An algebraic characterisation of ample type I groupoids

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Abstract

We give algebraic characterisations of the type I and CCR properties for locally compact second countable, ample Hausdorff groupoids in terms of subquotients of its Boolean inverse semigroup of compact open local bisections. It yields in turn algebraic characterisations of both properties for inverse semigroups with meets in terms of subquotients of their Booleanisation.

Keywords Inverse semigroup · Ample groupoid · Noncommutative stone duality · CCR · Type I

1 Introduction

Inverse semigroups provide an algebraic framework for studying partial dynamical systems and groupoids. This is particularly clearly described by noncommutative Stone duality, extending the classical duality between totally disconnected Hausdorff spaces and generalised Boolean algebras to a duality between ample groupoids and Boolean inverse semigroups. This cornerstone of the modern theory of inverse semigroups was introduced by Lawson and Lenz [13,14,17] following ideas of Resende [26], Lenz [20] and Exel [8]. In a nutshell noncommutative Stone duality associates to an ample groupoid $\mathcal{G}$ the Boolean inverse semigroup of its compact open local bisections $\Gamma(\mathcal{G})$. Various filter constructions can be employed to describe the converse operation [1,14,18].

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Since the discovery of noncommutative Stone duality an important aspect of this theory was to establish a dictionary between properties of ample groupoids and properties of Boolean inverse semigroups. A summary of the results obtained so far can be found in Lawson's survey article [16]. From an operator algebraic and representation theoretic point of view one fundamental side of groupoids and semigroups is the property of being CCR and the property of being type I. These notions arise upon considering groupoid and semigroup C^* -algebras. So far neither of these properties has been addressed by the dictionary of noncommutative Stone duality. Our first two main results fill this gap and establish algebraic characterisations of CCR and type I Boolean inverse semigroups, matching Clark and van Wyk's characterisations of the respective properties for groupoids [6,30,31].

Our characterisation takes the form of forbidden subquotients in analogy with the theme of forbidden minors in graph theory and other fields of combinatorics. The Boolean inverse semigroup B_{T_1} featuring in the next statement is introduced and described explicitly in Example 1. It is the algebraically simplest possible Boolean inverse semigroup which is not CCR. We refer the reader to Sects. 2.1 and 2.3 for further information about Boolean inverse semigroups, their corners and group quotients.

Theorem 1 *Let $\mathcal{G}$ be a second countable, ample Hausdorff groupoid. Then $\mathcal{G}$ is CCR if and only if the following two conditions are satisfied.*

- All group quotients of corners¹ of $\Gamma(\mathcal{G})$ are virtually abelian, and
- $\Gamma(\mathcal{G})$ does not have B_{T_1} as a subquotient²

Theorem 2 *Let $\mathcal{G}$ be a second countable, ample Hausdorff groupoid. Then $\mathcal{G}$ is type I if and only if the following two conditions are satisfied.*

- All group quotients of corners of $\Gamma(\mathcal{G})$ are virtually abelian, and
- $\Gamma(\mathcal{G})$ does not have an infinite, unital and 0-simplifying subquotient.

Historically the notation of CCR and type I C^* -algebras was motivated by problems in representation theory. Roughly speaking groups enjoying one of these properties have a well-behaved unitary dual. Originally studied in the context of Lie groups and algebraic groups [3,4,7,9], other classes of non-discrete groups were considered more recently [5,10]. The question which discrete groups are CCR and type I was answered conclusively by Thoma [29], characterising them as the virtually abelian groups. A more direct proof of Thoma's result was obtained by Smith [27] and his original proof was the basis for a Plancherel formula for general discrete groups recently obtained by Bekka [2]. The fundamental nature of CCR and type I C^* -algebras also led to the study of other group-like objects such as the aforementioned characterisation of groupoids with these properties by Clark and van Wyk [6,30,31]. Compared with Thoma's characterisation of discrete type I groups, our Theorems 1 and 2 can be considered as a generalisation to Boolean inverse semigroups. Naturally the question arises whether a similar characterisation can be obtained for inverse semigroups. The bridge between these structures is provided by the Booleanisation of an inverse semigroup [15,17]

¹ Corners are operator algebraic terminology, while semigroup theory knows them as local submonoids.

² In semigroup theory subquotients are called divisors.

which we denote by $B(S)$. Additional investigations of the passage from S to the universal groupoid $\mathcal{G}(S)$ going via $B(S)$ allow us to draw the following conclusion. In view of a direct algebraic construction of the Booleanisation we describe in Sect. 2.1, the next two results give an intrinsic characterisation of CCR and type I inverse semigroups.

Theorem 3 *Let S be a countable inverse semigroup with meets. Then S is CCR if and only if the following two conditions are satisfied.*

1. *All group subquotients of S are virtually abelian, and*
2. *$B(S)$ does not have B_{T_1} as a subquotient.*

Theorem 4 *Let S be a countable inverse semigroup with meets. Then S is type I if and only if the following two conditions are satisfied.*

1. *All group subquotients of S are virtually abelian, and*
2. *$B(S)$ does not have an infinite, unital, 0-simplifying subquotient.*

The Booleanisation of an inverse semigroup was first introduced in [15]. Our description is more natural from an operator algebraic point of view and equivalent to the original definition of Lawson.

This paper contains five sections. After the introduction we expose necessary preliminaries on (Boolean) inverse semigroups, ample groupoids and noncommutative Stone duality. In Sect. 3 we study separation properties of the orbit space of a groupoid and obtain algebraic characterisations of groupoids with T_1 and T_0 orbit spaces. In Sect. 4 we relate the isotropy groups of a groupoid to subquotients of the associated Boolean inverse semigroup. The proofs of our main results are collected in Sect. 5.

2 Preliminaries

In this section we recall all notions relevant to our work and introduce some elementary constructions that will be important throughout the text. For inverse semigroups the standard reference is [12]. The survey article [16] describes recent advances and the state of the art in inverse semigroup theory. Further [15] and [16, Sections 3 and 4] provide an introduction to Boolean inverse semigroups. It has to be pointed out that the definition of Boolean inverse semigroups used in the literature has changed over the years, so that care is due when consulting older material. The standard reference for groupoids and their C^* -algebras is [25]. For groupoids attached to inverse semigroups and Boolean inverse semigroups, we refer to [14, 21].

2.1 Inverse semigroups and Boolean inverse semigroups

In this section we recall the notions of inverse semigroups, Boolean inverse semigroups and the link between them provided by the universal enveloping Boolean inverse semigroup of an inverse semigroup, termed the Booleanisation.

An *inverse semigroup* S is a semigroup in which for every element $s \in S$ there is a unique element $s^* \in S$ satisfying $ss^*s = s$ and $s^*s s^* = s^*$. The set of idempotents

$E(S) \subseteq S$ forms a commutative meet-semilattice when endowed with the partial order $e \leq f$ if $ef = e$. We denote by $\text{supp } s = s^*s$ and $\text{im } s = ss^*$ the support and the image of an element $s \in S$, which are idempotents. The partial order on $E(S)$ extends to S by declaring $s \leq t$ if $t \text{supp } s = s$. Given an inverse semigroup S we denote by S_0 the inverse semigroup with zero obtained by formally adjoining an absorbing idempotent 0. In particular we will make use of groups with zero. A *character* on $E(S)$ is a non-zero semilattice homomorphism to $\{0, 1\}$. We will denote by $\widehat{E(S)}$ the space of characters on $E(S)$ equipped with the topology of pointwise convergence.

Let S be an inverse semigroup with zero. Two elements $s, t \in S$ (which are not necessarily idempotents) are called *orthogonal*, denoted $s \perp t$, if $st^* = 0 = s^*t$. A *Boolean inverse semigroup* is an inverse semigroup with zero whose semilattice of idempotents is a generalised Boolean algebra such that finite families of pairwise orthogonal elements have joins. Recall that a generalised Boolean algebra can be conveniently described as a non-unital Boolean ring. Given two Boolean inverse semigroups B and C and an inverse semigroup morphism $\phi : B \rightarrow C$ we say that ϕ is a morphism of Boolean inverse semigroups if it preserves the joins of orthogonal elements.

To every inverse semigroup S one associates the *enveloping Boolean inverse semigroup* or *Booleanisation* $S \subseteq B(S)$ which satisfies the universal property that for every Boolean inverse semigroup B and every semigroup homomorphism $S \rightarrow B$ there is a unique extension to $B(S)$ such that the following diagram commutes.

$$\begin{array}{ccc} S & \xrightarrow{\quad} & B \\ \downarrow & \nearrow & \\ B(S) & & \end{array}$$

The enveloping Boolean inverse semigroup is conveniently described as the left adjoint of the forgetful functor from Boolean inverse semigroups to inverse semigroups with zero [15]. We will use the following concrete description. Consider an inverse semigroup S . The semigroup algebra $I(S) = F_2[E(S)]$ is a non-unital Boolean ring whose characters (as a non-unital ring) are in one-to-one correspondence with characters of $E(S)$. We consider the following set of formal sums.

$$C(S) = \left\{ \sum_i s_i e_i \mid s_i \in S, e_i \in I(S) \text{ s.t. } (e_i)_i \text{ and } (s_i e_i s_i^*)_i \text{ are pairwise orthogonal} \right\}$$

We consider the equivalence relation given by the condition that $\sum_i s_i e_i \sim \sum_j t_j f_j$ if and only if $e_i f_j \neq 0$ implies that there is some $p \in E(S)$ such that $e_i f_j \leq p$ and $s_i p = t_j p$. One readily checks that the quotient of $C(S)$ by this equivalence relation is a Boolean inverse semigroup. Thanks to the existence of joins of orthogonal families every map of semigroups with zero into a Boolean inverse semigroup $S_0 \rightarrow B$ extends uniquely to a map $C(S)/\sim \rightarrow B$. By uniqueness of adjoint functors this shows that $C(S)/\sim \cong B(S)$ is the Booleanisation of S .

2.2 Ample groupoids

In this section we fix our notation for groupoids and recall some basic results. We recommend [25] and [23] as resources on the topic.

Given a groupoid $\mathcal{G}$, we denote its set of units by $\mathcal{G}^{(0)}$ and the range and source map by $r : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$ and $d : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$ respectively. Throughout the text $\mathcal{G}$ will be a *topological groupoid* meaning it is equipped with a topology making the multiplication and inversion continuous. A *local bisection* of a topological groupoid $\mathcal{G}$ is a subset $U \subseteq \mathcal{G}$ such that the restrictions $d|_U$ and $r|_U$ are homeomorphisms onto their images. We call $\mathcal{G}$ *étale* if its topology has a basis consisting of open local bisections. It is called *ample* if its topology has a basis consisting of compact open local bisections.

If $\mathcal{G}$ is a groupoid and $A \subseteq \mathcal{G}^{(0)}$ is a set of units we denote by $\mathcal{G}|_A = \{g \in \mathcal{G} \mid d(g), r(g) \in A\}$ the restriction of $\mathcal{G}$ to A . It does not need to be étale even if $\mathcal{G}$ is so. If $\mathcal{G}$ is étale and $A \subseteq \mathcal{G}^{(0)}$ is open, then $\mathcal{G}|_A$ is étale too.

The isotropy at $x \in \mathcal{G}^{(0)}$ is $\mathcal{G}|_x = \mathcal{G}|_{\{x\}}$. We denote by $\text{Iso}(\mathcal{G})$ the union of all isotropy groups considered as subsets of $\mathcal{G}$. Then $\mathcal{G}$ is *effective* if the interior of $\text{Iso}(\mathcal{G})$ equals $\mathcal{G}^{(0)}$. Given a unit $x \in \mathcal{G}^{(0)}$ its orbit is denoted by $\mathcal{G}x \subseteq \mathcal{G}^{(0)}$. We call $\mathcal{G}$ *minimal* if all its orbits are dense.

The set of orbits of a topological groupoid inherits a natural topology. We will be interested in separation properties of this *orbit space*. Recall that a topological space X is a T_0 -space if for every pair of points $x, y \in X$ there is an open set that contains exactly one of these points. It is a T_1 -space if for every pair of points $x, y \in X$ there is an open set containing x but not y . First we observe that the orbit space of a groupoid is a T_1 -space if and only if its orbits are closed. An analogous characterisation of groupoids whose orbit space is a T_0 -space is the subject of the Ramsey-Effros-Mackey dichotomy, which we now recall in the context of ample groupoids.

Proposition 1 *Let $\mathcal{G}$ be a second countable ample groupoid. Then the following statements are equivalent.*

- *The orbit space of $\mathcal{G}$ is T_0 .*
- *$\mathcal{G}$ has a self-accumulating orbit.*
- *The orbits of $\mathcal{G}$ are locally closed.*

Proof Observe that [24, p. 362] explains why the Hausdorffness assumption in [24] is redundant. So the equivalence between the first two items follows from [24, Theorem 2.1, (2) and (4)]. In order to prove the equivalence to the last item, we want to apply [24, Theorem 2.1, (4) and (5)]. To this end we have to show that the equivalence relation induced by $\mathcal{G}$ on $X = \mathcal{G}^{(0)}$ is an F_σ -subset of $X \times X$. The map $\mathcal{G} \rightarrow X \times X$ restricted to any compact open local bisection of $\mathcal{G}$ has a closed image. We conclude with the observation that there are only countably many compact open local bisections since $\mathcal{G}$ is second countable. $\square$

2.3 Noncommutative stone duality

Classical Stone duality establishes an equivalence of categories between locally compact totally disconnected Hausdorff topological spaces and generalised Boolean

algebras. If X is such a space the generalised Boolean algebra associated with it is $\text{CO}(X)$, the algebra of compact open subsets of X . Conversely, given a generalised Boolean algebra B , its spectrum $\widehat{B}$ is a locally compact totally disconnected Hausdorff topological space. Noncommutative Stone duality generalises this correspondence to an equivalence between ample Hausdorff groupoids and Boolean inverse semigroups. We refer the reader to [13, 14].

Given an ample groupoid $\mathcal{G}$, we denote by $\Gamma(\mathcal{G})$ the set of compact open local bisections of $\mathcal{G}$, which is a Boolean inverse semigroup. Conversely, given a Boolean inverse semigroup B , the set of ultrafilters for the natural order on B forms an ample groupoid $\mathcal{G}(B)$. Then $\mathcal{G}$ is Hausdorff if and only if $\Gamma(\mathcal{G})$ has meets. It follows from [14, Duality Theorem] that these two operations are dual to each other. See also [16, Theorem 4.4] and [19, Theorem 4.2]. We will not need to specify the morphisms of the categories involved in this duality. Invoking [19, Theorem 1.2] the universal groupoid of an inverse semigroup S can be naturally identified with $\mathcal{G}(B(S))$.

Noncommutative Stone duality establishes a dictionary between properties of Boolean inverse semigroups and those of ample Hausdorff groupoids. We recall some of its aspects that will be needed in this piece.

Corners and subgroupoids

Given an ample Hausdorff groupoid $\mathcal{G}$, idempotents in $\Gamma(\mathcal{G})$ correspond to compact open subsets of $\mathcal{G}$. Given such idempotent $p \in E(\Gamma(\mathcal{G}))$ corresponding to $U \subseteq \mathcal{G}^{(0)}$, the corner $p\Gamma(\mathcal{G})p$ is naturally isomorphic with $\Gamma(\mathcal{G}|_U)$. Further there is a one-to-one correspondence between open subgroupoids of $\mathcal{G}$ and Boolean inverse subsemigroups $E(\Gamma(\mathcal{G})) \subseteq B \subseteq \Gamma(\mathcal{G})$ assigning the groupoid $\bigcup B \subseteq \mathcal{G}$ to a Boolean inverse semigroup $B \subseteq \Gamma(\mathcal{G})$. Note in particular that if $E(\Gamma(\mathcal{G})) \subseteq B \subseteq \Gamma(\mathcal{G})$ is as above and $s, t \in B$, then their meet $s(s^*s \wedge s^*t)$ lies in B .

Morphisms

Noncommutative Stone duality does not cover all morphisms that one would naturally consider in the respective categories. This is why it is necessary and useful to note the following two statements. They ensure compatibility of noncommutative Stone duality with restriction maps. The next proposition is a reformulation of [19, Proposition 5.10]. See also [20].

Proposition 2 *Let $\mathcal{G}$ be an ample groupoid and let $A \subseteq \mathcal{G}^{(0)}$ be a closed $\mathcal{G}$ -invariant set. Then the restriction map $\text{CO}(\mathcal{G}^{(0)}) \rightarrow \text{CO}(A)$ extends to a unique homomorphism $\Gamma(\mathcal{G}) \rightarrow \Gamma(\mathcal{G}|_A)$ with the universal property that for every homomorphism $\pi : \Gamma(\mathcal{G}) \rightarrow B$ such that*

$$\begin{array}{ccc} \text{CO}(\mathcal{G}^{(0)}) & \xrightarrow{\pi|_{\text{CO}(\mathcal{G}^{(0)})}} & E(B) \\ \text{res}_A \downarrow & \nearrow & \\ \text{CO}(A) & & \end{array}$$

commutes, there is a unique extension to a commutative diagram

$$\begin{array}{ccc} \Gamma(\mathcal{G}) & \xrightarrow{\pi} & B \\ \text{res}_A \downarrow & \nearrow & \\ \Gamma(\mathcal{G}|_A) & & \end{array}$$

The following converse to Proposition 2 can be considered a reformulation of [19, Lemma 5.6].

Lemma 1 *Let $\mathcal{G}$ be an ample groupoid and $A \subseteq \mathcal{G}^{(0)}$ a closed subset such that the restriction $\text{res}_A : \text{CO}(\mathcal{G}^{(0)}) \rightarrow \text{CO}(A)$ extends to a map of inverse semigroups $\Gamma(\mathcal{G}) \rightarrow B$, for some inverse semigroup B . Then A is $\mathcal{G}$ -invariant.*

Proof Denote by $\pi : \Gamma(\mathcal{G}) \rightarrow B$ a map as in the statement of the lemma. Take $s \in \Gamma(\mathcal{G})$ satisfying $\text{supp}(s) \cap A = \emptyset$. Then the calculation

$$\text{res}_A(ss^*) = \pi(ss^*) = \pi(s)\pi(s^*) = \pi(s)\pi(s^*)\pi(s^*) = \pi(s)\text{res}_A(s^*s)\pi(s^*) = 0$$

shows that $\text{im}(s) \cap A = \emptyset$. So A is indeed $\mathcal{G}$ -invariant. $\square$

Minimal groupoids and 0-simplifying Boolean inverse semigroups

We introduce the algebraic notion corresponding to minimality of groupoids, following the reference [28].

Definition 1 Let B be a Boolean inverse semigroup. An ideal I of B is called *additive*, if it is closed under joins of orthogonal elements. Further B is called *0-simplifying* if its only additive ideals are $\{0\}$ and B .

The name of the notation 0-simplifying stems from the fact that additive ideals are exactly of the form $\pi^{-1}(0)$ for homomorphisms of Boolean inverse semigroups $B \rightarrow C$.

Proposition 3 ([28, Proposition 2.7]) *Let $\mathcal{G}$ be an ample groupoid. Then $\mathcal{G}$ is minimal if and only if $\Gamma(\mathcal{G})$ is 0-simplifying.*

2.4 CCR and type I groupoids

We refer the reader to [22] for the basic theory of C^* -algebras. To each of the objects considered in Sects. 2.1 and 2.2 one can associate a C^* -algebra. For groupoid C^* -algebras $C^*(\mathcal{G})$, we refer the reader to [25]. The C^* -algebras $C^*(S)$ associated with inverse semigroups are explained in [11, 23]. In particular it is known that $C^*(S) \cong C^*(\mathcal{G}(S))$ canonically. We refer to [22, Section 5.6] for details on the following two notions from representation theory of C^* -algebras.

Definition 2 Let A be a C^* -algebra. Then A is called *CCR* if the image of every irreducible $*$ -representation of A equals the compact operators. We say that A is *GCR* or of *type I* if the image of every irreducible $*$ -representation of A contains the compact operators.

Inverse semigroups and groupoids are called CCR or type I respectively, if their C^* -algebras have this property.

In this article, the notions of CCR and type I are accessed solely through the following special case of a result of Clark and van Wyk combined with the characterisation of discrete type I groups by Thoma.

Theorem 5 ([6], [30, Theorem 5.3], [31, Theorem 3.5] and [29]) *Let $\mathcal{G}$ be a second countable, étale, Hausdorff groupoid. Then $\mathcal{G}$ is CCR if and only if all the isotropy groups are virtually abelian and the orbit space of $\mathcal{G}$ is T_1 . Further $\mathcal{G}$ is type I if and only if all the isotropy groups are virtually abelian and the orbit space of $\mathcal{G}$ is T_0 .*

3 Separation properties of orbit spaces

In this section we consider separation properties of orbit spaces and provide algebraic characterisations of ample groupoids whose orbit space is a T_1 -space and a T_0 -space respectively. We start by introducing the simplest example of a groupoid whose orbit space is not T_1 .

Example 1 Let $\mathcal{G}_{T_1}$ be the groupoid associated with the equivalence relation on the one-point compactification $\mathbb{N} \cup \{\infty\}$ given by

$$\mathbb{N}^2 \cup \{(\infty, \infty)\}.$$

Denote by $B_{T_1} = \Gamma(\mathcal{G}_{T_1})$ the Boolean inverse semigroup of compact open local bisections of $\mathcal{G}_{T_1}$. Then B_{T_1} can be described as a set of partial bijections of $\mathbb{N} \cup \{\infty\}$. It is the union of all finite partial bijections of $\mathbb{N}$ together with the identity maps on sets $A \cup \{\infty\}$ where $A \subset \mathbb{N}$ runs through all cofinite subsets of the natural numbers. Further B_{T_1} admits a presentation with generators $(s_n)_{n \in \mathbb{N}}$ and f satisfying the relations

$$\begin{aligned} \text{im } s_n &= \text{supp } s_{n+1} && \text{for all } n, \\ \text{supp } s_n &\perp \text{supp } s_m && \text{for all } n \neq m, \text{ and} \\ f^2 &= f \geq \text{supp } s_n && \text{for all } n. \end{aligned}$$

The idempotents $\text{supp } s_n$ together with f generate a Boolean ring isomorphic with the algebra of compact open subsets $\text{CO}(\mathbb{N} \cup \{\infty\})$ and the elements s_n are identified with the compact open local bisections $\{(n, n+1)\}$ of $\mathcal{G}_{T_1}$.

Proposition 4 *Let $\mathcal{G}$ be an ample groupoid. Then the orbit space of $\mathcal{G}$ is not T_1 if and only if $\Gamma(\mathcal{G})$ has B_{T_1} as a subquotient.*

Proof Assume that the orbit space of $\mathcal{G}$ is not T_1 . Then by Proposition 1 there is some non-closed orbit, say $\mathcal{G}x \subseteq \mathcal{G}^{(0)}$. Let $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence from $\mathcal{G}x$ whose limit $x_\infty = \lim x_n$ does not lie in $\mathcal{G}x$. Since $\mathcal{G}$ is ample, there are local bisections $(s_n)_{n \in \mathbb{N}}$ in $\Gamma(\mathcal{G})$ such that $s_n \cap d^{-1}(x_n) \cap r^{-1}(x_{n+1}) \neq \emptyset$ for all $n \in \mathbb{N}$. Without loss of generality we may assume that $(\text{supp } s_n)_n$ are pairwise disjoint subsets of $\mathcal{G}^{(0)}$. Let $B \subseteq \Gamma(\mathcal{G})$ be the Boolean inverse semigroup generated by all $(s_n)_{n \in \mathbb{N}}$ together with the idempotents of $\Gamma(\mathcal{G})$. Denote by $\mathcal{H} = \bigcup B$ the open subgroupoid of $\mathcal{G}$ associated with B . The subset $A = \{x_n \mid n \in \mathbb{N} \cup \{\infty\}\} \subseteq \mathcal{G}^{(0)}$ is $\mathcal{H}$ -invariant, so that by Proposition 2 there is a quotient map of Boolean inverse semigroups $B \cong \Gamma(\mathcal{H}) \rightarrow \Gamma(\mathcal{H}|_A)$. It suffices to note that $\mathcal{H}|_A \cong \mathcal{G}_{T_1}$ so that $\Gamma(\mathcal{H}|_A) \cong B_{T_1}$ follows.

Assume now that B_{T_1} is a subquotient of $\Gamma(\mathcal{G})$. Denote by $(s_n)_{n \in \mathbb{N}}$ and f preimages in $\Gamma(\mathcal{G})$ of the generators of B_{T_1} . Replacing s_n by $s_n f$ we may suppose that $\text{supp } s_n \leq f$ holds for all $n \in \mathbb{N}$. Further, writing $p_n = \bigvee_{k < n} \text{supp } s_k$ and replacing s_n by $s_n(f - p_n)$, we may assume that $(\text{supp } s_n)_n$ are pairwise orthogonal. Write $t_n = s_{n-1} \cdots s_0$ and $q_n = t_n^*(\text{supp } s_n)t_n$. Then every finite subfamily of $(q_n)_{n \in \mathbb{N}}$ has a non-zero meet, since the same statement holds true for their images in B_{T_1} . Denoting by $U_n \subset \mathcal{G}^{(0)}$ the compact open subset corresponding to q_n it follows that $\bigcap_{n \in \mathbb{N}} U_n \neq \emptyset$. Choose x_0 in there and define $x_n = t_n x_0 \in \text{supp } s_n$. Since $\text{supp } s_n \leq f$ for all $n \in \mathbb{N}$, there is a convergent subsequence $(x_{n_k})_k$ of $(x_n)_n$. So the orbit $\mathcal{G}x_0$ has an accumulation point which proves that the orbit space of $\mathcal{G}$ is not T_1 . $\square$

Remark 1 Comparing the statement of Proposition 4 with Proposition 6, it is natural to ask whether it is possible to find a corner of $\Gamma(\mathcal{G})$ that has B_{T_1} as a quotient rather than finding B_{T_1} as a subquotient of $\Gamma(\mathcal{G})$. This is not possible as the following example shows. We consider the groupoid $\mathcal{G}$ arising from the equivalence relation on $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$, that relates all elements from $\mathbb{N}$ and nothing else. It is straight forward to check that $\mathcal{G}$ is an ample groupoid, since every point of $\mathbb{N}$ is isolated in $\beta\mathbb{N}$. Corners of $\Gamma(\mathcal{G})$ correspond to restrictions of $\mathcal{G}$ to compact open subsets as explained in Sect. 2.3. Compact open subsets of $\beta\mathbb{N}$ are exactly of the form $U = \overline{D}$ for subsets $D \subset \mathbb{N}$. Clearly if D is finite $\mathcal{G}_{T_1}$ cannot arise as a restriction of $\mathcal{G}|_U = \mathcal{G}|_D$. But every infinite subset of $\mathbb{N}$ has infinitely many accumulation points in $\beta\mathbb{N}$ so $\mathcal{G}_{T_1}$ cannot arise as a restriction of $\mathcal{G}$ at all.

Proposition 5 *Let $\mathcal{G}$ be a second countable, ample groupoid. Then the orbit space of $\mathcal{G}$ is not T_0 if and only if there is an infinite, unital and 0-simplifying subquotient of $\Gamma(\mathcal{G})$.*

Proof Assume first that the orbit space of $\mathcal{G}$ is not T_0 . By Proposition 1 there exists a self-accumulating orbit of $\mathcal{G}$. Denote its closure by A . Since $\mathcal{G}$ is étale, for every open subset $U \subset \mathcal{G}^{(0)}$ that intersects A non-trivially there is some element $s \in \Gamma(\mathcal{G})$ whose support and image are disjoint and both lie in U and intersect A non-trivially. Let $s_0 \in \text{CO}(\mathcal{G}^{(0)})$ be some trivial local bisection intersecting A non-trivially. If $s_0, \dots, s_n$ have been defined, choose some $s_{n+1} \in \Gamma(\mathcal{G})$ with disjoint support and image both

contained in $\text{supp } s_n$ and both intersecting A non-trivially. For $n \geq 1$ define

$$K_n = \bigsqcup_{(t_1, \dots, t_n) \in \prod_{i=1}^n \{s_0, s_i\}} t_1 \cdots t_n (\text{supp } s_n \cap A)$$

and observe that the open subgroupoid of $\mathcal{G}$ generated by $s_1, \dots, s_n$ leaves K_n invariant. Further, since the subsets $t_1 \cdots t_n (\text{supp } s_n \cap A)$ with $(t_1, \dots, t_n) \in \prod_{i=1}^n \{s_0, s_i\}$ are pairwise disjoint, all of its orbits in K_n have size 2^n . The non-empty compact sets $(K_n)_n$ are descending, and thus their intersection is non-empty. Denote the open subgroupoid of $\mathcal{G}$ generated by all $s_1, s_2, \dots$ with $\mathcal{H}$. Then $\bigcap_{n \geq 1} K_n$ is $\mathcal{H}$ -invariant and every $\mathcal{H}$ -orbit in $\bigcap_{n \geq 1} K_n$ is infinite. Let $K \subseteq \bigcap_{n \geq 1} K_n$ be any $\mathcal{H}$ -minimal subset. Then $\Gamma(\mathcal{H}|_K)$ is 0-simplifying by Proposition 3. We also observe that $\Gamma(\mathcal{H}|_K)$ is infinite, because K is infinite and it is unital, since K is compact. Since $\mathcal{H}$ is an open subgroupoid of $\mathcal{G}$, we obtain an inclusion of Boolean inverse semigroups $\Gamma(\mathcal{H}) \subseteq \Gamma(\mathcal{G})$. Moreover, since K is $\mathcal{H}$ -invariant by construction, Proposition 2 says that there is a quotient map $\Gamma(\mathcal{H}) \rightarrow \Gamma(\mathcal{H}|_K)$. So $\Gamma(\mathcal{H}|_K)$ is a subquotient of $\Gamma(\mathcal{G})$.

Let us now assume that there is an infinite, unital and 0-simplifying subquotient of $\Gamma(\mathcal{G})$. We will show that the orbit space of $\mathcal{G}$ is not T_0 . Write $\Gamma(\mathcal{G}) \supset C \twoheadrightarrow B$ for the given subquotient. Choosing an idempotent preimage $p \in E(C)$ of the unit of B we obtain a surjection $pCp \twoheadrightarrow B$. Let $V \subset \mathcal{G}^{(0)}$ be the compact open subset corresponding to p . Replacing C by pCp we thus find a unital inclusion $\Gamma(\mathcal{G}|_V) \supset C$ and a quotient map $\pi : C \twoheadrightarrow B$. Let $\mathcal{H}$ be the ample groupoid associated with C by noncommutative Stone duality, that is $\Gamma(\mathcal{H}) \cong C$. Write $X = \mathcal{H}^{(0)}$. Considering the restriction of π to idempotents we find a closed subset $A \subseteq X$ such that the following diagram commutes.

$$\begin{array}{ccc} \text{CO}(X) & \xrightarrow{\pi} & E(B) \\ \text{res}_A \downarrow & \nearrow \cong & \\ \text{CO}(A) & & \end{array}$$

Let $x \in A$. We will show that $\mathcal{H}x \cap A$ is self-accumulating. By noncommutative Stone duality there is an ample groupoid $\mathcal{K}$ such that $\Gamma(\mathcal{K}) \cong B$. From the fact that B is infinite, unital and 0-simplifying it follows that $\mathcal{K}$ is infinite, has a compact unit space and is minimal. In particular, the orbits of $\mathcal{K}$ are self-accumulating. So for every compact open neighbourhood $U \subseteq A$ of x there is some local bisection $t \in \Gamma(\mathcal{K})$ such that $x \in \text{supp } t$ and $x \notin \text{im } t \leq U$. Let $s \in \pi^{-1}(t)$ denote a preimage of t . Then $\text{supp } s \cap A = \pi(\text{supp } s) = \text{supp } \pi(s) = \text{supp } t$ and similarly $\text{im } s \cap A = \text{im } t$. It follows that $\mathcal{H}x \cap A$ and thus also the $\mathcal{H}$ -orbit of x is self-accumulating. Thus the orbit space of $\mathcal{H}$ is not T_0 . Consider now the surjective map $\varphi : V \rightarrow X$ which is dual to the inclusion $\text{CO}(X) \subset \text{CO}(V)$. Given $x, y \in X$ in the same $\mathcal{H}$ -orbit there is $s \in C$ such that $sx = y$. Let $u \in V$ be some preimage of x under φ . Considering s as a local bisection of $\mathcal{G}|_V$, we define $v = su$ which lies in the same $\mathcal{G}|_V$ -orbit as u and satisfies $\varphi(v) = y$. This shows that every $\mathcal{H}$ -orbit is contained in the image of

a $\mathcal{G}|_V$ -orbit. In particular, there is some orbit of $\mathcal{G}|_V$ that is not finite and hence not locally closed. So Proposition 1 says that the orbit space of $\mathcal{G}$ is not T_0 . $\square$

4 Group quotients and isotropy groups

In this section we relate the isotropy groups of an ample groupoid with certain subquotients of the Boolean inverse semigroup of its compact open local bisections. This result will allow us to translate the condition on isotropy groups from [6,30,31].

Proposition 6 *Let $\mathcal{G}$ be a second countable, ample groupoid whose orbit space is T_0 . Let $x \in \mathcal{G}^{(0)}$ be a unit, write $G = \mathcal{G}|_x$ for the isotropy group at x and denote by G_0 the associated group with zero. Then G_0 is a quotient of a corner of $\Gamma(\mathcal{G})$. Conversely, if $\mathcal{G}$ is any ample groupoid and G is a group such that G_0 is a quotient of a corner of $\mathcal{G}$ then G is a quotient of a point stabiliser of $\mathcal{G}$.*

Proof Since the orbit space of $\mathcal{G}$ is assumed to be T_0 and $\mathcal{G}$ is second countable, its orbits are locally closed by Proposition 1. Let $U \subseteq \mathcal{G}^{(0)}$ be a compact open neighbourhood of x such that $\mathcal{G}x \cap U$ is closed in U and hence compact. Since $\mathcal{G}$ is étale, we know that $\mathcal{G}x$ is countable so that $\mathcal{G}x \cap U$ is actually finite. We may thus shrink U so that $\mathcal{G}x \cap U = \{x\}$ holds. Denote by $p = U \in \Gamma(\mathcal{G})$ the idempotent local bisection associated with U . Then the corner of $\Gamma(\mathcal{G})$ can be identified as $p\Gamma(\mathcal{G})p = \Gamma(\mathcal{G}|_U)$. Since x is fixed by $\mathcal{G}|_U$, the restriction from U to $\{x\}$ induces a quotient of Boolean inverse semigroups $\Gamma(\mathcal{G}|_U) \rightarrow \Gamma(\mathcal{G}|_x) = (\mathcal{G}|_x)_0$ by Proposition 2.

Let us now assume that $\mathcal{G}$ is any ample groupoid, let G be a group and $p \in \Gamma(\mathcal{G})$ an idempotent for which there is a quotient map $\pi : p\Gamma(\mathcal{G})p \rightarrow G_0$. Denote by $U \subseteq \mathcal{G}^{(0)}$ the compact open subset corresponding to p . Then there is a natural isomorphism $p\Gamma(\mathcal{G})p \cong \Gamma(\mathcal{G}|_U)$. Since the algebra of idempotents of G_0 is trivial, $\pi|_{E(\Gamma(\mathcal{G}|_U))}$ is a character. By Stone duality there is $x \in U$ such that $\pi|_{E(\Gamma(\mathcal{G}|_U))} = \text{ev}_x$. In particular, $\{x\} \subseteq U$ is a $\mathcal{G}|_U$ -invariant subset by Lemma 1. So by the universal property of the restriction map described in Proposition 2 the homomorphism π factors through $\Gamma(\mathcal{G}|_U) \rightarrow \Gamma(\mathcal{G}|_x) = (\mathcal{G}|_x)_0$. So G_0 is a quotient of $(\mathcal{G}|_x)_0$ which implies that G is a quotient of $\mathcal{G}|_x$. $\square$

Example 2 It might be tempting to admit arbitrary subquotients of $\Gamma(\mathcal{G})$ in the statement of Proposition 6, however this does not even suffice under the condition that the orbit space of $\mathcal{G}$ is T_1 . Indeed, the topological full group of $\mathcal{G}$ is always a subgroup of $\Gamma(\mathcal{G})$, and it can be large even if $\mathcal{G}$ is effective. For example the topological full group associated with B_{T_1} is the group of all finitely supported permutations of $\mathbb{N}$.

We next formulate an appropriate version of Proposition 6 for inverse semigroups. Let us start with a short lemma relating quotients of an inverse semigroup and its Booleanisation.

Lemma 2 *Let S be an inverse semigroup and $B(S) \rightarrow G_0$ a group quotient of its Booleanisation. Then $S_0 \rightarrow G_0$ is surjective.*

Proof Denote the quotient map $B(S) \twoheadrightarrow G_0$ by π and let $g \in G$. Using the description of $B(S)$ presented in Sect. 2.1 there is some preimage $\sum_i t_i e_i \in B(S)$ of g . Since G_0 has only two idempotents, $\pi|_{E(B(S))}$ is a character. So there is a unique i_0 satisfying $\pi(e_{i_0}) = 1$. This implies $\pi(\sum_i t_i e_i) = \pi(t_{i_0})$ showing that $g \in \pi(S_0)$. $\square$

Proposition 7 *Let S be a countable inverse semigroup such that the orbit space of $\mathcal{G} = \mathcal{G}(S)$ is T_0 . Let $x \in \mathcal{G}^{(0)}$ and write $G = \mathcal{G}|_x$. Then the group with zero G_0 is a quotient of a corner of S_0 .*

Proof Fix $x \in \mathcal{G}^{(0)}$. Since $\mathcal{G}^{(0)} \cong \widehat{E(S)}$, there is $q \in E(S)$ such that $x(q) = 1$. Denote by $U \subseteq \mathcal{G}^{(0)}$ the compact open subset corresponding to q . As in the proof of Proposition 6 the fact that orbits of $\mathcal{G}$ are locally closed implies that $\mathcal{G}_x \cap U$ is finite. Fix an enumeration $x = x_0, x_1, \dots, x_n$ of $\mathcal{G}_x \cap U$. For $i \in \{1, \dots, n\}$ there is $q_i \in E(S)$ such that $x_0(q_i) = 1$ and $x_i(q_i) = 0$. Put $p = q \cdot q_1 \cdots q_n$ and let $V \subseteq \mathcal{G}^{(0)}$ be the compact open subset corresponding to p . Then the identification of corners of Boolean inverse semigroups says that

$$\mathcal{G}(pSp) = \mathcal{G}(B(pSp)) \cong \mathcal{G}(pB(S)p) \cong \mathcal{G}|_V.$$

Since $x \in V$ is $\mathcal{G}|_V$ -fixed, there is a quotient map $\Gamma(\mathcal{G}|_V) \rightarrow \Gamma(\mathcal{G}|_x) \cong (\mathcal{G}|_x)_0$. The identification $\Gamma(\mathcal{G}|_V) \cong B(pSp)$ shows that there is a surjection $B(pSp) \rightarrow (\mathcal{G}|_x)_0$. By Lemma 2 its restriction $pS_0p \cong (pSp)_0 \rightarrow (\mathcal{G}|_x)_0$ remains surjective. $\square$

5 Proof of the main results

We now obtain the proof of our main theorems. Thanks to the preparation made in the previous sections all proofs are rather similar and we spell out details only for Theorem 1.

Proof of Theorem 1 Since $\mathcal{G}$ is an étale, second countable Hausdorff groupoid, we may appeal to the results of Clark, van Wyk and Thoma described in Theorem 5. It follows that $\mathcal{G}$ is CCR if and only if all its isotropy groups are virtually abelian and its orbit space is T_1 . We can thus combine Proposition 6 and Proposition 4 to complete our proof. $\square$

Proof of Theorem 2 Upon replacing the reference to Proposition 4 by Proposition 5 the same argument as used in the proof of Theorem 1 can be applied. $\square$

Proof of Theorem 3 Since S has meets, it follows that $\mathcal{G}(S)$ is Hausdorff. So replacing the reference to Proposition 6 by Proposition 7, and alluding to the fact that $\Gamma(\mathcal{G}(S)) \cong B(S)$ the proof of Theorem 1 applies. $\square$

Proof of Theorem 4 Making the same adaptations as in the passage from the proof of Theorem 1 to 2, the proof of Theorem 3 applies. $\square$

Remark 2 In view of our construction of $B(S)$ described in Sect. 2.1, the Booleanisation of an inverse semigroup can be concretely calculated, so that the conditions on

$B(S)$ in Theorems 3 and 4 can be checked. Specifically for Theorem 4 we remark that the groupoid $\mathcal{G}(S)$ associated with an inverse semigroup always has a fixed point (whose isotropy group is the maximal group quotient of S). It corresponds to the trivial character on $E(S)$ that maps every idempotent to 1. Since 0-simplifying Boolean inverse semigroups correspond to minimal groupoids, it is not possible to directly translate the condition on $B(S)$ in Theorem 4 to an algebraic statement about S itself.

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