = \gamma - \alpha)$

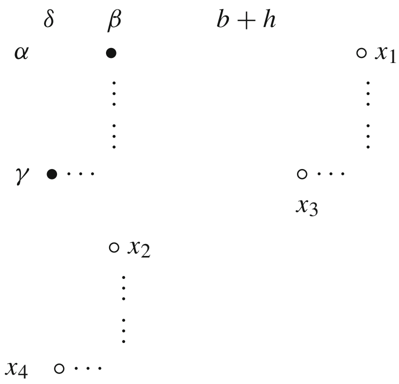


Diagram 9 $\delta = \beta, \gamma - \alpha > 0$

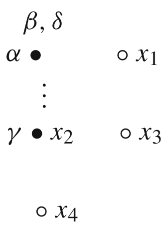
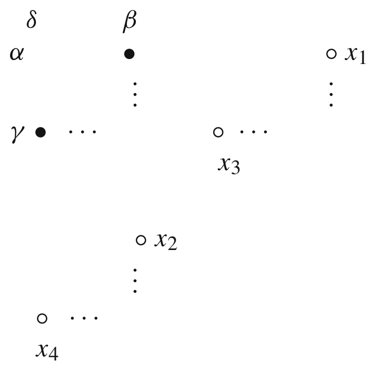


Diagram 10 $\delta - \beta = \gamma - \alpha > 0$



Example 2.6 $K = \{a_{\alpha_0\beta_0}\} \cup \{a_{\alpha_0+k_j,\beta_0+k_j} : j = 1, \dots \ell_0 - 1\} \cup K^{\sigma}_{\alpha_0+k_{\ell_0},\beta_0+k_{\ell_0}}$ is a subsemiheap of E, where $\alpha_0, \beta_0 \in \mathbb{Z}$, $\ell_0, \sigma, k_i \in \mathbb{N}$, $k_1 < k_2 < \dots < k_{\ell_0-1}$, and

$$K^{\sigma}_{\alpha_0+k_{\ell_0},\beta_0+k_{\ell_0}} = \{a_{\alpha_0+k_{\ell_0}+m\sigma,\beta_0+k_{\ell_0}+n\sigma} : m, n \in \mathbb{N}_0\}.$$

Proof Let $x_j = a_{\alpha_0+k_j,\beta_0+k_j}$ for $1 \leq k_j < \ell_0$ and $y_{mn} = a_{\alpha_0+\ell_0+m\sigma,\beta_0+\ell_0+n\sigma}$ for $m, n \in \mathbb{N}_0$.

The following eight products belong to K, as calculated by Lemma 2.1.

1. $x_j y^*_{mn} y_{pq} = \begin{cases} y_{n+p-m,q} & \text{if } p \geq m \quad \text{(Lemma 2.1(iii))} \\ y_{n,q+m-p} & \text{if } p \leq m \quad \text{(Lemma 2.1(iv))} \end{cases}$
2. $y_{mn} x^*_j y_{pq} = \begin{cases} y_{m,n+q-p} & \text{if } p \leq n \quad \text{(Lemma 2.1(i))} \\ y_{m+p-n,q} & \text{if } p \geq n \quad \text{(Lemma 2.1(ii))} \end{cases}$
3. $y_{mn} y^*_{pq} x_j = \begin{cases} y_{m,p+n-q} & \text{if } q \leq n \quad \text{(Lemma 2.1(i))} \\ y_{m+q-n,p} & \text{if } q \geq n \quad \text{(Lemma 2.1(iv)} \end{cases}$
4. $y_{mn} x^*_i x_j = y_{mn}$ (Lemma 2.1(i))
5. $x_i y^*_{mn} x_j = y_{nm}$ (Lemma 2.1(iv))
6. $x_i x^*_j y_{mn} = y_{mn}$ (Lemma 2.1(ii))
7. $x_i x^*_j x_\ell = x_{\max(i,j,\ell)}$ (Lemma 2.1(i)-(iv))
8. $y_{mn} y^*_{pq} y_{rs} = \begin{cases} y_{m,s+p+n-r-q} & \text{if } q \leq n \text{ and } r+q \leq p+n \quad \text{(Lemma 2.1(i))} \\ y_{m+r+q-p-n,s} & \text{if } q \leq n \text{ and } r+q \geq p+n \quad \text{(Lemma 2.1(ii))} \\ y_{m+q-n+r-p,s} & \text{if } q \geq n \text{ and } r \geq p \quad \text{(Lemma 2.1(iii))} \\ y_{m+q-n+p-r,s} & \text{if } q \geq n \text{ and } r \leq p \quad \text{(Lemma 2.1(iv))} \end{cases}$

We provide some details for cases 1 and 8.

For case 1, by Lemma 2.1(iii),

$$\begin{aligned} x_j y^*_{mn} y_{pq} &= a_{\alpha_0+k_j,\beta_0+k_j} a^*_{\alpha_0+\ell_0+m\sigma,\beta_0+\ell_0+n\sigma} a_{\alpha_0+\ell_0+p\sigma,\beta_0+\ell_0+q\sigma} \\ &= a_{\alpha_0+\ell_0+(n+p-m)\sigma,\beta_0+\ell_0+q\sigma}, \end{aligned}$$

if $p \geq m$ and $\ell_0 + n\sigma \geq k_j$; and by Lemma 2.1(iv),

$$a_{\alpha_0+k_j,\beta_0+k_j} a^*_{\alpha_0+\ell_0+m\sigma,\beta_0+\ell_0+n\sigma} a_{\alpha_0+\ell_0+p\sigma,\beta_0+\ell_0+q\sigma} \\ = a_{\alpha_0+\ell_0+n\sigma,\beta_0+\ell_0+(q+m-p)\sigma},$$

if $p \leq m$ and $\ell_0 + n\sigma \geq k_j$.

For case 8, with $q \leq n$, by Lemma 2.1(i),

$$a_{\alpha_0+\ell_0+m\sigma,\beta_0+\ell_0+n\sigma} a^*_{\alpha_0+\ell_0+p\sigma,\beta_0+\ell_0+q\sigma} a_{\alpha_0+\ell_0+r\sigma,\beta_0+\ell_0+s\sigma} \\ = a_{\alpha_0+\ell_0+m\sigma,\beta_0+\ell_0+(s+p+n-r-q)\sigma},$$

if $q \leq n$ and $r + q \leq p + n$, so that $s + p + n - r - q \in \mathbb{N}_0$; and by Lemma 2.1(ii),

$$a_{\alpha_0+\ell_0+m\sigma,\beta_0+\ell_0+n\sigma} a^*_{\alpha_0+\ell_0+p\sigma,\beta_0+\ell_0+q\sigma} a_{\alpha_0+\ell_0+r\sigma,\beta_0+\ell_0+s\sigma} \\ = a_{\alpha_0+\ell_0+(m+r+q-p-n)\sigma,\beta_0+\ell_0+s\sigma},$$

if $q \leq n$ and $r + q \geq p + n$, so that $m + r + q - p - n \in \mathbb{N}_0$.

The subcases of case 8 for which $q \geq n$ are as follows. By Lemma 2.1(iii),

$$a_{\alpha_0+\ell_0+m\sigma,\beta_0+\ell_0+n\sigma} a^*_{\alpha_0+\ell_0+p\sigma,\beta_0+\ell_0+q\sigma} a_{\alpha_0+\ell_0+r\sigma,\beta_0+\ell_0+s\sigma} \\ = a_{\alpha_0+\ell_0+(m+q-n+r-p)\sigma,\beta_0+\ell_0+s\sigma},$$

if $q \geq n$ and $r \leq p$, so that $m + q - n \in \mathbb{N}_0$; and by Lemma 2.1(iv),

$$a_{\alpha_0+\ell_0+m\sigma,\beta_0+\ell_0+n\sigma} a^*_{\alpha_0+\ell_0+p\sigma,\beta_0+\ell_0+q\sigma} a_{\alpha_0+\ell_0+r\sigma,\beta_0+\ell_0+s\sigma} \\ = a_{\alpha_0+\ell_0+(m+q-n)\sigma,\beta_0+\ell_0+(s+p-r)\sigma},$$

if $q \geq n$ and $r \leq p$, so that $m + q - n \in \mathbb{N}_0$ and $s + p - r \in \mathbb{N}_0$. □

Example 2.7 Let $K = \bigcup_{k\in A} K^p_{\alpha_0+k,\beta_0+k}$, where $\alpha_0, \beta_0 \in \mathbb{Z}$, $p > 0$, $A \subset \{0, 1, \dots, p-1\}$ and a_{α_0+k,β_0+k}, $k \in A$, denote the elements of K lying on the diagonal with $k < p$. (See the following Diagram and Proposition 3.15.) In fact, K is an inverse subsemigroup of E.

Proof We note first that setting $\alpha_0 = \beta_0 = 0$ for convenience (see Remark 2.8), and changing notation (see Remark 1.1),

$$K = \{(k + \ell p, k + mp) : k \in A, \ell, m \in \mathbb{N}_0\}$$

and it suffices to show that

$$(k_1 + \ell_1 p, k_1 + m_1 p)(k_2 + \ell_2 p, k_2 + m_2 p)^*(k_3 + \ell_3 p, k_3 + m_3 p)$$

belongs to K. We calculate this triple product using the four cases in Lemma 2.1.

By Lemma 2.1(i), if $k_2+m_2p \leq k_1+m_1p$, and $k_3+\ell_3p \leq k_1+(\ell_2+m_1-m_2)p$, then

$$(k_1+\ell_1p, k_1+m_1p)(k_2+\ell_2p, k_2+m_2p)^*(k_3+\ell_3p, k_3+m_3p)$$
$$= (k_1+\ell_1p, k_1+(m_3+\ell_2+m_1-m_2-\ell_3)p),$$

and it is required to show that $m_3+\ell_2+m_1-m_2-\ell_3 \geq 0$.

Following the argument in [2, Lemma 4.5], we have

$$k_1+(\ell_2+m_1-m_2-\ell_3)p \geq k_3 \geq 0$$

so that $(\ell_2+m_1-m_2-\ell_3)p \geq -k_1 > -p$ and therefore $\ell_2+m_1-m_2-\ell_3 \geq 0$ and $m_3+\ell_2+m_1-m_2-\ell_3 \geq 0$, as required.

By Lemma 2.1(ii), if $k_2+m_2p \leq k_1+m_1p$, and $k_3+\ell_3p \geq k_1+(\ell_2+m_1-m_2)p$, then

$$(k_1+\ell_1p, k_1+m_1p)(k_2+\ell_2p, k_2+m_2p)^*(k_3+\ell_3p, k_3+m_3p)$$
$$= (k_3+(\ell_1+\ell_3-\ell_2-m_1+m_2)p, k_3+m_3p)$$

and it is required to show that $\ell_3-\ell_2-m_1+m_2 \geq 0$.

Following the argument in [2, Lemma 4.5], we have

$$k_3+(\ell_3-\ell_2-m_1+m_2)p \geq k_1 \geq 0$$

so that $(\ell_3-\ell_2-m_1+m_2)p \geq -k_3 > -p$ and therefore $\ell_3-\ell_2-m_1+m_2 \geq 0$ and $\ell_1+\ell_3-\ell_2-m_1+m_2 \geq 0$, as required.

By Lemma 2.1(iii), if $k_3+\ell_3p \geq k_2+\ell_2p$, and $k_2+m_2p \geq k_1+m_1p$, then

$$(k_1+\ell_1p, k_1+m_1p)(k_2+\ell_2p, k_2+m_2p)^*(k_3+\ell_3p, k_3+m_3p)$$
$$= (k_3+(\ell_1+m_2-m_1+\ell_3-\ell_2)p, k_3+m_3p)$$

and it is required to show that $\ell_1+m_2-m_1+\ell_3-\ell_2 \geq 0$.

Following the argument in [2, Lemma 4.5], we have

$$k_3+(\ell_3-m_1+m_2-\ell_2)p \geq k_1 \geq 0$$

so that $(\ell_3-m_1+m_2-\ell_2)p \geq -k_3 > -p$ and therefore $\ell_3-m_1+m_2-\ell_2 \geq 0$ and $\ell_1+\ell_3-m_1+m_2-\ell_2 \geq 0$, as required.

By Lemma 2.1(iv), if $k_3+\ell_3p \leq k_2+\ell_2p$, and $k_2+m_2p \geq k_1+m_1p$, then

$$(k_1+\ell_1p, k_1+m_1p)(k_2+\ell_2p, k_2+m_2p)^*(k_3+\ell_3p, k_3+m_3p)$$
$$= (k_2+(\ell_1+m_2-m_1)p, k_2+(m_3+\ell_2-\ell_3)p)$$

and it is required to show that $\ell_1+m_2-m_1 \geq 0$. and $m_3+\ell_2-\ell_3 \geq 0$.

Following the argument in [2, Lemma 4.5], we have

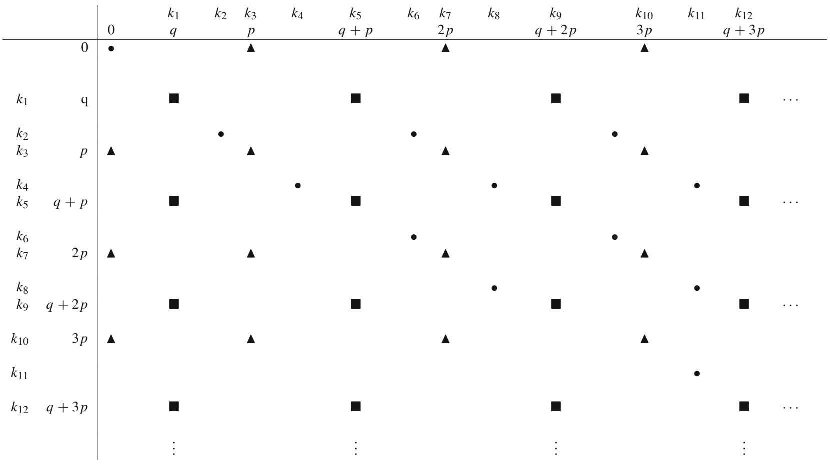
$$k_2 + (\ell_2 - \ell_3)p \geq k_3 \geq 0$$

so that $(\ell_2 - \ell_3)p \geq -k_3 > -p$ and therefore $\ell_2 - \ell_3 \geq 0$ and $m_3 + \ell_2 - \ell_3 \geq 0$, as required.

Following the argument in [2, Lemma 4.5], we have

$$k_2 + (m_2 - m_3)p \geq k_1 \geq 0$$

so that $(m_2 - m_1)p \geq -k_1 > -p$ and therefore $m_2 - m_1 \geq 0$ and $\ell_1 + m_2 - m_1 \geq 0$, as required. □



Remark 2.8 The adjoint operation $a_{ij} \mapsto a_{ij}^* = a_{ji}$ on the extended bicyclic semigroup E is an anti-isomorphism of a subsemiheap K of E onto the subsemiheap K^*, that is, $(ab^*c)^* = c^*ba^*$. As another application of Lemma 2.1, the translation map on the extended bicyclic semigroup is a triple isomorphism, that is, if $\varphi_{\alpha,\beta}(a_{ij}) = a_{i+\alpha,j+\beta}$, then

$$\varphi(a_{ij}a_{pq}^*a_{rs}) = \varphi(a_{ij})\varphi(a_{pq})^*\varphi(a_{rs}).$$

Hence, if K is a subsemiheap of $K_{\alpha,\beta}$, then $\varphi_{-\alpha,-\beta}(K)$ is a subsemiheap of the bicyclic semigroup $K_{0,0}$. At the very least, this fact can simplify notation in parts of this paper.

3 Subsemiheaps of the extended bicyclic semigroup

In this section, we shall determine all of the subsemiheaps of the extended bicyclic semigroup. We shall proceed as follows. First, for an arbitrary subsemiheap K of E, we define

$$\alpha_0 = \inf\{\alpha \in \mathbb{Z} : \exists \beta \in \mathbb{Z}, a_{\alpha\beta} \in K\},$$

and

$$\beta_0 = \inf\{\beta \in \mathbb{Z} : \exists \alpha \in \mathbb{Z}, a_{\alpha\beta} \in K\}.$$

We have four mutually exclusive and exhaustive cases, namely,

1. **Quadrant** $\alpha_0 \neq -\infty, \beta_0 \neq -\infty$
2. **Right Half Plane** $\alpha_0 = -\infty, \beta_0 \neq -\infty$
3. **Lower Half Plane** $\alpha_0 \neq -\infty, \beta_0 = -\infty$
4. **Full Plane** $\alpha_0 = -\infty, \beta_0 = -\infty$

Remark 3.1 We only need to find all of the subsemiheaps of E which are in case (1), since the other cases can be reduced to this case in steps, as follows, which shows that every subsemiheap of the extended bicyclic semigroup is the inductive limit of subsemiheaps in case (1) in the category of semiheaps and semiheap homomorphisms.[1]

- If a subsemiheap K of E is in case (2), then $K \subset \{a_{ij} : i \in \mathbb{Z}, j \geq \beta_0\}$ and $K = \cup_{\alpha\in\mathbb{Z}} K^\alpha$, where $K^\alpha = K \cap \{a_{ij} : i \geq \alpha, j \geq \beta_0\}$ (which we have denoted by K_{α,β_0}) is in case (1).
- If a subsemiheap K of E is in case (3), then $K \subset \{a_{ij} : i \geq \alpha_0, j \in \mathbb{Z}\}$ and $K = \cup_{\beta\in\mathbb{Z}} K_\beta$, where $K_\beta = K \cap \{a_{ij} : i \geq \alpha_0, j \geq \beta\}$ $(=K_{\alpha_0,\beta})$ is in case (1).
- If a subsemiheap K of E is in case (4), then $K \subset \{a_{ij} : i, j \in \mathbb{Z}\}$ and $K = \cup_{\alpha\in\mathbb{Z}} K_{(\alpha)}$, where $K_{(\alpha)} = K \cap \{a_{ij} : i \geq \alpha, j \in \mathbb{Z}\}$ is in case (3).
  Alternatively, if a subsemiheap K of E is in case (4), then $K = \cup_{\beta\in\mathbb{Z}} K^{(\beta)}$, where $K^{(\beta)} = K \cap \{a_{ij} : i \in \mathbb{Z}, j \geq \beta\}$ is in case (2).

Therefore we shall concentrate only on case (1). Suppose then that $\alpha_0 \neq -\infty$ and $\beta_0 \neq -\infty$. Then $K \subset K_{\alpha_0,\beta_0} = \{a_{pq} : p \geq \alpha_0, q \geq \beta_0\}$. We define three parameters as follows:

$$\begin{aligned}
\overline{\beta} &= \sup\{\beta \in \mathbb{Z} : a_{\alpha_0\beta} \in K\} \\
\overline{\alpha} &= \sup\{\alpha \in \mathbb{Z} : a_{\alpha\beta_0} \in K\} \\
\overline{\gamma} &= \sup\{k \in \mathbb{N}_0 : a_{\alpha_0+k,\beta_0+k} \in K\}.
\end{aligned}$$

.

We shall consider three primary cases:

$$\mathbf{1}.\ \overline{\beta} = \beta_0 \qquad \mathbf{2}.\ \beta_0 < \overline{\beta} < \infty \qquad \mathbf{3}.\ \overline{\beta} = \infty$$

Each of the cases 1, 2, 3, consists of three further subcases.

1.1 $\overline{\beta} = \beta_0, \overline{\alpha} = \alpha_0$ **1.2** $\overline{\beta} = \beta_0, \alpha_0 < \overline{\alpha} < \infty$ **1.3** $\overline{\beta} = \beta_0, \overline{\alpha} = \infty$.
2.1 $\overline{\beta} = \beta_0 < \infty, \overline{\alpha} = \alpha_0$ **2.2** $\beta_0 < \overline{\beta} < \infty, \alpha_0 < \overline{\alpha} < \infty$ **2.3** $\beta_0 < \overline{\beta} < \infty, \overline{\alpha} = \infty$.
3.1 $\overline{\beta} = \infty, \overline{\alpha} = \alpha_0$ **3.2** $\overline{\beta} = \infty, \alpha_0 < \overline{\alpha} < \infty$ **3.3** $\overline{\beta} = \infty, \overline{\alpha} = \infty$.

[1] In fact, it is an elementary inductive limit since the connecting maps are inclusions (see Theorem 1.2).

Each of these nine cases consists of three further subcases. Thus, in order to account for the quadrant case (1), and hence the other three cases, it will be necessary to consider 27 cases. We summarize the results in the table **Classification Scheme** below. It is worthy to note that by Diagram 2, if $\overline{\beta}$ is finite, by which we mean, $\beta_0 < \overline{\beta} < \infty$, then $a_{\alpha_0,\overline{\beta}}$ is the only point of K of the form $a_{\alpha_0,\beta}$. A similar statement holds for $\overline{\alpha}$. Also, if $\overline{\alpha} = \alpha_0$, or if $\overline{\beta} = \beta_0$, then $a_{\alpha_0,\beta_0} \in K$. Thus in cases **2.2, 2.3, 3.2**, and **3.3**, it is necessary to consider the two possibilities: $a_{\alpha_0,\beta_0} \in K$, and $a_{\alpha_0,\beta_0} \notin K$.

Classification Scheme

	Case	subcase	$\overline{\beta}$	$\overline{\alpha}$	$\overline{\gamma}$	Exists?	result
		1.1.1.	β_0	α_0	0	Yes	Lemma 3.2
	1.1	1.1.2.	β_0	α_0	Finite	Yes	Proposition 3.3
		1.1.3.	β_0	α_0	∞	Yes	Proposition 3.4
		1.2.1.	β_0	Finite	0	No	Lemma 3.7
1	1.2	1.2.2.	β_0	Finite	Finite	No	Lemma 3.7
		1.2.3.	β_0	Finite	∞	No	Lemma 3.7
		1.3.1.	β_0	∞	0	No	Lemma 3.7
	1.3	1.3.2.	β_0	∞	Finite	No	Lemma 3.7
		1.3.3.	β_0	∞	∞	No	Lemma 3.7
		2.1.1.	Finite	α_0	0	No	Lemma 3.8
	2.1	2.1.2.	Finite	α_0	Finite	No	Lemma 3.8
		2.1.3.	Finite	α_0	∞	No	Lemma 3.8
		2.2.1.	Finite	Finite	0	No	Lemma 3.9
2	2.2	2.2.2.	Finite	Finite	Finite	No	Lemma 3.9
		2.2.3.	Finite	Finite	∞	No	Lemma 3.9
		2.3.1.	Finite	∞	0	No	Lemma 3.10
	2.3	2.3.2.	Finite	∞	Finite	No	Lemma 3.10
		2.3.3.	Finite	∞	∞	No	Lemma 3.10
		3.1.1.	∞	α_0	0	No	Lemma 3.11
	3.1	3.1.2.	∞	α_0	Finite	No	Lemma 3.11
		3.1.3.	∞	α_0	∞	No	Lemma 3.11
		3.2.1.	∞	Finite	0	No	Lemma 3.11
3	3.2	3.2.2.	∞	Finite	Finite	No	Lemma 3.11
		3.2.3.	∞	Finite	∞	No	Lemma 3.11
		3.3.1.	∞	∞	0	No	Proposition 3.12
	3.3	3.3.2.	∞	∞	Finite	No	Proposition 3.12
		3.3.3.	∞	∞	∞	Yes	Proposition 3.18

We now proceed to analyze all 27 cases.

Lemma 3.2 *In case 1.1.1 ($\overline{\beta} = \beta_0$, $\overline{\alpha} = \alpha_0$, $\overline{\gamma} = 0$), we have $K = \{a_{\alpha_0\beta_0}\}$.*

Proof In this case, the diagram is the following, where the bullet represents the element $a_{\alpha_0\beta_0}$, and the circles indicate that no element of K occupies that position. (Ignore, for the moment, the symbols ■, ▲, △)

● o o o o o

o △ ■

o ▲ o

o o

o o

$\vdots$ $\ddots$

Suppose that a_{α_0+2,β_0+1}, denoted by ▲, belonged to K. Then by Diagram 3 applied to the points $a_{\alpha_0\beta_0}$ and ▲, the point a_{α_0+1,β_0+1}, denoted by △, would belong to K, a contradiction. So ▲ does not belong to K. By the same argument, no element of K resides in the second column of the diagram.

Suppose that a_{α_0+1,β_0+2}, denoted by ■, belonged to K. Then by Diagram 1 applied to the points $a_{\alpha_0\beta_0}$ and ■, the point a_{α_0+1,β_0+1}, denoted by △, would belong to K, a contradiction. So ■ does not belong to K. By the same argument, no element of K resides in the second row of the diagram.

Repetition of these two arguments shows that no element of K resides in any column or row of the diagram, other than the first row and column, and therefore $K = \{a_{\alpha_0\beta_0}\}$ contains exactly one element. □

Proposition 3.3 *In case 1.1.2 ($\overline{\beta} = \beta_0,\ \alpha_0 = \overline{\alpha},\ 0 < \overline{\gamma} < \infty$), we have*

$$K = \{a_{\alpha_0,\beta_0}\} \cup \{a_{\alpha_0+k_j,\beta_0+k_j} : 1 \leq j \leq n_0\},$$

where $1 \leq k_1 < k_2 < \cdots < k_{n_0} = \overline{\gamma}$ *and* $n_0 \in \mathbb{N}$.

Proof In this case, in Diagram 11, the bullets represent some of the elements of K residing on the diagonal, the circles indicate that no element of K occupies that position, and the dots represent both the finite number of points of K on the diagonal together with some positions on the diagonal not containing points of K (Ignore for the moment, the symbols ⊚, ⊡ which represent two elements of K lying on the diagonal, and the symbols ■, □, ▲, △). The symbol ♣ represents the element $a_{\alpha_0+k_{n_0},\beta_0+k_{n_0}}$. We shall show that all off-diagonal positions are not occupied by elements of K, which means that $K = \{a_{\alpha_0,\beta_0}\} \cup \{a_{\alpha_0+k_j,\beta_0+k_j} : 1 \leq j \leq n_0\}$. □

Suppose that for $k_j \leq \ell < k_{j+1}$, the point $a_{\alpha_0+k_{j+1},\beta_0+\ell}$, denoted by ■ in Diagram 11, belonged to K. Then by Diagram 2, starting with ■ and $a_{\alpha_0+k_{j+1},\beta_0+k_{j+1}}$, denoted by ⊡ in Diagram 11, shows that

$$K \supset K^{k_{j+1}-\ell}_{\alpha_0+k_{j+1},\beta_0+\ell} = \{a_{\alpha_0+k_{j+1}+m(k_{j+1}-\ell),\beta_0+\ell+n(k_{j+1}-\ell)} : m, n \in \mathbb{N}_0\}.$$

Then choosing $m = n - 1$, so that

$$k_{j+1} + m(k_{j+1} - \ell) = \ell + n(k_{j+1} - \ell)$$

and letting $n \to \infty$ shows that there are infinitely many points of K on the diagonal, a contradiction, so ■ $\notin K$. The same argument applies to every point on each row determined by $\alpha_0 + k_{j+1}$ to the left of $a_{\alpha_0+k_{j+1},\beta_0+k_{j+1}}$ for $0 \leq j \leq n_0 - 1$.

Suppose now that for $k_j < \ell \leq k_{j+1}$, the point $a_{\alpha_0+k_j,\beta_0+\ell}$, denoted by ▲ in Diagram 11, belonged to K. Then by Diagram 2, starting with $a_{\alpha_0+k_j,\beta_0+k_j}$, denoted by ⊚ in Diagram 11, and ▲, shows that

$$K \supset K^{\ell-k_j}_{\alpha_0+k_j,\beta_0+\ell} = \{a_{\alpha_0+k_j+m(\ell-k_j),\beta_0+k_j+n(\ell-k_j)} : m, n \in \mathbb{N}_0\}.$$

Then choosing $m = n$, and letting $n \to \infty$ show that there are infinitely many points of K on the diagonal, a contradiction, so $\blacktriangle \notin K$. The same argument applies to every point on each row $\alpha_0 + k_j$ to the right of $a_{\alpha_0+k_j,\beta_0+k_j}$ for $0 \leq j \leq n_0$.

Thus all rows containing an element of K on the diagonal do not contain any other elements of K, as in Diagram 12.

Diagram 11

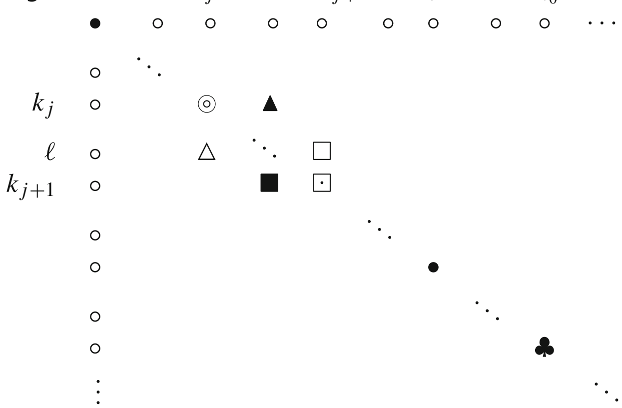
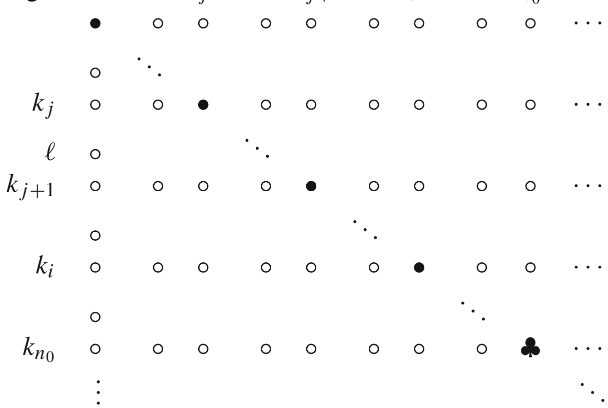


Diagram 12



A parallel argument, using Diagram 4 shows that all columns containing an element of K on the diagonal do not contain any other elements of K. For completeness, we include the details.

Suppose that for $k_j \leq \ell < k_{j+1}$, the point $a_{\alpha_0+\ell,\beta_0+k_{j+1}}$, denoted by □ in Diagram 11, belonged to K. Then by Diagram 4, starting with □ and $a_{\alpha_0+k_{j+1},\beta_0+k_{j+1}}$, denoted by ⊡, shows that

$$K \supset K^{k_{j+1}-\ell}_{\alpha_0+\ell,\beta_0+k_{j+1}} = \{a_{\alpha_0+\ell+m(k_{j+1}-\ell),\beta_0+k_{j+1}+n(k_{j+1}-\ell)} : m, n \in \mathbb{N}_0\}.$$

Then choosing $m = n + 1$, so that

$$k_{j+1} + n(k_{j+1} - \ell) = \ell + m(k_{j+1} - \ell)$$

and letting $n \to \infty$ shows that there are infinitely many points of K on the diagonal, a contradiction, so $\square \notin K$. The same argument applies to every point on each column determined by $\beta_0 + k_{j+1}$ above $a_{\alpha_0+k_{j+1},\beta_0+k_{j+1}}$ for $0 \le j \le n_0 - 1$.

Suppose now that for $k_j \le \ell < k_{j+1}$, the point $a_{\alpha_0+\ell,\beta_0+k_j}$, denoted by $\triangle$ in Diagram 11, belonged to K. Then by Diagram 4, starting with $\triangle$ and $a_{\alpha_0+k_j,\beta_0+k_j}$, denoted by $\circledcirc$, shows that

$$K \supset K^{\ell-k_j}_{\alpha_0+k_j,\beta_0+k_j} = \{a_{\alpha_0+k_j+m(\ell-k_j),\beta_0+k_j+n(\ell-k_j)} : m, n \in \mathbb{N}_0\},$$

Then choosing $m = n$, so that

$$k_j + n(\ell - k_j) = k_j + m(\ell - k_j)$$

and letting $n \to \infty$ shows that there are infinitely many points of K on the diagonal, a contradiction, so $\triangle \notin K$. The same argument applies to every point on each column determined by $\beta_0 + k_j$ above $a_{\alpha_0+k_j,\beta_0+k_j}$ for $0 \le j \le n_0$.

To complete the proof, we now show that no point a_{α_0+m,β_0+n}, with $m \ne n$ can belong to K. By what was just proved, it suffices to consider points which are not on a row or column containing a point of K, that is, $m \ne k_j$ for all j and $n \ne k_\ell$ for all ℓ.

We shall refer to the following diagram, which depicts the eight possible locations for the element a_{α_0+m,β_0+n}, reflecting the cases $m > n$ and $m < n$, namely,

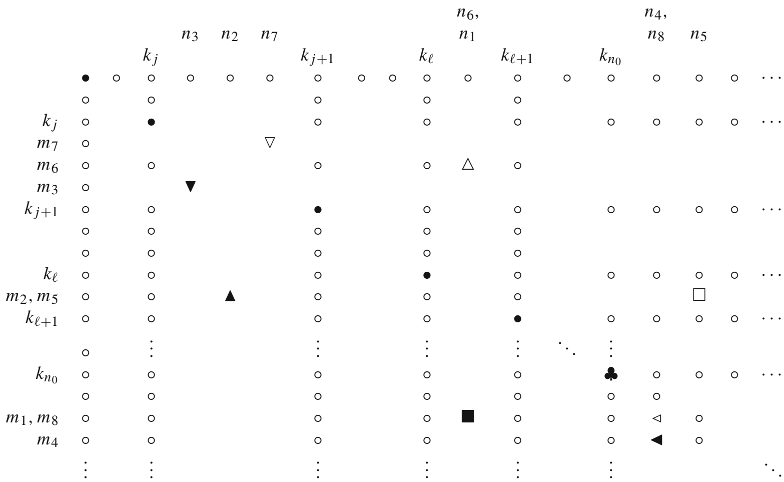
1. $m > k_{n_0} \ge k_{\ell+1} > n > k_\ell$, denoted by $\blacksquare$
2. $k_{\ell+1} > m > k_\ell \ge k_{j+1} > n > k_j$, denoted by $\blacktriangle$
3. $k_{j+1} > m > n > k_j$, denoted by $\blacktriangledown$
4. $m > n > k_{n_0}$, denoted by $\blacktriangleleft$
5. $n > k_{n_0} \ge k_{\ell+1} > m > k_\ell$, denoted by $\square$
6. $k_{\ell+1} > n > k_\ell \ge k_{j+1} > m > k_j$, denoted by $\triangle$
7. $k_{j+1} > n > m > k_j$, denoted by $\triangledown$
8. $n > m > k_{n_0}$, denoted by $\triangleleft$

Suppose first that $m > n$, for example case (2), $k_{\ell+1} > m_2 > k_\ell \ge k_{j+1} > n_2 > k_j$. We consider the two points $a_{\alpha_0+k_j,\beta_0+k_j}$ and $\blacktriangle = a_{\alpha_0+m_2,\beta_0+n_2}$. These two points are vertices of a triangle with height $h = m_2 - k_j$ greater than the base $b = n_2 - k_j$, so $h - b = m_2 - n_2$. Then by Lemma 2.2 (see Diagram 3), the point $x_2 = a_{\alpha_0+k_j+\beta_0+m_2-\alpha_0-k_j,\beta_o+m_2} = a_{\alpha_0+m_2,\beta_0+m_2}$ would belong to K, which is a contradiction since $k_\ell < m_2 < k_{\ell+1}$.

Suppose next that $m < n$, for example, case (6), $k_j < m_6 < k_{j+1} \le k_\ell < n_6 < k_{\ell+1}$ We again consider the two points $a_{\alpha_0+k_j,\beta_0+k_j}$ and $\triangle = a_{\alpha_0+m_6,\beta_0+n_6}$. These two points are vertices of a triangle with height $h = m_6 - k_j$ less than the base $b = n_6 - k_j$ and $b - h = n_6 - m_6$. Then by Lemma 2.2 (see Diagram 1), the point $x_1 = a_{\alpha_0+k_j+\beta_0+n_6-\beta_0-k_j,\beta_0+n_6} = a_{\alpha_0+n_6,\beta_0+n_6}$ would belong to K, which is a contradiction since $k_\ell < n_6 < k_{\ell+1}$.

The same two-part argument works in all the other cases, more precisely, as follows:

- For case (3), $k_j < n_3 < m_3 < k_{j+1}$, Diagram 3 applied to $a_{\alpha_0+k_j,\beta_0+k_j}$ and ▼ yields $x_2 = a_{\alpha_0+n_3,\beta_0+n_3}$
- For case (7), $k_j < m_7 < n_7 < k_{j+1}$, Diagram 1 applied to $a_{\alpha_0+k_j,\beta_0+k_j}$ and ▽ yields $x_3 = a_{\alpha_0+m_7,\beta_0+m_7}$
- For case (4), $m_4 > n_4 > k_{n_0}$, Diagram 3 applied to $a_{\alpha_0+k_{n_0},\beta_0+k_{n_0}}$ and ◀ yields $x_2 = a_{\alpha_0+n_4,\beta_0+n_4}$
- For case (8), $n_8 > m_8 > k_{n_0}$, Diagram 1 applied to $a_{\alpha_0+k_{n_0},\beta_0+k_{n_0}}$ and ◃ yields $x_3 = a_{\alpha_0+m_8,\beta_0+m_8}$
- For case (1), $m_1 > k_{n_0} \geq k_{\ell+1} > n_1 > k_\ell$, Diagram 3 applied to $a_{\alpha_0+k_\ell,\beta_0+k_\ell}$ and ■ yields $x_2 = a_{\alpha_0+n_1,\beta_0+n_1}$
- For case (5), $n_5 > k_{n_0} \geq k_{\ell+1} > m_5 > k_\ell$, Diagram 1 applied to $a_{\alpha_0+k_\ell,\beta_0+k_\ell}$ and □ yields $x_3 = a_{\alpha_0+m_5,\beta_0+m_5}$. This completes the proof of Proposition 3.3. □



Proposition 3.4 *In case 1.1.3: ($\overline{\beta} = \beta_0$, $\overline{\alpha} = \alpha_0$, $\overline{\gamma} = \infty$), either $K = \{a_{\alpha_0\beta_0}\} \cup \{a_{\alpha_0+k_j,\beta_0+k_j} : j \in \mathbb{N}\}$, or there exist $\sigma, \ell_0 \in \mathbb{N}$ such that (see Examples 2.6 and 2.7)*

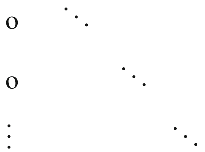
$$K = \{a_{\alpha_0\beta_0}\} \cup \{a_{\alpha_0+k_i,\beta_0+k_i} : i = 1, \dots \ell_0 - 1\} \cup K^\sigma_{\alpha_0+k_{\ell_0},\beta_0+k_{\ell_0}}, \text{ or} \tag{4}$$

$$K = \{a_{\alpha_0\beta_0}\} \cup \{a_{\alpha_0+k_i,\beta_0+k_i} : 1 \leq i < \ell_0\} \cup \left(\bigcup_{i=\ell_0}^{j} K^\sigma_{\alpha_0+k_i,\beta_0+k_i} \right) \tag{5}$$

where $1 \leq k_1 < k_2 < \cdots < k_i < \cdots < \infty$, and in (5), $\sigma = k_j - k_{\ell_0}$ where k_j is such that no point $a_{\alpha_0+k_{\ell_0},\beta_0+k_p}$ belongs to K for $1 \leq p < j$. If $\ell_0 = 1$, then the term $\{a_{\alpha_0+k_i,\beta_0+k_i} : 1 \leq i < \ell_0\}$ is missing in (4) and (5).

Proof In this case, the diagram is the following, where the bullet represents the element $a_{\alpha_0\beta_0}$, the circles indicate that no element of K occupies that position, and the dots indicate that infinitely many elements of K reside in the diagonal.

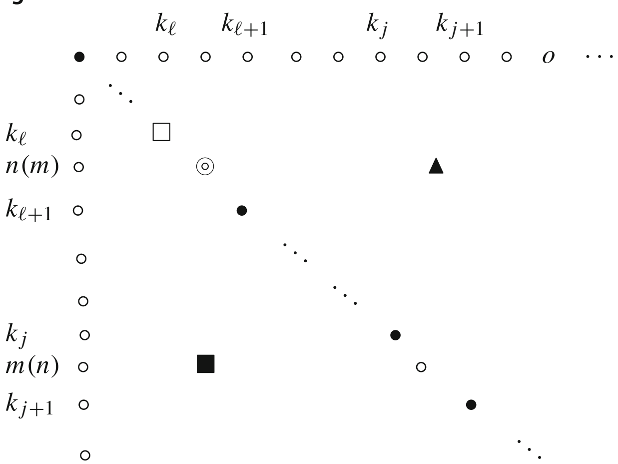
$\bullet \quad \circ \quad \circ \quad \cdots$



We consider a point a_{α_0+m,β_0+n}, denoted by ■ in Diagram 13, with $m \neq n$ and $m \neq k_j, n \neq k_j$ for every $j \geq 1$.

Suppose first that $m > n$, more precisely, $k_{j+1} > m > k_j \geq k_{\ell+1} > n > k_\ell$. We consider the two points $a_{\alpha_0+k_\ell,\beta_0+k_\ell}$ (an element of K on the diagonal), denoted by □ in Diagram 13, and $a_{\alpha_0+m,\beta_0+n} =$ ■. These two points are vertices of a right triangle with height $h = m - k_\ell$ greater than the base $b = n - k_\ell$, so $h - b = m - n$. Then by Lemma 2.2 (see Diagram 3), the point $x_2 = a_{\alpha_0+k_\ell+\beta_0+n-\beta_0-k_\ell,\beta_o+m} = a_{\alpha_0+n,\beta_0+n}$, denoted by ⊚, would belong to K, which is a contradiction since $k_\ell < n < k_{\ell+1}$. Therefore all elements below the diagonal which are neither located on a row nor on a column containing a diagonal point of K, do not belong to K.

Diagram 13



Suppose next that $m < n$, more precisely, $k_\ell < m < k_{\ell+1} \leq k_j < n < k_{j+1}$ We again consider the two points $a_{\alpha_0+k_\ell,\beta_0+k_\ell}$, denoted by □ in Diagram 13, and a_{α_0+m,β_0+n}, denoted by ▲ in Diagram 13. These two points are vertices of a right triangle with height $h = m - k_\ell$ smaller than the base $b = n - k_\ell$ and $b - h = n - m$. Then by Lemma 2.2 (see Diagram 1), the point $x_1 = a_{\alpha_0+k_l+\beta_0+n-\beta_0-k_\ell,\beta_0+n} = a_{\alpha_0+n,\beta_0+n}$, denoted by ⊚, would belong to K, which is a contradiction since $k_j < n < k_{j+1}$.

The same two-part argument works in the cases $k_j < m < n < k_{j+1}$ and $k_j < n < m < k_{j+1}$. Therefore all elements above or below the diagonal which are not located on a row or on a column containing a diagonal point of K, do not belong to K.

We now have Diagram 14. Thus the only off-diagonal points that can possibly belong to K are those that are located either on a row containing a diagonal point of K, indicated by $\cdots$, or on a column containing a diagonal point of K, indicated by vertical dots.

We next show that points which lie on a row or column, but not both, cannot belong to K.

Consider first, for any row determined by k_j and any n with $n \neq k_m$ for all m, the element $a_{\alpha_0+k_j,\beta_0+n}$, denoted by ■ in Diagram 15, and suppose it belonged to K. Then by Lemma 2.1(iii),

$$a_{\alpha_0\beta_0}a^*_{\alpha_0+k_j,\beta_0+n}a_{\alpha_0+k_j,\beta_0+n} = a_{\alpha_0+n,\beta_0+n} \text{ (denoted by}\square)$$

would belong to K, a contradiction since $n \neq k_m$ for every m.

Next, for any column determined by k_j and any n with $n \neq k_m$ for all m, consider the element $a_{\alpha_0+n,\beta_0+k_j}$, denoted by ▲ in Diagram 15, and suppose it belonged to K. Then by Lemma 2.1(i),

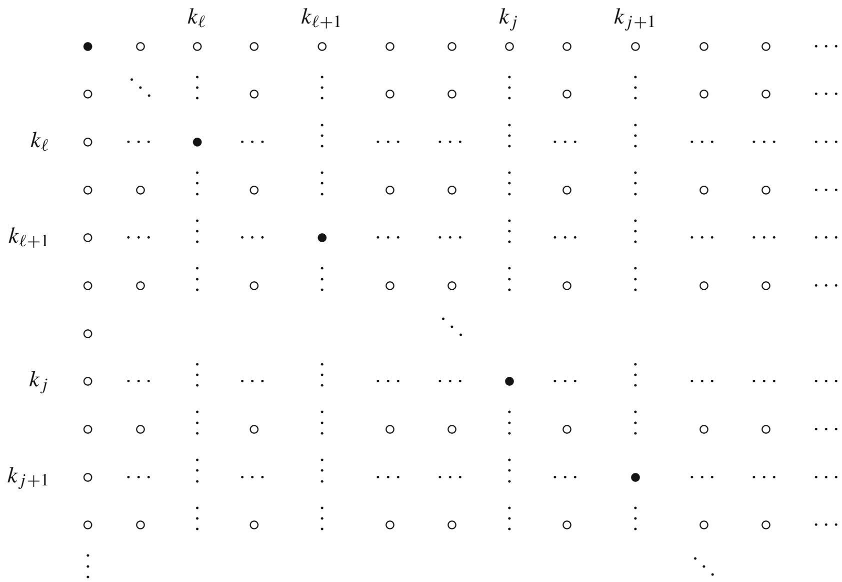
$$a_{\alpha_0+n,\beta_0+k_j}a^*_{\alpha_0+n,\beta_0+k_j}a_{\alpha_0\beta_0} = a_{\alpha_0+n,\beta_0+n} \text{ (denoted by}\square)$$

would belong to K, a contradiction since $n \neq k_m$ for every m.

We now have Diagram 16. Thus the only off-diagonal points that can possibly belong to K are those, denoted by ■, that are located simultaneously on a row containing a diagonal point of K and a column containing a diagonal point of K.

If none of the off-diagonal elements ■ belong to K, then obviously $K = \{a_{\alpha_0\beta_0}\} \cup \{a_{\alpha_0+k_j,\beta_0+k_j} : j \in \mathbb{N}\}$. We next consider the case that some of the elements ■ in Diagram 16 belong to K.

Diagram 14



A row determined by k_j for which no element other than $a_{\alpha_0+k_j,\beta_0+k_j}$ belongs to K, that is, $a_{\alpha_0+k_j,\beta_0+k_j} \in K$, and $a_{\alpha_0+k_j,\beta_0+k_p} \notin K$ for all $p \in \mathbb{N} - \{k_j\}$, will be called a *null* row. More precisely, a *right-null* (respectively *left-null*) row determined by k_j is one that satisfies $a_{\alpha_0+k_j,\beta_0+k_j} \in K$, and $a_{\alpha_0+k_j,\beta_0+k_p} \notin K$ for all $p > k_j$ (respectively $p < k_j$). The row determined by α_0 is a null row.

Similarly, a row determined by k_j which contains an element of K other than $a_{\alpha_0+k_j,\beta_0+k_j}$, that is, there exists $\ell > j$ (respectively $\ell < j$), such that $a_{\alpha_0+k_j,\beta_0+\ell} \in K$, will be called a *right-ample* (respectively *left-ample*) row. A row that is either left-ample or right-ample (or both), will be called simply *ample*. By Diagram 2, a left-ample row is also right-ample, but not conversely (see the sentence following Lemma 2.1). For the same reason, a right-null row is also left-null. As noted above, if all rows of K are null, then $K = \{a_{\alpha_0\beta_0}\} \cup \{a_{\alpha_0+k_j,\beta_0+k_j} : j \in \mathbb{N}\}$. Thus we have the following lemma.

Lemma 3.5 *If all the rows of K are right-null, then $K = \{a_{\alpha_0\beta_0}\} \cup \{a_{\alpha_0+k_j,\beta_0+k_j} : j \in \mathbb{N}\}$.*

Diagram 15

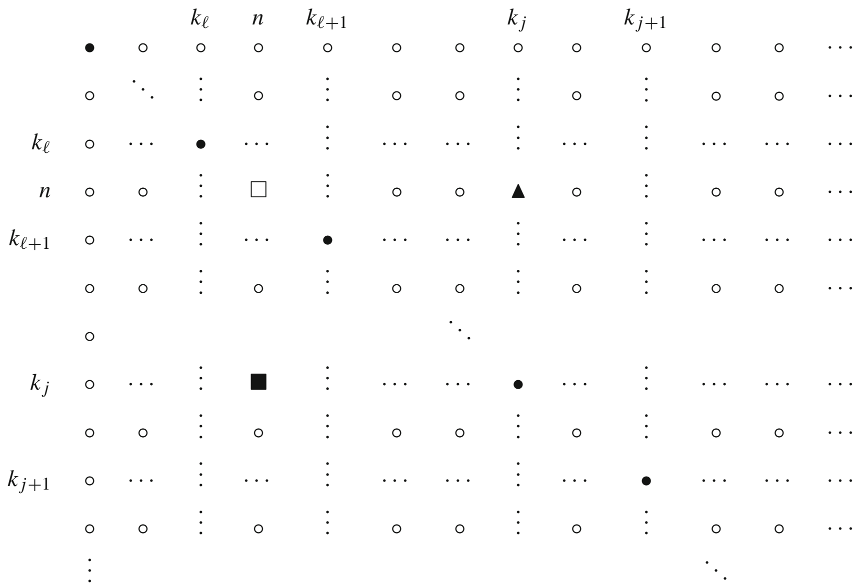
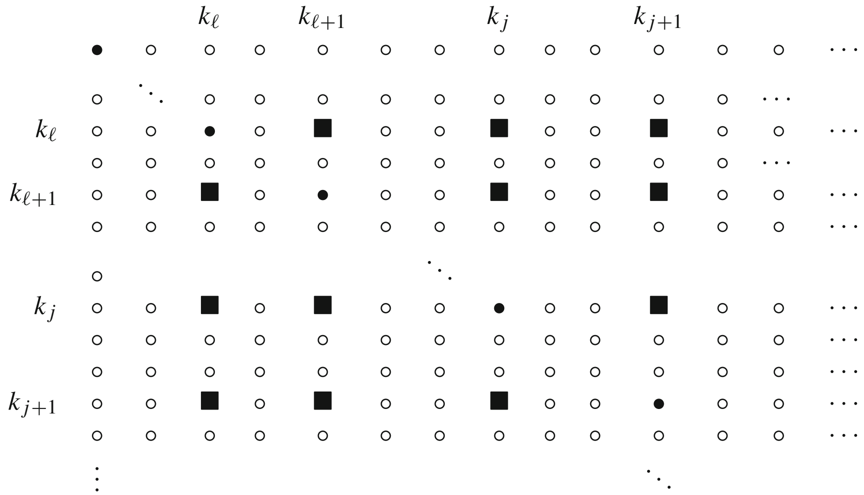
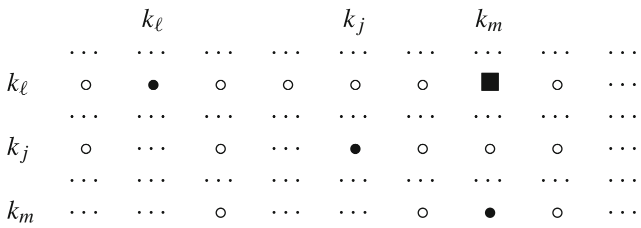


Diagram 16



Lemma 3.6 *All right-null rows lie above all ample rows. Hence, if there is at least one ample row, then there are only finitely many right-null rows.*

Proof Suppose k_j determines a right-null row, and k_ℓ determines an ample row, which we may assume to be right-ample, say containing the element $a_{\alpha_0+k_\ell,\beta_0+k_m}$, and suppose by way of contradiction that $\ell < j$. Since $\ell < j < m$, the diagram is the following, where ■ denotes the element $a_{\alpha_0+k_j,\beta_0+k_m}$.



The two points ■ and $a_{\alpha_0+k_j,\beta_0+k_j}$ are vertices of a triangle with base $b = k_m - k_j$ and height $h = k_j - k_\ell$. Then by Diagrams 6, 8, or 10, depending on the relative sizes of b and h, the point $x_3 = a_{\alpha_0+k_j,\beta_0+k_m+k_j-k_\ell}$ would belong to K, which is a contradiction since $k_m + k_j - k_\ell > k_j$. □

Let $\ell_0 \in \mathbb{N}$ be such that the first ample row is determined by k_{ℓ_0}, and assume without loss of generality, that this row is right-ample. Assume also, temporarily, that $\ell_0 > 1$. The rows lying above the row determined by $\alpha_0 + k_{\ell_0}$ do not contain any elements of K above the diagonal. It follows from Diagram 4 that the columns lying to the left of the column determined by $\beta_0 + k_{\ell_0}$ do not contain any elements of K below the diagonal. By Diagram 2, K contains infinitely many elements to the right of $a_{\alpha_0+k_{\ell_0},\beta_0+k_{\ell_0}}$, and then by Diagrams 2 and 4, K contains infinitely many elements below $a_{\alpha_0+k_{\ell_0},\beta_0+k_{\ell_0}}$. Since $\overline{\gamma} = \infty$, the subsemiheap $K \cap K_{\alpha_0+k_{\ell_0},\beta_0+k_{\ell_0}}$ falls into subcase 3.3.3 below (see Propositions 3.15 and 3.16, and Diagram 17), and therefore

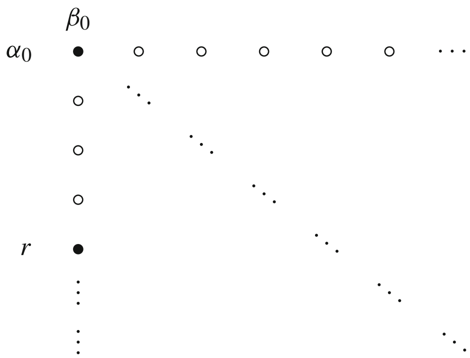
$$K = \{a_{\alpha_0\beta_0}\} \cup \{a_{\alpha_0+k_i,\beta_0+k_i} : 1 \le i < \ell_0\} \cup K^{k_j-k_{\ell_0}}_{\alpha_0+k_{\ell_0},\beta_0+k_{\ell_0}}, \tag{6}$$

or

$$K = \{a_{\alpha_0\beta_0}\} \cup \{a_{\alpha_0+k_i,\beta_0+k_i} : 1 \le i < \ell_0\} \cup \left(\bigcup_{i=\ell_0}^{j} K^{k_j-k_{\ell_0}}_{\alpha_0+k_i,\beta_0+k_i} \right) \tag{7}$$

where k_j is such that no point $a_{\alpha_0+k_{\ell_0},\beta_0+k_p}$ belongs to K for $1 \le p < j$. If $\ell_0 = 1$, then the term $\{a_{\alpha_0+k_i,\beta_0+k_i} : 1 \le i < \ell_0\}$ is missing in (6) and (7). (In Diagram 17, $\sigma = k_{\ell_0+1} + (k_j - k_{\ell_0})$). This completes the proof of Proposition 3.4 and hence of case 1.1. □

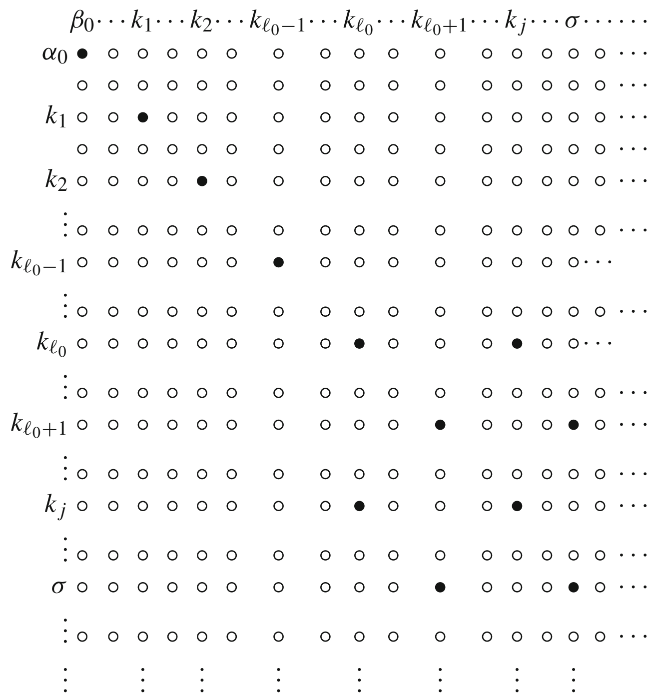
In each of the six subcases of cases 1.2 and 1.3, the diagram is the following:



Then applying Diagram 4 to the points $a_{\alpha_0\beta_0}$ and a_{α_0+r,β_0} shows that $x_1 = a_{\alpha_0,\beta_0+r} \in K$, a contradiction. Hence we have the following lemma.

Lemma 3.7 *Cases 1.2 and 1.3 do not occur.*

Diagram 17



Lemma 3.8 *Case 2.1 does not occur.*

Proof Diagram 18 includes each of the three subcases of case 2.1. Then applying Diagram 2 to the points $a_{\alpha_0\beta_0}$ and a_{α_0,β_0+p} shows that $x_1 = a_{\alpha_0+p,\beta_0} \in K$, a contradiction. □

Lemma 3.9 *Case 2.2 does not occur.*

Proof Note first that by Diagrams 2 and 4, only one element of K can reside on the row determined by α_0 or on the column determined by β_0. Thus, Diagram 19 depicts this case (if $r < p$).

Then by Diagrams 6 and 8 (depending on whether $r < p$ or $p \geq r$, K would contain elements $a_{\alpha_0,\beta_0+\ell}$ with $\ell > p$, a contradiction. □

Diagram 18

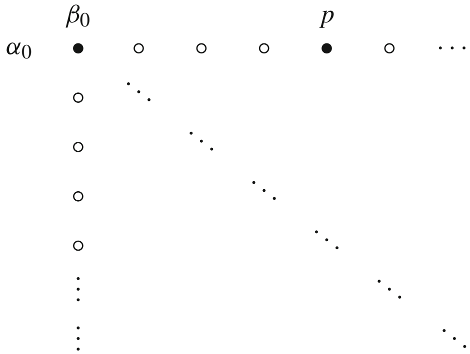
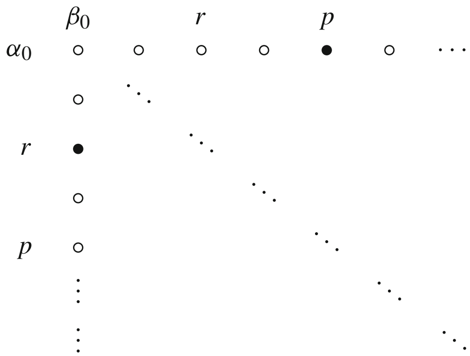
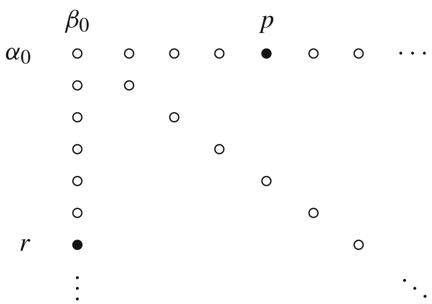


Diagram 19



Lemma 3.10 *Case 2.3 does not occur, hence case 2 does not occur.*

Proof In case 2.3.1 ($\beta_0 < \overline{\beta} < \infty, \overline{\alpha} = \infty, \overline{\gamma} = 0$), note that a_{α_0,β_0+p} is the only point of K on the row determined by α_0. From the following diagram we see that if $p < r$, we get a contradiction using Diagram 8, and if $r < p$ we get a contradiction using Diagram 6, whereas if $p = r$, we get a contradiction using Diagram 10.



The same proof applies to cases 2.3.2 ($\beta_0 < \overline{\beta} < \infty, \overline{\alpha} = \infty, 0 < \overline{\gamma} < \infty$) and 2.3.3 ($\beta_0 < \overline{\beta} < \infty, \overline{\alpha} = \infty, \overline{\gamma} = \infty$). □

Lemma 3.11 *Cases 3.1 and 3.2 do not occur.*

Proof Cases 3.1.1, 3.1.2, and 3.1.3 do not occur by Diagram 2. Cases 3.2.1, 3.2.2, and 3.2.3 do not occur by Diagram 6. □

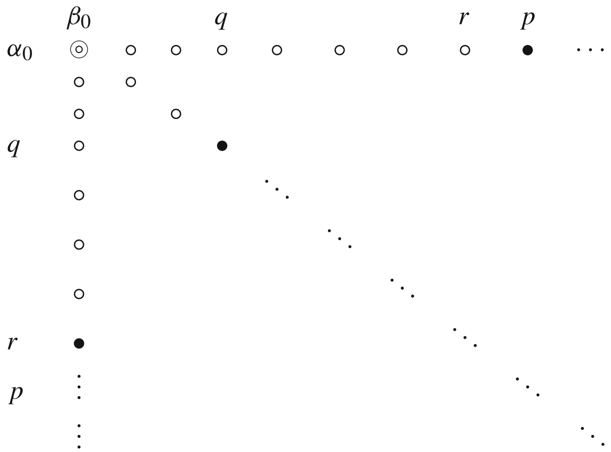
Proposition 3.12 *Cases 3.3.1 and 3.3.2 do not occur.*

Proof By Diagram 2 or 4, we may assume that $a_{\alpha_0,\beta_0} \notin K$. The subsets $K \cap K_{\alpha_0,\beta_0+k_1}$ and $K \cap K_{\alpha_0+\ell_1,\beta_0}$ are subsemiheaps of K which fall into case 3.3.3, which is described below in Proposition 3.18. However, as shown in the proof of Lemma 3.17 below, the four possible situations each lead to $a_{\alpha_0,\beta_0} \in K$. □

Case 3.3.3 ($\overline{\beta} = \infty, \overline{\alpha} = \infty, \overline{\gamma} = \infty$)

Let $a_{\alpha_0,\beta_0+p} \in K$ with $p \geq 1$ and $a_{\alpha_0,\beta_0+p'} \notin K$ for $1 \leq p' < p$. Similarly, let $a_{\alpha_0+r,\beta_0} \in K$ with $r \geq 1$ and $a_{\alpha_0+r',\beta_0} \notin K$ for $1 \leq r' < r$ and let $a_{\alpha_0+q,\beta_0+q} \in K$ with $q \geq 1$ and $a_{\alpha_0+q',\beta_0+q'} \notin K$ for $1 \leq q' < q$

In the diagram below, the bullets represent the three points of K which were just defined, the symbol ⊚ means that the element $a_{\alpha_0\beta_0}$ may or may not belong to K, and the circles indicate that no element of K occupies that position. The diagram represents just one of 13 possible cases (namely case (5) below), and is for illustration purposes only.



Of course, we must consider the various relations between the three elements $p, q, r \in \mathbb{N}$, some of which can be equal, of which there are six, namely

- $r \leq p \leq q$
- $p \leq r \leq q$
- $p \leq q \leq r$
- $r \leq q \leq p$
- $q \leq r \leq p$
- $q \leq p \leq r$

But for our purposes, it is necessary to distinguish 13 more refined cases, namely

1. $r < p < q$
2. $p < r < q$
3. $p < q < r$
4. $r < q < p$
5. $q < r < p$
6. $q < p < r$
7. $r = p < q$
8. $p = q < r$
9. $r = q < p$
10. $r < p = q$
11. $p < r = q$
12. $q < r = p$
13. $r = p = q$

Lemma 3.13 *Cases (3) to (11) do not occur. If $a_{\alpha_0,\beta_0} \in K$, then cases (1) and (2) do not occur. If $a_{\alpha_0\beta_0} \notin K$, then case (12) does not occur. In case (13), $a_{\alpha_0\beta_0} \in K$.*

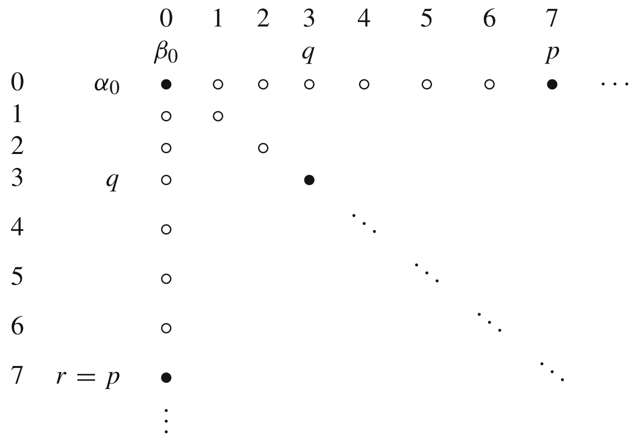
Proof By Lemma 2.1, we have

$$a_{\alpha_0,\beta_0+p}a^*_{\alpha_0+q,\beta_0+q}a_{\alpha_0+r,\beta_0} = \begin{cases} \text{(i) } a_{\alpha_0,\beta_0+p-r} & \text{if } r \leq p \text{ and } q \leq p \\ \text{(ii) } a_{\alpha_0+r-p,\beta_0} & \text{if } r \geq p \text{ and } q \leq p \\ \text{(iii) } a_{\alpha_0+r-p,\beta_0} & \text{if } r \geq q \text{ and } q \geq p \\ \text{(iv) } a_{\alpha_0+q-p,\beta_0+q-r} & \text{if } r \leq q \text{ and } q \geq p. \end{cases} \tag{8}$$

In case (1) with $a_{\alpha_0\beta_0} \in K$, we obtain a contradiction by Diagram 4.
In case (2) with $a_{\alpha_0\beta_0} \in K$, we obtain a contradiction by Diagram 2.
In case (3), we obtain a contradiction by (8(iii)).
In case (4), we obtain a contradiction by (8(i)).
In case (5), we obtain a contradiction by (8(i)).
In case (6), we obtain a contradiction by (8(ii)).
In case (7), we obtain a contradiction by (8(iv)).
In case (8), we obtain a contradiction by (8(ii)).
In case (9), we obtain a contradiction by (8(i)).
In case (10), we obtain a contradiction by (8(i)).
In case (11), we obtain a contradiction by (8(iii)).
In case (12) with $a_{\alpha_0\beta_0} \notin K$, we obtain a contradiction by (8(i)) or (ii).
In case (13), $a_{\alpha_0\beta_0} \in K$ by (8(iii)). □

It remains to consider cases (1) and (2), with $a_{\alpha_0\beta_0} \notin K$, and the cases (12) and (13), with $a_{\alpha_0\beta_0} \in K$. The latter two will be resolved in Propositions 3.15 and 3.16 and the former two in Lemma 3.17.

We start with some properties in case (12). The basic diagram for case (12) is the following.



Lemma 3.14 *In case (12), with (necessarily) $a_{\alpha_0\beta_0} \in K$,*

(a) *The rows* $1, 2, \ldots, q-1$ *contain no elements of* K *above the diagonal*
The columns $1, 2, \ldots, q-1$ *contain no elements of* K *below the diagonal*
(b) *The points* $a_{\alpha_0+q,\beta_0+q+i}$*, for* $1 \le i \le p-q$ *do not belong to* K.
The points $a_{\alpha_0+q+i,\beta_0+q}$*, for* $1 \le i \le p-q$ *do not belong to* K.
(c) *The points* a_{α_0+q,β_0+p}*, and* a_{α_0+p,β_0+q} *do not belong to* K.
(d) *The points* a_{α_0+q,β_0+j}*, for* $mp < j < mp+q$*, with* $m \in \mathbb{N}$ *do not belong to* K.
The points a_{α_0+i,β_0+q}*, for* $mp < i < mp+q$*, with* $m \in \mathbb{N}$ *do not belong to* K.
(e) $a_{\alpha_0+q,\beta_0+p+q}, a_{\alpha_0+p+q,\beta_0+q} \in K$.
(f) *The points* a_{α_0+q,β_0+j}*, for* $mp+q < j \le (m+1)p$*, with* $m \in \mathbb{N}$ *do not belong to* K.
The points a_{α_0+i,β_0+q}*, for* $mp+q < i \le (m+1)p$*, with* $m \in \mathbb{N}$ *do not belong to* K.
(g) *The points* a_{α_0,β_0+j}*, for* $mp < j < (m+1)p$*, with* $m \in \mathbb{N}$ *do not belong to* K.
The points a_{α_0+i,β_0}*, for* $mp < i < (m+1)p$*, with* $m \in \mathbb{N}$ *do not belong to* K.

Proof In what follows, we shall elaborate on the above diagram. In the next diagram, the symbols ■, ▲, ▼, ◄, ►, ♦, ♠, ♣, and their blank versions, represent points which, *a priori*, do not belong to K. They should be temporarily ignored. (The locations of (a)–(g) are indicated in the diagram preceding Proposition 3.16.)

(a) Consider the two points $a_{\alpha_0\beta_0}$ and a_{α_0+i,β_0+j}, the latter indicated by ■ in the next diagram, with $1 \le i < q$, $2 \le j < \infty$ and $i < j$, and suppose that a_{α_0+i,β_0+j} belongs to K. Then by Diagram 1, $x_3 = a_{\alpha+i,\beta_0+i}$, indicated by □, belongs to K, a contradiction. Therefore the rows $1, 2, 3, \ldots q-1$ contain no elements of K above the diagonal.

		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
		β_0				q			p				$p+q$			2p		
0	α_0	●	○	○	○	◁	▽	○	●			♣						…
1		○	○															…
2		○		□					◊			■						…
3		○			△													…
4	q	○		⊟		●		▼	◀		⧫		●					…
5		○																…
6		○				♠		⋱										…
7	$r = p$	●				▶			●									…
8					▲													…
9						⊞					⋱							…
10		★																…
11	$p+q$					●							●					…
12																		…
13																		…
14	$2p$								●							●		…

Consider the two points $a_{\alpha_0\beta_0}$ and a_{α_0+i,β_0+j}, the latter indicated by ▲ in the preceding diagram, with $1 \leq j < q$, $2 \leq i < \infty$ and $i > j$, and suppose that a_{α_0+i,β_0+j} belongs to K. Then by Diagram 3, $x_2 = a_{\alpha+j,\beta_0+j}$, indicated by △, belongs to K, a contradiction. Therefore columns $1, 2, 3, \ldots q-1$ contain no elements of K below the diagonal.

(b) Assuming that $a_{\alpha_0+q,\beta_0+q+j}$, indicated by ▼ in the preceding diagram, with $1 \leq j < p-q$, belongs to K, we have that

$$K \supset K^j_{\alpha_0+q,\beta_0+q} = \{a_{\alpha_0+q+\ell j,\beta_0+q+mj} : \ell, m \geq 0\}.$$

Then by Lemma 2.1(i),

$$a_{\alpha_0,\beta_0+p}a^*_{\alpha_0+q+\ell j,\beta_0+q+mj}a_{\alpha_0\beta_0} = a_{\alpha_0,\beta_0+(\ell-m)j+p} \in K,$$

provided that $0 \leq (\ell - m)j + p$ and $q + mj \leq p$. Then with $\ell = 0$ and $m = 1$, we have that a_{α_0,β_0-j+p}, indicated by ▽, belongs to K, which is a contradiction.

For the second statement of (b), the proof is the same, namely, assuming that $a_{\alpha_0+q+i,\beta_0+q}$, indicated by ♠, with $1 \leq i < p-q$, belongs to K, we have that

$$K \supset K^i_{\alpha_0+q,\beta_0+q} = \{a_{\alpha_0+q+\ell i,\beta_0+q+mi} : \ell, m \geq 0\}.$$

Then by Lemma 2.1(i),

$$a_{\alpha_0,\beta_0+p}a^*_{\alpha_0+q+\ell i,\beta_0+q+mi}a_{\alpha_0\beta_0} = a_{\alpha_0,\beta_0+(\ell-m)i+p} \in K,$$

provided that $0 \leq (\ell - m)i + p$ and $q + mi \leq p$. Then with $\ell = 0$ and $m = 1$, we have that a_{α_0,β_0-i+p}, indicated by ▽, belongs to K, which is a contradiction.

(c) Assuming that a_{α_0+q,β_0+p}, indicated by ◀, belongs to K, then by Lemma 2.1(iii),

$$a_{\alpha_0,\beta_0+p}a^*_{\alpha_0+q,\beta_0+p}a_{\alpha_0+q,\beta_0+q} = a_{\alpha_0,\beta_0+q},$$

indicated by $\triangleleft$, belongs to K, a contradiction. Assuming that a_{α_0+p,β_0+q}, indicated by $\blacktriangleright$, belongs to K, then applying Diagram 4 to the two points a_{α_0+q,β_0+q} and $\blacktriangleright$ we have $x_1 = a_{\alpha_0+q,\beta_0+p} \in K$, a contradiction to the previous paragraph.

(d) Suppose $mp < j < mp + q$ and assume that a_{α_0+q,β_0+j}, indicated by $\blacklozenge$ in the preceding diagram (with $m = 1$), belongs to K. Then by Lemma 2.1(iii),

$$a_{\alpha_0,\beta_0+mp}a^*_{\alpha_0+q,\beta_0+j}a_{\alpha_0+q,\beta_0+q} = a_{\alpha_0+j-mp,\beta_0+q},$$

indicated by $\lozenge$, belongs to K, a contradiction to (i), since $j - mp < q$.

Suppose $mp < i < mp + q$ and assume that a_{α_0+i,β_0+q}, indicated by $\boxplus$ in the preceding diagram (with $m = 1$), belongs to K. Then by Lemma 2.1(iv),

$$a_{\alpha_0+q,\beta_0+q}a^*_{\alpha_0+i,\beta_0+q}a_{\alpha_0+mp,\beta_0} = a_{\alpha_0+q,\beta_0+i-mp},$$

indicated by $\boxminus$, belongs to K, a contradiction to (i$'$), since $i - mp < q$.

(e) By Diagrams 6 or 8, applied to the vertices a_{α_0+q,β_0+q} and a_{α_0+p,β_0}, $x_1 = a_{\alpha_0+q,\beta_0+p+q} \in K$, and then by Diagram 7, $a_{\alpha_0+p+q,\beta_0+q} \in K$.

In the proofs of (f) and (g), and in the rest of this section, we can assume (by Remark 2.8), with no loss of generality, that $\alpha_0 = \beta_0 = 0$.

(f) Suppose that $(q, j) \in K$ with $mp + q < j \leq (m + 1)p$. By Lemma 2.1(iii), $(0, mp)(q, j)^*(q, q) = (j-mp, q)$. This is a contradiction to (b) since $p \geq j-mp > q$.

Suppose that $(i, q) \in K$ with $mp + q < i \leq (m + 1)p$. By Lemma 2.1(iv), $(q, q)(i, q)^*(mp, 0) = (q, i-mp)$. This is a contradiction to (b) since $p \geq i-mp > q$.

(g) Supposing that a_{α_0,β_0+j}, with $p < j < \infty$ and $j \notin \{2p, 3p, \ldots\}$, denoted by $\clubsuit$ in the preceding diagram, belongs to K, we apply Diagram 2 to a_{α_0,β_0+mp} and a_{α_0,β_0+j}, where $mp < j < (m+1)p$ to get $x_2 = a_{\alpha_0+j-mp,\beta_0+j} \in K$, a contradiction since $j-mp < p$. Hence no element of K occupies any position in the row determined by α_0 except for the points a_{α_0,β_0+mp} for $m \in \mathbb{N}_0$.

Supposing that a_{α_0+i,β_0}, with $p < i < \infty$ and $i \notin \{2p, 3p, \ldots\}$, denoted by $\bigstar$, belongs to K, we apply Diagram 4 to a_{α_0+mp,β_0} and a_{α_0+i,β_0}, where $mp < i < (m + 1)p$ to get a contradiction. Hence no element of K occupies any position in the column determined by β_0 except for the points $a_{\alpha_0+\ell p,\beta_0}$ for $\ell \in \mathbb{N}_0$. □

We now have the following diagram for case (12) with $a_{\alpha_0,\beta_0} \in K$, and it is clear that $K \cap K_{q,q}$ is also in subcase (12), so it follows that $K = \bigcup_{i=0}^{\infty} K^p_{q_i,q_i}$, where $a_{\alpha_0+q_i,\beta_0+q_i}$ are the points of K lying on the diagonal with

$$q = q_0 < q_1 < q_2 < \cdots < q_n < q_{n+1} < \cdots.$$

Proposition 3.15 *In case (12), with (necessarily) $a_{\alpha_0,\beta_0} \in K$, let $a_{\alpha_0+q_i,\beta_0+q_i}$, $0 \leq i < \infty$,be the points of K lying on the diagonal, such that*

$$q = q_0 < q_1 < q_2 < \cdots < q_n < p \quad \textit{and} \quad p < q_{n+1} < q_{n+2} < \cdots.$$

Then

$$K = \bigcup_{i=0}^{n} K^p_{q_i,q_i}.$$

Proof We know that $K = \bigcup_{i=0}^{\infty} K^p_{q_i,q_i}$. We need to show that $\bigcup_{i=n+1}^{\infty} K^p_{q_i,q_i} \subset \bigcup_{i=0}^{n} K^p_{q_i,q_i}$. For this it suffices to show that each q_{n+j} with $j \geq 1$ is congruent to some element of $\{q_0, q_1, \ldots q_n\}$, modulo p.

Let $(q_k + \ell p, q_k + mp) \in K^p_{q_k,q_k}$ for some $k \geq n+1$ with fixed ℓ, m, and let $(\ell' p, m' p) \in K^p_{0,0}$ with variable ℓ', m'. By Lemma 2.1(i),

$$(q_k + \ell p, q_k + mp)(\alpha_0, \beta_0)^*(\ell' p, m' p) = (q_k + \ell p, q_k + (m' + m - \ell')p) \in K$$

as long as $q_k + mp \geq 0$ and $\ell' p \leq q_k + mp$, We now choose ℓ' such that $q_k = (\ell' - m)p + d$, where $\ell' - m \geq 1$ and $0 \leq d < p$. To check that $\ell' p \leq q_k + mp$, we note that $(\ell' - m)p = q_k - d \leq q_k$. We now have

$$(q_k + \ell p, q_k + (m' + m - \ell')p) = (d + (\ell + \ell' - m)p, d).$$

Thus $(d + tp, d) \in K = \bigcup_{i=0}^{\infty} K^p_{q_i,q_i}$ for some $t \geq 0$, so that $(d + tp, d) = (q_i + rp, q_i + sp)$ for some $i, r, s \geq 0$. Hence $d + tp = q_i + rp$ and $d = q_i + sp$, so by subtraction $tp = (r - s)p$ and $d + (r - s)p = q_i + rp$ so that $d = q_i + sp$. Since $d < p$, $s = 0$ and $d = q_i$ with $i \leq n$. □

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	β_0				q			p				$p+q$			2p		
α_0	•	○	○	○	○	○	○	•	○	(g)	○	○	(g)	○	•	○	⋯
	○	○	(a)	○	○	○	○	○	○	○	○	○	○	○	○	○	⋯
	○	(a)	○	(a)	○	○	○	○	○	○	○	○	○	○	○	○	⋯
	○	○	(a)	○	(a)	○	○	○	○	○	○	○	○	○	○	○	⋯
q	○	○	○	(a)	•	(b)	○	(c)	○	(d)	○	•(e)	○	(f)	○	○	⋯
	○	○	○	○	(b)												⋯
	○	○	○	○	○		⋱										⋯
p	•	○	○	○	(c)			•							•		⋯
	○	○	○	○	○												⋯
	(g)	○	○	○	(d)					⋱							⋯
	○	○	○	○	○												⋯
	○	○	○	○	•(e)							•					⋯
	(g)	○	○	○	○								⋱				⋯
	○	○	○	○	(f)												⋯
$2p$	•	○	○	○	○			•							•		⋯
	○	○	○	○	○											⋱	⋯
	⋮	⋮	⋮	⋮	⋮			⋮				⋮			⋮		⋯

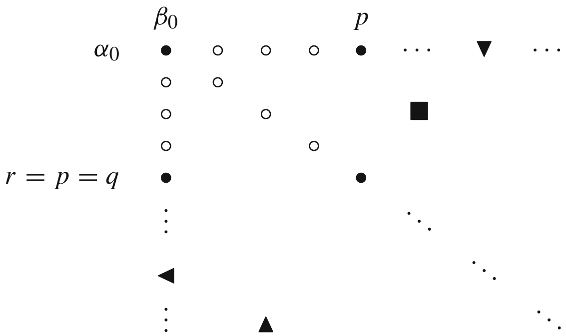
Proposition 3.16 *In case (13),* $K = K^p_{\alpha_0,\beta_0}$

Proof The diagram for case (13) is Diagram 20. (Temporarily ignore the symbols ■, ▲, ▼, ◀).

In the first place, we notice that by Diagrams 2 and 4,

$$K \supset K^p_{\alpha_0,\beta_0} = \{a_{\alpha_0+\ell p,\beta_0+mp} : \ell, m \in \mathbb{N}_0\}.$$

Diagram 20



The next four paragraphs refer to Diagram 20.

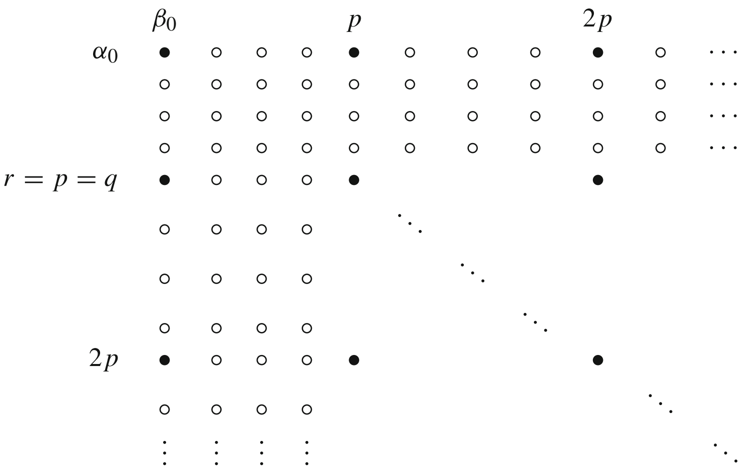
Supposing that a_{α_0+i,β_0+j} for $1 \leq i < p$ and $2 \leq j < \infty$, denoted by ■, belongs to K, we apply Diagram 1 to $a_{\alpha_0\beta_0}$ and a_{α_0+i,β_0+j} to get $x_3 = a_{\alpha_0+i,\beta_0+i} \in K$, a contradiction. Hence no element of K occupies any position above the diagonal in the rows determined by $\alpha_0 + i$, for $1 \leq i < p$.

Supposing that a_{α_0+i,β_0+j} for $2 \leq i < \infty$ and $1 \leq j < p$, denoted by ▲, belongs to K, we apply Diagram 3 to $a_{\alpha_0\beta_0}$ and a_{α_0+i,β_0+j} to get $x_2 = a_{\alpha_0+j,\beta_0+j} \in K$, a contradiction. Hence no element of K occupies any position below the diagonal in the columns determined by $\beta_0 + j$, for $1 \leq j < p$

Supposing that a_{α_0,β_0+j}, with $p < j < \infty$ and $j \notin \{2p, 3p, \ldots\}$, denoted by ▼, belongs to K, we apply Diagram 2 to a_{α_0,β_0+kp} and a_{α_0,β_0+j}, where $kp < j < (k+1)p$ to get $x_2 = a_{\alpha_0+j-kp,\beta_0+j} \in K$, a contradiction since $j - kp < p$. Hence no element of K occupies any position in the row determined by α_0 except for the points a_{α_0,β_0+mp} for $m \in \mathbb{N}_0$.

Supposing that a_{α_0+i,β_0}, with $p < i < \infty$ and $i \notin \{2p, 3p, \ldots\}$, denoted by ◀, belongs to K, we apply Diagram 4 to a_{α_0+kp,β_0} and a_{α_0+i,β_0}, where $kp < i < (k+1)p$ to get a contradiction. Hence no element of K occupies any position in the column determined by β_0 except for the points $a_{\alpha_0+\ell p,\beta_0}$ for $\ell \in \mathbb{N}_0$.

We now have



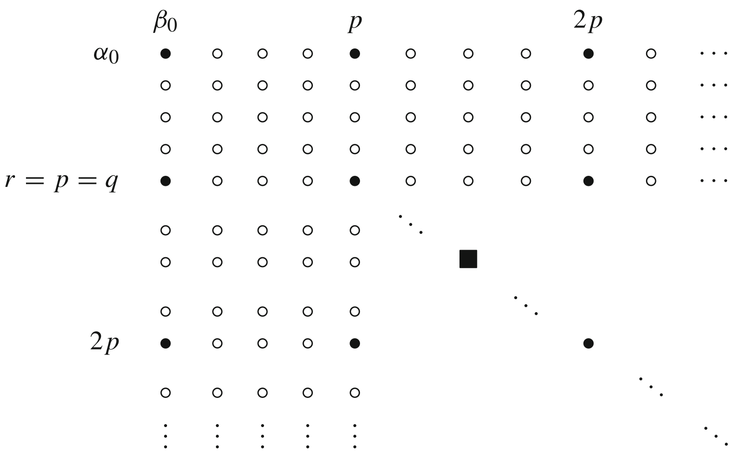
We next consider what happens in the row defined by $\alpha_0 + p$.

Supposing that $a_{\alpha_0+p,\beta_0+p+i}$ belongs to K, with $1 \leq i < p$, then applying Diagram 3 to a_{α_0,β_0+p} and $a_{\alpha_0+p,\beta_0+p+i}$ we obtain $x_2 = a_{\alpha_0+i,\beta_0+p+i} \in K$, which is a contradiction, and repeating this argument shows that no element of K occupies any position in the row determined by $\alpha_0 + p$ except for the points $a_{\alpha_0+p,\beta_0+mp}$ for $m \in \mathbb{N}_0$.

We next consider what happens in the column defined by $\beta_0 + p$.

Supposing that $a_{\alpha_0+p+i,\beta_0+p}$ belongs to K, with $1 \leq i < p$, then applying Diagram 1 to a_{α_0+p,β_0} and $a_{\alpha_0+p+i,\beta_0+p}$ we obtain $x_3 = a_{\alpha_0+p+i,\beta_0+i} \in K$, which is a contradiction, and repeating this argument shows that no element of K occupies any position in the column determined by $\beta_0 + p$ except for the points $a_{\alpha_0+\ell p,\beta_0+p}$ for $\ell \in \mathbb{N}_0$.

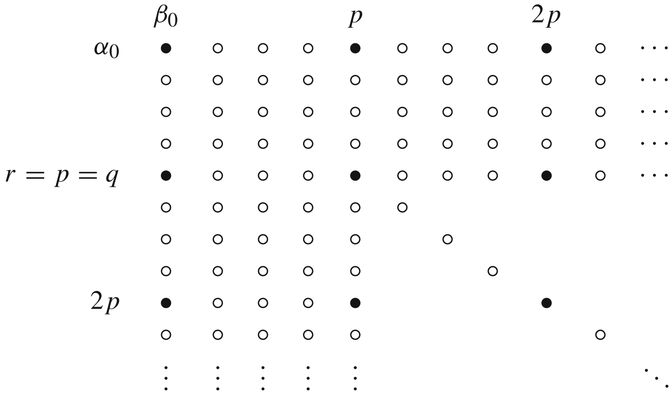
We now have (ignore temporarily the symbol ■)



Finally, we consider what happens along the diagonal.

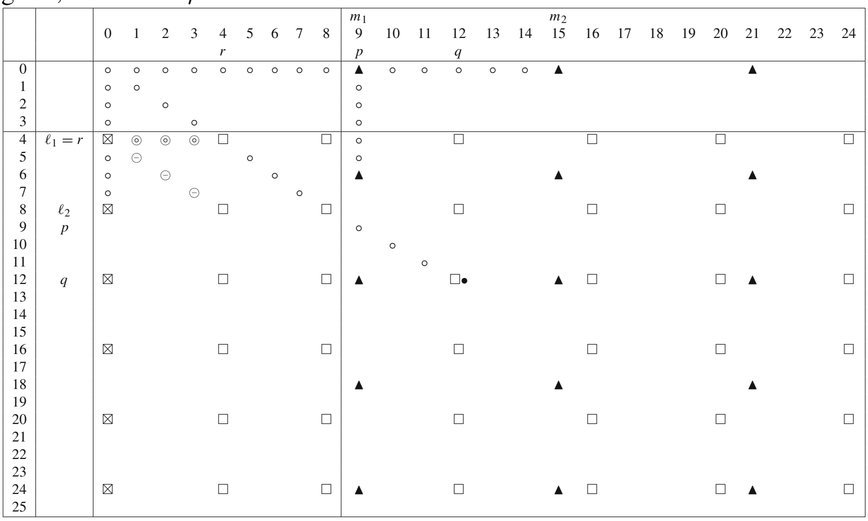
Supposing that $a_{\alpha_0+p+i,\beta_0+p+i}$, denoted by ■, with $1 \leq i < p$, belongs to K, we apply Diagram 1 to a_{α_0+p,β_0} and $a_{\alpha_0+p+i,\beta_0+p+i}$, we obtain $x_3 = a_{\alpha_0+p+i,\beta_0+i} \in K$, which is a contradiction, and repeating this argument shows that no element of K occupies any position in the diagonal except for the points $a_{\alpha_0+\ell p,\beta_0+\ell p}$ for $\ell \in \mathbb{N}_0$.

We now have



We are now in the position at the beginning of the proof, namely, the semiheap $K \cap K_{p,p}$ is in subcase (13) of case 3.3.3, and the result follows by applying successively what has already been proved. ☐

We shall now consider cases (1) and (2) with $a_{\alpha_0,\beta_0} \notin K$ (See Lemma 3.13), and assume with no loss of generality, that $\alpha_0 = \beta_0 = 0$. We consider the following diagram for case (1) and establish the following notation. The points of K on the row determined by α_0, indicated by ▲, are a_{α_0,β_0+m_i}, with $1 \le m_1 < m_2 < \cdots$, and the points on the column determined by β_0, indicated by ⊠, are $a_{\alpha_0+\ell_i,\beta_0}$, with $1 \le \ell_1 < \ell_2 < \cdots$. We denote $\sigma = m_2 - m_1$ and $\rho = \ell_2 - \ell_1$. For example, in that diagram, $\sigma = 6$ and $\rho = r = 4$.



We consider first $K_{r,0}$. By Diagram 2, the points (r, i), $1 < i < \rho$, indicated by ⊚, do not belong to K. By Diagram 4, $K \supset K^{\rho}_{r,0}$. The semiheap $K \cap K_{r,0}$ falls into case 3.3.3, more precisely, either cases (7), (12) or (13), but case (7) does not occur.

In case (13), the points $(r+j, j)$, $1 \leq j < \rho$, indicated by $\ominus$, do not belong to K, so by Proposition 3.16, $K \cap K_{r,0} = K^{\rho}_{r,0}$, and therefore in this case,

$$K_1 := K^{\rho}_{r,0} \cap K_{r,p} = \{(\alpha_0 + r + \ell\rho, \beta_0 + m\rho : \ell \in \mathbb{N}_0, m\rho \geq p\}.$$

By the same argument applied to $K_{0,p}$, assuming that $K \cap K_{0,p}$ is also in case (13), we have

$$K_2 := K^{\sigma}_{0,p} \cap K_{r,p} = \{(\alpha_0 + \ell'\sigma, \beta_0 + p + m'\sigma) : \ell'\sigma \geq r, m' \in \mathbb{N}_0\}.$$

K_1 is depicted by the symbols $\square$ in $K_{r,p}$ and K_2 is depicted by the symbols $\blacktriangle$ in $K_{r,p}$, and we must have $K_1 = K \cap K_{r,p} = K_2$.

As suggested by the diagram, we now show that $\sigma = \rho$, and that p and r are divisible by σ.

- Taking $m' = 0$ and $\ell'\sigma \geq r$, $(\ell'\sigma, p) \in K_2$ so that $(\ell'\sigma, p) = (r + \ell\rho, m\rho)$ for some $\ell, m \in \mathbb{N}_0$ with $m\rho \geq p$. Therefore $p = m\rho$ and $r = \ell'\sigma - \ell\rho$.
- Taking $\ell = 0$ and $m\rho \geq p$, $(r, m\rho) \in K_1$ so that $(r, m\rho) = (\ell'\sigma, p + m'\rho)$ for some $\ell', m' \in \mathbb{N}_0$ with $\ell'\sigma \geq r$. Therefore $r = \ell'\sigma$ and $m\rho = p + m'\sigma$.
  Thus p is divisible by ρ, say $p = m_0\rho$, and r is divisible by σ, say $r = \ell_0\sigma$.
- Taking $\ell = 0$ and $(m_0 + 1)\rho = p + \rho > p$, $(r, (m_0 + 1)\rho) \in K_1$ so that $(r, (m_0 + 1)\rho) = (\ell''\sigma, p + m''\rho)$ for some $\ell'', m'' \in \mathbb{N}_0$ with $\ell''\sigma \geq r$. Therefore $r = \ell''\sigma$ and $(m_0 + 1)\rho = p + m''\sigma$. So $p + \rho = p + m''\sigma$, and $\rho = m''\sigma$.
- Taking $m' = 0$ and $(\ell_0 + 1)\sigma = r + \sigma > r$, $((\ell_0 + 1)\sigma, p) \in K_2$ so that $((\ell_0 + 1)\sigma, p) = (r + \ell\rho, m\rho)$ for some $\ell, m \in \mathbb{N}_0$ with $m\rho \geq p$. Therefore $r + \sigma = r + \ell\rho$, so that $\sigma = \ell\rho$
  Thus ρ is divisible by σ and σ is divisible by ρ, hence $\sigma = \rho$.

Since p and r are each a multiple of ρ, it follows that $(r, p) \in K$, so that $(0, p)(r, p)^*(r, 0) = (0, 0) \in K$, which is a contradiction. We conclude that if both semiheaps $K \cap K_{r,0}$ and $K \cap K_{0,p}$ are in case (13), then case (1) does not occur.

It remains to show that case (1) does not occur in the three other possible cases, namely,

- $K \cap K_{r,0}$ is in case (12) and $K \cap K_{0,p}$ is in case (13)
- $K \cap K_{r,0}$ is in case (13) and $K \cap K_{0,p}$ is in case (12)
- $K \cap K_{r,0}$ is in case (12) and $K \cap K_{0,p}$ is in case (12)

Let us now suppose that $K \cap K_{r,0}$ is in case (12), and $K \cap K_{0,p}$ is in case (13) and refer to the following diagram. Recall that the points of K on the row determined by α_0, indicated by $\blacktriangle$, are $a_{\alpha_0, \beta_0 + m_i}$, with $1 \leq m_1 < m_2 < \cdots$, and the points on the column determined by β_0, indicated by $\boxtimes$, are $a_{\alpha_0 + \ell_i, \beta_0}$, with $1 \leq \ell_1 < \ell_2 < \cdots$. We denote $\sigma = m_2 - m_1$ and $\rho = \ell_2 - \ell_1$. For example, in that diagram, $\sigma = 6$ and $\rho = r = 4$.

Since $K \cap K_{r,0}$ is assumed in case (12), by Proposition 3.15, there exist $0 = j_0 < 1 \le j_1 < j_2 < \cdots < j_n < \rho$ such that

$$K \cap K_{r,0} = \bigcup_{i=0}^{n} K^{\rho}_{r+j_i,j_i},$$

and therefore in this case,

$$\begin{aligned} K_1 := K^{\rho}_{r,0} \cap K_{r,p} &= \bigcup_{i=0}^{n} \left(K^{\rho}_{r+j_i,j_i} \cap K_{r,p} \right) \\ &= \bigcup_{i=0}^{n} \{(r + j_i + \ell\rho,\, j_i + m\rho) : \ell, m \in \mathbb{N}_0,\, j_i + m\rho \ge p\}. \end{aligned}$$

In the diagram, we indicate the points of $K^{\rho}_{r+j_1,j_1} = K^{4}_{r+3,3}$ with the symbols ♡, and the points of $K^{\rho}_{r+j_2,j_2} = K^{4}_{r+6,6}$ with the symbols ⊕.

As before, assuming that $K \cap K_{0,p}$ is in case (13), we have

$$K_2 := K^{\sigma}_{0,p} \cap K_{r,p} = \{(\alpha_0 + \ell'\sigma,\, \beta_0 + p + m'\sigma) : \ell'\sigma \ge r, m' \in \mathbb{N}_0\}$$

Also in the diagram, K_1 is depicted by the symbols □, ♡, ⊕ in $K_{r,p}$ and K_2 is depicted by the symbols ▲ in $K_{r,p}$, and we must have $K_1 = K \cap K_{r,p} = K_2$.

		0	1	2	j_1 3	4 r	5	j_2 6	7	8	m_1 9 p	10	11	12 q	13	14	m_2 15	16	17	18	19	20	21	22	23	24	25
0		○	○	○	○	○	○	○	○	○	▲	○	○	○	○	○	▲						▲				
1		○	○								○																
2		○		○							○																
3		○			○						○																
4	$\ell_1 = r$	⊠				□				□	○			□				□				□				□	
5		○					○				○																
6		○						○			▲						▲						▲				
7	$r + j_1$	○			♡				♡○				♡				♡				♡				♡		
8	ℓ_2	⊠				□				□				□				□				□				□	
9	p										○																
10	$r + j_2$							⊕				⊕○				⊕				⊕				⊕			
11					♡				♡				♡○				♡				♡				♡		
12	q	⊠				□				□	▲			□•			▲	□				□	▲			□	
13																											
14								⊕				⊕				⊕				⊕				⊕			
15					♡				♡				♡				♡				♡				♡		
16		⊠				□				□				□				□				□				□	
17																											
18								⊕			▲	⊕				⊕	▲			⊕			▲	⊕			
19					♡				♡				♡				♡				♡				♡		
20		⊠				□				□				□				□				□				□	
21																											
22								⊕				⊕				⊕				⊕				⊕			
23					♡				♡				♡				♡				♡				♡		
24		⊠				□				□	▲			□			▲	□				□	▲			□	
25																											

We show first that $\rho = \sigma$. Since $K_1 \subset K_2$, for $\ell, m, i \in \mathbb{N}_0$, with $j_i + m\rho \ge p$, there exist $\ell', m' \in \mathbb{N}_0$ with $\ell'\sigma \ge r$ and

$$(r + j_i + \ell\rho,\, j_i + m\rho) = (\ell'\sigma,\, p + m'\sigma). \tag{9}$$

Fix i such that $j_i \geq p$. Then for all $\ell, m \in \mathbb{N}_0$, there exist $\ell', m' \in \mathbb{N}_0$ with $\ell'\sigma \geq r$ such that

$$r + j_i + \ell\rho = \ell'\rho \text{ and } j_i + m\rho = p + m'\sigma.$$

Eliminating j_i from these two equations results in

$$r + p = (m - \ell)\rho + (\ell' - m')\sigma, \tag{10}$$

with (ℓ', m') depending on (ℓ, m) and satisfying $\ell'\sigma \geq r$.

Since $K_2 \subset K_1$, for $\ell, m \in \mathbb{N}_0$, with $\ell\sigma \geq r$, there exist $\ell', m', i \in \mathbb{N}_0$ with $j_i + m'\rho \geq p$ such that

$$r + j_i + \ell'\rho = \ell\sigma \text{ and } j_i + m'\rho = p + m\sigma.$$

Eliminating j_i from these two equations results in

$$r + p = (m' - \ell')\rho + (\ell - m)\sigma, \tag{11}$$

with (ℓ', m') depending on (ℓ, m), provided $\ell\sigma \geq r$.

With $\ell \geq 0$ and $m \geq 0$, from (10), there exist ℓ_1, m_1 such that

$$r + p = (m - \ell)\rho + (\ell_1 - m_1)\sigma$$

and there exist ℓ_2, m_2 such that

$$r + p = (m + 1 - \ell)\rho + (\ell_2 - M_2)\sigma,$$

so by subtraction, $0 = \rho + [(\ell_2 - m_2) + (\ell_1 - m_1)]\sigma$ and σ divides ρ.

With $\ell\sigma \geq r$ and $m \geq 0$, from (11), there exist ℓ_3, m_3 such that

$$r + p = (m_3 - \ell_3)\rho + (\ell - m)\sigma$$

and there exist ℓ_4, m_4 such that

$$r + p = (m_4 - \ell_4)\rho + (\ell + 1 - m)\sigma,$$

so by subtraction, $0 = [(m_4 - \ell_4) - (m_3 - \ell_3)]\rho + \sigma$ and ρ divides σ.

Hence $\rho = \sigma$ and from (10) or (11), ρ divides $r + p$. Now, taking $i = 0, \ell = 0$ in (9), $r = \ell'\sigma$, so that also ρ divides p. Hence, as in the previous case, $(r, p) \in K$, so that $(0, p)(r, p)^*(r, 0) = (0, 0) \in K$, which is a contradiction. We conclude that if the semiheap $K \cap K_{r,0}$ is in case (12) and the semiheap $K \cap K_{0,p}$ is in case (13), then case (1) does not occur.

Since the adjoint mapping is an anti-isomorphism of the extended bicyclic semigroup (See Remark 2.8), it follows that if the semiheap $K \cap K_{r,0}$ is in case (13) and the semiheap $K \cap K_{0,p}$ is in case (12), then case (1) does not occur.

It remains to consider the case when both semiheaps $K \cap K_{r,0}$ and $K \cap K_{0,p}$ are in case (12). After this, again since the adjoint mapping is an anti-isomorphism, and case (1) has been shown to not occur, it will follow that case (2) also does not occur, so we will have the following lemma.

Lemma 3.17 *Cases (1) and (2) with (necessarily)* $(0, 0) \notin K$*, do not occur.*

Proof It suffices to show that if both semiheaps $K \cap K_{r,0}$ and $K \cap K_{0,p}$ are in case (12), then $(0, 0) \in K$ and therefore case (1) does not occur. We have

$$K_1 := K \cap K_{r,0} \cap K_{r,p} = \bigcup_{i=0}^{n} \{(r + j_i + \ell\rho,\ j_i + m\rho) : \ell, m \in \mathbb{N}_0,\ j_i + m\rho \geq p\}$$

and

$$K_2 := K \cap K_{0,p} \cap K_{r,p} = \bigcup_{i=0}^{n'} \{(k_i + \ell\sigma,\ p + k_i + m\sigma) : \ell, m \in \mathbb{N}_0, k_i + \ell\sigma \geq r\},$$

where $0 = k_0 < 1 \leq k_1 < k_2 < \cdots < k_{n'} < \sigma$.

Since $K_1 \subset K_2$, for $i, \ell, m \in \mathbb{N}_0$ with $j_i + m\rho \geq p$, there exist $i', \ell', m' \in \mathbb{N}_0$ satisfying $k_{i'} + \ell'\sigma \geq r$, such that

$$(r + j_i + \ell\rho,\ j_i + m\rho) = (k_{i'} + \ell'\sigma,\ p + k_{i'} + m'\sigma)$$

so that

$$r + j_i + \ell\rho = k_{i'} + \ell'\sigma \text{ and } j_i + m\rho = p + k_{i'} + m'\sigma$$

Fix i such that $j_i \geq p$. Then for every $\ell, m \in \mathbb{N}_0$, by subtraction, we have

$$r + p = (m - \ell)\rho + (\ell' - m')\sigma \tag{12}$$

with ℓ', m' depending only on $\ell, m \in \mathbb{N}_0$ (and ℓ' satisfying $k_{i'} + \ell'\sigma \geq r$ for some i').

Since $K_2 \subset K_1$, for $i, \ell, m \in \mathbb{N}_0$ with $k_i + \ell\sigma \geq r$, there exist $i', \ell', m' \in \mathbb{N}_0$ satisfying $j_{i'} + m'\rho \geq p$, such that

$$(k_i + \ell\sigma,\ p + k_i + m\sigma) = (r + j_{i'} + \ell'\rho,\ j_{i'} + m'\rho)$$

so that

$$k_i + \ell\sigma = r + j_{i'} + \ell'\rho \text{ and } p + k_i + m\sigma = j_{i'} + m'\rho$$

Fix i such that $k_i \geq r$. Then for every $\ell, m \in \mathbb{N}_0$, by subtraction, we have

$$r + p = (\ell - m)\sigma + (m' - \ell')\rho \tag{13}$$

with ℓ', m' depending only on $\ell, m \in \mathbb{N}_0$ (and m' satisfying $j_{i'} + m'\rho \geq p$ for some i').

With $\ell \geq 0$ and $m \geq 0$, from (12), there exist ℓ_1, m_1 such that

$$r + p = (m - \ell)\rho + (\ell_1 - m_1)\sigma$$

and there exist ℓ_2, m_2 such that

$$r + p = (m + 1 - \ell)\rho + (\ell_2 - M_2)\sigma,$$

so by subtraction, $0 = \rho + [(\ell_2 - m_2) + (\ell_1 - m_1)]\sigma$ and σ divides ρ.

With $\ell \geq 0$ and $m \geq 0$, from (13), there exist ℓ_1, m_1 such that

$$r + p = (\ell - m)\sigma + (m_1 - \ell_1)\rho$$

and there exist ℓ_2, m_2 such that

$$r + p = (\ell + 1 - m)\sigma + (m_2 - \ell_2)\rho,$$

so by subtraction, $0 = \sigma + [(m_2 - \ell_2) + (m_1 - \ell_1)]\rho$ and ρ divides σ.

Hence $\rho = \sigma$ and from (12) or (13), σ divides $p + r$. In fact, σ divides both p and r. Indeed, since $(q, q) \in K_1$ and $K_1 = K_2$, there exist ℓ, m, i and ℓ', m', i', such that

$$(q, q) = (r + j_i + \ell\sigma, j_i + \sigma) = (k_{i'} + \ell'\sigma, p + k_{i'} + m'\sigma),$$

so that $r + \ell\sigma = m\sigma$ and $p + m'\sigma = \ell'\sigma$. We now have $(r, p) \in K$, so that $(0, 0) = (0, p)(r, p)^*(r, 0) \in K$, a contradiction. □

We summarize the results of Lemma 3.13 to Lemma 3.17 in the following proposition.

Proposition 3.18 *If the semiheap K is in case 3.3.3, then either $K = K^p_{\alpha_0,\beta_0}$ for some $p > 0$, or there exist $p > 0$ and $q > 0$ such that*

$$K = \bigcup_{i=0}^{n} K^p_{q_i,q_i}.$$

where

$$q = q_0 < q_1 < q_2 < \cdots < q_n < p \quad \textit{and} \quad p < q_{n+1} < q_{n+2} < \cdots .$$

Proof By Lemma 3.17, cases (1) and (2) do not occur. By Lemma 3.13, cases (3)–(11) do not occur. Cases (12) and (13) are described in Propositions 3.15 and 3.16. □

4 Injectivity of W*-TROs

The notation for this section is the following.

S is an inverse semigroup with generalized inverse x^*.

K is a subset of S closed under the triple product xy^*z (semiheap).

π is the left regular representation of S on $H := \ell^2(S)$ so that S is an orthonormal basis for H and $\pi(x)$ is the partial isometry defined by $\pi(x)y = xy$ if $yy^* \leq x^*x$ and $\pi(x)y = 0$ otherwise.

$C^*_{\text{red}}(S)$ is the C*-algebra generated by $\{\pi(x) : x \in S\}$ and is the norm closure of span $\pi(S)$.

$TRO(K)$ is the TRO generated by $\pi(K)$ and is the norm closure of span $\pi(K)$.

$VN(S)$ is the von Neumann algebra generated by $\pi(S)$ and is the weak closure of $C^*_{\text{red}}(S)$

$VNTRO(K)$ is the W*-TRO generated by $\pi(K)$ and is the weak closure of $TRO(K)$.

Details of the left regular representation are as follows ([13, pp. 25–27]). We have

$$\pi(a_{ij})a_{pq} = \begin{cases} a_{ij}a_{pq}, \ a_{pq}a^*_{pq} \leq a^*_{ij}a_{ij} \\ \quad 0, \qquad \text{otherwise} \end{cases},$$

that is,

$$\pi(a_{ij})a_{pq} = \begin{cases} a_{ij}a_{pq}, \ a_{pp} \leq a_{jj} \\ \quad 0, \quad \text{otherwise} \end{cases},$$

or

$$\pi(a_{ij})a_{pq} = \begin{cases} a_{ij}a_{pq}, \quad p \geq j \\ \quad 0, \quad \text{otherwise} \end{cases},$$

or

$$\pi(a_{ij})a_{pq} = \begin{cases} a_{i+p-j,q}, \quad p \geq j \\ \quad 0, \quad \text{otherwise} \end{cases}$$

Define provisionally a linear map $\Phi_0 : \text{span}\,\pi(S) \to \text{span}\,\pi(K)$ as follows: $\Phi_0(0) = 0$, and for $x_1, \ldots x_n \in S$,

$$\Phi_0\left(\sum_{i=1}^{n} \lambda_i \pi(x_i)\right) = \sum_{x_i \in K} \lambda_i \pi(x_i).$$

Proposition 4.1 *The idempotent map Φ_0 is well-defined and contractive, and therefore extends to a contractive projection Φ on $C^*_{red}(S)$ with range $TRO(K)$. Moreover, Φ extends to a completely contractive projection on $VN(S)$ with range $VNTRO(K)$. Hence, if $VN(S)$ in an injective von Neumann algebra, then $VNTRO(K)$ is an injective operator space.*

Proof Let $a = \left\|\sum_{i=1}^{n} \lambda_i \pi(x_i)\right\|$ and $b = \left\|\sum_{x_i \in K} \lambda_i \pi(x_i)\right\|$. With $\xi = \sum_{z\in S}(\xi, z)z \in \ell^2(S)$,

$$\pi(x_i)\xi = \sum_{zz^* \leq x_i^* x_i} (\xi, z)x_i z$$

so that

$$b^2 = \sup_{\|\xi\|\leq 1} \left\| \sum_{x_i \in K} \lambda_i \pi(x_i)\xi \right\|^2 = \sup_{\|\xi\|\leq 1} \sum_{x_i\in K, z\in S, zz^* \leq x_i^* x_i} |\lambda_i(\xi, z)|^2$$

and by the same calculation

$$a^2 = \sup_{\|\xi\|\leq 1} \left\| \sum_{x_i \in S} \lambda_i \pi(x_i)\xi \right\|^2 = \sup_{\|\xi\|\leq 1} \sum_{x_i\in S, z\in S, zz^* \leq x_i^* x_i} |\lambda_i(\xi, z)|^2.$$

Therefore Φ_0 is contractive and extends to a contractive projection on $C^*_{\text{red}}(S)$ with range $TRO(K)$.

Let $A = C^*_{\text{red}}(S)$, $U = TRO(K)$, so that Φ^{**} is a contractive projection on the von Neumann algebra A^{**} with range U^{**}. By [10, Lemma], U^{**} is isomorphic to $VNTRO(K)$, and by [3, Theorem 2.5], Φ^{**} is a completely contractive projection with range $VNTRO(K)$. Therefore, the restriction $\overline{\Phi}$ of Φ^{**} to $VN(S)$ is a completely contractive projection of $VN(S)$ onto $VNTRO(K)$. If $VN(S)$ is injective, then there is a completely contractive projection P of $B(H)$ onto $VN(S)$, so that $\overline{\Phi} \circ P$ is a completely contractive projection with range $VNTRO(K)$. □

Example 4.2 Suppose that e and f are idempotents in the inverse semigroup S and that $K = eSf$, which is a subsemiheap of S. The corresponding induced map takes the form $\pi(x) \mapsto \pi(exf)$ (with $0 \to 0$) and is contractive since

$$\left\| \sum_i \lambda_i \pi(ex_i f) \right\| \leq \|\pi(e)\| \left\| \sum_i \lambda_i \pi(x_i) \right\| \|\pi(f)\| = \left\| \sum_i \lambda_i \pi(x_i) \right\|.$$

Hence Proposition 4.1 applies. This also applies to maps of the form $x \mapsto ex$ and $x \mapsto xf$.

The maximal subgroups of any inverse semigroup S are of the form

$$S_e^e = \{s \in S : ss^* = s^*s = e\}$$

for some idempotent e (See [13, p. 198]). Thus, the maximal subgroups of the extended bicyclic semigroup E reduce to one-element groups, so are trivially amenable and hence by [13, Theorem 4.5.2], $VN(E)$ is injective.[2]

Since $VN(E)$ is injective, where E is the extended bicyclic semigroup, it follows from Proposition 4.1 and Example 4.2, that $VNTRO(a_{ii}Ea_{jj})$ is an injective operator space, as are $VNTRO(Ea_{jj})$ and $VNTRO(a_{ii}E)$. More generally, we have

Corollary 4.3 *All of the subsemiheaps of the extended bicyclic semigroup E (which were determined in Theorem 1.2) give rise to injective W*-TROs.*

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[2] As pointed out to the authors by Alan Paterson, the proof of [13, Theorem 4.5.2] required the assumption that the universal groupoid of the inverse semigroup be Hausdorff. This assumption holds for the extended bicyclic semigroup E, because it is an E-unitary semigroup (see [9, Corollary 3.7] and [8, p.57]). In addition, it appears from [9] that the Hausdorff assumption can actually be dropped.

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