



\aleph_0 -categoricity of semigroups II

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Abstract

A countable semigroup is \aleph_0 -categorical if it can be characterised, up to isomorphism, by its first-order properties. In this paper we continue our investigation into the \aleph_0 -categoricity of semigroups. Our main results are a complete classification of \aleph_0 -categorical orthodox completely 0-simple semigroups, and descriptions of the \aleph_0 -categorical members of certain classes of strong semilattices of semigroups.

Keywords \aleph_0 -categorical · Semigroups · Rees matrix semigroups

1 Introduction

A countable structure is \aleph_0 -categorical if it is uniquely determined by its first-order properties, up to isomorphism. While the concept of \aleph_0 -categoricity arises naturally from model theory, it has a purely algebraic formulation thanks to the Ryll-Nardzewski theorem (RNT). Independently accredited to Engeler [3], Ryll-Nardzewski [28] and Svenonius [29], it states that the \aleph_0 -categoricity of a structure M is equivalent to there being only finitely many orbits in the natural action of $\text{Aut}(M)$ (the automorphism group of M) on M^n , for each $n \geq 1$. Significant results exist for both relational and algebraic structures from the point of view of \aleph_0 -categoricity, but, until recently, little was known in the context of semigroups. This article is the second of a pair initiating and developing the study of \aleph_0 -categorical semigroups. For background and motivation we refer the reader to Hodges [14] and Evans [5], and to our first article [9].

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We explore in [9] the behaviour of \aleph_0 -categoricity with respect to standard constructions, such as quotients and subsemigroups. For example, \aleph_0 -categoricity of a semigroup is inherited by both its maximal subgroups and its principal factors. Differences with the known theory for groups and rings emerged, for example, any \aleph_0 -categorical nil ring is nilpotent, but the same is not true for semigroups. While keeping the machinery at a low level, we were able to give, amongst other results, complete classifications of \aleph_0 -categorical primitive inverse semigroups and of E -unitary inverse semigroups with finite semilattices of idempotents.

For the work in this current article, it is helpful to develop some general strategies and then apply them in various contexts. In view of this, in Sect. 2, we introduce \aleph_0 -categoricity in the setting of (first-order) structures. Although we will mostly be working in the context of semigroups, this broader view will be useful for studying structures, such as graphs and semilattices, which naturally arise in our considerations of semigroups. Key results from Gould and Quinn-Gregson [9] are given in this setting. In particular, we formalise the previously defined concept of \aleph_0 -categoricity over a set of subsets; the \aleph_0 -categoricity of rectangular bands over any set of subrectangular bands acts as a useful example.

In Sect. 3 we construct a handy method for dealing with the \aleph_0 -categoricity of semigroups in which their automorphisms can be built from certain ingredients. This is then used in Sect. 4 to study the \aleph_0 -categoricity of strong semilattices of semigroups. The main results of this article are in Sect. 5, where we continue from [9] our study into the \aleph_0 -categoricity of completely 0-simple semigroups. We follow a method of Graham and Houghton by considering graphs arising from Rees matrix semigroups, which necessitated our study of \aleph_0 -categoricity in the general setting of structures.

We assume that all structures considered will be of countable cardinality.

2 The \aleph_0 -categoricity of a structure

We begin by translating a number of results in [9] to the general setting of (first-order) structures. Their proofs easily generalize, and as such we shall omit them, referencing only the corresponding result in [9].

A (first-order) structure is a set M together with a collection of constants \mathfrak{C} , finitary relations \mathfrak{R} , and finitary functions \mathfrak{F} defined on M . We denote the structure as $(M; \mathfrak{R}, \mathfrak{F}, \mathfrak{C})$, or simply M where no confusion may arise. Each constant element is associated with a constant symbol, each n -ary relation is associated with an n -ary relational symbol, and each n -ary function is associated with an n -ary function symbol. The collection L of these symbols is called the *signature* of M . We follow the usual convention of not distinguishing between the constants/relations/functions of M , and their corresponding abstract symbols in L .

Our main example is that of a semigroup (S, \cdot) , where S is a set together with a single (associative) binary operation \cdot , and so the associated signature consists of a single binary function symbol.

A property of a structure is *first-order* if it can be formulated within first-order predicate calculus. A (countable) structure is \aleph_0 -categorical if it can be uniquely classified by its first-order properties, up to isomorphism.

The central result in the study of \aleph_0 -categorical structures is the Ryll-Nardzewski Theorem, which translates the concept to the study of oligomorphic automorphism groups (see [14]). Before stating it, it is worth fixing some notation and definitions. Let $\phi : A \rightarrow B$ be a map, let $\bar{a} = (a_1, \dots, a_n)$ be an n -tuple of A and let $M \subseteq A$. Then we let $\bar{a}\phi$ denote the n -tuple of B given by $(a_1\phi, \dots, a_n\phi)$, and $M\phi$ denotes the subset $\{m\phi : m \in M\}$ of B .

Given a structure M , we say that a pair of n -tuples $\bar{a} = (a_1, \dots, a_n)$ and $\bar{b} = (b_1, \dots, b_n)$ of M are *automorphically equivalent* or *belong to the same n -automorphism type* if there exists an automorphism ϕ of M such that $\bar{a}\phi = \bar{b}$, that is, $a_i\phi = b_i$ for each $i \in \{1, \dots, n\}$. We denote this equivalence relation as $\bar{a} \sim_{M,n} \bar{b}$. We call $\text{Aut}(M)$ *oligomorphic* if $\text{Aut}(M)$ has only finitely many orbits in its action on M^n for each $n \geq 1$, that is, if each $|M^n / \sim_{M,n}|$ is finite.

Theorem 2.1 (The Ryll-Nardzewski theorem (RNT)) *A structure M is \aleph_0 -categorical if and only if $\text{Aut}(M)$ is oligomorphic.*

It follows from the RNT that every \aleph_0 -categorical structure is *uniformly locally finite* [14, Corollary 7.3.2], that is, there is a finite uniform bound on the size of the n -generated substructures, for each $n \geq 1$. In particular, an \aleph_0 -categorical semigroup is periodic, with bounded index and period.

Another immediate consequence of the RNT is that any characteristic substructure inherits \aleph_0 -categoricity, where a subset/substructure is called *characteristic* if it is invariant under automorphisms of the structure. However, key subsemigroups of a semigroup such as maximal subgroups and principal ideals are not necessarily characteristic, and a more general definition is required:

Definition 2.2 Let M be a structure and, for some fixed $t \in \mathbb{N}$, let $\{\bar{X}_i : i \in I\}$ be a collection of t -tuples of M . Let $\{A_i : i \in I\}$ be a collection of subsets of M with the property that for any automorphism ϕ of M such that there exists $i, j \in I$ with $\bar{X}_i\phi = \bar{X}_j$, then $\phi|_{A_i}$ is a bijection from A_i onto A_j . Then we call $\mathcal{A} = \{(A_i, \bar{X}_i) : i \in I\}$ a system of t -pivoted pairwise relatively characteristic (t -pivoted p.r.c.) subsets (or, substructure, if each A_i is a substructure) of M . The t -tuple \bar{X}_i is called the *pivot* of A_i ($i \in I$). If $|I| = 1$ then, letting $A_1 = A$ and $\bar{X}_1 = \bar{X}$, we write $\{(A, X)\}$ simply as (A, X) , and call A an \bar{X} -pivoted relatively characteristic (\bar{X} -pivoted r.c.) subset/substructure of M .

In [9], Definition 2.2 was shown to be of use in regard to, for example, Green's relations. In particular, $\{(H_e, e) : e \in E(S)\}$ forms a system of 1-pivoted p.r.c. subgroups of a semigroup S . It then followed from the proposition below that maximal subgroups inherit \aleph_0 -categoricity, and moreover there exists only finitely many non-isomorphic maximal subgroups in an \aleph_0 -categorical semigroup.

Proposition 2.3 [9, Proposition 3.3] *Let M be an \aleph_0 -categorical structure and $\{(A_i, \bar{X}_i) : i \in I\}$ a system of t -pivoted p.r.c. subsets of M . Then $\{|A_i| : i \in I\}$ is finite. If, further, each A_i forms a substructure of M , then $\{A_i : i \in I\}$ is finite, up to isomorphism, with each A_i \aleph_0 -categorical.*

We use the RNT in conjunction with [9, Lemma 2.8] to prove that a structure M is \aleph_0 -categorical in the following way. For each $n \in \mathbb{N}$, let $\gamma_1, \dots, \gamma_r$ be a finite list of equivalence relations on M^n such that M^n/γ_i is finite for each $1 \leq i \leq r$ and

$$\gamma_1 \cap \gamma_2 \cap \dots \cap \gamma_r \subseteq \sim_{M,n}.$$

A consequence of the two aforementioned results is that M is \aleph_0 -categorical. This result will often be drawn upon in a less formal way as follows. Suppose that we have an equivalence relation σ on M^n that arises from different ways in which a given condition may be fulfilled; if M^n/σ is finite, then we say the condition has *finitely many choices*.

Example 2.4 Recalling [9, Example 2.10], consider the equivalence $\natural_{X,n}$ on n -tuples of a set X given by

$$(a_1, \dots, a_n) \natural_{X,n} (b_1, \dots, b_n) \text{ if and only if } [a_i = a_j \Leftrightarrow b_i = b_j, \text{ for each } i, j]. \tag{2.1}$$

A pair of n -tuples \bar{a} and \bar{b} are $\natural_{X,n}$ -equivalent if and only if there exists a bijection $\phi : \{a_1, \dots, a_n\} \rightarrow \{b_1, \dots, b_n\}$ such that $a_i\phi = b_i$, and the number of $\natural_{X,n}$ -classes of X^n is finite, for each $n \in \mathbb{N}$. Note also that if M is a structure then any pair of n -automorphically equivalent tuples are clearly $\natural_{M,n}$ -equivalent.

Let M be a structure and $\mathcal{A} = \{A_i : i \in I\}$ a collection of subsets of M . We may extend the signature of M to include the unary relations A_i ($i \in I$). We denote the resulting structure as $\underline{M} = (M; \mathcal{A})$, which we call a *set extension* of M . If $\mathcal{A} = \{A_1, \dots, A_n\}$ is finite, then we may simply write \underline{M} as $(M; A_1, \dots, A_n)$.

Notice that automorphisms of \underline{M} are simply those automorphisms of M which fix each A_i setwise, that is automorphisms ϕ such that $A_i\phi = A_i$ ($i \in I$). The set of all such automorphisms will be denoted $\text{Aut}(M; \mathcal{A})$, and clearly forms a subgroup of $\text{Aut}(M)$. The \aleph_0 -categoricity of \underline{M} is therefore equivalent to our previous notion of M being \aleph_0 -categorical over \mathcal{A} in [9].

Lemma 2.5 [9, Lemma 5.2] *Let M be a structure with a system of t -pivoted p.r.c. subsets $\{(A_i, \bar{X}_i) : i \in I\}$. Then $(M; \{A_i : i \in I\})$ is \aleph_0 -categorical if and only if M is \aleph_0 -categorical and I is finite.*

Lemma 2.6 [9, Lemma 5.3] *Let M be a structure, let $t, r \in \mathbb{N}$, and for each $k \in \{1, \dots, r\}$ let $\bar{X}_k \in M^t$. Suppose also that A_k is an \bar{X}_k -pivoted relatively characteristic subset of M for $1 \leq k \leq r$. Then $(M; A_1, \dots, A_r)$ is \aleph_0 -categorical if and only if M is \aleph_0 -categorical.*

Consequently, if S is an \aleph_0 -categorical semigroup and G_1, \dots, G_n is a collection of maximal subgroups of S then $(S; G_1, \dots, G_n)$ is \aleph_0 -categorical.

However, note that not every \aleph_0 -categorical set extension of a semigroup requires the subsets to be relatively characteristic. We claim that any set extension of a rectangular band by a finite set of subrectangular bands is \aleph_0 -categorical. This result is of particular use in the next section when considering the \aleph_0 -categoricity of normal bands.

Recall that every rectangular band can be written as a direct product of a left zero and right zero semigroup. The following isomorphism theorem for rectangular bands will be vital for proving our claim, and follows immediately from Howie [17, Corollary 4.4.3]:

Lemma 2.7 *Let $B_1 = L_1 \times R_1$ and $B_2 = L_2 \times R_2$ be a pair of rectangular bands. If $\phi_L : L_1 \rightarrow L_2$ and $\phi_R : R_1 \times R_2$ are a pair of bijections, then the map $\phi : B_1 \rightarrow B_2$ given by $(l, r)\phi = (l\phi_L, r\phi_R)$ is an isomorphism, denoted $\phi = \phi_L \times \phi_R$. Conversely, every isomorphism can be constructed this way.*

Theorem 2.8 *If B is a rectangular band and B_1, \dots, B_r is a finite list of subrectangular bands of B , then $\underline{B} = (B; B_1, \dots, B_r)$ is \aleph_0 -categorical. In particular, a rectangular band is \aleph_0 -categorical.*

Proof Let $B = L \times R$, where L is a left zero semigroup and R is a right zero semigroup. For each $1 \leq k \leq r$, let $L_k \subseteq L$ and $R_k \subseteq R$ be such that $B_k = L_k \times R_k$. Define a pair of equivalence relations σ_L and σ_R on L and R , respectively, by

$$i \sigma_L j \Leftrightarrow [i \in L_k \Leftrightarrow j \in L_k, \text{ for each } k],$$

$$i \sigma_R j \Leftrightarrow [i \in R_k \Leftrightarrow j \in R_k, \text{ for each } k].$$

The equivalence classes of σ_L are simply the set $L \setminus \bigcup_{1 \leq k \leq r} L_k$ together with certain intersections of the sets L_k . Since r is finite, it follows that L/σ_L is finite, and similarly R/σ_R is finite. Let $\bar{a} = ((i_1, j_1), \dots, (i_n, j_n))$ and $\bar{b} = ((k_1, \ell_1), \dots, (k_n, \ell_n))$ be a pair of n -tuples of B under the four conditions that

- (1) $i_s \sigma_L k_s$ for each $1 \leq s \leq n$,
- (2) $j_s \sigma_R \ell_s$ for each $1 \leq s \leq n$,
- (3) $(i_1, \dots, i_n) \mathfrak{h}_{L,n}(k_1, \dots, k_n)$,
- (4) $(j_1, \dots, j_n) \mathfrak{h}_{R,n}(\ell_1, \dots, \ell_n)$,

where $\mathfrak{h}_{L,n}$ and $\mathfrak{h}_{R,n}$ are the equivalence relations given by (2.1). By conditions (3) and (4), there exists bijections

$$\phi_L : \{i_1, \dots, i_n\} \rightarrow \{k_1, \dots, k_n\} \text{ and } \phi_R : \{j_1, \dots, j_n\} \rightarrow \{\ell_1, \dots, \ell_n\}$$

given by $i_s \phi_L = k_s$ and $j_s \phi_R = \ell_s$ for each $1 \leq s \leq n$. By condition (1), we can pick a bijection Φ_L of L which extends ϕ_L and fixes each σ_L -classes setwise, and similarly construct Φ_R . Then $\Phi = \Phi_L \times \Phi_R$ is an automorphism of B . Moreover, if $(i, j) \in B_k$ then $i \in L_k$ and as $i \sigma_L (i\Phi_L)$ we have $i\Phi_L \in L_k$. Dually, $j \in R_k$ and as $j \sigma_R (j\Phi_R)$ we

have $j\Phi_R \in R_k$. Hence there exists $\ell \in L$ and $r \in R$ such that $(i\Phi_L, r)$ and $(\ell, j\Phi_R)$ are in B_k , so that

$$(i\Phi_L, r)(\ell, j\Phi_R) = (i\Phi_L, j\Phi_R) \in B_k$$

as B_k is a subrectangular band. We have thus shown that $(i, j)\Phi = (i\Phi_L, j\Phi_R) \in B_k$, and so $B_k\Phi \subseteq B_k$. We observe that $\Phi^{-1} = \Phi_L^{-1} \times \Phi_R^{-1}$ is also an automorphism of B with Φ_L^{-1} and Φ_R^{-1} setwise fixing the σ_L -classes and σ_R -classes, respectively. Following our previous argument we have $B_k\Phi^{-1} \subseteq B_k$, and so $B_k\Phi = B_k$ for each k . Thus Φ is an automorphism of \underline{B} , and is such that

$$(i_s, j_s)\Phi = (i_s\Phi_L, j_s\Phi_R) = (i_s\phi_L, j_s\phi_R) = (k_s, \ell_s)$$

for each $1 \leq s \leq n$, so that $\bar{a} \sim_{B, n} \bar{b}$. Hence, as each of the four conditions on \bar{a} and \bar{b} have finitely many choices, it follows that \underline{B} is \aleph_0 -categorical. \square

Note that any set can be considered as a structure with no relations, functions or constants. Every bijection of the set is therefore an automorphism, and as such all sets are easily shown to be \aleph_0 -categorical. In fact a simplification of the proof of Theorem 2.8 gives:

Corollary 2.9 *Let M be a set, and M_1, \dots, M_r be a finite list of subsets of M . Then $(M; M_1, \dots, M_r)$ is \aleph_0 -categorical.*

3 A new method: $(M, M'; \underline{N}; \Psi)$ -systems

For many of the structures we will consider, automorphisms can be built from isomorphisms between their components. For example, for a strong semilattice of semigroups $S = [Y; S_\alpha; \psi_{\alpha, \beta}]$, we can construct automorphisms of S from certain isomorphisms between the semigroups S_α . In this example we also require an automorphism of the semilattice Y , which acts as an indexing set for the semigroups S_α . We now extend this idea by setting up some formal machinery to deal with structures in which the automorphisms are built from a collection of data.

Notation 3.1 Given a pair of structures M and M' , we let $\text{Iso}(M; M')$ denote the set of all isomorphisms from M onto M' .

Definition 3.2 Let M be an L -structure with fixed substructure M' . Let $\mathcal{A} = \{M_i : i \in N\}$ be a set of substructures of M' indexed by some K -structure N such that $M' = \bigcup_{i \in N} M_i$. Let N_1, \dots, N_r be a finite partition of N , and set $\underline{N} = (N; N_1, \dots, N_r)$. For each $i, j \in N$, let $\Psi_{i, j}$ be a subset of $\text{Iso}(M_i; M_j)$ under the conditions that

- (3.1) if $i, j \in N_k$ for some $1 \leq k \leq r$ then $\Psi_{i, j} \neq \emptyset$,
- (3.2) if $\phi \in \Psi_{i, j}$ and $\phi' \in \Psi_{j, \ell}$ then $\phi\phi' \in \Psi_{i, \ell}$,

- (3.3) if $\phi \in \Psi_{i,j}$ then $\phi^{-1} \in \Psi_{j,i}$,
- (3.4) if $\pi \in \text{Aut}(\underline{N})$ and $\phi_i \in \Psi_{i,i\pi}$ for each $i \in N$, then there exists an automorphism of M extending the ϕ_i .

Letting $\Psi = \bigcup_{i,j \in N} \Psi_{i,j}$, then, under the conditions above, we call $\mathcal{A} = \{M_i : i \in N\}$ an $(M, M'; \underline{N}; \Psi)$ -system (in M). If $M' = M$ then we may simply refer to this as an $(M; \underline{N}; \Psi)$ -system.

By Condition (3.1) if $i, j \in N_k$ for some k , then $M_i \cong M_j$. Hence the number of isomorphism types in \mathcal{A} is bounded by r . Moreover, it follows from Conditions (3.1)–(3.3) that $\Psi_{i,i}$ is a subgroup of $\text{Aut}(M_i)$, for each $i \in N$. If the sets M_i are not pairwise disjoint, then Condition (3.4) should be met with caution. Indeed, if $x \in M_i \cap M_j$ then by taking π to be the identity map of \underline{N} , we have that $x\phi_i, x\phi_j \in M_i \cap M_j$ for all $\phi_i \in \text{Aut}(M_i)$ and $\phi_j \in \text{Aut}(M_j)$ by Condition (3.4). However, for our work the sets M_i will mostly be pairwise disjoint, or will all intersect at an element which is fixed by every isomorphism between the M_i . For example, M could be a semigroup containing a zero, and 0 is the intersection of each of the sets M_i .

Note also that no link needs to exist between the signatures L and K . For most of our examples they will be the signature of semigroups and the signature of sets (the empty signature), respectively.

Given an $(M; M'; \underline{N}; \Psi)$ -system $\mathcal{A} = \{M_i : i \in N\}$ in M , we aim to show that, if N is \aleph_0 -categorical and each M_i possess a stronger notion of \aleph_0 -categoricity, then M is \aleph_0 -categorical. The stronger notion that we require comes from the following definition, which generalises the notion of \aleph_0 -categoricity of set extensions.

Definition 3.3 Let M be a structure and Ψ a subgroup of $\text{Aut}(M)$. Then we say that that M is \aleph_0 -categorical over Ψ if Ψ has only finitely many orbits in its action on M^n for each $n \geq 1$. We denote the resulting equivalence relation on M^n as $\sim_{M, \Psi, n}$.

By taking Ψ to be those automorphisms which fix certain subsets of M we recover our original definition of \aleph_0 -categoricity of a set extension. Similarly, by taking Ψ to be those automorphisms which preserve a fixed equivalence relation, or those which fix certain equivalence classes, we obtain a pair of notions defined in [9].

Lemma 3.4 Let M be a structure, and $\mathcal{A} = \{M_i : i \in N\}$ be an $(M, M'; \underline{N}; \Psi)$ -system. If \underline{N} is \aleph_0 -categorical and each M_i is \aleph_0 -categorical over $\Psi_{i,i}$ then

$$|(M')^n / \sim_{M,n} | < \aleph_0$$

for each $n \geq 1$.

Proof Let $\underline{N} = (N; N_1, \dots, N_r)$ and, for each $1 \leq k \leq r$, fix some $m_k \in N_k$. For each $i \in N_k$, let $\theta_i \in \Psi_{i,m_k}$, noting that such an element exists by Condition (3.1) on Ψ . Let $\bar{a} = (a_1, \dots, a_n)$ and $\bar{b} = (b_1, \dots, b_n)$ be a pair of n -tuples of M' , with $a_t \in M_{i_t}$ and $b_t \in M_{j_t}$, and such that $(i_1, \dots, i_n) \sim_{\underline{N}, n} (j_1, \dots, j_n)$ via $\pi \in \text{Aut}(\underline{N})$, say. For

each $1 \leq k \leq r$, let $i_{k1}, i_{k2}, \dots, i_{kn_k}$ be the entries of (i_1, \dots, i_n) belonging to N_k , where $k1 < k2 < \dots < kn_k$, and set

$$\bar{a}_k = (a_{k1}, \dots, a_{kn_k}) \in (M')^{n_k}.$$

We similarly form each \bar{b}_k , observing that as $i_t\pi = j_t$ for each $1 \leq t \leq n$ and π fixes the sets N_j setwise ($1 \leq j \leq r$) the elements $j_{k1}, j_{k2}, \dots, j_{kn_k}$ are precisely the entries of (j_1, \dots, j_n) belonging to N_k , so that $\bar{b}_k = (b_{k1}, \dots, b_{kn_k})$ for some $b_{kt} \in M'$. Notice that as N_1, \dots, N_r partition N we have $n = n_1 + n_2 + \dots + n_r$. Since $i_{kt}, j_{kt} \in N_k$ for each $1 \leq t \leq n_k$, we have that $a_{kt}\theta_{i_{kt}}$ and $b_{kt}\theta_{j_{kt}}$ are elements of M_{m_k} . We may thus suppose further that for each $1 \leq k \leq r$,

$$(a_{k1}\theta_{i_{k1}}, \dots, a_{kn_k}\theta_{i_{kn_k}}) \sim_{M_{m_k}, \Psi_{m_k, m_k, n_k}} (b_{k1}\theta_{j_{k1}}, \dots, b_{kn_k}\theta_{j_{kn_k}})$$

via $\sigma_k \in \Psi_{m_k, m_k}$, say (where if \bar{a}_k is a 0-tuple, then we take σ_k to be the identity of N_{m_k}). For each $1 \leq k \leq r$ and each $i \in N_k$, let

$$\phi_i = \theta_i\sigma_k\theta_{i\pi}^{-1} : M_i \rightarrow M_{i\pi},$$

noting that $\phi_i \in \Psi_{i, i\pi}$ by Conditions (3.2) and (3.3) on Ψ , since θ_i, σ_k and $\theta_{i\pi}$ are elements of Ψ . Hence, by Condition (3.4) on Ψ , there exists an automorphism ϕ of M extending each ϕ_i . For any $1 \leq k \leq r$ and any $1 \leq t \leq n_k$ we have

$$a_{kt}\phi = a_{kt}\phi_{i_{kt}} = a_{kt}\theta_{i_{kt}}\sigma_k\theta_{i_{kt}\pi}^{-1} = b_{kt}\theta_{j_{kt}}\theta_{j_{kt}}^{-1} = b_{kt},$$

and so $\bar{a} \sim_{M, n} \bar{b}$ via ϕ . Since \underline{N} is \aleph_0 -categorical and each M_i are \aleph_0 -categorical over $\Psi_{i, i}$, the conditions imposed on the tuples \bar{a} and \bar{b} have finitely many choices, and so $|(M')^n / \sim_{M, n}|$ is finite. □

By Corollary 2.9, the structure N in the lemma above can simply be a set. In most cases we take $M' = M$, and the result simplifies accordingly by the RNT as follows.

Corollary 3.5 *Let M be a structure, and $\mathcal{A} = \{M_i : i \in N\}$ be an $(M; \underline{N}; \Psi)$ -system. If \underline{N} is \aleph_0 -categorical and each M_i is \aleph_0 -categorical over $\Psi_{i, i}$, then M is \aleph_0 -categorical.*

Example 3.6 The corollary above could be used to efficiently prove the interplay of \aleph_0 -categoricity and the greatest 0-direct decomposition of a semigroup with zero [9, Theorem 4.8]. Indeed, if $S = \bigsqcup_{i \in I}^0 S_i$ is the greatest 0-direct decomposition of S , and I_1, \dots, I_n is a finite partition of I corresponding to the isomorphism types of the summands of S , then it is a simple exercise to show that $\mathcal{S} = \{S_i : i \in I\}$ is an $(S; (I; I_1, \dots, I_n); \Psi)$ -system, where Ψ is the collection of all isomorphisms between summands. Since $(I; I_1, \dots, I_n)$ is \aleph_0 -categorical, it follows by Corollary 3.5 that S is \aleph_0 -categorical if each S_i is \aleph_0 -categorical (over $\Psi_{i, i} = \text{Aut}(S_i)$).

4 Strong semilattices of semigroups

In this section we study the \aleph_0 -categoricity of strong semilattices of semigroups by making use of our most recent methodology. We are motivated by the work of the author in [22] and [23], where the homogeneity of bands and inverse semigroups are shown to depend heavily on the homogeneity of strong semilattices of rectangular bands and groups, respectively. Recall that a structure is *homogeneous* if every isomorphism between finitely generated substructures extend to an automorphism. A uniformly locally finite homogeneous structure is \aleph_0 -categorical [19, Corollary 3.1.3]. Consequently, each homogeneous band is \aleph_0 -categorical, although the same is not true for homogeneous inverse semigroups.

While there has not yet been a general study into \aleph_0 -categorical semilattices, a complete classification of countable homogeneous semilattices was completed in [6] and [7]. Since semilattices are uniformly locally finite, this provides us with a countably infinite collection of \aleph_0 -categorical semilattices. For example, the linear order \mathbb{Q} is a homogeneous semilattice, and all \aleph_0 -categorical linear orders are classified in [27].

Let Y be a semilattice. To each $\alpha \in Y$ associate a semigroup S_α , and assume that $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$. For each pair $\alpha, \beta \in Y$ with $\alpha \geq \beta$, let $\psi_{\alpha,\beta} : S_\alpha \rightarrow S_\beta$ be a morphism such that $\psi_{\alpha,\alpha}$ is the identity mapping and if $\alpha \geq \beta \geq \gamma$ then $\psi_{\alpha,\beta}\psi_{\beta,\gamma} = \psi_{\alpha,\gamma}$. On the set $S = \bigcup_{\alpha \in Y} S_\alpha$ define a multiplication by

$$a * b = (a\psi_{\alpha,\alpha\beta})(b\psi_{\beta,\alpha\beta})$$

for $a \in S_\alpha, b \in S_\beta$, and denote the resulting structure by $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$. Then S is a semigroup, and is called a *strong semilattice Y of the semigroups S_α* ($\alpha \in Y$). The semigroups S_α are called the *components* of S . We follow the convention of denoting an element a of S_α as a_α .

The idempotents of $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ are given by $E(S) = \bigcup_{\alpha \in Y} E(S_\alpha)$, and if $E(S)$ forms a subsemigroup of S then

$$E(S) = [Y; E(S_\alpha); \psi_{\alpha,\beta}|_{E(S_\alpha)}].$$

We build automorphisms of strong semilattices of semigroups in a natural way using the following well known result. A proof can be found in [21].

Theorem 4.1 *Let $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ be a strong semilattices of semigroups. Let $\pi \in \text{Aut}(Y)$ and, for each $\alpha \in Y$, let $\theta_\alpha : S_\alpha \rightarrow S_{\alpha\pi}$ be an isomorphism. Assume further that for any $\alpha \geq \beta$, the diagram*

$$\begin{array}{ccc}
 S_\alpha & \xrightarrow{\theta_\alpha} & S_{\alpha\pi} \\
 \downarrow \psi_{\alpha,\beta} & & \downarrow \psi_{\alpha\pi,\beta\pi} \\
 S_\beta & \xrightarrow{\theta_\beta} & S_{\beta\pi}
 \end{array} \tag{4.1}$$

commutes. Then the map $\theta = \bigcup_{\alpha \in Y} \theta_\alpha$ is an automorphism of S , denoted $\theta = [\theta_\alpha, \pi]_{\alpha \in Y}$.

We denote the diagram (4.1) by $[\alpha, \beta; \alpha\pi, \beta\pi]$. The map π is called the *induced (semilattice) automorphism* of Y , denoted θ^Y .

Unfortunately, not all automorphisms of strong semilattices of semigroups can be constructed as in Theorem 4.1. We shall call a strong semilattice of semigroups S *automorphism-pure* if every automorphism of S can be constructed as in Theorem 4.1. For example, every strong semilattice of completely simple semigroups is automorphism-pure [20, Lemma IV.1.8], and so both strong semilattices of groups (Clifford semigroups) and strong semilattices of rectangular bands (normal bands) are automorphism-pure.

Let $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ be a strong semilattice of semigroups. We denote the equivalence relation on Y corresponding to isomorphism types of the semigroups S_α by η_S , so that $\alpha \eta_S \beta \Leftrightarrow S_\alpha \cong S_\beta$. We let Y^S denote the set extension of Y given by $Y^S := (Y; Y/\eta_S)$.

Proposition 4.2 *Let $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ be automorphism-pure and \aleph_0 -categorical. Then each S_α is \aleph_0 -categorical and Y^S is \aleph_0 -categorical, with Y/η_S finite.*

Proof For each $\alpha \in Y$ fix some $x_\alpha \in S_\alpha$. We claim that $\{(S_\alpha, x_\alpha) : \alpha \in Y\}$ forms a system of 1-pivoted p.r.c. subsemigroups of S . Indeed, let θ be an automorphism of S such that $x_\alpha \theta = x_\beta$ for some $\alpha, \beta \in Y$. Since S is automorphism-pure, there exists $\pi \in \text{Aut}(Y)$ and isomorphisms $\theta_\alpha : S_\alpha \rightarrow S_{\alpha\pi}$ ($\alpha \in Y$) such that $\theta = [\theta_\alpha, \pi]_{\alpha \in Y}$. Hence $S_\alpha \theta = S_\beta$, and the claim follows. Consequently, by the \aleph_0 -categoricity of S and Proposition 2.3, each S_α is \aleph_0 -categorical and Y/η_S is finite.

Let $\bar{a} = (\alpha_1, \dots, \alpha_n)$ and $\bar{b} = (\beta_1, \dots, \beta_n)$ be a pair of n -tuples of Y such that there exists $a_{\alpha_k} \in S_{\alpha_k}$ and $b_{\beta_k} \in S_{\beta_k}$ with $(a_{\alpha_1}, \dots, a_{\alpha_n}) \sim_{S,n} (b_{\beta_1}, \dots, b_{\beta_n})$ via $[\theta'_\alpha, \pi']_{\alpha \in Y} \in \text{Aut}(S)$, say. Since $\pi' \in \text{Aut}(Y)$ and $S_\alpha \cong S_{\alpha\pi'}$ for each $\alpha \in Y$, it follows that $\pi' \in \text{Aut}(Y^S)$. Moreover, $\alpha_k \pi' = \beta_k$ for each k , so that $\bar{a} \sim_{Y^S,n} \bar{b}$ via π' . We have thus shown that

$$|(Y^S)^n / \sim_{Y^S,n}| \leq |S^n / \sim_{S,n}| < \aleph_0,$$

as S is \aleph_0 -categorical. Hence Y^S is \aleph_0 -categorical. □

A natural question arises: how can we build an \aleph_0 -categorical strong semilattice of semigroups from an \aleph_0 -categorical semilattice and a collection of \aleph_0 -categorical semigroups? In this paper we will only be concerned with the \aleph_0 -categoricity of strong semilattices of semigroups in which all connecting morphisms are injective or all are constant. For arbitrary connecting morphisms, the problem of assessing \aleph_0 -categoricity appears to be difficult to capture in a reasonable way. Examples of more complex \aleph_0 -categorical strong semilattices of semigroups arise from Quinn-Gregson [22], where the *universal* normal band is shown to have surjective but not injective connecting morphisms. We first study the case where each connecting morphism is a constant map.

Suppose that Y is a semilattice and, for each $\alpha \in Y$, S_α is a semigroup containing an idempotent e_α . For each $\alpha \in Y$ let $\psi_{\alpha,\alpha}$ be the identity automorphism of S_α , and for $\alpha > \beta$ let $\psi_{\alpha,\beta}$ be the constant map with image $\{e_\beta\}$. We follow the notation of

Worawiset [30] and let $\psi_{\alpha,\beta} := C_{\alpha,e_\beta}$ for each $\alpha > \beta$ in Y . It is easy to check that $\psi_{\alpha,\beta}\psi_{\beta,\gamma} = \psi_{\alpha,\gamma}$ for all $\alpha \geq \beta \geq \gamma$ in Y , so that $S = [Y; S_\alpha; C_{\alpha,e_\beta}]$ forms a strong semilattice of semigroups. We call S a *constant strong semilattice of semigroups*.

Definition 4.3 If $S = [Y; S_\alpha; C_{\alpha,e_\beta}]$ is a constant strong semilattice of semigroups, then we denote the subset of $\text{Iso}(S_\alpha; S_\beta)$ consisting of those isomorphisms which map e_α to e_β as $\text{Iso}(S_\alpha; S_\beta)^{[e_\alpha; e_\beta]}$. Notice that the set $\text{Iso}(S_\alpha; S_\alpha)^{[e_\alpha; e_\alpha]}$ is simply the subgroup $\text{Aut}(S_\alpha; \{e_\alpha\})$ of $\text{Aut}(S_\alpha)$. We may then define a relation v_S on Y by

$$\alpha v_S \beta \Leftrightarrow \text{Iso}(S_\alpha; S_\beta)^{[e_\alpha; e_\beta]} \neq \emptyset,$$

so that $v_S \subseteq \eta_S$.

The relation v_S is reflexive since $1_{S_\alpha} \in \text{Aut}(S_\alpha; \{e_\alpha\})$ for each $\alpha \in Y$, and it easily follows that v_S forms an equivalence relation on Y .

Proposition 4.4 Let $S = [Y; S_\alpha; C_{\alpha,e_\beta}]$ be such that $Y/v_S = \{Y_1, \dots, Y_r\}$ is finite, $\mathcal{Y} = (Y; Y_1, \dots, Y_r)$ is \aleph_0 -categorical and each S_α is \aleph_0 -categorical. Then S is \aleph_0 -categorical.

Proof We prove that $\{S_\alpha : \alpha \in Y\}$ forms an $(S; \mathcal{Y}; \Psi)$ -system for some Ψ . For each $\alpha, \beta \in Y$, let $\Psi_{\alpha,\beta} = \text{Iso}(S_\alpha; S_\beta)^{[e_\alpha; e_\beta]}$ and fix $\Psi = \bigcup_{\alpha,\beta \in Y} \Psi_{\alpha,\beta}$. Then Conditions (3.1)–(3.3) are seen to be satisfied since v_S forms an equivalence relation on Y . Let $\pi \in \text{Aut}(\mathcal{Y})$ and, for each $\alpha \in Y$, let $\theta_\alpha \in \Psi_{\alpha,\alpha\pi}$. We claim that $\theta = [\theta_\alpha, \pi]_{\alpha \in Y}$ is an automorphism of S . Indeed, for any $s_\alpha \in S_\alpha$ and any $\beta < \alpha$ we have

$$s_\alpha C_{\alpha,e_\beta} \theta_\beta = e_\beta \theta_\beta = e_{\beta\pi} = s_\alpha \theta_\alpha C_{\alpha\pi,e_{\beta\pi}}$$

so that the diagram $[\alpha, \beta; \alpha\pi, \beta\pi]$ commutes. Moreover $[\alpha, \alpha; \alpha\pi, \alpha\pi]$ commutes as

$$s_\alpha 1_{S_\alpha} \theta_\alpha = s_\alpha \theta_\alpha = s_\alpha \theta_\alpha 1_{S_{\alpha\pi}},$$

and the claim follows by Theorem 4.1. Since θ extends each θ_α , we have that $\{S_\alpha : \alpha \in Y\}$ is an $(S; \mathcal{Y}; \Psi)$ -system. Moreover, as S_α is \aleph_0 -categorical, it is \aleph_0 -categorical over $\Psi_{\alpha,\alpha} = \text{Aut}(S_\alpha; \{e_\alpha\})$ by [9, Lemma 2.6]. Hence S is \aleph_0 -categorical by Corollary 3.5. □

Examining our two main classes of automorphism-pure strong semilattices of semigroups: Clifford semigroups and normal bands, the result above reduces accordingly. If $S = [Y; G_\alpha; C_{\alpha,e_\beta}]$ is a constant strong semilattice of groups, then e_α is the identity of G_α , and so $\text{Iso}(G_\alpha; G_\beta) = \text{Iso}(G_\alpha; G_\beta)^{[e_\alpha; e_\beta]}$ for each $\alpha, \beta \in Y$. On the other hand, if $S = [Y; B_\alpha; C_{\alpha,e_\beta}]$ is a constant strong semilattice of rectangular bands, then it follows from Lemma 2.7 that $\text{Iso}(B_\alpha; B_\beta) \neq \emptyset$ if and only if $\text{Iso}(B_\alpha; B_\beta)^{[e_\alpha; e_\beta]} \neq \emptyset$, for any $e_\alpha \in B_\alpha, e_\beta \in B_\beta$. In both cases we therefore have $v_S = \eta_S$. Moreover, each rectangular band B_α is \aleph_0 -categorical by Theorem 2.8, and the following result is then immediate by Propositions 4.2 and 4.4.

Corollary 4.5 *Let $S = [Y; S_\alpha; C_{\alpha, \epsilon_\beta}]$ be a constant strong semilattice of rectangular bands (groups). Then S is \aleph_0 -categorical if and only if Y^S is \aleph_0 -categorical, with Y/η_S finite (and each group S_α is \aleph_0 -categorical).*

We now consider the \aleph_0 -categoricity of a strong semilattice of semigroups $S = [Y; S_\alpha; \psi_{\alpha, \beta}]$ such that each connecting morphism is injective. For each $\alpha > \beta$ in Y , we abuse notation somewhat by denoting the isomorphism $\psi_{\alpha, \beta}^{-1}|_{\text{Im } \psi_{\alpha, \beta}}$ simply by $\psi_{\alpha, \beta}^{-1}$. We observe that if $\alpha > \beta > \gamma$ and $x_\gamma \in \text{Im } \psi_{\alpha, \gamma}$, say $x_\gamma = x_\alpha \psi_{\alpha, \gamma}$, then

$$x_\gamma \psi_{\alpha, \gamma}^{-1} \psi_{\alpha, \beta} = x_\alpha \psi_{\alpha, \gamma} \psi_{\alpha, \gamma}^{-1} \psi_{\alpha, \beta} = x_\alpha \psi_{\alpha, \beta} = x_\gamma \psi_{\beta, \gamma}^{-1}.$$

Hence, on the restricted domain $\text{Im } \psi_{\alpha, \gamma}$, we have

$$\psi_{\alpha, \gamma}^{-1} \psi_{\alpha, \beta} = \psi_{\beta, \gamma}^{-1}. \tag{4.2}$$

If Y has a zero (i.e. a minimum element under the natural order) we may define an equivalence relation ξ_S on Y by $\alpha \xi_S \beta$ if and only if $S_\alpha \psi_{\alpha, 0} = S_\beta \psi_{\beta, 0}$. If $\alpha \xi_S \beta$ then $\psi_{\alpha, 0} \psi_{\beta, 0}^{-1}$ is an isomorphism from S_α onto S_β , and so $\xi_S \subseteq \eta_S$.

Proposition 4.6 *Let $S = [Y; S_\alpha; \psi_{\alpha, \beta}]$ be such that each $\psi_{\alpha, \beta}$ is injective. Let Y be a semilattice with zero and $Y/\xi_S = \{Y_1, \dots, Y_r\}$ be finite, with*

$$\{S_\alpha \psi_{\alpha, 0} : \alpha \in Y\} = \{T_1, \dots, T_r\}.$$

Then S is \aleph_0 -categorical if both $\mathcal{Y} = (Y; Y_1, \dots, Y_r)$ and $\mathcal{S}_0 = (S_0; T_1, \dots, T_r)$ are \aleph_0 -categorical. Moreover, if S is automorphism-pure and \aleph_0 -categorical, then conversely both \mathcal{Y} and \mathcal{S}_0 are \aleph_0 -categorical.

Proof Suppose first that both \mathcal{Y} and \mathcal{S}_0 are \aleph_0 -categorical. Let $\bar{a} = (a_{\alpha_1}, \dots, a_{\alpha_n})$ and $\bar{b} = (b_{\beta_1}, \dots, b_{\beta_n})$ be n -tuples of S with $(\alpha_1, \dots, \alpha_n) \sim_{\mathcal{Y}, n} (\beta_1, \dots, \beta_n)$ via $\pi \in \text{Aut}(\mathcal{Y})$, say. Suppose further that

$$(a_{\alpha_1} \psi_{\alpha_1, 0}, \dots, a_{\alpha_n} \psi_{\alpha_n, 0}) \sim_{\mathcal{S}_0, n} (b_{\beta_1} \psi_{\beta_1, 0}, \dots, b_{\beta_n} \psi_{\beta_n, 0})$$

via $\theta_0 \in \text{Aut}(\mathcal{S}_0)$, say. Then for each $\alpha \in Y$ we have $S_\alpha \psi_{\alpha, 0} = S_{\alpha\pi} \psi_{\alpha\pi, 0}$, and so we can take an isomorphism $\theta_\alpha : S_\alpha \rightarrow S_{\alpha\pi}$ given by

$$\theta_\alpha = \psi_{\alpha, 0} \theta_0 \psi_{\alpha\pi, 0}^{-1}.$$

For each $\alpha \geq \beta$ in Y , the diagram $[\alpha, \beta; \alpha\pi, \beta\pi]$ commutes as

$$\begin{aligned} \psi_{\alpha, \beta} \theta_\beta &= \psi_{\alpha, \beta} (\psi_{\beta, 0} \theta_0 \psi_{\beta\pi, 0}^{-1}) = \psi_{\alpha, 0} \theta_0 \psi_{\beta\pi, 0}^{-1} \\ &= \psi_{\alpha, 0} \theta_0 (\psi_{\alpha\pi, 0}^{-1} \psi_{\alpha\pi, \beta\pi}) = \theta_\alpha \psi_{\alpha\pi, \beta\pi}, \end{aligned}$$

where the penultimate equality is due to (4.2) as $\text{Im } \psi_{\alpha\pi, 0} = \text{Im } \psi_{\alpha, 0} = (\text{Im } \psi_{\alpha, 0})\theta_0$. Hence $\theta = [\theta_\alpha, \pi]_{\alpha \in Y}$ is an automorphism of S by Theorem 4.1. Furthermore,

$$a_{\alpha_k} \theta = a_{\alpha_k} \theta_{\alpha_k} = a_{\alpha_k} \psi_{\alpha_k,0} \theta_0 \psi_{\alpha_k,0}^{-1} = b_{\beta_k} \psi_{\beta_k,0} \psi_{\beta_k,0}^{-1} = b_{\beta_k}$$

for each $1 \leq k \leq n$, so that $\bar{a} \sim_{S,n} \bar{b}$ via θ . We thus have that

$$|S^n / \sim_{S,n}| \leq |\mathcal{Y}^n / \sim_{\mathcal{Y},n}| \cdot |S_0^n / \sim_{S_0,n}| < \aleph_0$$

and so S is \aleph_0 -categorical.

Conversely, suppose S is automorphism-pure and \aleph_0 -categorical. For each $1 \leq k \leq r$, fix some $\gamma_k \in Y_k$, where we assume without loss of generality that $S_{\gamma_k} \psi_{\gamma_k,0} = T_k$. For each $\alpha \in Y$, fix some $x_\alpha \in S_\alpha$. Let $\bar{a} = (\alpha_1, \dots, \alpha_n)$ and $\bar{b} = (\beta_1, \dots, \beta_n)$ be n -tuples of Y such that

$$(x_{\alpha_1}, \dots, x_{\alpha_n}, x_{\gamma_1}, \dots, x_{\gamma_r}) \sim_{S,n+r} (x_{\beta_1}, \dots, x_{\beta_n}, x_{\gamma_1}, \dots, x_{\gamma_r}),$$

via $\theta \in \text{Aut}(S)$, say. Since S is automorphism-pure there exists $\pi \in \text{Aut}(Y)$ and $\theta_\alpha \in \text{Iso}(S_\alpha; S_{\alpha\pi})$ such that $\theta = [\theta_\alpha, \pi]_{\alpha \in Y}$. The automorphism π fixes each γ_k , so that $S_{\gamma_k} \theta = S_{\gamma_k}$. Hence, as the diagram $[\gamma_k, 0; \gamma_k, 0]$ commutes for each k , we have

$$T_k = S_{\gamma_k} \psi_{\gamma_k,0} = (S_{\gamma_k} \theta_{\gamma_k}) \psi_{\gamma_k,0} = S_{\gamma_k} \psi_{\gamma_k,0} \theta_0 = T_k \theta_0 = T_k \theta.$$

If $\alpha \in Y_k$ then, by the commutativity of the diagram $[\alpha; 0; \alpha\pi, 0]$, we therefore have

$$S_\alpha \psi_{\alpha,0} = T_k = T_k \theta_0 = S_\alpha \psi_{\alpha,0} \theta_0 = S_\alpha \theta_\alpha \psi_{\alpha\pi,0} = S_{\alpha\pi} \psi_{\alpha\pi,0},$$

and so $\pi \in \text{Aut}(\mathcal{Y})$. We have shown that

$$|\mathcal{Y}^n / \sim_{\mathcal{Y},n}| \leq |S^{n+r} / \sim_{S,n+r}| < \aleph_0$$

and so \mathcal{Y} is \aleph_0 -categorical. Now suppose \bar{c} and \bar{d} are n -tuples of S_0 such that

$$(\bar{c}, x_{\gamma_1}, \dots, x_{\gamma_r}) \sim_{S,n+r} (\bar{d}, x_{\gamma_1}, \dots, x_{\gamma_r}),$$

via $\theta' = [\theta'_\alpha, \pi']_{\alpha \in Y} \in \text{Aut}(S)$, say. Then arguing as before we have that $T_k \theta' = T_k$ for each k , and it follows that $\theta'_\alpha \in \text{Aut}(S_0)$, with $\bar{c} \theta'_0 = \bar{d}$. Hence

$$|S_0^n / \sim_{S_0,n}| \leq |S^{n+r} / \sim_{S,n+r}| < \aleph_0$$

and so S_0 is \aleph_0 -categorical. □

Note that if Y is finite, then the meet of all the elements of Y is a zero. Moreover, as Y is finite, it is \aleph_0 -categorical over any set of subsets by the RNT, and so the result above simplifies accordingly in this case:

Corollary 4.7 *Let $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ be such that Y is finite and each $\psi_{\alpha,\beta}$ is injective. If $S_0 = (S_0; \{S_\alpha \psi_{\alpha,0} : \alpha \in Y\})$ is \aleph_0 -categorical then S is \aleph_0 -categorical. Conversely, if S is automorphism-pure and \aleph_0 -categorical then S_0 is \aleph_0 -categorical.*

For a Clifford semigroup S , the property that the connecting morphisms are injective is equivalent to S being E -unitary, that is, such that for all $e \in E(S)$

and all $s \in S$, if $es \in E$ then $s \in E(S)$ [17, Exercise 5.20]. Since Clifford semigroups are automorphism-pure, we therefore have the following simplification of Proposition 4.6.

Corollary 4.8 *Let $S = [Y; G_\alpha; \psi_{\alpha,\beta}]$ be an E -unitary Clifford semigroup. Let Y be a semilattice with zero and Y/ξ_S be finite. Then S is \aleph_0 -categorical if and only if $(Y; Y/\xi_S)$ and $(S_0; \{S_\alpha \psi_{\alpha,0} : \alpha \in Y\})$ are \aleph_0 -categorical. In particular, if Y is finite then S is \aleph_0 -categorical if and only if $(S_0; \{S_\alpha \psi_{\alpha,0} : \alpha \in Y\})$ is \aleph_0 -categorical.*

Example 4.9 We use the work of Apps [1] to construct examples of \aleph_0 -categorical E -unitary Clifford semigroups as follows. Let G be an \aleph_0 -categorical group and $H_1 < H_2 < \dots$ a characteristic series in G , so that each H_i is a characteristic subgroup of G and H_i is a subgroup of H_{i+1} . Apps proved that such a series must be finite, and there exists a characteristic series $\{1\} = G_0 < G_1 < G_2 < \dots < G_n = G$ with each G_i/G_{i-1} a characteristically simple \aleph_0 -categorical group. For each $0 \leq i \leq n$, let $K_i = G_i \times \{i\}$ be an isomorphic copy of G_i . For each $0 \leq i < j \leq n$, let $\psi_{i,j} : K_i \rightarrow K_j$ be the map given by $(x, i)\psi_{i,j} = (x, j)$. Then we may form a strong semilattice of the groups K_i by taking $S = [Y; K_i; \psi_{i,j}]$, where Y is the set $\{0, 1, \dots, n\}$ with the reverse ordering $0 > 1 > 2 > \dots > n$. Notice that S is E -unitary as each connecting morphism is injective. Moreover, each $K_i \psi_{i,n} = G_i \times \{n\}$ is a characteristic subgroup of $K_n = G_n \times \{n\}$. Hence, by Lemma 2.6, $(K_n; \{K_i \psi_{i,n} : 1 \leq i \leq n\})$ is \aleph_0 -categorical. Since Y is finite, we have that $(Y; Y/\xi_S)$ is \aleph_0 -categorical, and so S is \aleph_0 -categorical by Corollary 4.8.

If $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ is such that each connecting morphism is an isomorphism, then $Y/\xi_S = \{Y\}$, and so the result above simplifies accordingly. However we can prove a more general result directly (without the condition that Y has a zero) with aid of the following proposition. The result is folklore, but a proof can be found in [21].

Proposition 4.10 *Let $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ be such that each $\psi_{\alpha,\beta}$ is an isomorphism. Then $S \cong S_\alpha \times Y$ for any $\alpha \in Y$. Conversely, if T is a semigroup and Z is a semilattice then $T \times Z$ is isomorphic to a strong semilattice of semigroups such that each connecting morphism is an isomorphism.*

Corollary 4.11 *Let $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ be such that each $\psi_{\alpha,\beta}$ is an isomorphism. If S_α and Y are \aleph_0 -categorical, then S is \aleph_0 -categorical. Moreover, if S is automorphism-pure then the converse holds.*

Proof By Proposition 4.10, S is isomorphic to $S_\alpha \times Y$ for any $\alpha \in Y$. The first half of the result then follows as \aleph_0 -categoricity is preserved by finite direct products [12].

If S is automorphism-pure then the converse holds by Proposition 4.2, as $(Y; Y/\eta_S)$ being \aleph_0 -categorical clearly implies Y is \aleph_0 -categorical. □

5 \aleph_0 -categorical Rees matrix semigroups

A semigroup S is called simple (0-simple) if it has no proper ideals (if its only proper ideal is $\{0\}$ and $S^2 \neq \{0\}$). A simple (0-simple) semigroup is called *completely simple* (*completely 0-simple*) if contains a primitive idempotent, i.e. a non-zero idempotent e such that for any non-zero idempotent f of S ,

$$ef = fe = f \Rightarrow e = f.$$

Since an \aleph_0 -categorical semigroup is periodic, it follows that every \aleph_0 -categorical (0-)simple semigroup is completely (0-)simple (see the proof of Theorem 3.12 of [9]). By Rees theorem [25], to study the \aleph_0 -categoricity of a completely 0-simple semigroup, it is sufficient to consider Rees matrix semigroups:

Theorem 5.1 (The Rees Theorem) *Let G be a group, let I and Λ be non-empty index sets and let $P = (p_{\lambda,i})$ be an $\Lambda \times I$ matrix with entries in $G \cup \{0\}$. Suppose no row or column of P consists entirely of zeros (that is, P is regular). Let $S = (I \times G \times \Lambda) \cup \{0\}$, and define multiplication $*$ on S by*

$$(i, g, \lambda) * (j, h, \mu) = \begin{cases} (i, gp_{\lambda,j}h, \mu) & \text{if } p_{\lambda,j} \neq 0 \\ 0 & \text{else} \end{cases}$$

$$0 * (i, g, \lambda) = (i, g, \lambda) * 0 = 0 * 0 = 0.$$

Then S is a completely 0-simple semigroup, denoted $\mathcal{M}^0[G; I, \Lambda; P]$, and is called a (regular) Rees matrix semigroup (over G). Conversely, every completely 0-simple semigroup is isomorphic to a Rees matrix semigroup.

The matrix P is called the *sandwich matrix* of S . If P has no zero entries, then $I \times G \times \Lambda$ forms a subsemigroup of $\mathcal{M}^0[G; I, \Lambda, P]$, called a *Rees matrix semigroup without zero* and denoted $\mathcal{M}[G; I, \Lambda; P]$. Every completely simple semigroup is isomorphic to a Rees matrix semigroup without zero [17, Section 3.3].

Lemma 5.2 *Let G be a group and P be a $\Lambda \times I$ matrix with entries from G . Then $\mathcal{M}[G; I, \Lambda; P]$ is \aleph_0 -categorical if and only if $\mathcal{M}^0[G; I, \Lambda; P]$ is \aleph_0 -categorical.*

Proof The result is immediate from Gould and Quinn-Gregson [9, Corollary 2.12] since $\mathcal{M}^0[G; I, \Lambda; P]$ is isomorphic to $\mathcal{M}[G; I, \Lambda; P]$ with a zero adjoined. \square

As a consequence, to examine the \aleph_0 -categoricity of both completely simple and completely 0-simple semigroups, it suffices to study Rees matrix semigroups.

A fundamental discovery in [9] was that to understand the \aleph_0 -categoricity of an arbitrary semigroup, it is necessary to study \aleph_0 -categorical completely (0-)simple semigroups. Indeed, they arise as principal factors of an \aleph_0 -categorical semigroup, as well as giving examples of 0-direct indecomposable summands in a semigroup with zero.

In [9] the \aleph_0 -categoricity of Rees matrix semigroups over identity matrices (known as *Brandt semigroups*) were determined, although we deferred the general case to this current article. Countable homogeneous completely simple semigroups have been classified (modulo our understanding of homogeneous groups) in [24], which gives rise to more complex examples of \aleph_0 -categorical completely (0-)simple semigroups.

Given a Rees matrix semigroup $S = \mathcal{M}^0[G; I, \Lambda; P]$ with $P = (p_{\lambda,i})$, we let $G(P)$ denote the subset of G of all non-zero entries of P , that is, $G(P) := \{p_{\lambda,i} : p_{\lambda,i} \neq 0\}$. The idempotents of S are easily described [17, Page 71]:

$$E(S) = \{(i, p_{\lambda,i}^{-1}, \lambda) : p_{\lambda,i} \neq 0\}.$$

Since there exists a simple isomorphism theorem for Rees matrix semigroups [17, Theorem 3.4.1] (see Theorem 5.10), we should be hopeful of achieving a thorough understanding of \aleph_0 -categorical Rees matrix semigroups via the RNT. However, from the isomorphism theorem it is not clear how the \aleph_0 -categoricity of the semigroup $\mathcal{M}^0[G; I, \Lambda; P]$ affects the sets I and Λ . We instead follow a technique of Graham [10] and Houghton [15] of constructing a bipartite graph from the sets I and Λ .

A *bipartite graph* is a (simple) graph whose vertices can be split into two disjoint non-empty sets L and R such that every edge connects a vertex in L to a vertex in R . The sets L and R are called the *left set* and the *right set*, respectively. Formally, a bipartite graph is a triple $\Gamma = \langle L, R, E \rangle$ such that L and R are non-empty trivially intersecting sets and

$$E \subseteq \{\{x, y\} : x \in L, y \in R\}.$$

We call $L \cup R$ the set of *vertices* of Γ and E the set of *edges*. An isomorphism between a pair of bipartite graphs $\Gamma = \langle L, R, E \rangle$ and $\Gamma' = \langle L', R', E' \rangle$ is a bijection $\psi : L \cup R \rightarrow L' \cup R'$ such that $L\psi = L'$, $R\psi = R'$, and $\{l, r\} \in E$ if and only if $\{l\psi, r\psi\} \in E'$. We are therefore regarding bipartite graphs in the signature $L_{BG} = \{Q_L, Q_R, E\}$, where Q_L and Q_R are unary relations, which correspond to the sets L and R , respectively, and E is a binary relation corresponding to the edge relation (here we abuse the notation somewhat by letting E denote the edge relation *and* the set of edges).

Let $\Gamma = \langle L, R, E \rangle$ be a bipartite graph. Then Γ is called *complete* if, for all $x \in L, y \in R$, we have $\{x, y\} \in E$. If $E = \emptyset$ then Γ is called *empty*. If each vertex of Γ is incident to exactly one edge, then Γ is called a *perfect matching*. The *complement* of Γ is the bipartite graph $\langle L, R, E' \rangle$ with

$$E' = \{\{x, y\} : x \in L, y \in R, \{x, y\} \notin E\}.$$

Hence an empty bipartite graph is the complement of a complete bipartite graph, and vice-versa. We call Γ *random* if, for each $k, \ell \in \mathbb{N}$, and for every distinct $x_1, \dots, x_k, y_1, \dots, y_\ell$ in L (in R) there exists infinitely many $u \in R$ ($u \in L$) such that $\{u, x_i\} \in E$ but $\{u, y_j\} \notin E$ for each $1 \leq i \leq k$ and $1 \leq j \leq \ell$.

Clearly, for each pair $n, m \in \mathbb{N}^* = \mathbb{N} \cup \{\aleph_0\}$, there exists a unique (up to isomorphism) complete bipartite graph with left set of size n and right set of size m , which

we denote as $K_{n,m}$. There also exists a unique, up to isomorphism, perfect matching with left and right sets of size n , denoted P_n . Similar uniqueness holds for the empty bipartite graph $E_{n,m}$ with left set of size n and right set of size m , and the complement of the perfect matching P_n , which we denote as CP_n . Less obviously, any pair of random bipartite graphs are isomorphic [4].

Theorem 5.3 [8] *A countable bipartite graph is homogeneous if and only if it is isomorphic to either $K_{n,m}$, $E_{n,m}$, P_n , CP_n for some $n, m \in \mathbb{N}^*$, or the random bipartite graph.*

Since bipartite graphs are relational structures with finitely many relations, homogeneous bipartite graphs are uniformly locally finite,¹ and thus \aleph_0 -categorical. Unfortunately, no full classification of \aleph_0 -categorical bipartite graphs exists.

Let $\Gamma = \langle L, R, E \rangle$ be a bipartite graph. A *path* \mathfrak{p} in Γ is a finite sequence of vertices

$$\mathfrak{p} = (v_0, v_1, \dots, v_n)$$

such that v_i and v_{i+1} are adjacent for each $0 \leq i \leq n - 1$. For example, if $\{x, y\}$ is an edge in E then both (x, y) and (y, x) are paths in Γ . A pair of vertices x and y are *connected*, denoted $x \bowtie y$, if and only if $x = y$ or there exists a path (v_1, v_2, \dots, v_n) in Γ such that $v_1 = x$ and $v_n = y$. It is clear that \bowtie is an equivalence relation on the set of vertices of Γ , and we call the equivalence classes the *connected components* of Γ . Each connected component is a sub-bipartite graph of Γ under the induced structure, and we let $\mathcal{C}(\Gamma)$ denote the set of connected components of Γ .

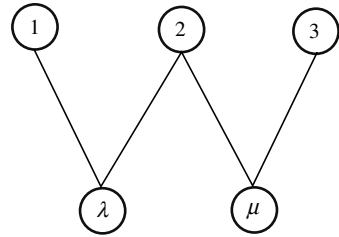
Let Γ be a bipartite graph with $\mathcal{C}(\Gamma) = \{\Gamma_i : i \in A\}$. For any automorphism ϕ of Γ and $x, y \in \Gamma$ we have that $(x, v_2, \dots, v_{n-1}, y)$ is a path in Γ if and only if $(x\phi, v_2\phi, \dots, v_{n-1}\phi, y\phi)$ is a path in Γ , since ϕ preserves edges and non-edges. Hence $x \bowtie y$ if and only if $x\phi \bowtie y\phi$, and so there exists a bijection π of A such that $\Gamma_i\phi = \Gamma_{i\pi}$ for each $i \in I$. We have thus proven the reverse direction of the following result, the forward being immediate.

Proposition 5.4 *Let $\Gamma = \langle L, R, E \rangle$ be a bipartite graph with $\mathcal{C}(\Gamma) = \{\Gamma_i : i \in A\}$. Let π be a bijection of A and $\phi_i : \Gamma_i \rightarrow \Gamma_{i\pi}$ an isomorphism for each $i \in A$. Then $\bigcup_{i \in I} \phi_i$ is an automorphism of Γ . Conversely, every automorphism of Γ can be constructed in this way.*

Proposition 5.5 *Let $\Gamma = \langle L, R, E \rangle$ be a bipartite graph with $\mathcal{C}(\Gamma) = \{\Gamma_i : i \in A\}$. Then Γ is \aleph_0 -categorical if and only if each connected component is \aleph_0 -categorical and $\mathcal{C}(\Gamma)$ is finite, up to isomorphism.*

¹ This differs from the graph theoretical notion of being uniformly locally finite, i.e. such that the degrees of the vertices are bounded above by some finite value.

Fig. 1 Induced bipartite graph



Proof (\Rightarrow) By Proposition 5.4 we have that, for any choice of $x_i \in \Gamma_i$ ($i \in A$), the set $\{(\Gamma_i, x_i) : i \in A\}$ forms a system of 1-pivoted p.r.c. sub-bipartite graphs of Γ . The result then follows from Proposition 2.3.

(\Leftarrow) First we show that $\mathcal{C}(\Gamma)$ forms a $(\Gamma; \underline{A}; \Psi)$ -system in Γ for some \underline{A} and Ψ . Let A_1, \dots, A_r be the finite partition of A corresponding to the isomorphism types of the connected components of Γ , that is, $\Gamma_i \cong \Gamma_j$ if and only if $i, j \in A_k$ for some k . Fix $\underline{A} = (A; A_1, \dots, A_r)$. For each $i, j \in A$, let $\Psi_{i,j} = \text{Iso}(\Gamma_i; \Gamma_j)$ and fix $\Psi = \bigcup_{i,j \in A} \Psi_{i,j}$. Then Ψ clearly satisfy Conditions (3.1)–(3.3). Let $\pi \in \text{Aut}(\underline{A})$ and, for each $i \in A$, let $\phi_i \in \Psi_{i,\pi i}$. Then by Proposition 5.4, $\phi = \bigcup_{i \in A} \phi_i$ is an automorphism of Γ , and so Ψ satisfies Condition (3.4). Hence $\mathcal{C}(\Gamma)$ forms an $(\Gamma; \underline{A}; \Psi)$ -system. Each Γ_i is \aleph_0 -categorical (over $\Psi_{i,i} = \text{Aut}(\Gamma_i)$) and \underline{A} is \aleph_0 -categorical by Corollary 2.9, and so Γ is \aleph_0 -categorical by Corollary 3.5. \square

Definition 5.6 Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a Rees matrix semigroup with $P = (p_{\lambda,i})$. Then we form a bipartite graph $\Gamma(P) = \langle I, \Lambda, E \rangle$ with edge set

$$E = \{\{i, \lambda\} : p_{\lambda,i} \neq 0\},$$

which we call the *induced bipartite graph of S*.

The above construct has long been fundamental to the study of Rees matrix semigroups, and has its roots in a paper by Graham in [10]. Here, it is used to describe the maximal nilpotent subsemigroups of a Rees matrix semigroup, where a semigroup is *nilpotent* if some power is equal to $\{0\}$. All maximal subsemigroups of a finite Rees matrix semigroup were described in the same paper, a result which was later extended in [11] to arbitrary finite semigroups. In [16], Howie used the induced bipartite graph to describe the subsemigroup of a Rees matrix semigroup generated by its idempotents. Finally, in [15], Houghton described the homology of the induced bipartite graph, and a detailed overview of his work is given in [26].

Example 5.7 Let $S = \mathcal{M}^0[G; \{1, 2, 3\}, \{\lambda, \mu\}; P]$ where

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} \lambda \\ \mu \end{matrix} & \begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix} \end{matrix}.$$

Then the induced bipartite graph of S is given in Fig. 1.

Example 5.8 Let $S = \mathcal{M}^0[G;I, \Lambda;P]$ be such that P has no zero entries, so that S is isomorphic to a completely simple semigroup with zero adjoined. Then $\Gamma(P)$ is a complete bipartite graph.

Notation 5.9 Let $S = \mathcal{M}^0[G;I, \Lambda;P]$ be a Rees matrix semigroup. For an n -tuple $\bar{a} = ((i_1, g_1, \lambda_1), \dots, (i_n, g_n, \lambda_n))$ of S^* , we write $\Gamma(\bar{a})$ for the $2n$ -tuple $(i_1, \lambda_1, \dots, i_n, \lambda_n)$ of $\Gamma(P)$.

Following [2], we adapt the isomorphism theorem for Rees matrix semigroups to explicitly highlight the role of the induced bipartite graph:

Theorem 5.10 Let $S_1 = \mathcal{M}^0[G_1;I_1, \Lambda_1;P_1]$ and $S_2 = \mathcal{M}^0[G_2;I_2, \Lambda_2;P_2]$ be a pair of Rees matrix semigroups with sandwich matrices $P_1 = (p_{\lambda,i})$ and $P_2 = (q_{\mu,j})$, respectively. Let $\psi \in \text{Iso}(\Gamma(P_1); \Gamma(P_2))$, $\theta \in \text{Iso}(G_1; G_2)$, and $u_i, v_\lambda \in G_2$ for each $i \in I_1, \lambda \in \Lambda_1$. Then the mapping $\phi : S_1 \rightarrow S_2$ given by

$$(i, g, \lambda)\phi = (i\psi, u_i(g\theta)v_\lambda, \lambda\psi)$$

is an isomorphism if and only if $p_{\lambda,i}\theta = v_\lambda \cdot q_{\lambda\psi, i\psi} \cdot u_i$ whenever $p_{\lambda,i} \neq 0$. Moreover, every isomorphism from S_1 to S_2 can be described in this way.

The isomorphism ϕ will be denoted as $(\theta, \psi, (u_i)_{i \in I_1}, (v_\lambda)_{\lambda \in \Lambda_1})$. We also denote the induced group isomorphism θ as ϕ_{G_1} , and the induced bipartite graph isomorphism ψ as $\phi_{\Gamma(P_1)}$, so that $\phi = (\phi_{G_1}, \psi_{\Gamma(P_1)}, (u_i)_{i \in I_1}, (v_\lambda)_{\lambda \in \Lambda_1})$. Note that the induced group isomorphism is not uniquely defined by ϕ . That is, there may exist $\theta' \in \text{Iso}(G_1; G_2)$ and $u'_i, v'_\lambda \in G_2$, such that $\theta' \neq \theta$ but $\phi = (\theta', \psi, (u'_i)_{i \in I_1}, (v'_\lambda)_{\lambda \in \Lambda_1})$. Examples of this phenomenon will occur throughout this work.

The composition and inverses of isomorphisms between Rees matrix semigroups behave in a natural way as follows, and a proof can be found in [21].

Corollary 5.11 Let $S_k = \mathcal{M}^0[G_k;I_k, \Lambda_k;P_k]$ ($k = 1, 2, 3$) be Rees matrix semigroups. Then for any pair of isomorphisms $\phi = (\theta, \psi, (u_i)_{i \in I_1}, (v_\lambda)_{\lambda \in \Lambda_1}) \in \text{Iso}(S_1; S_2)$ and $\phi' = (\theta', \psi', (u'_j)_{j \in I_2}, (v'_\mu)_{\mu \in \Lambda_2}) \in \text{Iso}(S_2; S_3)$ we have:

- (i) $\phi\phi' = (\theta\theta', \psi\psi', (u'_{i\psi}(u_i\theta'))_{i \in I_1}, ((v_\lambda\theta')v'_{\lambda\psi})_{\lambda \in \Lambda_1})$;
- (ii) $\phi^{-1} = (\theta^{-1}, \psi^{-1}, ((u_{i\psi^{-1}})^{-1}\theta^{-1})_{i \in I_2}, ((v_{\lambda\psi^{-1}})^{-1}\theta^{-1})_{\lambda \in \Lambda_2})$.

Let $\Gamma = \langle L, R, E \rangle$ be a bipartite graph. For each $n \in \mathbb{N}$, we let $\sigma_{\Gamma, n}$ be the equivalence relation on Γ^n given by

$$(x_1, \dots, x_n) \sigma_{\Gamma, n} (y_1, \dots, y_n) \Leftrightarrow [x_i \in L \Leftrightarrow y_i \in L, \text{ for each } 1 \leq i \leq n].$$

Since each entry of an n -tuple of Γ lies in either L or R we have that

$$|\Gamma^n / \sigma_{\Gamma,n}| = 2^n,$$

for each n . Moreover, as the automorphisms of Γ fixes the sets L and R , it easily follows that $\sim_{\Gamma,n} \subseteq \sigma_{\Gamma,n}$.

Proposition 5.12 *If $S = \mathcal{M}^0[G;I, \Lambda;P]$ is \aleph_0 -categorical, then G and $\Gamma(P)$ are \aleph_0 -categorical.*

Proof Since G is isomorphic to the non-zero maximal subgroups of S , it is \aleph_0 -categorical by Gould and Quinn-Gregson [9, Corollary 3.7]. Now let $\bar{a} = (a_1, \dots, a_n)$ and $\bar{b} = (b_1, \dots, b_n)$ be a pair of $\sigma_{\Gamma(P),n}$ -related n -tuples of $\Gamma(P)$. Let $i_1 < i_2 < \dots < i_s$ and $j_1 < j_2 < \dots < j_t$ be the indexes of entries of \bar{a} lying in I and Λ , respectively (noting that the same is true for \bar{b} as $\bar{a} \sigma_{\Gamma,n} \bar{b}$). Suppose further that there exists $i \in I, \lambda \in \Lambda$ such that the n -tuples

$$\begin{aligned} &((a_{i_1}, 1, \lambda), \dots, (a_{i_s}, 1, \lambda), (i, 1, a_{j_1}), \dots, (i, 1, a_{j_t})) \quad \text{and} \\ &((b_{i_1}, 1, \lambda), \dots, (b_{i_s}, 1, \lambda), (i, 1, b_{j_1}), \dots, (i, 1, b_{j_t})), \end{aligned}$$

are automorphically equivalent via $\phi \in \text{Aut}(S)$, say. By Theorem 5.10, $a_{i_r} \phi_{\Gamma(P)} = b_{i_r}$ and $a_{j_{r'}} \phi_{\Gamma(P)} = b_{j_{r'}}$ for each $1 \leq r \leq s$ and $1 \leq r' \leq t$. Hence $\bar{a} \sim_{\Gamma(P),n} \bar{b}$ via $\phi_{\Gamma(P)}$, and we have thus shown that

$$|\Gamma(P)^n / \sim_{\Gamma(P),n}| \leq 2^n \cdot |S^n / \sim_{S,n}|.$$

Hence $\Gamma(P)$ is \aleph_0 -categorical by the \aleph_0 -categoricity of S . □

However, the converse to the proposition above does not hold in general (even in the completely simple case).

Example 5.13 Let $G = \{1, a\}$ be the group of size 2 and let $I = \{i_0, i_1, \dots, \}$ and $\Lambda = \{\lambda_0, \lambda_1, \dots, \}$ be infinite sets. Let P be the $\Lambda \times I$ matrix in which $p_{\lambda_k, i_\ell} = a$ if and only if $k \geq \ell \geq 1$, that is,

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 & \dots \\ 1 & a & 1 & 1 & \dots & 1 & \dots \\ 1 & a & a & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 1 & \dots \\ 1 & a & \dots & a & a & 1 & \dots \\ 1 & a & \dots & a & a & a & \dots \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let $S = \mathcal{M}[G;I, \Lambda;P]$. Then $\Gamma(P)$ is a complete bipartite graph, and thus \aleph_0 -categorical. However, $\{((i_0, 1, \lambda_0), (i_k, 1, \lambda_k)) : k \in \mathbb{N}\}$ can be shown to be an infinite set of distinct 2-automorphism types of S . Alternatively, we will show at the end of the section that S is not \aleph_0 -categorical by Proposition 5.29.

5.1 Connected Rees components

Let $S_k = \mathcal{M}^0[G; I_k, \Lambda_k; P_k]$ ($k \in A$) be a collection of Rees matrix semigroups with $P_k = (p_{\lambda,i}^{(k)})$ and $S_k \cap S_\ell = \{0\}$ for each $k, \ell \in A$. Then we may form a single Rees matrix semigroup $S = \mathcal{M}^0[G; I, \Lambda; P]$, where $I = \bigcup_{k \in A} I_k$, $\Lambda = \bigcup_{k \in A} \Lambda_k$ and $P = (p_{\lambda,i})$ is the Λ by I matrix defined by

$$p_{\lambda,i} = \begin{cases} p_{\lambda,i}^{(k)} & \text{if } \lambda, i \in \Gamma(P_k), \text{ for some } k \\ 0 & \text{else.} \end{cases}$$

That is, P is the block matrix

$$P = \begin{bmatrix} P_1 & 0 & 0 & \dots \\ 0 & P_2 & 0 & \dots \\ 0 & 0 & P_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}. \tag{5.1}$$

We denote S by $\bigotimes_{k \in A}^G S_k$. The subsemigroups S_k of S are called *Rees components of S* . Notice that each $\Gamma(P_k)$ is a union of connected components of $\Gamma(P)$. The subsemigroup S_k will be called a *connected Rees component of S* if $\Gamma(P_k)$ is connected (and is therefore a connected component of $\Gamma(P)$).

Conversely, for any Rees matrix semigroup $S = \mathcal{M}^0[G; I, \Lambda; P]$ there exists partitions $\{I_k : k \in A\}$ and $\{\Lambda_k : k \in A\}$ of I and Λ , respectively, such that $\mathcal{C}(\Gamma(P)) = \{\Lambda_k \cup I_k : k \in A\}$. Consequently, for each $k \in A$, the subsemigroup $S_k = \mathcal{M}^0[G; I_k, \Lambda_k; P_k]$ of S is a connected Rees component, where P_k is the $\Lambda_k \times I_k$ submatrix of P , and are such that $S_k S_\ell = 0$ for all $k \neq \ell$. Following the work of Graham [10], we may then permute the rows and columns of P if necessary to assume without loss of generality that P is a block matrix of the form (5.1).

Note that if S is a Rees matrix semigroup with connected Rees components $\{S_k : k \in A\}$ then clearly

$$E(S) = \bigcup_{k \in A} E(S_k). \tag{5.2}$$

Using the fact that automorphisms of $\Gamma(P)$ arise as collections of isomorphisms between its connected components, we obtain an alternative description of automorphisms of a Rees matrix semigroups. The proof is a simple exercise, and can be found in [21].

Corollary 5.14 *Let $S = \bigotimes_{k \in A}^G S_k = \mathcal{M}^0[G; I, \Lambda; P]$ be a Rees matrix semigroup such that each $S_k = \mathcal{M}^0[G; I_k, \Lambda_k; P_k]$ is a connected Rees component of S . Let π be a bijection of A and, for each $k \in A$, let $\phi_k = (\theta, \psi_k, (u_i^{(k)})_{i \in I_k}, (v_\lambda^{(k)})_{\lambda \in \Lambda_k})$ be an isomorphism from S_k to $S_{k\pi}$. Then $\phi = (\theta, \psi, (u_i)_{i \in I}, (v_\lambda)_{\lambda \in \Lambda})$ is an automorphism of S , where $\psi = \bigcup_{k \in A} \psi_k$, and if $i, \lambda \in \Gamma(P_k)$ then $u_i = u_i^{(k)}$ and $v_\lambda = v_\lambda^{(k)}$. Moreover, every automorphism of S can be described in this way.*

We observe that the induced group automorphisms of the isomorphisms ϕ_k above must all be equal.

Recall that if $S = \mathcal{M}^0[G; I, \Lambda; P]$ is \aleph_0 -categorical, then $\Gamma(P)$ is \aleph_0 -categorical by Proposition 5.12, and thus $\mathcal{C}(\Gamma(P))$ is finite, up to isomorphism, with each connected component being \aleph_0 -categorical by Proposition 5.5. We extend this result to the set of all connected Rees components of S as follows:

Proposition 5.15 *Let $S = \bigotimes_{k \in A}^G S_k$ be an \aleph_0 -categorical Rees matrix semigroup such that each S_k is a connected Rees component of S . Then each S_k is \aleph_0 -categorical and S has finitely many connected Rees components, up to isomorphism.*

Proof We claim that $\{(S_k, a_k) : k \in A\}$ is a system of 1-pivoted p.r.c. subsemigroups of S for any $a_k \in S_k^*$, to which the result follows by Proposition 2.3. Indeed, let ϕ be an automorphism of S such that $a_k \phi = a_l$ for some k, l . Then, by Corollary 5.14, there exists a bijection π of A with $S_k \phi = S_{k\pi} = S_l$ as required. \square

Our interest is now in attaining a converse to the proposition above, since it would provide us with a method for building ‘new’ \aleph_0 -categorical Rees matrix semigroups from ‘old’. With the aid of Lemma 3.4, we shall prove that a converse exists in the class of Rees matrix semigroups over finite groups. The case where the maximal subgroups are infinite is an open problem.

Given a pair $S = \mathcal{M}^0[G; I, \Lambda; P]$ and $S' = \mathcal{M}^0[G; I', \Lambda'; Q]$ of Rees matrix semigroups over a group G , we denote $\text{Iso}(S; S')(1_G)$ as the set of isomorphisms between S and S' with trivial induced group isomorphism. That is, $\text{Iso}(S; S')(1_G)$ is the subset of $\text{Iso}(S; S')$ given by

$$\{\phi : \exists \psi \in \text{Iso}(\Gamma(P); \Gamma(Q)) \text{ and } u_i, v_\lambda \in G \text{ such that } \phi = (1_G, \psi, (u_i)_{i \in I}, (v_\lambda)_{\lambda \in \Lambda})\}.$$

If $S = S'$ we denote this simply as $\text{Aut}(S)(1_G)$, and notice that $\text{Aut}(S)(1_G)$ is a subgroup of $\text{Aut}(S)$ by Corollary 5.11.

Lemma 5.16 *Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a Rees matrix semigroup over a finite group G . Then S is \aleph_0 -categorical if and only if S is \aleph_0 -categorical over $\text{Aut}(S)(1_G)$.*

Proof Let S be \aleph_0 -categorical with $G = \{g_1, \dots, g_r\}$ finite. Let \bar{a} and \bar{b} be a pair of n -tuples of S . For some fixed $p_{\mu,j} \neq 0$, let \bar{g} be the r -tuple of S given by $\bar{g} = ((j, g_1, \mu), \dots, (j, g_r, \mu))$, and suppose that $(\bar{a}, \bar{g}) \sim_{S, n+r} (\bar{b}, \bar{g})$ via $\phi = (\theta, \psi, (u_i)_{i \in I}, (v_\lambda)_{\lambda \in \Lambda})$, say. Then, for each $1 \leq k \leq r$, we have

$$(j, g_k, \mu)\phi = (j\psi, u_j(g_k\theta)v_\mu, \mu\psi) = (j, g_k, \mu),$$

so that $g_k\theta = u_j^{-1}g_kv_\mu^{-1}$. For each $i \in I, \lambda \in \Lambda$, let $\bar{u}_i = u_iu_j^{-1}$ and $\bar{v}_\lambda = v_\mu^{-1}v_\lambda$. Then

$$\begin{aligned} (i\psi, \bar{u}_i g_k \bar{v}_\lambda, \lambda\psi) &= (i\psi, (u_i u_j^{-1}) g_k (v_\mu^{-1} v_\lambda), \lambda\psi) \\ &= (i\psi, u_i (g_k \theta) v_\lambda, \lambda\psi) \\ &= (i, g_k, \lambda)\phi, \end{aligned}$$

for any $(i, g_k, \lambda) \in S$, so that $\phi = (1_G, \psi, (\bar{u}_i)_{i \in I}, (\bar{v}_\lambda)_{\lambda \in \Lambda}) \in \text{Aut}(S)(1_G)$. Consequently, $(\bar{a}, \bar{g}) \sim_{S, \text{Aut}(S)(1_G), n+r} (\bar{b}, \bar{g})$ and in particular $\bar{a} \sim_{S, \text{Aut}(S)(1_G), n} \bar{b}$. We have thus shown that

$$|S^n / \sim_{S, \text{Aut}(S)(1_G), n}| \leq |S^{n+r} / \sim_{S, n+r}| < \aleph_0,$$

as S is \aleph_0 -categorical. Hence S is \aleph_0 -categorical over $\text{Aut}(S)(1_G)$.

The converse is immediate. □

We are now able to prove our desired converse to Proposition 5.15 in the case where the maximal subgroups are finite.

Theorem 5.17 *Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a Rees matrix semigroup such that G is finite. Then S is \aleph_0 -categorical if and only if each connected Rees component of S is \aleph_0 -categorical and S has only finitely many connected Rees components, up to isomorphism.*

Proof (\Rightarrow) Immediate from Proposition 5.15.

(\Leftarrow) Since S is regular with finite maximal subgroups, to prove S is \aleph_0 -categorical, it suffices by [9, Corollary 3.14] to show that $|E(S)^n / \sim_{S, n}|$ is finite, for each $n \in \mathbb{N}$. Let $\{S_k : k \in A\}$ be the set of connected Rees components of S , which is finite up to isomorphism and with each S_k being \aleph_0 -categorical. Define a relation η on A by $i \eta j$ if and only if $\text{Iso}(S_i; S_j)(1_G) \neq \emptyset$. By Corollary 5.11 we have that η is an equivalence relation.

We first prove that A/η is finite. Suppose for contradiction that there exists an infinite set X of pairwise η -inequivalent elements of A . Since S has finitely many connected components up to isomorphism, there exists an infinite subset $\{i_r : r \in \mathbb{N}\}$ of X such that $S_{i_n} \cong S_{i_m}$ for each n, m . Fix an isomorphism $\phi_{i_n} : S_{i_n} \rightarrow S_{i_1}$ for each $n \in \mathbb{N}$. Then as $\text{Aut}(G)$ is finite there exists distinct n, m such that $\phi_{i_n}^G = \phi_{i_m}^G$, and so $\phi_{i_n} \phi_{i_m}^{-1} \in \text{Iso}(S_{i_n}; S_{i_m})(1_G)$ by Corollary 5.11. Hence $i_n \eta i_m$, a contradiction, and so A/η is finite.

Let $S' = \bigcup_{k \in A} S_k$, noting that S' is the 0-direct union of the S_k , and in particular is a subsemigroup of S . Let $A/\eta = \{A_1, \dots, A_r\}$ and set $\underline{A} = (A; A_1, \dots, A_r)$. For each $i, j \in A$, let $\Psi_{i,j} = \text{Iso}(S_i; S_j)(1_G)$ and fix $\Psi = \bigcup_{i,j \in A} \Psi_{i,j}$. We prove that $\{S_k : k \in A\}$ forms an $(S; S'; \underline{A}; \Psi)$ -system in S . First, by our construction, if $i, j \in A_m$ for some m then $\Psi_{i,j} \neq \emptyset$, and so Ψ satisfies Condition (3.1). Furthermore, it follows immediately from Corollary 5.11 that Ψ satisfies Conditions (3.2) and (3.3). Finally, take any $\pi \in \text{Aut}(\underline{A})$ and, for each $k \in A$, let $\phi_k \in \Psi_{k, k\pi}$. Then as $\phi_k^G = 1_G$ for each $k \in A$, we may construct an automorphism ϕ of S from the set of isomorphisms $\{\phi_k : k \in A\}$ by Corollary 5.14. Hence, as ϕ extends each ϕ_k by construction, we have that $\{S_k : k \in A\}$ forms an $(S; S'; \underline{A}; \Psi)$ -system as required. Since S_k is \aleph_0 -categorical, it is \aleph_0 -categorical over $\Psi_{k,k} = \text{Aut}(S_k)(1_G)$ by Lemma 5.16. By Corollary 2.9 \underline{A} is \aleph_0 -categorical, and so

$$|(S')^n / \sim_{S, n}| < \aleph_0$$

by Lemma 3.4. Given that $E(S) \subseteq S'$ by (5.2), we therefore have that

$$|E(S)^n / \sim_{S,n} | \leq |(S')^n / \sim_{S',n} | < \aleph_0.$$

Hence S is \aleph_0 -categorical. □

Open Problem 5.18 Does Theorem 5.17 hold if G is allowed to be any \aleph_0 -categorical group?

5.2 Labelled bipartite graphs

In Example 5.13, the problem which arose was that by shifting from the sandwich matrix $P = (p_{\lambda,i})$ to the induced bipartite graph $\Gamma(P)$ we have “forgotten” the value of the entries $p_{\lambda,i}$. In this subsection we extend the construction of the induced bipartite graph of a Rees matrix semigroup to attempt to rectifying this problem, as well as to build classes of \aleph_0 -categorical Rees matrix semigroups. Further examples of \aleph_0 -categorical Rees matrix semigroups can then be built using Theorem 5.17.

Definition 5.19 Let $\Gamma = \langle L, R, E \rangle$ be a bipartite graph, Σ a set, and $f : E \rightarrow \Sigma$ a surjective map. Then the triple (Γ, Σ, f) is called a Σ -labeled (by f) bipartite graph, which we denote as Γ^f .

A pair of Σ -labeled bipartite graphs $\Gamma^f = (\Gamma, \Sigma, f)$ and $\Gamma^{f'} = (\Gamma', \Sigma, f')$ are *isomorphic* if there exists an isomorphism $\psi : \Gamma \rightarrow \Gamma'$ which preserves labels, that is, such that

$$\{x, y\}f = \sigma \Leftrightarrow \{x\psi, y\psi\}f' = \sigma.$$

This gives rise to a natural signature in which to consider Σ -labeled bipartite graphs as follows. For each $\sigma \in \Sigma$, take a binary relation symbol E_σ and let

$$L_{BG\Sigma} = L_{BG} \cup \{E_\sigma : \sigma \in \Sigma\}.$$

Then we call $L_{BG\Sigma}$ the *signature of Σ -labeled bipartite graphs*, where $(x, y) \in E_\sigma$ if and only if $\{x, y\} \in E$ and $\{x, y\}f = \sigma$.

Let Γ^f be a Σ -labeled bipartite graph. Then for any set Σ' and bijection $g : \Sigma \rightarrow \Sigma'$, we can form a Σ' -labeling of Γ simply by taking Γ^{fg} , which we call a *relabeling of Γ^f* . Notice that if ψ is an automorphism of Γ , then $\psi \in \text{Aut}(\Gamma^f)$ if and only if $\psi \in \text{Aut}(\Gamma^{fg})$. Indeed, if $\psi \in \text{Aut}(\Gamma^f)$ then for any edge $\{x, y\}$ of Γ we have

$$\{x, y\}fg = \sigma' \Leftrightarrow \{x, y\}f = \sigma'g^{-1} \Leftrightarrow \{x\psi, y\psi\}f = \sigma'g^{-1} \Leftrightarrow \{x\psi, y\psi\}fg = \sigma',$$

since g is a bijection. The converse is proven similarly, and the following result is then immediate from the RNT.

Lemma 5.20 *Let Γ^f be a Σ -labeling of a bipartite graph Γ . Then Γ^f is \aleph_0 -categorical if and only if any relabeling of Γ^f is \aleph_0 -categorical.*

Lemma 5.21 *If $\Gamma^f = (\Gamma, \Sigma, f)$ is an \aleph_0 -categorical labeled bipartite graph then Σ is finite and Γ is \aleph_0 -categorical.*

Proof For each $\sigma \in \Sigma$, let $\{x_\sigma, y_\sigma\}$ be an edge in Γ such that $\{x_\sigma, y_\sigma\}f = \sigma$. Then $\{(x_\sigma, y_\sigma) : \sigma \in \Sigma\}$ is a set of distinct 2-automorphism types of Γ^f , and so Σ is finite by the RNT. Since automorphisms of Γ^f induce automorphisms of Γ , the final result is immediate from the RNT. \square

A consequence of the previous pair of lemmas is that, in the context of \aleph_0 -categoricity, it suffices to consider finitely labeled bipartite graphs, with labeling set $\mathbf{m} = \{1, 2, \dots, m\}$ for some $m \in \mathbb{N}$.

Lemma 5.22 *Let $\Gamma^f = (\langle L, R, E \rangle, \mathbf{m}, f)$ be an \mathbf{m} -labeled bipartite graph such that either L or R are finite. Then Γ^f is \aleph_0 -categorical.*

Proof Without loss of generality assume that $L = \{l_1, l_2, \dots, l_r\}$ is finite. Define a relation τ on R by $y \tau y'$ if and only if y and y' are adjacent to the same elements in L and $\{l_i, y\}f = \{l_i, y'\}f$ for each such $l_i \in L$. Note that since both L and \mathbf{m} are finite, R has finitely many τ -classes, say R_1, \dots, R_r . Considering R simply as a set, fix $\mathcal{A} = (R; R_1, \dots, R_r)$.

Since L is finite, to prove the \aleph_0 -categoricity of Γ^f it suffices to show that $(\Gamma^f \setminus L)^n = R^n$ has finitely many $\sim_{\Gamma^f, n}$ -classes for each $n \in \mathbb{N}$ by a simple generalization of Gould and Quinn-Gregson [9, Proposition 2.11]. Let $\bar{a} = (r_1, \dots, r_n)$ and $\bar{b} = (r'_1, \dots, r'_n)$ be n -tuples of R such that $\bar{a} \sim_{\mathcal{A}, n} \bar{b}$ via $\psi \in \text{Aut}(\mathcal{A})$, say. We claim that the map $\hat{\psi} : \Gamma^f \rightarrow \Gamma^f$ which fixes L and is such that $\hat{\psi}|_R = \psi$ is an automorphism of Γ^f . Indeed, as ψ setwise fixes the τ -classes, we have $(r, r\psi) \in \tau$ for each $r \in R$. Hence r and $r\psi$ are adjacent to the same elements in L , and so

$$\{l_i, r\} \in E \Leftrightarrow \{l_i, r\psi\} \in E \Leftrightarrow \{l_i \hat{\psi}, r\hat{\psi}\} \in E,$$

so that $\hat{\psi}$ is an automorphism of Γ . Similarly $\{l_i, r\}f = \{l_i, r\psi\}f = \{l_i \hat{\psi}, r\hat{\psi}\}f$, so that $\hat{\psi}$ preserves labels. This proves the claim.

For each $1 \leq k \leq n$ we have $r_k \hat{\psi} = r_k \psi = r'_k$, so that $\bar{a} \sim_{\Gamma^f, n} \bar{b}$. Consequently,

$$|(\Gamma^f \setminus L)^n / \sim_{\Gamma^f, n}| \leq |\mathcal{A}^n / \sim_{\mathcal{A}, n}|.$$

The set extension \mathcal{A} is \aleph_0 -categorical by Corollary 2.9, and so $|\mathcal{A}^n / \sim_{\mathcal{A}, n}|$ is finite for each $n \geq 1$. Hence Γ^f is \aleph_0 -categorical. \square

Lemma 5.23 *Let $\Gamma^f = (\langle L, R, E \rangle, \mathbf{m}, f)$ be such that there exists $p \in \mathbf{m}$ with $\{x, y\}f = p$ for all but finitely many edges in Γ . Then Γ^f is \aleph_0 -categorical if and only if Γ is \aleph_0 -categorical.*

Proof Suppose Γ is \aleph_0 -categorical, and that $\{l_1, r_1\}, \dots, \{l_t, r_t\}$ are precisely the edges of Γ such that $\{l_k, r_k\}f \neq p$, where $l_k \in L$ and $r_k \in R$. Let \bar{a} and \bar{b} be n -tuples of Γ^f such that

$$(\bar{a}, l_1, r_1, \dots, l_t, r_t) \sim_{\Gamma, n+2t} (\bar{b}, l_1, r_1, \dots, l_t, r_t)$$

via $\psi \in \text{Aut}(\Gamma)$, say. We claim that ψ is an automorphism of Γ^f . For each $1 \leq k \leq t$ we have $l_k\psi = l_k$ and $r_k\psi = r_k$ so that

$$\{l_k, r_k\}f = \{l_k\psi, r_k\psi\}f.$$

It follows that $\{l, r\}f = p$ if and only if $\{l\psi, r\psi\}f = p$, and so ψ preserves all labels, thus proving the claim. Consequently, $\bar{a} \sim_{\Gamma^f, n} \bar{b}$ via ψ , so that

$$|(\Gamma^f)^n / \sim_{\Gamma^f, n}| \leq |\Gamma^{n+2t} / \sim_{\Gamma, n+2t}| < \aleph_0$$

by the \aleph_0 -categoricity of Γ . Hence Γ^f is \aleph_0 -categorical.

The converse is immediate from Lemma 5.21. □

Definition 5.24 Given a Rees matrix semigroup $S = \mathcal{M}^0[G; I, \Lambda; P]$, we form a $G(P)$ -labeling of the induced bipartite graph $\Gamma(P) = \langle I, \Lambda, E \rangle$ of S in the natural way by taking the labeling $f : E \rightarrow G(P)$ given by

$$\{i, \lambda\}f = p_{\lambda, i}.$$

We denote the labeled bipartite graph by $\Gamma(P)^l$, which we call the *induced labeled bipartite graph* of S .

Note that, unlike the corresponding case for the induced bipartite graph $\Gamma(P)$, there exist isomorphic Rees matrix semigroups with non-isomorphic induced labeled bipartite graphs. For example, let G be a non-trivial group and P and Q be 1×2 matrices over $G \cup \{0\}$ given by

$$P = \begin{pmatrix} 1 & a \end{pmatrix} \quad Q = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

where $a \notin \{0, 1\}$. Let $S = \mathcal{M}^0[G; \mathbf{2}, \mathbf{1}; P]$ and $T = \mathcal{M}^0[G; \mathbf{2}, \mathbf{1}; Q]$, noting that $\Gamma(P) = \Gamma(Q)$ (and are isomorphic to $K_{2,1}$). Then $(1_G, 1_{\Gamma(P)}, (u_i)_{i \in \mathbf{2}}, (v_\lambda)_{\lambda \in \mathbf{1}})$ is an isomorphism from S to T , where $u_1 = 1 = v_1$, and $u_2 = a$. However, since $\Gamma(P)^l$ and $\Gamma(Q)^l$ have different labeling sets, they are not isomorphic.

Proposition 5.25 *Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a Rees matrix semigroup such that G and $\Gamma(P)^l$ are \aleph_0 -categorical. Then S is \aleph_0 -categorical.*

Proof Since $\Gamma(P)^l$ is \aleph_0 -categorical, the set $G(P)$ is finite by Lemma 5.21, say $G(P) = \{x_1, \dots, x_r\}$. Consider a pair of n -tuples $\bar{a} = ((i_1, g_1, \lambda_1), \dots, (i_n, g_n, \lambda_n))$ and $\bar{b} = ((j_1, h_1, \mu_1), \dots, (j_n, h_n, \mu_n))$ of S^* under the pair of conditions that

- (1) $(g_1, \dots, g_n, x_1, \dots, x_r) \sim_{G, n+r} (h_1, \dots, h_n, x_1, \dots, x_r)$,
- (2) $\Gamma(\bar{a}) \sim_{\Gamma(P)^l, 2n} \Gamma(\bar{b})$,

via $\theta \in \text{Aut}(G)$ and $\psi \in \text{Aut}(\Gamma(P)^l)$, respectively (noting the use of Notation 5.9 here). We claim that $\phi = (\theta, \psi, (1)_{i \in I}, (1)_{\lambda \in \Lambda})$ is an automorphism of

S . Indeed, if $p_{\lambda,i} \neq 0$ for some $i \in I, \lambda \in \Lambda$, then $p_{\lambda,i} = x_k$ for some k , so that $\{i, \lambda\}f = \{i\psi, \lambda\psi\}f = x_k$. Consequently,

$$p_{\lambda,i}\theta = x_k\theta = x_k = p_{\lambda\psi,i\psi},$$

and the claim follows by Theorem 5.10. Hence

$$(i_t, g_t, \lambda_t)\phi = (i_t\psi, g_t\theta, \lambda_t\psi) = (j_t, h_t, \mu_t)$$

for each $1 \leq t \leq n$, so that

$$|(S^*)^n / \sim_{S,n}| \leq |G^{n+r} / \sim_{G,n+r}| \cdot |(\Gamma(P)^l)^{2n} / \sim_{\Gamma(P)^l, 2n}| < \aleph_0,$$

as G and $\Gamma(P)^l$ are \aleph_0 -categorical. Hence S is \aleph_0 -categorical by Gould and Quinn-Gregson [9, Proposition 2.11]. \square

The proposition above enables us to produce concrete examples of \aleph_0 -categorical Rees matrix semigroups. For example, the result below is immediate from Lemma 5.22.

Corollary 5.26 *Let S be a Rees matrix semigroup over an \aleph_0 -categorical group having sandwich matrix P with finitely many rows or columns, and $G(P)$ being finite. Then S is \aleph_0 -categorical.*

Similarly, Lemma 5.23 may be used in conjunction with Proposition 5.25 to obtain:

Corollary 5.27 *Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a Rees matrix semigroup such that G and $\Gamma(P)$ are \aleph_0 -categorical, and all but finitely many of the non-zero entries of P are the identity of G . Then S is \aleph_0 -categorical.*

However, the converse to Proposition 5.25 fails to hold in general, and a counterexample will be constructed later in the next subsection. The idea is that any $\mathcal{M}^0[G; I, \Lambda; P]$ in which $G(P)$ is infinite forces $\Gamma(P)^l$ to be non \aleph_0 -categorical by Lemma 5.25.

Open Problem 5.28 Does there exist an \aleph_0 -categorical connected Rees matrix semigroup with $G(P)$ finite which is not isomorphic to a Rees matrix semigroup with \aleph_0 -categorical induced labeled bipartite graph?

We prove that the open problem has a negative answer for the case of completely simple semigroups. Given a completely simple semigroup $\mathcal{M}[G; I, \Lambda; P]$, we call P *normal* if there exist $i \in I$ and $\lambda \in \Lambda$ such that $p_{\mu,i} = p_{\lambda,j} = 1$ for every $j \in I$ and $\mu \in \Lambda$. Every completely simple semigroup is isomorphic to a Rees matrix semigroup without zero in which the sandwich matrix is normal [17].

Proposition 5.29 *Let $\mathcal{M}[G;I, \Lambda;P]$ be an \aleph_0 -categorical completely simple semigroup in which P is normal and $G(P)$ is finite. Then $\Gamma(P)^l$ is \aleph_0 -categorical.*

Proof Suppose P is normalised via $i^* \in I$ and $\lambda^* \in \Lambda$. Since $G(P)$ is finite we may fix some finite subsets $I' = \{x_1, \dots, x_p\} \subseteq I$ and $\Lambda' = \{y_1, \dots, y_q\} \subseteq \Lambda$ such that the $\Lambda' \times I'$ submatrix of P contains every element of $G(P)$. Let \bar{x} be the pq -tuple of S given by

$$((x_1, 1, y_1), (x_1, 1, y_2), \dots, (x_1, 1, y_q), (x_2, 1, y_1), \dots, (x_p, 1, y_q)),$$

Using the notation of Proposition 5.12, let $\bar{a} = (a_1, \dots, a_n)$ and $\bar{b} = (b_1, \dots, b_n)$ be a pair of $\sigma_{\Gamma(P)^l, n}$ -related n -tuples of $\Gamma(P)^l$. Let $i_1 < i_2 < \dots < i_s$ and $j_1 < j_2 < \dots < j_t$ be the indexes of entries of \bar{a} (and thus \bar{b}) lying in I and Λ , respectively. Suppose further that there exists $i \in I$ and $\lambda \in \Lambda$ such that the $n + pq + 1$ -tuples

$$\begin{aligned} &((a_{i_1}, 1, \lambda), \dots, (a_{i_s}, 1, \lambda), (i, 1, a_{j_1}), \dots, (i, 1, a_{j_t}), \bar{x}, (i^*, 1, \lambda^*)) \text{ and} \\ &((b_{i_1}, 1, \lambda), \dots, (b_{i_s}, 1, \lambda), (i, 1, b_{j_1}), \dots, (i, 1, b_{j_t}), \bar{x}, (i^*, 1, \lambda^*)) \end{aligned}$$

are automorphically equivalent via $\phi = [\theta, \psi, (u_i)_{i \in I}, (v_\lambda)_{\lambda \in \Lambda}] \in \text{Aut}(S)$, say. Then ψ is an automorphism of $\Gamma(P)$ which maps \bar{a} to \bar{b} . We aim to show that ψ preserves labels, i.e., $p_{\lambda, i} = p_{\lambda\psi, i\psi}$ for every $i \in I, \lambda \in \Lambda$. Since ϕ fixes $(i^*, 1, \lambda^*)$ we have by Gould and Quinn-Gregson [24, Corollary 4.9] that there exists $g \in G$ with $u_i = g$ and $v_\lambda = g^{-1}$ for every $i \in I, \lambda \in \Lambda$. Since ψ fixes $x_1, \dots, x_p, y_1, \dots, y_q$ we have

$$P_{y_k x_\ell} \theta = v_{y_k} P_{y_k \psi x_\ell} \psi u_{x_\ell} = g^{-1} P_{y_k x_\ell} g.$$

Consequently, as every $p_{\lambda, i}$ is equal to some $P_{y_k x_\ell}$, we have $p_{\lambda, i} \theta = g^{-1} p_{\lambda, i} g$ for every $i \in I, \lambda \in \Lambda$. However, $p_{\lambda, i} \theta = v_\lambda P_{\lambda \psi, i\psi} u_i = g^{-1} p_{\lambda \psi, i\psi} g$, and hence ψ preserves labels as required. We have thus shown that

$$|\Gamma(P)^l|^n / \sim_{\Gamma(P)^l, n} \leq |S^{n+pq+1}| / \sim_{S, n+pq+1} < \aleph_0$$

as S is \aleph_0 -categorical. □

The sandwich matrix P of a Rees matrix semigroup $\mathcal{M}^0[G;I, \Lambda;P]$ can also always be normalised, but it is necessarily more complex. We can restate Open Problem 5.28 as follows:

Open Problem 5.30 *If $\mathcal{M}^0[G;I, \Lambda;P]$ is \aleph_0 -categorical, where P is normal and $G(P)$ is finite, then is $\Gamma(P)^l$ \aleph_0 -categorical?*

Notice that in Example 5.13, the labeled bipartite graph is clearly not \aleph_0 -categorical since each i_k is adjacent to exactly k vertices in which the edge is labeled by a . By construction the matrix P is normal via row α_0 and column i_0 , and hence S is not \aleph_0 -categorical by the proposition above.

5.3 Pure completely 0-semigroups

Following Jackson and Volkov [18], we call a completely 0-simple semigroup S *pure* if it is isomorphic to a Rees matrix semigroup with sandwich matrix over $\{0, 1\}$. Houghton [15] considered *trivial cohomology classes* of Rees matrix semigroups, a property which is proven in Sect. 2 of his article to be equivalent to being pure. Hence, by Houghton [15, Theorem 5.1], a completely 0-simple semigroup is pure if and only if, for each $a, b \in S$,

$$[a, b \in \langle E(S) \rangle \text{ and } a \mathcal{H} b] \Rightarrow a = b.$$

It follows that all orthodox completely 0-simple semigroups are necessarily pure, but the converse is not true in general. Indeed, a completely 0-simple semigroup is orthodox if and only if it is isomorphic to a Rees matrix semigroup with sandwich matrix over $\{0, 1\}$ and with induced bipartite graph a disjoint union of complete bipartite graphs [13, Theorem 6]. Hence, in this case, it can be easily shown that the isomorphism types of the connected Rees components depends only on the isomorphism types of the induced (complete) bipartite graphs.

We observe that if the sandwich matrix of a Rees matrix semigroup is over $\{0, 1\}$ then $\Gamma(P)^l$ is simply labeled by $\{1\}$. Therefore all automorphisms of $\Gamma(P)$ automatically preserve the labeling, and so $\Gamma(P)^l$ is \aleph_0 -categorical if and only if $\Gamma(P)$ is \aleph_0 -categorical. The equivalence of statements (1), (3), and (4) in the result below therefore follow from Propositions 5.12 and 5.25. For the interest of the reader we give an alternative proof of (4) \Rightarrow (1) using results in [9].

Lemma 5.31 *Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a pure Rees matrix semigroup. Then the following are equivalent:*

- (1) S is \aleph_0 -categorical;
- (2) G and $\langle E(S) \rangle$ are \aleph_0 -categorical;
- (3) G and $\Gamma(P)$ are \aleph_0 -categorical;
- (4) G and $\mathcal{M}^0[\{1\}; I, \Lambda; P]$ are \aleph_0 -categorical.

Proof (1) \Rightarrow (2) If S is \aleph_0 -categorical then so is G by Proposition 5.12. Clearly $E(S)$ is preserved by automorphisms of S , and hence $\langle E(S) \rangle$ is a characteristic subsemigroup of S , and thus inherits \aleph_0 -categoricity.

(2) \Rightarrow (3) Suppose that $\langle E(S) \rangle = \langle \{(i, 1, \lambda) : p_{\lambda, i} \neq 0\} \cup \{0\} \rangle$ is \aleph_0 -categorical. Let $S_k = \mathcal{M}^0[G; I_k, \Lambda_k; P_k]$ ($k \in A$) be the connected Rees components of S , where P_k is the $\Lambda_k \times I_k$ submatrix of P . Then $\langle E(S) \rangle$ is isomorphic to the 0-direct union of the semigroups $E_k = \langle E(S_k) \rangle$, and since each P_k is regular it is a simple exercise to show that $E_k = \mathcal{M}^0[\{1\}; I_k, \Lambda_k; P_k]$. By Gould and Quinn-Gregson [9, Corollary 4.9] $\langle E(S) \rangle$ is \aleph_0 -categorical if and only if each E_k is \aleph_0 -categorical and $\{E_k : k \in A\}$ is finite, up to isomorphism. By Proposition 5.12 each $\Gamma(P_k)$ is \aleph_0 -categorical, and by Theorem 5.10 $\mathcal{C}(\Gamma(P)) = \{\Gamma(P_k) : k \in A\}$ is finite, up to isomorphism. Hence $\Gamma(P)$ is \aleph_0 -categorical by Proposition 5.5.

(3) \Rightarrow (4) Immediate from Corollary 5.27.

(4) \Rightarrow (1) The elements of the combinatorial Rees matrix semigroup $T = \mathcal{M}^0[\{1\}; I, \Lambda; P]$ can be identified² with the set $(I \times \Lambda) \cup \{0\}$. Since \aleph_0 -categoricity is preserved by finite direct products [12], the semigroup $U = G \times T$ is \aleph_0 -categorical. The set $I = \{(g, 0) : g \in G\}$ is an ideal of U , and the Rees quotient U/I is a principal factor of U . Hence U/I is \aleph_0 -categorical by Gould and Quinn-Gregson [9, Theorem 3.12]. Moreover, the map $\phi : U/I \rightarrow S$ given by $0\phi = 0$ and $(g, (i, \lambda))\phi = (i, g, \lambda)$ ($g \in G, i \in I, \lambda \in \Lambda$) is an isomorphism, to which the result follows. □

Furthermore, since complete bipartite graphs are \aleph_0 -categorical by Theorem 5.3, a disjoint union of complete bipartite graphs is \aleph_0 -categorical if and only if it has finitely many connected components, up to isomorphism, by Proposition 5.5. The corollary above thus reduces in the orthodox case as follows.

Corollary 5.32 *Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be an orthodox Rees matrix semigroup. Then the following are equivalent:*

- (1) S is \aleph_0 -categorical;
- (2) G and $E(S)$ are \aleph_0 -categorical;
- (3) G is \aleph_0 -categorical and $\Gamma(P)$ has finitely many connected components, up to isomorphism;
- (4) G and $\mathcal{M}^0[\{1\}; I, \Lambda; P]$ are \aleph_0 -categorical.

In [9] we studied inverse completely 0-simple semigroups, that is, Brandt semigroups. These are necessarily orthodox, and are isomorphic to a Rees matrix semigroup of the form $\mathcal{M}^0[G; I, I; P]$ where P is the identity matrix, that is, $p_{ii} = 1$ and $p_{ij} = 0$ for each $i \neq j$ in I , and are denoted $\mathcal{B}^0[G; I]$. Since the induced bipartite graph of a Brandt semigroup is a perfect matching, it is \aleph_0 -categorical by Theorem 5.3. Corollary 5.32 then simplifies to obtain our classification of \aleph_0 -categorical Brandt semigroups [9, Theorem 4.2], which states that a Brandt semigroup over a group G is \aleph_0 -categorical if and only if G is \aleph_0 -categorical.

We are now able to construct a simple counterexample to the converse of Proposition 5.25. Let $G = \{g_i : i \in \mathbb{N}\}$ be an infinite \aleph_0 -categorical group. Let

$$S = \mathcal{M}^0[G; \mathbb{N}, \mathbb{N}; P] = \mathcal{B}^0[G; \mathbb{N}] \text{ and } T = \mathcal{M}^0[G; \mathbb{N}, \mathbb{N}; Q],$$

where $Q = (q_{i,j})$ is such that $q_{i,i} = g_i$ and $q_{i,j} = 0$ for each $i \neq j$. Then $\Gamma(P) = \Gamma(Q)$ (and are isomorphic to $P_{\mathbb{N}}$) and $(1_G, 1_{\Gamma(P)}, (g_i^{-1})_{i \in \mathbb{N}}, (1)_{\lambda \in \mathbb{N}})$ is an isomorphism from S to T by Theorem 5.10 since

$$p_{i,i}1_G = 1 = g_i g_i^{-1} = 1 \cdot q_{i,i} \cdot g_i^{-1},$$

² Semigroups of this form are known as *rectangular 0-bands*.

for each $i \in \mathbb{N}$. Since S is \aleph_0 -categorical by the \aleph_0 -categoricity of G , the same is true of T . However, $\Gamma(Q)^i$ is a G -labeling, and is thus not \aleph_0 -categorical by Lemma 5.21. Hence T is our desired counterexample.

5.4 Alternative directions

To further incorporate the link between the induced bipartite graph of a Rees matrix semigroup and the entries of the sandwich matrix, we could instead introduce the stronger notion of an *induced group labeled bipartite graph*. A group labeled bipartite graph is a G -labeled bipartite graph $\Gamma^f = ((L, R, E), G, f)$, for some group G , where an automorphism of Γ^f is a pair $(\psi, \theta) \in \text{Aut}(\Gamma) \times \text{Aut}(G)$ such that, for each $\ell \in L, r \in R$,

$$(\ell, r)f = g \Leftrightarrow (\ell\psi, r\psi)f = g\theta.$$

However, group labeled bipartite graphs do not appear to be first-order structures.

Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be such that $G(P)$ forms a subgroup of G . Then we may define the *induced group labeled bipartite graph* of S as the $G(P)$ -labeled bipartite graph $\Gamma(P)^f$, with automorphisms being pairs $(\psi, \theta) \in \text{Aut}(\Gamma) \times \text{Aut}(G(P))$ such that $p_{\lambda\psi, i\psi} = p_{\lambda, i}\theta$ for each $i \in I, \lambda \in \Lambda$. Notice that if (ψ, θ) is an automorphism of the induced group labeled bipartite graph of S and is such that θ extends to an automorphism θ' of G , then $(\theta', \psi, (1)_{i \in I}, (1)_{\lambda \in \Lambda})$ is clearly an automorphism of S . However, we do not in general obtain all automorphisms of S in this way. Similar problems therefore arise in regard to when \aleph_0 -categoricity of S passes to its induced group labeled bipartite graph (by which we mean the induced group labeled bipartite graph has an oligomorphic automorphism group).

An alternative next step could be to extend the scope of this section by considering the \aleph_0 -categoricity of Rees matrix semigroups over semigroups (or monoids), denoted $\mathcal{M}^0[S; I, \Lambda; P]$, where again we assume P is regular. Similarly we may define $\mathcal{M}[S; I, \Lambda; P]$. However, this task is as difficult as considering the \aleph_0 -categoricity of all semigroups. Indeed, if S is a semigroup then $T = \mathcal{M}^0[S^1; \{i\}, \{\lambda\}; (1)]$ is isomorphic to S with both a zero and an identity adjoined, and by Gould and Quinn-Gregson [9, Corollary 2.12] S is \aleph_0 -categorical if and only if T is \aleph_0 -categorical. A second problem that arises is that the vital Theorem 5.10 only holds in the forwards direction for Rees matrix semigroups over semigroups. As such we do not have an explicit description of the automorphism group of $\mathcal{M}^0[S; I, \Lambda; P]$ via its components, and many of the proofs of this section do not seem to be easily extendable. In fact the \aleph_0 -categoricity of a Rees matrix semigroup over a semigroup S does not necessarily pass to S , unlike for groups as shown in Proposition 5.12. For example, take any semigroup S with zero element e , and consider $M = \mathcal{M}[S; \{i\}, \{\lambda\}; (e)]$. Then M is isomorphic to a null semigroup with zero element (i, e, λ) , which is \aleph_0 -categorical by Gould and Quinn-Gregson [9, Example 2.7]; taking S to be non \aleph_0 -categorical gives our desired example. On the other hand, it can be easily shown that Proposition 5.25 can be extended to Rees matrix semigroups over monoids. This allows us to build chains of \aleph_0 -categorical semigroups as follows. Let M be an \aleph_0 -categorical monoid, and let P be a $\Lambda \times I$ matrix over $\{0, 1\}$ in which $\Gamma(P)$ is \aleph_0 -categorical. Take

$M_1 = \mathcal{M}^0[M; I, \Lambda; P]$, and inductively define $M_k = \mathcal{M}^0[M_{k-1}^1; I, \Lambda; P]$ for $k > 1$. Then each M_k is \aleph_0 -categorical, and M_{k-1} embeds into M_k , for each $k \in \mathbb{N}$.

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