Correction to: Davenport constant for semigroups

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Introduction

The Davenport constant of a finite abelian group G , denoted $D(G)$, is defined as the smallest positive integer ℓ such that every sequence of terms from G of length at least ℓ contains one or more terms whose product is the identity element of G . This invariant was popularized by H. Davenport in the 1960's, notably for its significance in algebraic number theory (as mentioned in [5]). Though attributed to H. Davenport, who proposed the study of this constant in 1965, Rogers [6] in 1963 pioneered the investigation of this combinatorial invariant. This invariant together with the celebrated EGZ Theorem obtained by Erdős et al. [2] in 1961 led to the emergence of a branch in additive group theory, called zero-sum theory (see [3] for a survey). In 2008, the author and Gao [9] formulated the definition of the Davenport constant for finite commutative semigroups.

Definition A [9] Let S be a finite commutative semigroup with identity. Let T be a sequence of terms from the semigroup S . We call T reducible if T contains a proper subsequence T' ($T' \neq T$) such that the sum of all terms of T' equals the sum of all terms of T . Define the *Davenport constant of the semigroup* S , denoted $D(S)$, to be the smallest positive integer ℓ such that every sequence T of length at least ℓ of terms from S is reducible.

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The above definition can be also applied to infinite commutative semigroups with little modification, see [8].

Regretfully, one result in [9] (Theorem B below) on the Davenport constant for the multiplicative semigroup of a direct product of residue class rings has turned out to be incorrect. On the other hand, since the definition was proposed, several results (see [1, 7, 8, 10, 11]) have been obtained on the Davenport constant of commutative semigroups. In the present paper, we will fix this mistake by proving a weaker result (Theorem 1) by a different method.

Theorem B [9] *For integers $r \geq 1$, $n_1, \dots, n_r > 1$, let $R = \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_r}$. Let $\mathcal{S}_R$ be the multiplicative semigroup of the ring R and $U(R)$ be the group of units of R . Then $D(\mathcal{S}_R) - D(U(R)) = \#\{i \in [1, r] : 2 \parallel n_i\}$.*

Theorem 1 *For integers $r \geq 1$, $n_1, \dots, n_r > 1$, let $R = \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_r}$. Then*

$$\#\{i \in [1, r] : 2 \parallel n_i\} \leq D(\mathcal{S}_R) - D(U(R)) \leq \#\{i \in [1, r] : 2 \mid n_i\}.$$

Preliminaries

For integers $m, n \in \mathbb{Z}$, we set $[m, n] = \{d \in \mathbb{Z} : m \leq d \leq n\}$. For a prime p and an integer n , let $v_p(n)$ be the p -adic valuation of n , i.e., the largest integer k such that p^k divides n if $n \neq 0$, and $v_p(0) = \infty$.

- In the rest of this paper, we always let $\mathcal{S}$ be a finite commutative semigroup with identity $0_{\mathcal{S}}$.

The operation on $\mathcal{S}$ is denoted as $+$. Let $U(\mathcal{S}) = \{a \in \mathcal{S} : a + a' = 0_{\mathcal{S}} \text{ for some } a' \in \mathcal{S}\}$ be the group of units of $\mathcal{S}$. For any element $c \in \mathcal{S}$, let

$$\text{St}(c) = \{a \in U(\mathcal{S}) : a + c = c\}$$

denote the stabilizer of c in the group $U(\mathcal{S})$. Green's preorder on the semigroup $\mathcal{S}$, denoted $\leq_{\mathcal{H}}$, is defined by

$$a \leq_{\mathcal{H}} b \Leftrightarrow a = b + c \text{ for some } c \in \mathcal{S}.$$

Green's congruence on $\mathcal{S}$, denoted $\mathcal{H}$, is defined by:

$$a \mathcal{H} b \Leftrightarrow a \leq_{\mathcal{H}} b \text{ and } b \leq_{\mathcal{H}} a.$$

We write $a <_{\mathcal{H}} b$ to mean that $a \leq_{\mathcal{H}} b$ but $a \mathcal{H} b$ does not hold.

A sequence T of terms from $\mathcal{S}$ is denoted by $T = a_1 a_2 \cdot \dots \cdot a_{\ell} = \bullet_{a \in \mathcal{S}} a^{[v_a(T)]}$, where $v_a(T)$ is a nonnegative integer and means that the element a occurs $v_a(T)$ times in the sequence T . By $\cdot$ we denote the operation of sequence concatenation. By

$|T|$ we denote the length of the sequence, i.e., $|T| = \sum_{a \in \mathcal{S}} v_a(T) = \ell$. By ε we denote the empty sequence with $|\varepsilon| = 0$. We call T' a subsequence of T if $v_a(T') \leq v_a(T)$ for each element $a \in \mathcal{S}$, denoted by $T' \mid T$, moreover, we write $T'' = T \cdot T'^{[-1]}$ to mean the unique subsequence of T with $T' \cdot T'' = T$. We call T' a proper subsequence of T provided that $T' \mid T$ and $T' \neq T$. In particular, the empty sequence ε is a proper subsequence of every nonempty sequence. Let $\sigma(T) = a_1 + \cdots + a_\ell$ be the sum of all terms from T . In particular, $\sigma(\varepsilon) = 0_{\mathcal{S}}$. We call a sequence T reducible if $\sigma(T') = \sigma(T)$ for some proper subsequence T' of T , and irreducible otherwise.

In what follows, we put

$$R = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$$

where $r \geq 1, n_1, \dots, n_r > 1$ are integers, and denote by $\mathcal{S}_R$ the multiplicative semigroup of the ring R . For any element $x \in \mathcal{S}_R$, we denote by $\theta_x = (x_1, \dots, x_r) \in [1, n_1] \times \cdots \times [1, n_r]$ the unique r -tuple of integers such that $(\bar{x}_1, \dots, \bar{x}_r)$ is the corresponding form of the element x in the ring R . Let $\kappa_i(\theta_x) = x_i$ where $i \in [1, r]$. As stated above, the operation of the semigroup $\mathcal{S}_R$ will be denoted $+$ instead of the notation $*$ used for the multiplication of the ring R . Then we observe that, for any elements a, b and c of $\mathcal{S}_R$,

$$a = b + c \text{ holds in } \mathcal{S}_R \Leftrightarrow \kappa_i(\theta_a) \equiv \kappa_i(\theta_b) * \kappa_i(\theta_c) \pmod{n_i} \text{ for each } i \in [1, r]. \quad (1)$$

- Note that the group $U(R)$ of units of the ring R coincides with the group $U(\mathcal{S}_R)$, and that the semigroup $\mathcal{S}_R$ consists of all elements of R , including the zero element $0_R = (\bar{n}_1, \dots, \bar{n}_r)$ of R . In [9], we had a slip of the pen and wrote ‘let $R^\times = R \setminus \{0\}$ be the multiplicative semigroup of R .’

To prove Theorem 1, we will need the following two lemmas.

Lemma 1 ([4], Lemma 6.1.3) *Let G be a finite abelian group, and let H be a subgroup of G . Then, $D(G) \geq D(G/H) + D(H) - 1$.*

Lemma 2 *Let a and b be two elements of $\mathcal{S}_R$. Then the following conclusions hold:*

(i) *If $a \leq_{\mathcal{H}} b$, then $\gcd(\kappa_i(\theta_b), n_i) \mid \gcd(\kappa_i(\theta_a), n_i)$ for each $i \in [1, r]$, and $\text{St}(b) \subseteq \text{St}(a)$.*

(ii) *$a \mathcal{H} b$ if and only if $\gcd(\kappa_i(\theta_b), n_i) = \gcd(\kappa_i(\theta_a), n_i)$ for each $i \in [1, r]$.*

(iii) *Suppose $a <_{\mathcal{H}} b$. If there exists some index $t \in [1, r]$ such that*

$$v_p(\gcd(\kappa_t(\theta_b), n_t)) < v_p(\gcd(\kappa_t(\theta_a), n_t)) \text{ for some odd prime } p$$

or

$$v_2(\gcd(\kappa_t(\theta_b), n_t)) < v_2(\gcd(\kappa_t(\theta_a), n_t)) < v_2(n_t),$$

then $\text{St}(b) \subsetneq \text{St}(a)$.

Proof (i) Since $a \leq_{\mathcal{H}} b$, we have

$$a = b + c \quad \text{for some } c \in \mathcal{S}_R. \quad (2)$$

By (1) and (2), we derive that $\kappa_i(\theta_a) \equiv \kappa_i(\theta_b) * \kappa_i(\theta_c) \pmod{n_i}$, which implies

$$\gcd(\kappa_i(\theta_b), n_i) \mid \gcd(\kappa_i(\theta_a), n_i)$$

for each $i \in [1, r]$. The fact $\text{St}(b) \subseteq \text{St}(a)$ follows from (2) readily.

(ii) The necessity follows from Conclusion (i) immediately. We need only to show the sufficiency. Assume $\gcd(\kappa_i(\theta_b), n_i) = \gcd(\kappa_i(\theta_a), n_i)$ for each $i \in [1, r]$. Note that there exist integers $c_i, c'_i \in [1, n_i]$ such that

$$\kappa_i(\theta_a) * c_i \equiv \kappa_i(\theta_b) \pmod{n_i} \quad \text{and} \quad \kappa_i(\theta_b) * c'_i \equiv \kappa_i(\theta_a) \pmod{n_i} \quad (3)$$

where $i \in [1, r]$. Take elements c and c' of $\mathcal{S}_R$ such that $\kappa_i(\theta_c) = c_i$ and $\kappa_i(\theta_{c'}) = c'_i$ for each $i \in [1, r]$. It follows from (1) and (3) that $a + c = b$ and $b + c' = a$, and so $a \mathcal{H} b$.

(iii) Let q be the largest prime with

$$v_q(\gcd(\kappa_t(\theta_b), n_t)) < v_q(\gcd(\kappa_t(\theta_a), n_t)). \quad (4)$$

Let

$$h = v_q(\gcd(\kappa_t(\theta_a), n_t)). \quad (5)$$

Take an element $d \in \mathcal{S}_R$ with

$$\kappa_i(\theta_d) = 1 \quad \text{for each } i \in [1, r] \setminus \{t\}, \quad (6)$$

and with

$$\kappa_t(\theta_d) \equiv \begin{cases} 2 \frac{n_t}{q^h} + 1 \pmod{n_t} & \text{if } q > 2 \text{ and } \gcd\left(2 \frac{n_t}{q^h} + 1, n_t\right) = 1; \\ \frac{n_t}{q^h} + 1 \pmod{n_t} & \text{otherwise.} \end{cases} \quad (7)$$

Then we have the following.

Claim A $d \in \text{U}(\mathcal{S}_R)$.

Proof of Claim A By (6), it suffices to establish that $\gcd(\kappa_t(\theta_d), n_t) = 1$.

Suppose $q = 2$. Then $\kappa_t(\theta_d) \equiv \frac{n_t}{q^h} + 1 \pmod{n_t}$ and thus

$$\gcd\left(\kappa_t(\theta_d), \frac{n_t}{q^h}\right) = \gcd\left(\frac{n_t}{q^h} + 1, \frac{n_t}{q^h}\right) = 1. \quad (8)$$

By the maximality of q , we have $v_2(\gcd(\kappa_t(\theta_d), n_t)) < v_2(n_t)$. Combined with (5), we have that $q \mid \frac{n_t}{q^h}$ and so $\gcd(\kappa_t(\theta_d), q) = \gcd(\frac{n_t}{q^h} + 1, q) = 1$. Combined with (8), we have that $\gcd(\kappa_t(\theta_d), n_t) = 1$ when $q = 2$.

Suppose $q > 2$. By (7), it remains to show that $\gcd(\frac{n_t}{q^h} + 1, n_t) = 1$ or $\gcd(2\frac{n_t}{q^h} + 1, n_t) = 1$. Assume to the contrary that $\gcd(\frac{n_t}{q^h} + 1, n_t) > 1$ and $\gcd(2\frac{n_t}{q^h} + 1, n_t) > 1$. Since $w \nmid \frac{n_t}{q^h} + 1$ and $w \nmid 2\frac{n_t}{q^h} + 1$ for every prime divisor w of n_t with $w \neq q$, it follows that $q \mid \frac{n_t}{q^h} + 1$ and $q \mid 2\frac{n_t}{q^h} + 1$. It follows that $q \mid 2(\frac{n_t}{q^h} + 1) - (2\frac{n_t}{q^h} + 1) = 1$, which is absurd. This proves Claim A. $\square$

By (4), (5), (6) and (7), we check that $\kappa_i(\theta_d) * \kappa_i(\theta_a) \equiv \kappa_i(\theta_a) \pmod{n_i}$ for each $i \in [1, r]$, and $\kappa_i(\theta_d) * \kappa_i(\theta_b) \not\equiv \kappa_i(\theta_b) \pmod{n_i}$. By (1), we have that $d + a = a$ and $d + b \neq b$. Combined with Claim A, we have $d \in \text{St}(a) \setminus \text{St}(b)$. Combined with (i), we have $\text{St}(b) \subsetneq \text{St}(a)$, completing the proof. $\square$

The proof of Theorem 1

Proof of Theorem 1 Recall that the group $U(R)$ coincides with the group $U(\mathcal{S}_R)$ and so $D(U(R)) = D(U(\mathcal{S}_R))$. Denote $P_2 = \#\{i \in [1, r] : 2 \parallel n_i\}$ and $\delta = \#\{i \in [1, r] : 2 \mid n_i\}$.

We first show that $D(\mathcal{S}_R) \geq D(U(\mathcal{S}_R)) + P_2$. Assume without loss of generality that

$$\{i \in [1, r] : 2 \parallel n_i\} = [1, P_2]. \quad (9)$$

Take an irreducible sequence A of terms from $U(\mathcal{S}_R)$ of length $D(U(\mathcal{S}_R)) - 1$. Let

$$B = A \cdot \left(\begin{smallmatrix} \bullet \\ i \in [1, P_2] \end{smallmatrix} b_i \right) \quad (10)$$

where $\theta_{b_i} = (1, \dots, 1, 2, 1, \dots, 1)$ with 2 appearing at the i -th component for each $i \in [1, P_2]$. Now we show that the sequence B is irreducible. Suppose to the contrary that B contains a proper subsequence B' with

$$\sigma(B') = \sigma(B). \quad (11)$$

Since $\sigma(B') \mathcal{H} \sigma(B)$ and $\gcd(\kappa_i(\theta_x), n_i) = 1$ for any term x of A and any $i \in [1, r]$, it follows from Lemma 2 (ii) that $\left(\begin{smallmatrix} \bullet \\ i \in [1, P_2] \end{smallmatrix} b_i \right) \mid B'$, say

$$B' = A' \cdot \left(\begin{smallmatrix} \bullet \\ i \in [1, P_2] \end{smallmatrix} b_i \right) \quad (12)$$

where A' is a proper subsequence of A . By (10), (11) and (12), we derive that

$$2\kappa_i(\theta_{\sigma(A')}) \equiv \kappa_i(\theta_{\sigma(B')}) = \kappa_i(\theta_{\sigma(B)}) \equiv 2\kappa_i(\theta_{\sigma(A)}) \pmod{n_i} \quad \text{for each } i \in [1, P_2] \quad (13)$$

and

$$\kappa_j(\theta_{\sigma(A')}) = \kappa_j(\theta_{\sigma(B')}) = \kappa_j(\theta_{\sigma(B)}) = \kappa_j(\theta_{\sigma(A)}) \quad \text{for each } j \in [P_2 + 1, r]. \quad (14)$$

Since both $\kappa_i(\theta_{\sigma(A')})$ and $\kappa_i(\theta_{\sigma(A)})$ are odd, it follows from (9) and (13) that $\kappa_i(\theta_{\sigma(A')}) \equiv \kappa_i(\theta_{\sigma(A)}) \pmod{n_i}$, where $i \in [1, P_2]$. Combined with (14), we have that $\sigma(A') = \sigma(A)$, which is a contradiction. This proves that B is irreducible, and so $D(\mathcal{S}_R) \geq |B| + 1 = D(U(\mathcal{S}_R)) + P_2$.

It remains to show that $D(\mathcal{S}_R) \leq D(U(\mathcal{S}_R)) + \delta$. Let $T = a_1 \cdot a_2 \cdot \dots \cdot a_\ell$ be an arbitrary sequence of terms from the semigroup $\mathcal{S}_R$ of length $\ell = D(U(\mathcal{S}_R)) + \delta$. It suffices to show that T contains a *proper* subsequence T' with $\sigma(T') = \sigma(T)$. Take a shortest subsequence V of T such that

$$\sigma(V) \mathcal{H} \sigma(T). \quad (15)$$

We may assume without loss of generality that

$$V = a_1 \cdot a_2 \cdot \dots \cdot a_t \quad \text{where } t \in [0, \ell].$$

If $t = 0$, i.e., $V = \varepsilon$, then $\sigma(V) = 0_{\mathcal{S}_R}$, which implies that $\sigma(T) \in U(\mathcal{S}_R)$ by (15). It follows that T is a sequence of terms from the group $U(\mathcal{S}_R)$ and of length $|T| = D(U(\mathcal{S}_R)) + \delta \geq D(U(\mathcal{S}_R))$, and thus, T is reducible, and we are done. Hence, we assume that

$$t > 0.$$

By the minimality of $|V|$, we derive that

$$0_{\mathcal{S}_R} >_{\mathcal{H}} a_1 >_{\mathcal{H}} (a_1 + a_2) >_{\mathcal{H}} \dots >_{\mathcal{H}} \sum_{i=1}^t a_i. \quad (16)$$

Recall that an empty sum of elements of $\mathcal{S}_R$ is taken equal to $0_{\mathcal{S}_R}$. Denote $K_i = \text{St}(\sum_{j=1}^i a_j)$ where $i \in [0, t]$. Note that K_i is a subgroup of $U(\mathcal{S}_R)$ for each $i \in [0, t]$. Combined with (16) and Lemma 2 (i), we have that

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots \subseteq K_t. \quad (17)$$

Moreover, we have the following. □

Claim B *There exists a subset $M \subseteq [0, t - 1]$ with $|M| \geq t - \delta$ such that $K_i \subsetneq K_{i+1}$ for each $i \in M$.*

Proof of Claim B Let $v \in [0, t - 1]$ be an arbitrary index with $K_v = K_{v+1}$. Let $\gamma = \sum_{i=1}^{v+1} a_i$ and $\beta = \sum_{i=1}^v a_i$. Since $\gamma <_{\mathcal{H}} \beta$, it follows from Lemma 2 that $\gcd(\kappa_j(\theta_\beta), n_j) \mid \gcd(\kappa_j(\theta_\gamma), n_j)$ for each $j \in [1, r]$, and moreover,

$$v_2(\gcd(\kappa_w(\theta_\beta), n_w)) < v_2(\gcd(\kappa_w(\theta_\gamma), n_w)) = v_2(n_w)$$

for some $w \in [1, r]$. Since the choice of the index $v \in [0, t - 1]$ is arbitrary, we have proved Claim B. $\square$

For each $m \in M$, since $\frac{U(\mathcal{S}_R)}{K_{m+1}} \cong \frac{U(\mathcal{S}_R)/K_m}{K_{m+1}/K_m}$ and $D(K_{m+1}/K_m) \geq 2$, it follows from Lemma 1 that

$$\begin{aligned} D(U(\mathcal{S}_R)/K_{m+1}) &= D\left(\frac{U(\mathcal{S}_R)/K_m}{K_{m+1}/K_m}\right) \\ &\leq D(U(\mathcal{S}_R)/K_m) - (D(K_{m+1}/K_m) - 1) \\ &\leq D(U(\mathcal{S}_R)/K_m) - 1. \end{aligned} \quad (18)$$

By (17), (18) and Claim B, we conclude that

$$\begin{aligned} 1 \leq D(U(\mathcal{S}_R)/K_t) &\leq D(U(\mathcal{S}_R)/K_0) - |M| \\ &\leq D(U(\mathcal{S}_R)) - (t - \delta) \\ &= (\ell - \delta) - (t - \delta) \\ &= \ell - t \\ &= |T \cdot V^{[-1]}|. \end{aligned} \quad (19)$$

By (15) and Lemma 2 (ii), we have

$$\gcd(\kappa_i(\theta_{\sigma(V)}), n_i) = \gcd(\kappa_i(\theta_{\sigma(T)}), n_i) \quad \text{for each } i \in [1, r]. \quad (20)$$

Let

$$\mathcal{P}_i = \{p : p \text{ is prime with } v_p(\gcd(\kappa_i(\theta_{\sigma(V)}), n_i) = v_p(n_i) > 0\} \quad \text{for each } i \in [1, r].$$

Let x be an arbitrary term of $T \cdot V^{[-1]}$. By (20), we have that, for each $i \in [1, r]$,

$$q \nmid \kappa_i(\theta_x) \quad \text{for any prime divisor } q \text{ of } n_i \text{ with } q \notin \mathcal{P}_i. \quad (21)$$

By the Chinese Remainder Theorem, we can choose integers $\tilde{x}_i \in [1, n_i]$ such that

$$\tilde{x}_i \equiv 1 \pmod{p^{v_p(n_i)}} \quad \text{for any prime } p \in \mathcal{P}_i \quad (22)$$

and

$$\tilde{x}_i \equiv \kappa_i(\theta_x) \pmod{q^{v_q(n_i)}} \quad \text{for any prime divisor } q \text{ of } n_i \text{ with } q \notin \mathcal{P}_i, \quad (23)$$

where $i \in [1, r]$. Let $\tilde{x}$ be the element of $\mathcal{S}_R$ with $\kappa_i(\theta_{\tilde{x}}) = \tilde{x}_i$ for each $i \in [1, r]$. By (21), (22) and (23), we conclude that $\gcd(\kappa_i(\theta_{\tilde{x}}), n_i) = 1$ for each $i \in [1, r]$, i.e.,

$$\tilde{x} \in U(\mathcal{S}_R). \quad (24)$$

It follows that

$$\tilde{x}_i * \kappa_i(\theta_{\sigma(V)}) \equiv \tilde{x}_i * 0 \equiv 0 \equiv \kappa_i(\theta_x) * 0 \equiv \kappa_i(\theta_x) * \kappa_i(\theta_{\sigma(V)}) \pmod{p^{v_p(n_i)}}$$

for any prime $p \in \mathcal{P}_i$, and now (23) implies that

$$\tilde{x}_i * \kappa_i(\theta_{\sigma(V)}) \equiv \kappa_i(\theta_x) * \kappa_i(\theta_{\sigma(V)}) \pmod{q^{v_q(n_i)}}$$

for any prime divisor q of n_i with $q \notin \mathcal{P}_i$, that is,

$$\sigma(V) + \tilde{x} = \sigma(V) + x. \quad (25)$$

Since the choice of the above term $x \mid T \cdot V^{[-1]}$ is arbitrary, it follows from (19) and (24) that $\bullet_{x \mid T \cdot V^{[-1]}} \tilde{x}$ is a nonempty sequence of terms from $U(\mathcal{S}_R)$ of length $|\bullet_{x \mid T \cdot V^{[-1]}} \tilde{x}| = |T \cdot V^{[-1]}| \geq D(U(\mathcal{S}_R)/K_i)$. It follows that there exists a *nonempty* subsequence $W \mid T \cdot V^{[-1]}$ such that $\sigma(\bullet_{x \mid W} \tilde{x}) \in K_i$, which implies

$$\sigma(V) + \sigma(\bullet_{x \mid W} \tilde{x}) = \sigma(V). \quad (26)$$

By (25) and (26), we conclude that

$$\begin{aligned} \sigma(T) &= \sigma(T \cdot W^{[-1]} \cdot V^{[-1]}) + (\sigma(V) + \sigma(W)) \\ &= \sigma(T \cdot W^{[-1]} \cdot V^{[-1]}) + (\sigma(V) + \sigma(\bullet_{x \mid W} \tilde{x})) \\ &= \sigma(T \cdot W^{[-1]} \cdot V^{[-1]}) + \sigma(V) \\ &= \sigma(T \cdot W^{[-1]}), \end{aligned}$$

and $T' = T \cdot W^{[-1]}$ is the desired proper subsequence of T . This completes the proof of the theorem. $\square$

Concluding remarks

If $v_2(n_i) \in \{0, 1\}$ for each $i \in [1, r]$, then Theorem 1 gives the equality

$$D(\mathcal{S}_R) - D(U(R)) = \#\{i \in [1, r] : 2 \parallel n_i\} = \#\{i \in [1, r] : 2 \mid n_i\}.$$

Otherwise, we remark that the upper bound in Theorem 1 is attainable in some cases. For example, take $R = \mathbb{Z}_8^{r_1} \oplus \mathbb{Z}_4^{r_2} \oplus \mathbb{Z}_2^{r_3}$ where $r_1, r_2, r_3 \geq 0$ and $r = r_1 + r_2 + r_3 > 0$. Note that $U(R) \cong C_2^{2r_1+r_2}$ is the direct sum of $2r_1 + r_2$ cyclic groups of order two. It is well known that $D(U(R)) = D(C_2^{2r_1+r_2}) = 2r_1 + r_2 + 1$ (see [3], Theorem 3.1). For each $i \in [1, r]$, we take $a_i \in \mathcal{S}_R$ with $\theta_{a_i} = (1, \dots, 1, 2, 1, \dots, 1)$ where the only 2 occurs at the i -th component. It is not hard to check that $T = (\bullet_{i \in [1, r_1]} a_i^{[3]}) \cdot (\bullet_{j \in [r_1+1, r_1+r_2]} a_j^{[2]}) \cdot (\bullet_{k \in [r_1+r_2+1, r]} a_k)$ is an irreducible sequence of terms from the semigroup $\mathcal{S}_R$. It follows that $D(\mathcal{S}_R) \geq |T| + 1 = 3r_1 + 2r_2 + r_3 + 1 = D(U(R)) + r$, and thus, $D(\mathcal{S}_R) - D(U(R)) = \#\{i \in [1, r] : 2 \mid n_i\}$. So it would be interesting to determine the precise value of $D(\mathcal{S}_R) - D(U(R))$. For some progress on this problem, we refer the reader to [1].

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