The Lawson number of a semitopological semilattice

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Abstract

For a Hausdorff topologized semilattice X its *Lawson number* $\bar{\Lambda}(X)$ is the smallest cardinal κ such that for any distinct points $x, y \in X$ there exists a family $\mathcal{U}$ of closed neighborhoods of x in X such that $|\mathcal{U}| \leq \kappa$ and $\bigcap \mathcal{U}$ is a subsemilattice of X that does not contain y . It follows that $\bar{\Lambda}(X) \leq \bar{\psi}(X)$, where $\bar{\psi}(X)$ is the smallest cardinal κ such that for any point $x \in X$ there exists a family $\mathcal{U}$ of closed neighborhoods of x in X such that $|\mathcal{U}| \leq \kappa$ and $\bigcap \mathcal{U} = \{x\}$. We prove that a compact Hausdorff semitopological semilattice X is Lawson (i.e., has a base of the topology consisting of subsemilattices) if and only if $\bar{\Lambda}(X) = 1$. Each Hausdorff topological semilattice X has Lawson number $\bar{\Lambda}(X) \leq \omega$. On the other hand, for any infinite cardinal λ we construct a Hausdorff zero-dimensional semitopological semilattice X such that $|X| = \lambda$ and $\bar{\Lambda}(X) = \bar{\psi}(X) = \text{cf}(\lambda)$. A topologized semilattice X is called (i) ω -*Lawson* if $\bar{\Lambda}(X) \leq \omega$; (ii) *complete* if each non-empty chain $C \subseteq X$ has $\inf C \in \overline{C}$ and $\sup C \in \overline{C}$. We prove that for any complete subsemilattice X of an ω -Lawson semitopological semilattice Y , the partial order $\leq_X = \{(x, y) \in X \times X : xy = x\}$ of X is closed in $Y \times Y$ and hence X is closed in Y . This implies that for any continuous homomorphism $h : X \rightarrow Y$ from a complete topologized semilattice X to an ω -Lawson semitopological semilattice Y the image $h(X)$ is closed in Y .

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1 Introduction

In this paper we introduce a new cardinal invariant $\bar{\Lambda}(X)$ of a Hausdorff topologized semilattice X , called the Lawson number of X . This was motivated by studying the closedness properties of complete topologized semilattices, see [1–6]. It turns out that complete semitopological semilattices share many common properties with compact topological semilattices, in particular their continuous homomorphic images in Hausdorff topological semilattices are closed.

A *semilattice* is any commutative semigroup of idempotents (an element x of a semigroup is called an *idempotent* if $xx = x$).

Each semilattice X carries the natural partial order $\leq_X$ defined by $x \leq_X y$ iff $xy = x = yx$. Many properties of semilattices are defined in the language of the natural partial order. In particular, for any point $x \in X$ we can consider its upper and lower sets

$$\uparrow x := \{y \in X : xy = x\} \text{ and } \downarrow x := \{y \in X : xy = y\}$$

in the partially ordered set $(X, \leq_X)$.

A subset C of a semilattice X is called a *chain* if $xy \in \{x, y\}$ for any $x, y \in C$. A semilattice X is called *chain-finite* if each chain in X is finite. A semilattice is called *linear* if it is a chain in itself.

A semilattice endowed with a topology is called a *topologized semilattice*. A topologized semilattice X is called a (*semi*) *topological semilattice* if the semigroup operation $X \times X \rightarrow X$, $(x, y) \mapsto xy$, is (separately) continuous.

In [16] Stepp proved that for any homomorphism $h : X \rightarrow Y$ from a chain-finite semilattice to a Hausdorff topological semilattice Y , the image $h(X)$ is closed in Y . In [1], the first authors improved this result of Stepp proving the following theorem.

Theorem 1 (Banach, Bardyla) *For any homomorphism $h : X \rightarrow Y$ from a chain-finite semilattice to a Hausdorff semitopological semilattice Y , the image $h[X]$ is closed in Y .*

Topological generalizations of the notion of chain-finiteness are the notions of chain-compactness and completeness, discussed in [5].

A topologized semilattice X is called

- *chain-compact* if each closed chain in X is compact;
- *complete* if each non-empty chain $C \subseteq X$ has $\inf C \in \overline{C}$ and $\sup C \in \overline{C}$.

Here $\overline{C}$ stands for the closure of C in X . Chain-compact and complete topologized semilattices appeared to be very helpful in studying the closedness properties of topologized semilattices, see [1–6, 11]. By Theorem 3.1 [1], a Hausdorff semitopological

semilattice is chain-compact if and only if it is complete (see also Theorem 4.3 [5] for generalization of this characterization to topologized posets). In [1] the first two authors proved the following closedness property of complete topologized semilattices.

Theorem 2 (Banakh, Bardyla) *For any continuous homomorphism $h : X \rightarrow Y$ from a complete topologized semilattice X to a Hausdorff topological semilattice Y , the image $h[X]$ is closed in Y .*

Theorems 1 and 2 motivate the following (still) open problem.

Problem 1 Assume that $h : X \rightarrow Y$ is a continuous homomorphism from a complete topologized semilattice X to a Hausdorff semitopological semilattice Y . Is $h[X]$ closed in Y ?

In [4] the first two authors gave the following partial answer to Problem 1.

Theorem 3 (Banakh, Bardyla) *For any continuous homomorphism $h : X \rightarrow Y$ from a complete topologized semilattice X to a sequential Hausdorff semitopological semilattice Y , the image $h[X]$ is closed in Y .*

Another partial result to Problem 1 was given in [6].

Theorem 4 (Banakh, Bardyla, Ravsky) *For any continuous homomorphism $h : X \rightarrow Y$ from a complete topologized semilattice X to a functionally Hausdorff semitopological semilattice Y , the image $h[X]$ is closed in Y .*

Let us recall that a topological space X is *functionally Hausdorff* if for any distinct points $x, y \in X$ there exists a continuous map $f : X \rightarrow \mathbb{R}$ such that $f(x) \neq f(y)$.

In fact, in [6] Theorem 4 was derived from the following closedness property of the partial order of a complete subsemilattice of a functionally Hausdorff semitopological semilattice.

Theorem 5 (Banakh, Bardyla, Ravsky) *For any complete subsemilattice X of a functionally Hausdorff semitopological semilattice Y , the partial order $\leq_X$ of X is a closed subset of $Y \times Y$.*

In this paper we shall show that the answer to Problem 1 is affirmative under the additional condition that the semitopological semilattice Y is ω -Lawson. We shall also prove a counterpart of Theorem 5 for complete subsemilattices of ω -Lawson semitopological semilattices.

We define a topologized semilattice X to be ω -Lawson if for any distinct points $x, y \in X$ there exists a countable family $\mathcal{U}$ of closed neighborhoods of x such that $\bigcap \mathcal{U}$ is a subsemilattice of X that does not contain y . A topologized semilattice X is ω -Lawson if and only if it is Hausdorff and has at most countable Lawson number $\bar{\Lambda}(X)$.

The *Lawson number* $\bar{\Lambda}(X)$ of a Hausdorff topologized semilattice X is defined as the smallest cardinal κ such that for any distinct points $x, y \in X$ there exists a family $\mathcal{U}$ of closed neighborhoods of x such that $|\mathcal{U}| \leq \kappa$ and $\bigcap \mathcal{U}$ is a subsemilattice of X that does not contain y .

The Lawson number will be studied in more details in Sect. 2. In that section we shall prove that every Hausdorff topological semilattice X has $\bar{\Lambda}(X) \leq \omega$. On the other hand, for any infinite cardinal λ we shall construct a Hausdorff zero-dimensional semitopological semilattice X of cardinality $|X| = \lambda$ and Lawson number $\bar{\Lambda}(X) = \text{cf}(\lambda)$. In Sect. 4 we prove the main result of this paper:

Theorem 6 *For any complete subsemilattice X of an ω -Lawson semitopological semilattice Y , the natural partial order $\leq_X$ of X is a closed subset of $Y \times Y$. Consequently, X is closed in Y .*

Remark 1 By [7] (and [8]), there exists a metrizable (Lawson) semitopological semilattice X whose partial order $\leq_X$ is not closed in $X \times X$.

Since the completeness is preserved by continuous homomorphisms into Hausdorff semitopological semilattices (see Lemma 1), Theorem 6 implies the following corollary giving a partial answer to Problem 1.

Corollary 1 *For any continuous homomorphism $h : X \rightarrow Y$ from a complete topologized semilattice to an ω -Lawson semitopological semilattice Y the image $h[X]$ is closed in Y .*

Problem 2 Let X be a complete subsemilattice of an ω_1 -Lawson semitopological semilattice Y . Is X closed in Y ? Is the natural partial order $\leq_X$ of X closed in $X \times X$?

2 The Lawson number of a Hausdorff topologized semilattice

In this section we study the Lawson number of a topologized semilattice in more details. We start with recalling the principal definition.

Definition 1 The *Lawson number* $\bar{\Lambda}(X)$ of a Hausdorff topologized semilattice X is the smallest cardinal κ such that for any distinct points $x, y \in X$ there exists a family $\mathcal{U}$ of closed neighborhoods of x such that $|\mathcal{U}| \leq \kappa$ and the intersection $\bigcap \mathcal{U}$ is a subsemilattice of X that does not contain y .

For any Hausdorff topologized semilattice X the Lawson number $\bar{\Lambda}(X)$ is well-defined and does not exceed the *closed pseudocharacter* $\bar{\psi}(X)$ of X , defined as the smallest cardinal κ such that for any point $x \in X$ there exists a family $\mathcal{U}$ of closed neighborhoods of x such that $|\mathcal{U}| \leq \kappa$ and $\bigcap \mathcal{U} = \{x\}$. Therefore, $\bar{\Lambda}(X) \leq \bar{\psi}(X)$ for any Hausdorff topologized semilattice X .

Observe that a topologized semilattice X is ω -Lawson if and only if X is Hausdorff and has $\bar{\Lambda}(X) \leq \omega$.

Definition 2 A topologized semilattice X is defined to be κ -Lawson for a cardinal κ if X is Hausdorff and $\bar{\Lambda}(X) \leq \kappa$.

The Lawson number admits the following simple characterization.

Proposition 1 *The Lawson number of a Hausdorff topologized semilattice X is equal to the smallest cardinal κ such that for any distinct points $x, y \in X$ there exist a closed subsemilattice L in X and a family $\mathcal{V}$ of closed neighborhoods of x such that $|\mathcal{V}| \leq \kappa$ and $\bigcap \mathcal{V} \subseteq L \subseteq X \setminus \{y\}$.*

Proof We should prove that $\bar{\Lambda}(X) = \Lambda(X)$ where $\Lambda(X)$ is the smallest cardinal κ such that for any distinct points $x, y \in X$ there exist a closed subsemilattice L in X and a family $\mathcal{V}$ of closed neighborhoods of x such that $|\mathcal{V}| \leq \kappa$ and $\bigcap \mathcal{V} \subseteq L \subseteq X \setminus \{y\}$.

To see that $\Lambda(X) \leq \bar{\Lambda}(X)$, for any distinct points $x, y \in X$ use the definition of $\bar{\Lambda}(X)$ and find a family $\mathcal{U}$ of closed neighborhoods of x such that $|\mathcal{U}| \leq \bar{\Lambda}(X)$ and $L := \bigcap \mathcal{U}$ is a closed subsemilattice of X that does not contain y . Then $\bigcap \mathcal{U} = L \subseteq X \setminus \{y\}$, witnessing that $\Lambda(X) \leq \bar{\Lambda}(X)$.

To see that $\bar{\Lambda}(X) \leq \Lambda(X)$, for any distinct points $x, y \in X$ use the definition of $\Lambda(X)$ and find a closed subsemilattice L of X and a family $\mathcal{V}$ of closed neighborhoods of x such that $|\mathcal{V}| \leq \Lambda(X)$ and $\bigcap \mathcal{V} \subseteq L \subseteq X \setminus \{y\}$. Then $\mathcal{U} := \{V \cup L : V \in \mathcal{V}\}$ is a family of closed neighborhoods of x such that $|\mathcal{U}| \leq |\mathcal{V}| \leq \Lambda(X)$ and $\bigcap \mathcal{U} = (\bigcap \mathcal{V}) \cup L = L$ is a closed subsemilattice of X that does not contain y and witnesses that $\bar{\Lambda}(X) \leq \Lambda(X)$. $\square$

The notion of a 1-Lawson semilattice extends the well-known notion of a Lawson semilattice (or else a topologized semilattice with small subsemilattices), introduced and studied by Lawson [13] (see also [10, Chapter 2]). Following [10, p. 12], we define a topologized semilattice X to be

- *Lawson* if it has a base of the topology consisting of open subsemilattices;
- a *V-semilattice* if for any points $x \in X$ and $y \in X \setminus \uparrow x$ there exists a point $v \in X \setminus \downarrow y$ such that $\uparrow v$ is a neighborhood of x in X ;
- *$\mathbb{I}$ -separated* if for any distinct points $x, y \in X$ there exists a continuous homomorphism $f : X \rightarrow \mathbb{I}$ such that $f(x) \neq f(y)$.

Here by $\mathbb{I}$ we denote the unit interval $[0, 1]$ endowed with the semilattice operation $\min$.

Proposition 2 *For a Hausdorff semitopological semilattice X , consider the following conditions:*

1. X is 1-Lawson;
2. X is Lawson;
3. X is a V-semilattice;
4. X is $\mathbb{I}$ -separated.

Any of the conditions (2),(3),(4) implies (1). If the space X is compact, then the conditions (1)–(4) are equivalent.

Proof To prove that X is 1-Lawson, fix any distinct points $x, y \in X$. We need to find a closed subsemilattice L in X that contains x in its interior but does not contain y .

(2) $\Rightarrow$ (1) Assume that X is Lawson. Since X is Hausdorff, there exists a closed neighborhood $N_x \subseteq X$ of x such that $y \notin N_x$. Since X is Lawson, there exists an open subsemilattice $V \subseteq X$ such that $x \in V \subseteq N_x$. Since X is a semitopological semilattice, the closure $\bar{V}$ of the semilattice V is a closed subsemilattice that contains x in its interior but does not contain y . Therefore, the semilattice X is 1-Lawson.

(3) $\Rightarrow$ (1) Assume that X is a V-semilattice. If $x \notin \downarrow y$, then $y \notin \uparrow x$ and there exists an element $v \in X \setminus \downarrow y$ such that the upper set $\uparrow v$ contains x in its interior. Since X is a Hausdorff semitopological semilattice the upper set $\uparrow v = \{z \in X : zv = v\}$

is closed. Then the closed subsemilattice $\uparrow v$ is a neighborhood of x that does not contain y . If $x \in \downarrow y$, then $x \notin \uparrow y$. Since X is a V -semilattice, there exists an element $u \in X \setminus \downarrow x$ such that the upper set $\uparrow u$ contains y in its interior. Observe that the complement $U := X \setminus \uparrow u$ is an open subsemilattice of X , containing x . Then $\overline{U}$ is a closed subsemilattice of X , which is a neighborhood of x that does not contain y .

(4) $\Rightarrow$ (1) Assuming that X is $\mathbb{I}$ -separated, we can find a continuous homomorphism $f : X \rightarrow \mathbb{I}$ such that $f(x) \neq f(y)$. Choose any closed neighborhood $N \subset \mathbb{I}$ of $f(x)$ such that $f(y) \notin N$. Then $f^{-1}(N)$ is a closed subsemilattice in X that contains x in its interior but does not contain y .

Now assume that X is compact. In this case the conditions (1)–(4) are equivalent by Theorem 7.1 in [2]. In fact, the equivalence of the conditions (2) and (4) is a classical result of Lawson [13, 14]. $\square$

Example 1 The topological semilattice $\mathbb{Z}^\omega$ with the Tychonoff product topology and coordinatewise operation of minimum is Lawson and $\mathbb{I}$ -separated but not a V -semilattice.

By [10, Example 2.21], there exists a metrizable compact topological semilattice, which is not Lawson and hence is not 1-Lawson. However, such a semilattice necessarily is ω -Lawson as shown by the following simple proposition.

Proposition 3 *Each Hausdorff topological semilattice X is ω -Lawson.*

Proof Given two distinct points $x, y \in X$, choose a decreasing sequence $(U_n)_{n \in \omega}$ of open neighborhoods of x such that $y \notin \overline{U_0}$ and $U_n \cdot U_n \subseteq U_{n-1}$ and hence $\overline{U_n} \cdot \overline{U_n} \subseteq \overline{U_{n-1}}$ for all $n \in \mathbb{N}$. The choice of U_0 is possible by the Hausdorff property of X , and the choice of the neighborhoods U_n is possible by the continuity of the semilattice operation at (x, x) . It follows that the intersection $\bigcap_{n \in \omega} \overline{U_n}$ is a closed subsemilattice of X containing x but not y . $\square$

Corollary 2 *Each compact Hausdorff semitopological semilattice is ω -Lawson.*

Proof By [14], each compact Hausdorff semitopological semilattice is a topological semilattice and by Proposition 3, is ω -Lawson. $\square$

Let us also notice the following trivial (but useful) fact.

Proposition 4 *Each (Hausdorff) linear topologized semilattice X is Lawson (and 1-Lawson).*

Let us recall that a topological space X is *Urysohn* if any distinct points in X have disjoint closed neighborhoods. It is clear that each Urysohn space is Hausdorff.

We define a topological space X to be κ -*Urysohn* for a cardinal κ if for any distinct points $x, y \in X$ there are families $\mathcal{U}_x$ and $\mathcal{U}_y$ of open sets on X such that $\max\{|\mathcal{U}_x|, |\mathcal{U}_y|\} \leq \kappa$, $x \in \bigcap \mathcal{U}_x$, $y \in \bigcap \mathcal{U}_y$ and the sets $\bigcap_{U \in \mathcal{U}_x} \overline{U}$ and $\bigcap_{V \in \mathcal{U}_y} \overline{V}$ are disjoint.

It is easy to see that a topological space is Urysohn if and only if it is 1-Urysohn.

Example 2 For every infinite cardinal κ , there exists a Hausdorff Lawson topological semilattice, which is not κ -Urysohn.

Proof Take any ordinal λ of cofinality $\text{cf}(\lambda) > \kappa$ (for example, put $\lambda := \kappa^+$). Consider the set $L = \{x_\alpha\}_{\alpha \leq \lambda} \cup \{z\} \cup \{y_\alpha\}_{\alpha \leq \lambda}$ of pairwise distinct points endowed with the linear order in which $x_\alpha < x_\beta < z < y_\beta < y_\alpha$ for any ordinals $\alpha < \beta \leq \lambda$. Let $\check{L} := L \setminus \{x_\lambda, y_\lambda\}$. On the set

$$X = (\check{L} \times [0, \lambda)) \cup (\{x_\lambda, y_\lambda\} \times \{\lambda\})$$

consider the semilattice operation

$$(x, \alpha) \cdot (y, \beta) := \begin{cases} (\min\{x, y\}, \min\{\alpha, \beta\}) & \text{if } \alpha, \beta < \lambda; \\ (\min\{x, z\}, \alpha), & \text{if } \alpha < \lambda = \beta; \\ (\min\{z, y\}, \beta), & \text{if } \beta < \lambda = \alpha; \\ (\min\{x, y\}, \lambda), & \text{if } \alpha = \lambda = \beta. \end{cases}$$

Endow X with the topology τ consisting of all sets $U \subseteq X$ satisfying the following three conditions:

- if $(z, \alpha) \in U$ for some $\alpha \in [0, \lambda)$, then $\{(x_\gamma, \alpha), (y_\gamma, \alpha) : \beta < \gamma < \lambda\} \subseteq U$ for some $\beta \in [0, \lambda)$;
- if $(x_\lambda, \lambda) \in U$, then $\{(x_\beta, \gamma) : \beta, \gamma \in [\alpha, \lambda)\} \subseteq U$ for some $\alpha \in [0, \lambda)$;
- if $(y_\lambda, \lambda) \in U$, then $\{(y_\beta, \gamma) : \beta, \gamma \in [\alpha, \lambda)\} \subseteq U$ for some $\alpha \in [0, \lambda)$.

Taking into account that $\text{cf}(\lambda) > \kappa$, we can show that (X, τ) is a required Hausdorff Lawson topological semilattice which is not κ -Urysohn. $\square$

Now we construct Hausdorff zero-dimensional semitopological semilattices having an arbitrarily large Lawson number. We recall that a topological space is *zero-dimensional* if it has a base of the topology consisting of open-and-closed sets.

Example 3 For any infinite cardinal λ there exists a Hausdorff zero-dimensional semitopological semilattice X such that $|X| = \lambda$ and $\bar{\Lambda}(X) = \bar{\psi}(X) = \text{cf}(\lambda)$.

Proof Consider the set

$$X := \{A \subseteq \lambda : A = \lambda \text{ or } A \text{ is finite}\}$$

endowed with the semilattice operation of union. This semilattice has cardinality $|X| = \lambda$. Here we identify the cardinal λ with the set $[0, \lambda)$ of all ordinals smaller than λ .

Now the trick is to introduce an appropriate topology on the semilattice X . For this we define several kinds of sets in λ .

A finite subset $A \subseteq \lambda$ is defined to be *sparse* if $|A \cap [\alpha, \alpha + \omega)| \leq 1$ for any ordinal $\alpha \in \lambda$.

For a set $A \in X$ and an ordinal $\alpha \in \lambda$ consider the set

$$S[A; \alpha] := \{B \in X : B \cap [0, \alpha) = A \cap [0, \alpha) \text{ and } B \cap [\alpha, \lambda) \text{ is sparse}\},$$

and observe that $\lambda \notin S[A; \alpha]$.

Let $\alpha \in \lambda$ be an ordinal, n be a finite ordinal and ε be a positive real number. A subset $A \subset \lambda$ is called (α, n, ε) -fat if there exists a limit ordinal $\beta \in [\alpha, \lambda)$ and a finite ordinal $m > n$ such that

- (i) the set $A \cap [\beta + \omega, \lambda)$ is sparse;
- (ii) $[\beta, \beta + \omega) \cap A = [\beta, \beta + m]$;
- (iii) the set $[0, \beta) \cap A$ is finite and has cardinality $< \varepsilon \cdot m$.

The conditions (i),(ii) ensure that the ordinal β is unique.

Consider the subset

$$F[\alpha, n, \varepsilon] := \{\lambda\} \cup \{A \in X : A \text{ is } (\alpha, n, \varepsilon)\text{-fat}\}$$

of X .

Now we define a topology τ on the semilattice X . This topology consists of the sets $U \subseteq X$ satisfying the following two conditions:

- (a) for any finite subset $A \in U$ of λ there exists an ordinal $\alpha \in \lambda$ such that $A \in S[A, \alpha] \subseteq U$;
- (b) if $\lambda \in U$, then there exist ordinals $\alpha \in \lambda$, $k \in \omega$, and a positive real number ε such that $F[\alpha, k, \varepsilon] \subseteq U$.

It is easy to see that τ is a well-defined topology on X . Now we show that this topology is Hausdorff and zero-dimensional. Subsets $U \in \tau$ of X will be called τ -open. $\square$

Claim 1 For any element $A \in X$ and ordinal $\alpha \in \lambda$ the set $S[A; \alpha]$ is open in (X, τ) .

Proof To see that $S[A; \alpha]$ is open, take any element $B \in S[A; \alpha]$ and observe that $B \cap [0, \alpha) = A \cap [0, \alpha)$ and $B \cap [\alpha, \lambda)$ is sparse. Then $B \in S[B; \alpha] \subseteq S[A; \alpha] \subseteq X \setminus \{\lambda\}$ and the set $S[A; \alpha]$ is open by the definition of the topology τ . $\square$

Claim 2 For any ordinals $\alpha \in \lambda$, $n \in \omega$ and a positive real number ε , the set $F[\alpha, n, \varepsilon]$ is open and closed in (X, τ) .

Proof Given any element $A \in F[\alpha, n, \varepsilon] \setminus \{\lambda\}$, find a unique limit ordinal $\beta \geq \alpha$ witnessing that A is (α, n, ε) -fat. Then $S[A; \beta + \omega] \subseteq F[\alpha, n, \varepsilon]$, witnessing that the set $F[\alpha, n, \varepsilon]$ is open in (X, τ) .

To see that this set is closed in (X, τ) , choose any set $A \in X \setminus F[\alpha, n, \varepsilon]$. It follows that A is a finite subset of λ , which is not (α, n, ε) -fat. Let β be the smallest limit ordinal such that the intersection $A \cap [\beta + \omega, \lambda)$ is sparse. Let $m \in \omega$ be the smallest finite ordinal such that $A \cap [\beta, \beta + \omega) \subseteq [\beta, \beta + m]$. Since A is not (α, n, ε) -fat, one of the following conditions holds:

1. $\beta < \alpha$;
2. $m \leq n$;
3. $\beta \geq \alpha$ and $m > n$ but $[\beta, \beta + m] \not\subseteq A$;
4. $\beta \geq \alpha$, $m > n$, $[\beta, \beta + m] \subseteq A$ but $|A \cap [0, \beta)| \geq \varepsilon \cdot m$.

In all these cases $S[A; \beta + \omega]$ is a τ -open neighborhood of A such that $S[A; \beta + \omega] \cap F[\alpha, n, \varepsilon] = \emptyset$. $\square$

Claim 3 For any finite set $A \in X$ of λ and any ordinal $\alpha \in \lambda$, the set $S[A; \alpha]$ is closed in (X, τ) .

Proof Take any element $B \in X \setminus S[A; \alpha]$. If $B = \lambda$, then $F[\alpha, 1, 1]$ is a neighborhood of B , disjoint with $S[A; \alpha]$.

If $B \neq \lambda$, then $B \notin S[A; \alpha]$ implies that either $B \cap [\alpha, \lambda)$ is not sparse or $B \cap [\alpha, \lambda)$ is sparse but $B \cap [0, \alpha) \neq A \cap [0, \alpha)$. In the latter case $S[B; \alpha]$ is a τ -open neighborhood of B , disjoint with $S[A; \alpha]$. So, we assume that $B \cap [\alpha, \lambda)$ is not sparse. In this case we can choose any ordinal $\beta \in \lambda$ with $B \subset [0, \beta)$ and observe that $S[B, \beta]$ is a τ -open neighborhood of B such that $S[B; \beta] \cap S[A; \alpha] = \emptyset$. $\square$

Claim 4 The topology τ is Hausdorff.

Proof Take any distinct elements $A, B \in X$. If A and B are finite subsets of X , then we can find an ordinal $\alpha \in \lambda$ such that $A \cup B \subseteq [0, \alpha)$ and observe that $S[A; \alpha]$ and $S[B; \alpha]$ are disjoint τ -open neighborhoods of the elements A and B , respectively.

If $B = \lambda$, then A is a finite set, contained in $[0, \alpha)$ for some ordinal $\alpha \in \lambda$. In this case $S[A; \alpha]$ and $F[\alpha, 1, 1]$ are disjoint τ -open neighborhoods of A and B , respectively.

The case $A = \lambda$ can be considered by analogy. $\square$

Claims 1–4 show that the topology τ is Hausdorff and zero-dimensional.

Claim 5 The topologized semilattice (X, τ) is semitopological.

Proof Given any element $a \in X$, we should prove that the shift $s_a : X \rightarrow X$, $s_a : x \mapsto ax$, is continuous. If $a = \lambda$, then $s_a(X) = \{\lambda\}$ is a singleton, so the continuity of s_a is trivial. So, we assume that a is a finite subset of λ . To check the continuity of the shift s_a at a point $x \in X$, fix any neighborhood $O_{ax} \in \tau$ of the point $ax = x \cup a$.

If $x \neq \lambda$, then $ax \neq \lambda$ and by the definition of the topology τ , there exists an ordinal $\alpha \in \lambda$ such that $ax \in S[ax; \alpha] \subseteq O_{ax}$. Replacing α by a larger ordinal, we can assume that $ax \subseteq [0, \alpha)$. Then $O_x := S[x; \alpha]$ is a τ -open neighborhood of x such that $s_a(O_x) \subseteq S[ax; \alpha] \subseteq O_{ax}$.

If $x = \lambda$, then $ax = \lambda$ and by the definition of the topology τ , there exist $\alpha \in \lambda$, $k \in \omega$ and $\varepsilon > 0$ such that $F[\alpha, k, \varepsilon] \subseteq O_{ax}$. Replacing α by a larger ordinal, we can assume that $a \subset [0, \alpha)$. Replacing k by a larger number, we can assume that $|a| \leq \frac{1}{2}\varepsilon k$. In this case $O_x := F[\alpha, k, \frac{1}{2}\varepsilon]$ is a τ -open neighborhood of $x = \lambda$ such that $s_a(O_x) \subseteq F[\alpha, k, \varepsilon] \subseteq O_{ax}$. $\square$

Claim 6 Let $\mathcal{U} \subseteq \tau$ be a family of open sets and L be a τ -closed subsemilattice in X such that $|\mathcal{U}| < \text{cf}(\lambda)$ and $\emptyset \neq \bigcap \mathcal{U} \subseteq L$. Then $\lambda \in L$.

Proof Fix any element $x \in \bigcap \mathcal{U}$. If $x = \lambda$, then $\lambda = x \in \bigcap \mathcal{U} \subseteq L$ and we are done. So, we assume that x is a finite subset of λ . Since the set L is τ -closed, the inclusion $\lambda \in L$ will follow as soon as we show that each neighborhood $O_\lambda \in \tau$ of λ meets the set L . By Claim 2, there exist ordinals $\alpha \in \lambda$, $k \in \omega$ and a positive real number ε such that $F[\alpha, k, \varepsilon] \subseteq O_\lambda$.

By Claim 1 and $|\mathcal{U}| \leq \kappa < \text{cf}(\lambda)$, there exists a limit ordinal $\beta \in [\alpha, \lambda)$ such that $x \in [0, \beta)$ and $x \in S[x; \beta] \subseteq \bigcap \mathcal{U} \subseteq L$. Choose a finite ordinal $n > k$ such that $|x| < \varepsilon n$. Observe that for every ordinal $\gamma \in [\beta, \beta + n]$ the set $x \cup \{\gamma\}$ belongs to the semilattice $L \supseteq S[x; \beta]$. Since L is a subsemilattice, the (β, k, ε) -fat set $x \cup [\beta, \beta + n]$ belongs to $L \cap F[\beta, k, \varepsilon] \subseteq L \cap \mathcal{O}_\lambda$. $\square$

Claim 7 The semitopological semilattice (X, τ) has $\bar{\Lambda}(X, \tau) = \bar{\psi}(X, \tau) = \text{cf}(\lambda)$.

Proof Claim 6 implies that $\text{cf}(\lambda) \leq \bar{\Lambda}(X, \tau)$. Since $\bar{\Lambda}(X, \tau) \leq \bar{\psi}(X, \tau)$, it remains to prove that $\bar{\psi}(X, \tau) \leq \text{cf}(\lambda)$.

Choose a cofinal subset $C \subseteq \lambda$ of cardinality $|C| = \text{cf}(\lambda)$. To see that $\bar{\psi}(X, \tau) \leq \text{cf}(\lambda)$, take any $A \in X$. If A is a finite subset of λ , then $A \subseteq [0, \alpha)$ for some ordinal $\alpha \in \lambda$. Then $\{A\} = \bigcap_{\alpha \leq \gamma \in C} S[A; \gamma]$. If $A = \lambda$, then $\{A\} = \{\lambda\} = \bigcap_{\gamma \in C} F[\gamma; 1, 1]$. In both cases the singleton $\{A\}$ is the intersection of $\text{cf}(\lambda)$ many closed neighborhoods of A , witnessing that $\bar{\psi}(X, \tau) \leq \text{cf}(\lambda)$. $\square$

3 Complete topologized semilattices

In this section we recall some known properties and characterizations of complete topologized semilattices.

By a *poset* we understand a set endowed with a partial order. A *topologized poset* is a poset endowed with a topology. So, each topologized semilattice is a topologized poset.

A subset D of a poset $(X, \leq)$ is called

- a *chain* if any elements $x, y \in D$ are comparable in the sense that $x \leq y$ or $y \leq x$;
- *up-directed* if for any $x, y \in D$ there exists $z \in D$ such that $x \leq z$ and $y \leq z$;
- *down-directed* if for any $x, y \in D$ there exists $z \in D$ such that $z \leq x$ and $z \leq y$.

It is clear that each chain in a poset is both up-directed and down-directed.

A topologized poset X is defined to be

- *up-complete* if any nonempty up-directed subset $U \subseteq X$ has the least upper bound $\sup U \in \bar{U}$ in X ;
- *down-complete* if any nonempty down-directed subset $D \subseteq X$ has the greatest lower bound $\inf D \in \bar{D}$ in X .

The proof of the following classical characterization can be found in [9,12,15] or [5, 2.2].

Proposition 5 For a topologized poset X the following conditions are equivalent:

1. X is up-complete;
2. Each non-empty chain $C \subseteq X$ has the least upper bound $\sup C \in \bar{C}$ in X .

Proposition 5 implies the following useful characterization of completeness in topologized semilattices.

Corollary 3 A topologized semilattice X is complete if and only if it is up-complete and down-complete.

This corollary implies that each closed subsemilattice of a complete topologized semilattice has the smallest element.

A topologized semilattice Y is called $\uparrow$ -closed if for every $y \in Y$ the upper set $\uparrow y = \{x \in Y : xy = y\}$ is closed in Y . It is easy to see that each T_1 semitopological semilattice is $\uparrow$ -closed.

The following lemma (that can be derived from Corollary 3) is proved in [[5], Lemma 5.3].

Lemma 1 *Let $h : X \rightarrow Y$ be a continuous surjective homomorphism between topologized semilattices. If X is complete and Y is $\uparrow$ -closed, then the topologized semilattice Y is complete.*

4 Proof of Theorem 6 and Corollary 1

The proof of Theorem 6 is based on the following lemma.

Lemma 2 *Let X be a complete subsemilattice of a semitopological semilattice Y . Let a pair $(x, y) \in Y \times Y$ belong to the closure of the natural partial order $\leq_X$ of X in $Y \times Y$, and let $\{U_n\}_{n \in \omega}$, $\{V_n\}_{n \in \omega}$ be sequences of closed neighborhoods of the points x and y in Y , respectively. Then there exist points $x' \in X \cap \bigcap_{n \in \omega} U_n$ and $y' \in X \cap \bigcap_{n \in \omega} V_n$ such that $x' \leq y'$.*

Proof Replacing each set U_n by $\bigcap_{i \leq n} U_i$, we can assume that $U_{n+1} \subseteq U_n$ for all $n \in \omega$. By the same reason, we can assume that the sequence $(V_n)_{n \in \omega}$ is decreasing. For every $n \in \omega$ denote by U_n° and V_n° the interiors of the sets U_n and V_n in Y .

By induction we shall construct sequences $(x_n)_{n \in \omega}$ and $(y_n)_{n \in \omega}$ of points of X such that for every $n \in \omega$ the following conditions are satisfied:

- (1_n) $x_n \leq y_n$;
- (2_n) $\{x_i \cdots x_n, x_i \cdots x_n x\} \subseteq U_i^\circ$ for all $i \leq n$;
- (3_n) $\{y_i \cdots y_n, y_i \cdots y_n y\} \subseteq V_i^\circ$ for all $i \leq n$.

To choose the initial points x_0, y_0 , use the separate continuity of the semilattice operation and find neighborhoods $U'_0 \subseteq U_0^\circ$ and $V'_0 \subseteq V_0^\circ$ of x and y in Y such that $U'_0 x \subseteq U_0^\circ$ and $V'_0 y \subseteq V_0^\circ$. By our assumption, there are points $x_0 \in X \cap U'_0$ and $y_0 \in X \cap V'_0$ such that $x_0 \leq y_0$. The choice of the neighborhoods U'_0 and V'_0 ensures that the conditions (2₀) and (3₀) are satisfied.

Now assume that for some $n \in \mathbb{N}$ points $x_0, \dots, x_{n-1}$ and $y_0, \dots, y_{n-1}$ of X are chosen so that the conditions (1_{n-1})–(3_{n-1}) are satisfied. The condition (2_{n-1}) implies that for every $i \leq n$ we have the inclusion $x_i \cdots x_{n-1} x = x_i \cdots x_{n-1} x \in U_i^\circ$ (if $i = n$, then we understand that $x_i \cdots x_{n-1} x = x$). Using the continuity of the shift $s_x : Y \rightarrow Y$, $s_x : z \mapsto xz$, we can find a neighborhood $U'_n \subseteq Y$ of x such that $x_i \cdots x_{n-1} \cdot (U'_n \cup U'_n x) \subseteq U_i^\circ$ for every $i \leq n$. By analogy, we can find a neighborhood $V'_n \subseteq Y$ of y such that $y_i \cdots y_{n-1} \cdot (V'_n \cup V'_n y) \subseteq V_i^\circ$ for every $i \leq n$. By our assumption, there are points $x_n \in X \cap U'_n$ and $y_n \in X \cap V'_n$ such that $x_n \leq y_n$. The choice of the neighborhoods U'_0 and V'_0 ensures that the conditions (2_n) and (3_n) are satisfied. This completes the inductive step.

Now for every $i \in \omega$ consider the chain $C_i = \{x_i \cdots x_n : n \geq i\} \subseteq U_i^\circ$ in X . By the completeness of X , this chain has $\inf C_i \in X \cap \overline{C_i} \subseteq X \cap \overline{U_i^\circ} \subseteq X \cap U_i$. Observing that $\inf C_i \leq x_i x_{i+1} \cdots x_n \leq x_{i+1} \cdots x_n$ for all $i > n$, we see that $\inf C_i$ is a lower bound of the chain C_{i+1} and hence $\inf C_i \leq \inf C_{i+1}$. By the completeness of X , for every $i \in \omega$ the chain $D_i := \{\inf C_j : j \geq i\} \subseteq U_i$ has $\sup D_i \in X \cap \overline{D_i} \subseteq X \cap U_i$. Since the sequence $(\inf C_i)_{i \in \omega}$ is increasing, we get $\sup D_0 = \sup D_i \in X \cap U_i$ for all $i \in \omega$. Consequently, $\sup D_0 \in X \cap \bigcap_{i \in \omega} U_i$.

By analogy, for every $k \in \omega$ consider the chain $E_i = \{y_i \cdots y_n : n \geq i\} \subseteq V_i^\circ$ in X . By the completeness of X , this chain has $\inf E_i \in X \cap \overline{E_i} \subseteq X \cap \overline{V_i^\circ} \subseteq X \cap V_i$. By the completeness of X , for every $i \in \omega$ the chain $F_i := \{\inf E_j : j \geq i\} \subseteq V_i$ has $\sup F_i \in X \cap \overline{F_i} \subseteq X \cap \overline{V_i} = X \cap V_i$. Since the sequence $(\inf E_i)_{i \in \omega}$ is increasing, we get $\sup F_0 = \sup F_i \in X \cap V_i$ for all $i \in \omega$. Consequently, $\sup F_0 \in X \cap \bigcap_{i \in \omega} V_i$.

To finish the proof of Lemma 2, it suffices to show that $\sup D_0 \leq \sup F_0$. The inductive conditions (1_n) , $n \in \omega$, imply that $\inf C_i \leq \inf E_i$ for all $i \in \omega$ and $\sup D_0 = \sup\{\inf C_i : i \in \omega\} \leq \sup\{\inf E_i : i \in \omega\} = \sup F_0$. $\square$

The following two lemmas imply Theorem 6.

Lemma 3 *Let Y be an ω -Lawson semitopological semilattice. For any complete subsemilattice $X \subseteq Y$ the natural partial order $\leq_X$ of X is closed in $Y \times Y$.*

Proof By Corollary 3, the complete semitopological semilattice X is both up-complete and down-complete. To show that the partial order $\leq_X := \{(x, y) \in X \times X : x \leq y\}$ is closed in $Y \times Y$, take any pair (y_1, y_2) in the closure of the set $\leq_X$ in $Y \times Y$. For every $i \in \{1, 2\}$, let $\mathcal{U}_i$ be the set of all countable families $\mathcal{U}$ of closed neighborhoods of y_i in Y such that $\bigcap \mathcal{U}$ is a subsemilattice of Y . By Lemma 2, for any $\mathcal{U}_1 \in \mathcal{U}_1$ and $\mathcal{U}_2 \in \mathcal{U}_2$ there are points $x_1 \in X \cap \bigcap \mathcal{U}_1$ and $x_2 \in X \cap \bigcap \mathcal{U}_2$ such that $x_1 \leq x_2$. In particular, the closed subsemilattice $X \cap \bigcap \mathcal{U}_1$ is not empty and has the smallest element $\inf(X \cap \bigcap \mathcal{U}_1) \in X$ (by the down-completeness of X). Denote this smallest element by $x(\mathcal{U}_1)$. It follows that $x(\mathcal{U}_1) := \inf(X \cap \bigcap \mathcal{U}_1) \leq x_1 \leq x_2$. Consequently, the closed subsemilattice $(\uparrow x(\mathcal{U}_1)) \cap (X \cap \bigcap \mathcal{U}_2) \ni x_2$ is not empty and has the smallest element (by down-completeness of X), which will be denoted by $y(\mathcal{U}_1, \mathcal{U}_2)$. Observe that $x(\mathcal{U}_1) \in X \cap \bigcap \mathcal{U}_1$, $y(\mathcal{U}_1, \mathcal{U}_2) \in X \cap \bigcap \mathcal{U}_2$ and $x(\mathcal{U}_1) \leq y(\mathcal{U}_1, \mathcal{U}_2)$. For any families $\mathcal{U}_1 \in \mathcal{U}_1$ and $\mathcal{U}_2, \mathcal{U}'_2 \in \mathcal{U}_2$ with $\mathcal{U}_2 \subseteq \mathcal{U}'_2$ we have $y(\mathcal{U}_1, \mathcal{U}_2) \leq y(\mathcal{U}_1, \mathcal{U}'_2)$. Therefore, the set $\{y(\mathcal{U}_1, \mathcal{U}_2) : \mathcal{U}_2 \in \mathcal{U}_2\} \subseteq X$ is up-directed and by the up-completeness of X , it has the smallest upper bound in X , which will be denoted by $y(\mathcal{U}_1)$. It follows that $x(\mathcal{U}_1) \leq y(\mathcal{U}_1)$. We claim that $y(\mathcal{U}_1) = y_2$. In the opposite case we can use the ω -Lawson property of Y and choose a countable family $\mathcal{U}'_2 \in \mathcal{U}_2$ such that $y(\mathcal{U}_1) \notin \bigcap \mathcal{U}'_2$. Taking into account that the set $\{y(\mathcal{U}_1, \mathcal{U}_2 \cap \mathcal{U}'_2) : \mathcal{U}_2 \in \mathcal{U}_2\}$ is cofinal in $\{y(\mathcal{U}_1, \mathcal{U}_2) : \mathcal{U}_2 \in \mathcal{U}_2\}$, we conclude that

$$y(\mathcal{U}_1) = \sup\{y(\mathcal{U}_1, \mathcal{U}_2) : \mathcal{U}_2 \in \mathcal{U}_2\} = \sup\{y(\mathcal{U}_1, \mathcal{U}_2 \cap \mathcal{U}'_2) : \mathcal{U}_2 \in \mathcal{U}_2\} \in \bigcap \mathcal{U}'_2,$$

which contradicts the choice of the family $\mathcal{U}'_2$. This contradiction shows that $y_2 = y(\mathcal{U}_1) \in X$. Now we see that $x(\mathcal{U}_1) \leq y(\mathcal{U}_1) = y_2$ for every $\mathcal{U}_1 \in \mathcal{U}_1$. By the up-completeness of the semitopological semilattice X , the up-directed subset $\{x(\mathcal{U}_1) :$

$\mathcal{U}_1 \in \mathfrak{U}_1\}$ has the smallest upper bound $x \in X$. It follows from $x(\mathcal{U}_1) \leq y_2$ for all $\mathcal{U}_1 \in \mathfrak{U}_1$ that $x \leq y_2$.

It remains to check that $x = y_1$. In the opposite case, using the ω -Lawson property of X , we can find a countable family $\mathcal{U}'_1 \in \mathfrak{U}_1$ such that $x \notin \bigcap \mathcal{U}'_1$. Taking into account that the family $\{x(\mathcal{U}_1 \cap \mathcal{U}'_1) : \mathcal{U}_1 \in \mathfrak{U}_1\}$ is cofinal in $\{x(\mathcal{U}_1) : \mathcal{U}_1 \in \mathfrak{U}_1\}$, we conclude that

$$x = \sup\{x(\mathcal{U}_1) : \mathcal{U}_1 \in \mathfrak{U}_1\} = \sup\{x(\mathcal{U}_1 \cap \mathcal{U}'_1) : \mathcal{U}_1 \in \mathfrak{U}_1\} \in \bigcap \mathcal{U}'_1,$$

which contradicts the choice of $\mathcal{U}'_1$. This contradiction shows that $y_1 = x \in X$. Therefore we obtain that $(y_1, y_2) \in X \times X$ and $y_1 = x \leq y_2$, which means that $(y_1, y_2) \in \leq_X$. $\square$

Lemma 4 *Each complete subsemilattice X of an ω -Lawson semitopological semilattice Y is closed in Y .*

Proof By Lemma 3, the partial order $\leq_X := \{(x, y) \in X \times X : xy = x\}$ is a closed subset of $Y \times Y$. By Corollary 3, the complete semilattice X has the smallest element $\min X \in X$. Consider the continuous map $f : Y \rightarrow Y \times Y$, $f : y \mapsto (\min X, y)$, and observe that $X = f^{-1}(\leq_X)$ is a closed subset of X , being the preimage of the closed set $\leq_X$ under the continuous map f . $\square$

Finally, we prove Corollary 1.

Lemma 5 *For every continuous homomorphism $h : X \rightarrow Y$ from a complete topologized semilattice X to an ω -Lawson semitopological semilattice Y , the image $h[X]$ is closed in Y .*

Proof Observe that the ω -Lawson property of Y implies that the semitopological semilattice Y is Hausdorff and hence $\uparrow$ -closed. By Lemma 1, the semitopological semilattice $h[X]$ of Y is complete and by Lemma 4, $h[X]$ is closed in Y . $\square$

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References

1. Banakh, T., Bardyla, S.: Characterizing chain-finite and chain-compact topological semilattices. *Semigroup Forum* **98**(2), 234–250 (2019)
2. Banakh, T., Bardyla, S.: The interplay between weak topologies on topological semilattices. *Topol. Appl.* **259**, 134–154 (2019)

3. Banach, T., Bardyla, S.: Completeness and absolute H -closedness of topological semilattices. *Topol. Appl.* **260**, 189–202 (2019)
4. Banach, T., Bardyla, S.: On images of complete subsemilattices in sequential semitopological semilattices. *Semigroup Forum* **100**, 662–670 (2020)
5. Banach, T., Bardyla, S.: Complete topologized posets and semilattices. *Topol. Proc.* **57**, 177–196 (2021)
6. Banach, T., Bardyla, S., Ravsky, A.: The closedness of complete subsemilattices in functionally Hausdorff semitopological semilattices. *Topol. Appl.* **267**, 106874 (2019)
7. Banach, T., Bardyla, S., Ravsky, A.: A metrizable semitopological semilattice with non-closed partial order. *Top. Algebra Appl.* **8**(1), 67–75 (2020)
8. Banach, T., Bardyla, S., Ravsky, A.: A metrizable Lawson semitopological semilattice with non-closed partial order. *Proc. Intern. Geom. Center* **13**(3), 10–17 (2020)
9. Bruns, G.: A lemma on directed sets and chains. *Arch. Math.* **18**(6), 561–563 (1967)
10. Carruth, J.H., Hildebrandt, J.A., Koch, R.J.: *The Theory of Topological Semigroups*, vol. II. Marcel Dekker Inc, New York and Basel (1986)
11. Gutik, O., Repovš, D.: On linearly ordered H -closed topological semilattices. *Semigroup Forum* **77**(3), 474–481 (2008)
12. Iwamura, T.: A lemma on directed sets. *Zenkoku Shijo Sugaku Danwakai* **262**, 107–111 (1944)
13. Lawson, J.D.: Topological semilattices with small semilattices. *J. London Math. Soc. (2)* **1**(1), 719–724 (1969)
14. Lawson, J.D.: Joint continuity in semitopological semigroups. *Illinois J. Math.* **18**(2), 275–285 (1974)
15. Markowsky, G.: Chain-complete posets and directed sets with applications. *Algebra Univers.* **6**(1), 53–68 (1976)
16. Stepp, J.W.: Algebraic maximal semilattices. *Pac. J. Math.* **58**(1), 243–248 (1975)

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