Some results on $\mathcal{L}$ -commutative semigroups

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Abstract

We prove first that every $\mathcal{H}$ -commutative semigroup is stable. Using this result [and some results from the standard text (Nagy, Special classes of semigroups, Kluwer, Dordrecht, 2001)], we give two equivalent conditions for a semigroup to be an archimedean $\mathcal{H}$ -commutative semigroup containing an idempotent element. It turns out that this result can be partially extended to $\mathcal{L}$ -commutative semigroups and quasi-commutative semigroups.

Keywords $\mathcal{H}$ -commutative semigroup · $\mathcal{L}$ -commutative semigroup · Quasi-commutative semigroup · Archimedean semigroup · Stable semigroup · Nil-extension

1 Introduction and preliminaries

For undefined terminology of semigroup theory, the reader is referred to the books [3, 8, 9].

One of the most important decompositions in the theory of semigroups is a decomposition of a semigroup (induced by a semilattice congruence) into archimedean semigroups. In this field, one of the biggest contributions is due to Putcha, see [9, Chapter 2] (recall that a semigroup is a Putcha semigroup if and only if it is a semilattice of archimedean semigroups). Therefore, it is worth investigating certain classes of Putcha semigroups (especially, with idempotents), see, e.g., [9, Chapters 4–12]. On the other hand, many naturally arising semigroups are semilattice unions of archimedean semigroups. A good example is the semigroup of all $n \times n$ upper triangular matrices over a field, and more generally, the Zariski closure of any solvable linear algebraic group. Moreover, semilattices of archimedean semigroups

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arise naturally in the theory of finite semigroups and languages, where the pseudo-variety is denoted by **DS**.

Denote the set of idempotents of a semigroup S by E_S , and the set of its regular elements by $\text{Reg}(S)$.

Let S be a semigroup and let $A \subseteq S$. Put

$$\sqrt{A} = \{s \in S : (\exists n \in \mathbb{Z}_+)(s^n \in A)\}.$$

Also, for each $a \in S$, let us put as usual: $J(a) = S^1 a S^1$.

Recall that a semigroup S is *archimedean* if for every ideal I of S , we have

$$S = \sqrt{I}.$$

Remark 1 Observe that if in addition $E_S \neq \emptyset$, then E_S is contained in each ideal of S . Hence $\text{Reg}(S)$ is a subset of a kernel of S (by a *kernel* of a semigroup A we mean (if it exists) the least ideal of A ; this ideal is then denoted by K_A).

We say that a congruence ρ on a semigroup S is a *semilattice* congruence if the semigroup S/ρ is a semilattice. It is clear that the least semilattice congruence on an arbitrary semigroup always exists. Denote it by η . It is also clear (and well-known) that in any semigroup $\mathcal{J} \subseteq \eta$.

Let $\mathcal{C}$ be some fixed class of semigroups (call its elements *$\mathcal{C}$ -semigroups*). A semigroup S is said to be a *semilattice of $\mathcal{C}$ -semigroups* if there exists a semilattice congruence ρ on S such that every ρ -class of S is a $\mathcal{C}$ -semigroup (for example, S is a *semilattice of groups* if there is a semilattice congruence ρ on S such that every ρ -class is a group). One can modify this definition for any variety of bands.

The following important result is due to Putcha (see [9, Theorem 2.1]).

Result 1.1 *A semigroup S is a semilattice of archimedean semigroups if and only if it is a Putcha semigroup, i.e., $J(a) \subseteq \sqrt{J(a^2)}$ for all $a \in S$. In that case, the corresponding semilattice congruence is equal to η . Moreover,*

$$\eta = \{(a, b) \in S \times S : (\exists m, n \in \mathbb{Z}_+)(a^m \in SbS \text{ \& } b^n \in SaS)\}.$$

Recall that a semigroup is *unipotent* if it has exactly one idempotent, and that a *nil-semigroup* is just a (unipotent) semigroup with zero in which every element has a power equal to zero.

It is clear that for an arbitrary ideal I of a semigroup S , the relation

$$\rho_I = (I \times I) \cup 1_S,$$

where 1_S is the identity relation on S , is a congruence on S (called the *Rees congruence* induced by I). We denote the (Rees) quotient semigroup induced by this congruence by S/I . In that case, we also say that S is an *ideal extension* of the semigroup I by the semigroup S/I . Moreover, every extension of a semigroup S by its ideal I such that I is a retract of S is called a *retract ideal extension*.

Recall from [9] that a semigroup S is *$\mathcal{H}$ -commutative* if

$$(\forall a, b \in S)(\exists x \in S^1)(ab = bxa).$$

Also, S is:

(a) $\mathcal{R}$ -commutative if

$$(\forall a, b \in S)(\exists x \in S^1)(ab = bax);$$

(b) $\mathcal{L}$ -commutative if

$$(\forall a, b \in S)(\exists x \in S^1)(ab = xba).$$

Recall further from [2] that a semigroup S is *stable* if for all $a, b \in S$,

(a) $(Sa \subseteq Sab) \implies (Sa = Sab)$, and

(b) $(aS \subseteq baS) \implies (aS = baS)$.

We say that S is *left stable* if it meets the condition (a), and S is *right stable* if it satisfies the condition (b).

Recently in [1], the authors studied $\mathcal{H}$ -commutative semigroups. In particular, they proved the following result (see [1, Theorem 4.10]).

Result 1.2 *Every $\mathcal{H}$ -commutative archimedean semigroup containing an idempotent element is saturated.*

We prove first that every $\mathcal{H}$ -commutative semigroup is stable. Using this result (and some results from [9] and [2]), we give two equivalent conditions for an arbitrary semigroup to be an archimedean $\mathcal{H}$ -commutative semigroup containing an idempotent element. Also, a weaker such characterization for $\mathcal{L}$ -commutative semigroups is contained in Section 3. The last section brings some concluding remarks on $\mathcal{H}(\mathcal{L})$ -commutative regular semigroups and archimedean quasi-commutative semigroups.

Recall also that a semigroup S is *intra-regular* if $J(a) = J(a^2)$ for all $a \in S$. It is easily seen (and this is also well-known) that in this case, $\mathcal{J}$ is a semilattice congruence on S , whence $\eta = \mathcal{J}$ in any intra-regular semigroup. This implies that each principal factor J_a^0 is not null. Thus every $\mathcal{J}$ -class J_a of S is simple. In fact, the following well-known result of Croisot is valid (see [3, Theorem 4.4]).

Result 1.3 *A semigroup S is intra-regular if and only if it is the semilattice $S/\mathcal{J}$ of the simple semigroups J_a ($a \in S$).*

Recall from [8] that a semigroup is *completely regular* (i.e., a (disjoint) union of groups) if and only if it is a semilattice of completely simple semigroups.

Corollary 1.4 *Each intra-regular stable semigroup is completely regular.*

Proof Let S be an intra-regular stable semigroup and let $a \in S$. Then $a = xaay$ for some $x, y \in S^1$. If $x = 1$ and $y \neq 1$, then

$$a = aay = a(aay)y = aa^2y^2 \in Sa^2S.$$

The same conclusion we obtain in other cases. The statement of the corollary now follows from the above remark, Result 1.3 and [2, Corollary 1.1 (7)]. $\square$

Further, the following result will be very useful (see [9, Theorems 5.1–2]).

Result 1.5 *A semigroup is both $\mathcal{R}$ -commutative and $\mathcal{L}$ -commutative if and only if it is $\mathcal{H}$ -commutative.*

Also, a semigroup S is $\mathcal{H}(\mathcal{L})$ -commutative if and only if $\mathcal{H}(\mathcal{L})$ is a commutative congruence on S .

Finally, the following result has been proved independently in [1, Proposition 2.1] and in [6, Proposition 2.1].

Result 1.6 *The set of idempotents of an arbitrary $\mathcal{H}$ -commutative semigroup S is contained in the center of S . In particular, $E_S = \emptyset$ or E_S forms a semilattice.*

2 Some results on $\mathcal{H}$ -commutative semigroups

To begin, we have the following useful proposition.

Proposition 2.1 *The following three statements are valid:*

- (a) *Every $\mathcal{L}$ -commutative semigroup is left stable;*
- (b) *Every $\mathcal{R}$ -commutative semigroup is right stable;*
- (c) *Every $\mathcal{H}$ -commutative semigroup is stable.*

Proof

- (a) Let S be an $\mathcal{L}$ -commutative semigroup and let $Sa \subseteq Sab$, where $a, b \in S$. Take also an arbitrary $x \in Sab$. Then $x = (sa)b$ for some $s \in S$. As S is $\mathcal{L}$ -commutative, there is $y \in S^1$ such that $(sa)b = yb(sa) = (ybs)a$. Hence $x \in Sa$. Thus $Sa = Sab$, so S is left stable.
- (b) This follows from (a) by duality.
- (c) This follows from (a), (b) and the first sentence of Result 1.5.

$\square$

Remark 2 Let us note that the concept of stability considered in this article is stronger than that of [4, Section 6.6] (c.f. also [4, Exercise 1] for 6.6). Hence the above proposition holds under either interpretation.

Recall that a semigroup is *simple* if it has no proper ideals. It is clear that if a semigroup has a simple ideal, then it is its kernel. Note finally that a kernel of any semigroup is a simple semigroup.

Observe that any group is $\mathcal{H}$ -commutative.

Corollary 2.2 *The kernel of an arbitrary $\mathcal{H}$ -commutative semigroup S is a group.*

Proof Indeed, we know that S is stable. In the light of [2, Corollary 2.5], K_S is completely simple. On the other hand, by Result 1.6, K_S contains exactly one idempotent. Thus K_S is a group. $\square$

Recall from [8] that a semigroup S with zero (denoted by 0) is a *0-group* if $S \setminus \{0\}$ is a group.

Notice further that in an arbitrary $\mathcal{H}$ -commutative semigroup S , we have

$$aS = Sa$$

for all $a \in S$ ([1, Proposition 2.2], c.f. also [9, Lemma 5.2] and the first sentence of Result 1.5).

Proposition 2.3 *Every 0-simple semigroup S , in which for any $a \in S$:*

$$aS = Sa,$$

is a 0-group, provided that S has at least three elements.

Proof Let $a \in S \setminus \{0\}$. Then $aS = Sa$ is a *non-zero* ideal of S (otherwise, $\{0, a\}$ is a proper ideal of S). Hence $aS = Sa = S$ for all $a \in S \setminus \{0\}$. Thus S is a 0-group by [8, Proposition 1.1.1]. $\square$

Theorem 2.4 *The following conditions on a semigroup S are equivalent:*

- (a) S is an archimedean $\mathcal{H}$ -commutative semigroup with $E_S \neq \emptyset$;
- (b) S is a retract ideal extension of a group by a commutative nil-semigroup;
- (c) S is a subdirect product of a group and a commutative nil-semigroup.

Proof (a) $\implies$ (b). Indeed, by Remark 1 and Corollary 2.2, S has a kernel which is a group. It is clear that the semigroup S/K_S is an $\mathcal{H}$ -commutative nil-semigroup. As in any nil-semigroup, $\mathcal{H}$ is the identity relation, S/K_S is a commutative nil-semigroup (see the second sentence of Result 1.5). Finally, take any $a \in S$ and denote the identity of the group K_S by e . Then $ea, ae \in K_S$. Hence

$$ae = e(ae) = (ea)e = ea.$$

Thus the map $a \rightarrow ea$ ($a \in S$) is a retraction of S onto K_S , as required.

(b) $\implies$ (c). Let φ be a retraction of S onto its group kernel G . Denote the congruence induced by this retraction by ρ . Then

$$S/\rho \cong G, \quad \rho \cap \rho_G = 1_S.$$

Hence S is a subdirect product of the group G and the commutative nil-semigroup S/G .

(c) $\implies$ (a). Let S be a subdirect product of a group A with identity 1 and a commutative nil-semigroup X with zero 0 (we may assume that $S \subseteq A \times X$). Then there is $a \in A$ such that $(a, 0) \in S$. Also, $(a^{-1}, x) \in S$ for some $x \in X$. Hence $(1, 0) \in S$. Consequently,

$$A \cong A \times \{0\} \subseteq S$$

is a kernel of S . It is clear that S is archimedean. As $(a, x) \mathcal{H}^S (b, y)$ if and only if $a \mathcal{H}^A b$ and $x \mathcal{H}^X y$, $\mathcal{H}$ is a commutative congruence on S (since in any group Green's relation $\mathcal{H}$ is the universal relation and X is commutative). Thus (a) follows from the above and the second sentence of Result 1.5. $\square$

3 Some results on $\mathcal{L}$ -commutative semigroups

In this section, we give some equivalent condition for an $\mathcal{L}$ -commutative semigroup S to be an archimedean semigroup with $E_S \neq \emptyset$.

First of all, the following result will be useful.

Proposition 3.1 *In an arbitrary $\mathcal{L}$ -commutative semigroup S ,*

$$\mathcal{L} = \mathcal{D} = \mathcal{J}, \quad \mathcal{H} = \mathcal{R}.$$

Proof Indeed, let $a, b \in S$ and $a \neq b$. Suppose also that $a = xby$ for some $x, y \in S^1$. If $y = 1$ or $by = yb$, then $a \in Sb$; otherwise $by = zyb$ for some $z \in S$. Hence $a = xzyb \in Sb$. We may conclude that $\mathcal{J} = \mathcal{L}$. The rest of the result follows from the structure of the lattice of Green's relations. $\square$

We are now able to show the following result.

Theorem 3.2 *The following conditions on a semigroup S are equivalent:*

- (a) S is an archimedean $\mathcal{L}$ -commutative semigroup with $E_S \neq \emptyset$;
- (b) S is an ideal extension of a left group by a commutative nil-semigroup.

Proof (a) $\implies$ (b). Indeed, by Remark 1 and Proposition 3.1, S has a kernel which is a left group. It is clear that the semigroup S/K_S is $\mathcal{L}$ -commutative nil-semigroup. As in any nil-semigroup, $\mathcal{L}$ is the identity relation, S/K_S is a commutative nil-semigroup (see the second sentence of Result 1.5).

(b) $\implies$ (a). Let S be an ideal extension of a left group A by a commutative nil-semigroup B . As A is a simple ideal of S , $A = K_S$. This implies that S is archimedean. We show that S is $\mathcal{L}$ -commutative. For this, let $a, b \in S$. Suppose first that

$a, b \notin K_S$. If $ab \notin K_S$, then $ba \notin K_S$ and $ab = ba$ (as B is commutative); otherwise $ab, ba \in K_S$. As K_S is left simple $ab = xba$ for some $x \in K_S$. Hence S is $\mathcal{L}$ -commutative. On the other hand, if a or b belongs to K_S , then $ab, ba \in K_S$ and so (as above) S is $\mathcal{L}$ -commutative. $\square$

Remark 3 Observe that in any archimedean semigroup A with a kernel,

$$\mathcal{J} = (K_A \times K_A) \cup 1_A.$$

Thus the following statements are also valid:

S is an archimedean semigroup with $E_S \neq \emptyset$ and $\mathcal{L} = \mathcal{J}$ if and only if S is an ideal extension of a left group by a nil-semigroup.

S is an archimedean semigroup with $E_S \neq \emptyset$ and $\mathcal{H} = \mathcal{J}$ if and only if S is a retract ideal extension of a group by a nil-semigroup.

Some well-known facts on epigroups will be helpful. First of all, a semigroup S is an *epigroup* if for every $a \in S$ there is a positive integer n and $e \in E_S$ such that $a^n \in H_e$. In that case, denote by $g(a)$ the least positive integer n for which $a^n \in H_e$ and recall that $a^m \in H_e$ for all natural numbers $m \geq g(a)$. Finally, for any $e \in E_S$, the set

$$K_e = \{a \in S : a^{g(a)} \in H_e\}$$

is called the *unipotency class* of S generated by e . By Green's Theorem, $E_S \cap K_e = \{e\}$, and by above, $\{K_e : e \in E_S\}$ is a partition of S . Denote the equivalence relation induced by this partition by $\mathcal{K}$ and the only idempotent in K_a by e_a ($e_a = e_b$ if and only if $a \mathcal{K} b$).

The following example shows that there is an archimedean $\mathcal{L}$ -commutative semigroup S with $E_S \neq \emptyset$ which is *not* a *retract* extension of a left group by a commutative nil-semigroup.

Example 3.3 Indeed, consider the following well-known semigroup:

$$L_{3,1} = \langle a, f : a^2 f = a^2, fa = f^2 = f \rangle = \{e = a^2, f, g = af, a\}.$$

This semigroup has the following multiplication table:

$\cdot$	e	f	g	a
e	e	e	e	e
f	f	f	f	f
g	g	g	g	g
a	e	g	e	e

Clearly, $K_{L_{3,1}} = E_{L_{3,1}}$, so $L_{3,1}$ is archimedean. Observe also that $\mathcal{L}$ is a commutative congruence on $L_{3,1}$ since $L_{3,1}/\mathcal{L} = \{\{e, f, g\}, \{a\}\}$. Hence $L_{3,1}$ is $\mathcal{L}$ -commutative. On the other hand, $K_{L_{3,1}}$ is *not* a retract ideal of $L_{3,1}$ since $\mathcal{K}$ is *not* a congruence on $L_{3,1}$.

Further, we have the following theorem.

Theorem 3.4 *The following conditions on a semigroup S are equivalent:*

- (a) S is a retract ideal extension of a left group by a commutative nil-semigroup;
- (b) S is a subdirect product of a left group and a commutative nil-semigroup;
- (c) S is an $\mathcal{L}$ -commutative left zero band of unipotent epigroups and $\text{Reg}(S)$ is an ideal of S .

Proof (a) $\implies$ (b). This is (now) clear.

(b) $\implies$ (c). Let S be a subdirect product of a left group A and a commutative nil-semigroup X with zero 0 (we may assume that $S \subseteq A \times X$). Take further an arbitrary idempotent e of A . Then there exists $x \in X$ such that $(e, x) \in S$. Also, $x^n = 0$ for some positive integer n . Hence

$$(e, 0) = (e^n, x^n) = (e, x)^n \in S.$$

Thus $E_S \cong E_A$. As, $ae_a = a$ for any $a \in A$ and $\text{Reg}(X) = \{0\}$, we conclude that

$$A \cong A \times \{0\} = \text{Reg}(S) \subseteq S$$

is a kernel of S (since A is simple). It is now obvious that S is archimedean. Hence S is an epigroup. Finally, the proof that $\mathcal{L}$ is a commutative congruence on S is closely similar to the corresponding proof in Theorem 2.4. Consequently, S is $\mathcal{L}$ -commutative (Result 1.5).

Next, define the relation ρ on S by

$$(a, x) \rho (b, y) \iff a \mathcal{H}^A b.$$

Since $\mathcal{H}^A$ is a left zero band congruence on A , ρ is a left zero band congruence on S . Note also that $(a, x) \rho (e_a, 0)$. From the above, the definition of ρ and Green's Theorem follows that every ρ -class is unipotent epigroup (as $(a, x) \rho (a^n, x^n)$ for every positive integer n).

(c) $\implies$ (a). First of all, every idempotent of S is primitive. Hence, by [7, Theorem 2.5],

$$\text{Reg}(S) = K_S$$

is a completely simple semigroup. Also, E_S is a left zero semigroup. Indeed, (ρ is below the corresponding left zero congruence on S from the above condition (c)) $e \rho ef$ for all $e, f \in E_S$ and $ef \in \text{Reg}(S)$. Hence

$$(ef)^2 = (ef \cdot e)f = eff = ef = e.$$

It is now clear that S is a nil-extension of the left group K_S by the commutative nil-semigroup S/K_S . Finally, K_S is a retract ideal of S by [6, Theorem 3.3]. $\square$

Denote the least band congruence on an arbitrary semigroup by β and observe that if an epigroup is a band of unipotent semigroups, then the corresponding band congruence is equal to $\mathcal{K}$, i.e., $\mathcal{K} = \beta$ (as clearly $\mathcal{K} \subseteq \beta$ in any epigroup, whence $\beta = \mathcal{K}^\sharp$, here $\mathcal{K}^\sharp$ denotes the least congruence containing the relation $\mathcal{K}$).

We have shown above that an archimedean $\mathcal{L}$ -commutative semigroup S with $E_S \neq \emptyset$ is an *epigroup*. For brevity, any epigroup in which $\mathcal{K}$ is a congruence will be called a $\mathcal{K}$ -*epigroup*.

Corollary 3.5 *The following conditions on a semigroup S are equivalent:*

- (a) S is an archimedean $\mathcal{L}$ -commutative $\mathcal{K}$ -epigroup;
- (b) S is a retract ideal extension of a left group by a commutative nil-semigroup;
- (c) S is a subdirect product of a left group and a commutative nil-semigroup;
- (d) S is an $\mathcal{L}$ -commutative left zero band of unipotent epigroups and $\text{Reg}(S)$ is an ideal of S ;
- (e) S is an $\mathcal{L}$ -commutative left zero band of unipotent epigroups.

Proof (a) $\implies$ (b). We know that S is an ideal extension of the left group K_S by a commutative nil-semigroup. Further, define the map from S onto K_S by $a \rightarrow e_a a$. As E_S is a left zero band and $e_a a = ae_a$, we have $(a, b \in S)$:

$$(e_a a)(e_b b) = (ae_a)(e_b b) = a(e_a e_b)b = ae_a b = e_a ab = (e_a e_b)ab,$$

so S is a retract ideal extension (since $\mathcal{K}$ is a congruence on S) of the left group K_S by the commutative nil-semigroup S/K_S .

(b) $\implies$ (c) $\implies$ (d) $\implies$ (b). This follows from the above proofs.

(e) $\implies$ (a). Indeed, any idempotent of S is primitive. Hence, by [7, Theorem 2.5],

$$\text{Reg}(S) = K_S$$

is a completely simple semigroup. It is now obvious that S is an archimedean $\mathcal{K}$ -epigroup. $\square$

Remark 4 Let S be a left zero band of the semigroups S_i ($i \in I$). Then each S_i is a right ideal of S .

Moreover, we have the following proposition [see above conditions (d), (e)].

Proposition 3.6 *Let S be an $\mathcal{L}$ -commutative left zero band S/ρ of the semigroups $a\rho$ ($a \in S$). Then each ρ -class is an $\mathcal{L}$ -commutative right ideal of S .*

Proof Suppose that S is $\mathcal{L}$ -commutative and let $c \in S$; $a, b \in c\rho$. Then either $ab = ba$ or $ab = xba$ for some $x \in S$. As $ab = xba \rho x$, $x \in c\rho$. The statement now follows from the above remark. $\square$

Further, from the above two results we obtain the following corollary.

Corollary 3.7 *Every archimedean $\mathcal{L}$ -commutative $\mathcal{K}$ -epigroup S is a left zero band of $\mathcal{L}$ -commutative retract ideal extensions of a left group by a commutative nil-semigroup.*

Remark 5 One can modify Corollary 3.5 for an archimedean $\mathcal{K}$ -epigroup in which $\mathcal{L} = \mathcal{J}$ (see Remarks 3, 1).

4 Concluding remarks

We give a simpler proof of the following result (see [1, Theorem 2.6(a)]).

Result 4.1 *The following conditions on a semigroup S are equivalent:*

- (a) S is $\mathcal{H}$ -commutative and regular;
- (b) S is $\mathcal{H}$ -commutative and $S/\mathcal{H}$ is regular;
- (c) S is a semilattice of groups;
- (d) $\mathcal{H} = \eta$.

Proof (a) $\implies$ (b). This is clear.

(b) $\implies$ (c). As $T = S/\mathcal{H}$ is commutative and regular, T is a semilattice of groups. Thus T is a semilattice (since it is combinatorial). The thesis follows now from Green's Theorem.

(c) $\implies$ (d). In the light of Green's Theorem, it is clear (and well-known) that $\mathcal{H} \subseteq \eta$ (in an arbitrary semigroup). The converse is obvious.

(d) $\implies$ (a). Let $a, b \in S$. Again in a view of Green's Theorem, S is a semilattice of groups. Hence the elements ab, ba are contained in some subgroup of S (say G). Thus $ab = xba$ and $ab = bay$ for some $x, y \in G$. Consequently, (a) holds by the first sentence of Result 1.5. $\square$

Remark 6 Note that [1, Theorem 2.6 (b)] has been shown independently in [5, Corollary 3.5]. In fact, the author have proved the following more general result (see [5, Proposition 3.4]).

Result 4.2 *The set of regular elements of an arbitrary $\mathcal{R}$ -commutative semigroup S is either empty or a semilattice of right groups. In the latter case, E_S is a right regular band.*

Remark 7 [1, Theorem 2.10] is a particular case of Result 1.1 (above); c.f. [1, Proposition 2.2] and [9, Theorem 5.3] with its proof on page 71.

Furthermore, notice that in any Putcha semigroup (see again Result 1.1) with central idempotents, each η -class contains at most one idempotent (see [6, Propositions 2.2]).

Recall the well-known fact due to Munn that any simple epigroup is *completely simple* (i.e., it is simple and its idempotents are primitive), see [8, Theorem 3.2.11].

It is also well-known that a semigroup S is completely simple if and only if it is the rectangular band $S/\mathcal{H}$ of the groups $H_e (e \in E_S)$.

Using the above methods, we are able to prove the following theorem (c.f. [10, Theorem 3.39]).

Theorem 4.3 *The following conditions on a semigroup S are equivalent:*

- (a) S is an archimedean $\mathcal{K}$ -epigroup;
- (b) S is a retract ideal extension of a completely simple semigroup by a nil-semigroup;
- (c) S is a subdirect product of a completely simple semigroup and a nil-semigroup;
- (d) S is a rectangular band of unipotent epigroups and $\text{Reg}(S)$ is an ideal of S ;
- (e) S is a rectangular band of unipotent epigroups.

Proof (a) $\implies$ (b). By Remark 1, S has a kernel that contains $\text{Reg}(S)$. Let $a \in K_S$. Then the pseudo-inverse $(e_a a)^{-1}$ of a (see [10, Section 2.1]) belongs to K_S (since $e_a a \in H_{e_a}$). Hence K_S is a simple epigroup. Thus K_S is completely simple by the above remark. This implies that S/K_S is a nil-semigroup. Also, $\mathcal{K}$ is the least rectangular band congruence on S . Consequently, S is a retract ideal extension of a completely simple semigroup by a nil-semigroup (see [6, Theorem 3.3]).

(b) $\implies$ (c). This is clear.

(c) $\implies$ (d). The proof is closely similar to the proof of implication “(b) $\implies$ (c)” of Theorem 3.4.

(e) $\implies$ (a). The proof is essentially identical to the corresponding proof of Corollary 3.5. $\square$

Further, we have the following result (c.f. [1, Theorem 2.5 (b)]).

Proposition 4.4 *Let S be a semigroup and let $\mathcal{X} \in \{\mathcal{H}, \mathcal{L}, \mathcal{R}, \mathcal{J}\}$ be a congruence on S . Then $S/\mathcal{X}$ is $\mathcal{X}$ -trivial. Moreover, if ρ is any congruence on S such that S/ρ is $\mathcal{X}$ -trivial, then*

$$\mathcal{X} \subseteq \rho.$$

Proof Let $H_a \mathcal{H}^{S/\mathcal{H}} H_b$ ($a, b \in S$). As $\mathcal{H}$ is a congruence on S , there are elements $x, y \in S^1$ such that $a \mathcal{H} x b$ and $a \mathcal{H} b y$. This implies that $a \in S^1 b$ and $a \in b S^1$. By symmetry, we may conclude that $H_a = H_b$. The second part of the proposition is clear.

The proof of remaining cases is similar. $\square$

Remark 8 Let $\mathcal{X} \in \{\mathcal{H}, \mathcal{L}, \mathcal{R}, \mathcal{J}\}$ be a commutative congruence on a semigroup S . Then $S/\mathcal{X}$ is a commutative semigroup in which all Green’s relations are trivial.

Consider an arbitrary completely simple semigroup S and let x, y be arbitrary idempotents of S such that

$$xy = x.$$

Then $yx \in E_S$ and $yx = yxy \in ySy = H_y$. Thus

$$yx = y.$$

In fact, using [7, Theorem 2.5], one can show (as above) the following proposition (recall that a semigroup S is E -inverse if for every $a \in S$ there exists $x \in E_S$ such that $ax \in E_S$).

Proposition 4.5 *Let e be an arbitrary idempotent and let f be an arbitrary primitive idempotent of an E -inverse semigroup S . Then*

$$(ef = e) \implies (fe = f).$$

In that case, $e \mathcal{L} f$. In particular, $e \in K_S = J(f) = J_f$ and e is primitive.

Note that the dual of the above proposition is valid.

Moreover, we have the following theorem.

Theorem 4.6 *The following conditions on a semigroup S are equivalent:*

- (a) S is $\mathcal{L}$ -commutative and regular;
- (b) S is $\mathcal{L}$ -commutative, every η -class of S contains an idempotent and $S/\mathcal{L}$ is regular;
- (c) S is a semilattice of left groups;
- (d) $\mathcal{L} = \eta$ and every η -class of S contains an idempotent element;
- (e) S is completely regular and $efe = ef$ for all $e, f \in E_S$ (i.e., E_S is a left regular band);
- (f) S is completely regular and $efe = ef$ for each $e, f \in E_A$, where A is an arbitrary η -class of S ;
- (g) S is regular and $aS \subseteq Sa$ for any $a \in S$.

Proof (a) $\implies$ (b). This is clear.

(b) $\implies$ (d). As above, $S/\mathcal{L}$ is a semilattice of groups. Hence $S/\mathcal{L}$ is a semilattice (since it is combinatorial as $\mathcal{L}$ -trivial). Thus $\mathcal{L} = \eta$.

(d) $\implies$ (c). Indeed, S is the semilattice S/η of left simple semigroups (see [3, Section 4.1]). Consider an arbitrary η -class A of S . As A is left simple and contains an idempotent, A is a regular semigroup and E_A is a left zero semigroup. Consequently, A is a left group.

(c) $\implies$ (a). This is clear. We have just proved that the conditions (a), (b), (c), (d) are equivalent.

(c) $\implies$ (e). As any left group is a union of groups, S is completely regular. Finally, S is $\mathcal{L}$ -commutative, so $efe = ef$ for all $e, f \in E_S$ by the dual of Result 4.2.

(e) $\implies$ (f). This is trivial.

(f) $\implies$ (c). Let $e, f \in E_A$, where A is an arbitrary η -class of S (note that A is a completely simple semigroup). Then $ef, e \in E_A$ and $ef \cdot e = ef$. In the light of the second part of Remark 8, $e(ef) = e$, i.e., $ef = e$ for all $e, f \in E_A$. Hence A is a left group.

(g) $\implies$ (a). Indeed, let $a, b \in S$. As S is *regular*, $ab \in abS \subseteq Sab$. Hence S is $\mathcal{L}$ -commutative. The converse implication is clear. $\square$

Recall from [9, Chapter 8] that a semigroup S is:

(a) *left quasi-commutative* if

$$(\forall a, b \in S)(\exists n \in \mathbb{Z}_+)(ab = b^n a);$$

(b) *right quasi-commutative* if

$$(\forall a, b \in S)(\exists n \in \mathbb{Z}_+)(ab = ba^n);$$

(c) σ -*reflexive* if for all $a, b \in S$ and any subsemigroup A of S , the following implication is valid:

$$(ab \in A) \implies (ba \in A).$$

Remark 9 Observe that any left (right) quasi-commutative semigroup is $\mathcal{L}(\mathcal{R})$ -commutative. Recall also from [9, Theorem 8.2] that the above three conditions are equivalent in an arbitrary semigroup, and that a semigroup is *quasi-commutative* if meets one (and hence all) of the above conditions. Consequently, every quasi-commutative semigroup is $\mathcal{H}$ -commutative. In particular, $\mathcal{H} = \mathcal{J}$ in any quasi-commutative semigroup.

Also, recall from [9, Chapter 8] that a *non-commutative* group is called *hamiltonian* if every its subgroup is normal (c.f. also [9, Theorem 8.1]). As mentioned in the proof of [9, Theorem 8.1] (last line on page 111), each hamiltonian group is periodic (i.e., torsion).

Moreover, it is obvious that any subsemigroup of a quasi-commutative semigroup is quasi-commutative. By the above remarks, *an arbitrary subsemigroup of a quasi-commutative group G is a normal subgroup of G .*

Further, it is clear (and well-known) that a subgroup H of a group G is normal if and only if for all $a, b \in G$, $ab \in H$ implies that $ba \in H$.

Finally, note that each non-commutative quasi-commutative semigroup contains an idempotent element (see [9, Lemma 8.3]).

From the above remark and Proposition 2.1 we have the following result.

Corollary 4.7 *Each quasi-commutative semigroup is stable.*

Also, we have the following theorem (c.f. [9, Theorem 8.5]).

Theorem 4.8 *The following conditions on a semigroup S are equivalent:*

(a) S is an archimedean quasi-commutative semigroup;

- (b) S is a retract ideal extension of a hamiltonian group by a commutative nil-semigroup.

Proof (a) $\implies$ (b). This follows from Remarks 9, 3.

(b) $\implies$ (a). It is clear that S is archimedean. Consider further an arbitrary subsemigroup A of S and let $a, b \in S$ with $ab \in A$. If

- (1) $a, b \in K_S$, as $H = A \cap K_S$ is a normal subgroup of K_S , then $ba \in A$ (we use here the above remarks);
- (2) $a \in K_S, b \notin K_S$, then $ab \in H = A \cap K_S$ and $ba \in K_S$. Let φ be the given retraction of S onto K_S . Then

$$ab = (ab)\varphi = (a\varphi)(b\varphi) \in H,$$

so (as K_S is σ -reflexive)

$$ba = (ba)\varphi = (b\varphi)(a\varphi) \in H;$$

- (3) $a, b \notin K_S$. Suppose first that $ab \notin K_S$. Then $ba \notin K_S$ and $ab = ba$ (since S/K_S is commutative). On the other hand, if $ab \in K_S$, then $ba \in K_S$ (as S/K_S is commutative). Hence $ba \in A$ [see the proof of (2)]. $\square$

Corollary 4.9 *Each non-commutative archimedean component of a quasi-commutative semigroup is a periodic retract ideal extension of a hamiltonian group by a commutative nil-semigroup.*

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