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RESEARCH ARTICLE

Congruences on ample semigroups

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Abstract
Congruences on an ample semigroup S are investigated. For any admissible congruence ρ on S, the minimum σ_ρ (the maximum μ_ρ) admissible congruence on S whose restriction to $E(S)$ is the trace of ρ is obtained. An expression for a congruence which is contained in (contains) any admissible congruence of the same kernel is given. The concept of congruence pairs for S is introduced. It is shown that for any admissible congruence ρ of S; $(\operatorname{tr}\rho, \ker\rho)$ is a congruence pair for S. Conversely, the congruence of S which contains (is contained in) any admissible congruence associated with a given congruence pair for S is formulated.

Keywords Abundant semigroups · Ample semigroups · Admissible congruences · Trace · Kernel · Congruence pair

1 Introduction

An *ample semigroup* is an adequate semigroup S in which $eS^1 \cap aS^1 = eaS^1$ and $S^1e \cap S^1a = S^1ae$ for any $a \in$ and $e^2 = e$ in S. A semigroup S is *adequate* if each $\mathscr{L}^*$-class and each $\mathcal{R}^*$-class in S contains an idempotent and the idempotents commute. Here two elements a, b of a semigroup S are $\mathcal{L}^*$-*related* (respectively $\mathcal{R}^*$-*related*) if and only if they are related by the Green's relation $\mathcal{L}$ (respectively $\mathcal{R}$) in an oversemigroup of S, and $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$.

The class of ample (formerly type A) semigroups was first studied by Fountain [10]. This class of semigroups contains inverse semigroups, full subsemigroups of an inverse semigroup, and semilattices of cancellative monoids. Fountain [10], Armostrong [1] and Lawson [18] extended some basic properties of inverse semigroups to ample semigroups. Fountain [9] initiated a study of one sided ample semigroups. For an extended abstract of the subject, the reader may look at [11] and [13]. Further, several

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authors (see for example; [3,4,6–8] and [19]) have made use of ample semigroups in characterizing classes of abundant semigroups in a way analogous to that in which inverse semigroups are used in characterizing the corresponding classes of regular semigroups.

There is now a class more general than the class of abundant (such as semiabundant) semigroups where results emerging from the studies of both inverse and ample semigroups are accumulated (see [12]). For a survey, the reader may consult Hollings [16]. Congruences play an important role in the investigation of properties of inverse semigroups and there are now deep and well developed theories for congruences on inverse semigroups. For a survey, we refer the reader to [2,17,20] or [22]. It seems natural, therefore, to extend the results concerning congruences on inverse semigroups to ample semigroups. The present paper is devoted to this task. In particular, we extend the results of Green [14], Petrich [21] and Reilly and Scheiblich [24] to ample semigroups. It has been proven that the trace and kernel approach is successful in studying congruences on inverse semigroups (see [21], [23] and [25]) We shall adopt this approach in investigating congruences on ample semigroups.

We start in Sect. 2 by giving the definition and basic properties of ample semigroups S and describing the maximum congruence μ included in $\mathcal{H}^*$ on S and the minimum cancellative congruence σ on S. We state necessary and sufficient condition for a semigroup to be a full subdirect product of a cancellative monoid and a fundamental ample semigroup. Normal congruences on the semilattice E of an ample semigroup S form the subject of Sect. 3. We obtain expressions for the minimum and the maximum admissible congruences on S whose restriction to E is a given normal congruence on E. The former has been introduced earlier in [13]. For any congruence ρ on S, the *trace* of ρ is the restriction of ρ on E. In Sect. 4 we characterize the relationship between two admissible congruences on S having the same trace. We draw several interesting consequences of this characterization. In Sect. 5 we determine the kernels of the minimum and the maximum admissible congruences on S having the same trace. This leads to a specific description of the kernels of σ and μ. The concept of a normal subsemigroup of an ample semigroup S is introduced in Sect. 6. The kernels of admissible congruences on S are normal subsemigroups of S. The maximum congruence on S whose kernel is a given normal subsemigroup is determined, as well as the least congruence whose kernel is a given normal subsemigroup N. The concept of congruence pairs for ample semigroups is the subject of the last section where we see the trace and the kernel of admissible congruence on an ample semigroup S form a congruence pair for S, but a congruence pair of S does not determine a unique admissible congruence. However, a description is given of those congruences on an ample semigroup whose trace and kernel form a given congruence pair.

Any undefined notion and terminology can be in [10] or [17].

2 Ample semigroups

In this section we review briefly the basic facts and concepts connected to ample semigroups. For more details we refer the reader to [1,10,13] and [18]. We denote the set of idempotents in any semigroup S by $E(S)$. If $E(S)$ is a commutative subsemigroup of S, then S is said to be an *E-semigroup*. An *ample semigroup S* is an E-semigroup equipped with unary operations * and † such that the following conditions hold:

(i) For all elements a of S; $aa^* = a$ and $a^\dagger a = a$,
(ii) For all elements a of S and $x, y \in S^1$,

$$ax = ay \quad \Longrightarrow \quad a^*x = a^*y \quad \text{and} \quad xa = ya \quad \Longrightarrow \quad xa^\dagger = ya^\dagger,$$

(iii) For all elements a and idempotents e of S,

$$ea = a(ea)^* \quad \text{and} \quad ae = (ae)^\dagger a.$$

This definition of ample semigroups is equivalent to that stated in the introduction (see [10]). It follows that the elements a^*, $a^\dagger$ are idempotents for any a in S and that $e^* = e = e^\dagger$ for any idempotent e. In fact, a^* (respectively $a^\dagger$) is the unique idempotent in the $\mathcal{L}^*$-class (respectively $\mathcal{R}^*$-class) of a, where, as we recall, two elements a, b of S are *$\mathcal{L}^*$-related* (respectively *$\mathcal{R}^*$-related*) if and only if they are related by Green's relation $\mathcal{L}$ (respectively, $\mathcal{R}$) in some oversemigroup of S. The intersection $\mathcal{L}^* \cap \mathcal{R}^*$ is denoted by $\mathcal{H}^*$. From now on, S will denote an ample semigroup with semilattice and $E(S) = E$ of idempotents.

At this stage we remind the reader of some elementary properties of ample semigroups which will be used frequently.

Lemma 2.1 *Let a, b be elements of an adequate semigroup S. Then:*

(i) $a\mathcal{L}^*b$ *if and only if* $a^* = b^*$ *and* $a\mathcal{R}^*b$ *if and only if* $a^\dagger = b^\dagger$.
(ii) $(ab)^* = (a^*b)^*$ *and* $(ab)^\dagger = (ab^\dagger)^\dagger$;
(iii) $(ab)^*b^* = (ab)^*$ *and* $a^\dagger(ab)^\dagger = (ab)^\dagger$.

An *admissible* congruence on the adequate semigroup S is a semigroup congruence ρ which satisfies the implications;

$$a\,x \;\; \rho \;\; a\,y \Longrightarrow a^*x \;\; \rho \;\; a^*y \quad \text{and} \quad x\,a \;\; \rho \;\; y\,a \Longrightarrow x\,a^\dagger \;\; \rho \;\; y\,a^\dagger$$

for all elements a of S and $x, y \in S^1$ [5].

Lemma 2.2 *If ρ is an admissible congruence on the adequate semigroup S and if a, b are elements of S such that $a \;\rho\; b$, then $a^* \;\rho\; b^*$ and $a^\dagger \;\rho\; b^\dagger$.*

Proof Since ρ is a congruence, we have $a\,a^* \;\; \rho \;\; b\,a^*$, that is, $a \;\; \rho \;\; b\,a^*$ and hence $b \;\; \rho \;\; b\,a^*$ whence $b^* \;\; \rho \;\; b^*\,a^*$ since ρ is admissible. Similary, $a^* \;\; \rho \;\; a^*\,b^*$ and as $a^*\,b^* = b^*\,a^*$ we have $a^* \;\rho\; b^*$ as required. The argument for $a^\dagger \;\rho\; b^\dagger$ is similar. □

It is noteworthy that the conditions in Lemma 2.2 are not strong enough to imply admissibility. For example, the admissible congruences on a cancellative monoid C are precisely the cancellative congruences on C. But when we take $a^* = 1 = a^\dagger$ for all a in C, then any congruence on C satisfies the conditions of Lemma 2.2.

We remark that if ρ is an admissible congruence on the ample semigroup S, then S/ρ is an ample semigroup when $*$ and $\dagger$ are defined on S/ρ by putting

$$(a\ \rho)^* = a^*\ \rho \quad \textit{and} \quad (a\ \rho)^\dagger = a^\dagger\ \rho$$

Moreover, (see [5]), if $x\rho$ is an idempotent in S/ρ,then there exists an idempotent e in S such that $(x, e) \in \rho$.

The natural homomorphism from S onto S/ρ is an admissible homomorphism in the following sense.

A homomorphism $\theta : S \longrightarrow T$ of adequate semigroups is admissible if

$$a\,\mathcal{L}^*(S)\,b \ \text{ implies } \ a\,\theta\,\mathcal{L}^*(T)\,b\,\theta \text{ and } \ a\,\mathcal{R}^*(S)\,b \ \text{ implies } \ a\,\theta\,\mathcal{R}^*(T)\,b\,\theta.$$

The relation μ on the ample semigroup S is defined by the following rule:

$$(a, b) \in \mu \ \text{ if and only if } \ (e\,a)^* = (e\,b)^* \ \text{ and } \ (a\,e)^\dagger = (b\,e)^\dagger \ \text{ for all } \ e \in E.$$

In [10] it is shown that μ is the maximum congruence contained in $\mathcal{H}^*$. From [5] we conclude that μ is an admissible congruence on any ample semigroup. It follows from Sect. 3 of this paper that μ is the maximum idempotent-separating admissible congruence on S. The ample semigroup S is said to be *fundamental* if μ is the identity relation on S. For further details on admissible congruences, we may refer the reader to [15].

The relation σ on the ample semigroup S is defined by the following rule:

$$(a, b) \in \sigma \ \text{ if and only if } \ a\,e = b\,e \ \text{ for some } \ e \in E.$$

In [18] it is shown that σ is the minimum cancellative congruence on S. As S is an ample semigroup, $e\,a = a\,(e\,a)^*$ and $a\,e = (a\,e)^\dagger a$ for any $a \in S$, $e \in E$, then—alternatively—σ can be given as:

$$(a, b) \in \sigma \ \text{ if and only if } \ f\,a = f\,b \ \text{ for some } \ f \in E.$$

We conclude this section by the structure of an ample semigroup on which the intersection of $\sigma \cap \mu$ is the identity relation. This can be considered as a generalization of Theorem 4.5 of [18].

Theorem 2.3 *A semigroup P is a full subdirect product of a cancellative monoid and a fundamental ample semigroup if and only if P is an ample semigroup on which* $\sigma \cap \mu = i$.

Proof Let P be a full subdirect product of a cancellative monoid M and a fundamental ample semigroup T. Denote the identity element of M by 1. Clearly, P is ample and

$$E(P) = \{(1, e) \in M \times T : e \in E(T)\}.$$

Notice that for any $(m, t) \in P$,

$$(m, t)^* = (1, t^*) \text{ and } (m, t)^\dagger = (1, t^\dagger).$$

For any (m, t), (n, s) in P such that $((m, t), (n, s)) \in \sigma \cap \mu$, that is, for any $(1, e) \in E(P)$, and thus for any idempotent e in T;

$$\begin{aligned}&(m, t)\ (1, e) = (n, s)\ (1, e),\\ &[(1, e)\ (m, t)]^* = [(1, e)\ (n, s)]^* \text{ and}\\ &[(m, t)\ (1, e)]^\dagger = [(n, s)\ (1, e)]^\dagger.\end{aligned}$$

It follows that:

$$m = n, (1, \ (e\,t)^*) = (1, (e\,s)^*) \text{ and } (1, \ (t\,e)^\dagger) = (1, \ (s\,e)^\dagger).$$

Therefore $m = n$ and $(t, s) \in \mu(T)$. As by the hypothesis, T is fundamental, so $\mu(T) = i$ and $t = s$. Hence $(m, t) = (n, s)$ and $\sigma \cap \mu = i$ on P.

Conversely, let P be an ample semigroup. Recall that P/σ is a cancellative monoid and P/μ is fundamental, so that the mapping: $\psi : P \longrightarrow P/\sigma \times P/\mu$ defined by $x\,\psi = (x\,\sigma, x\,\mu)$ is a semigroup homomorphism. If $a\,\mu$ is an idempotent of P/μ, then by the admissibility of μ, there exists $e \in E(P)$ such that $(e, a) \in \mu$. But also $e\ \sigma$ is the identity element of P/σ. Therefore, $e\,\psi = (e\ \sigma, a\ \mu)$ and thus im ψ is a full subdirect product of $P/\sigma \times P/\mu$. Further, if $\sigma \cap \mu = i$, then ψ is one-to-one and $P \cong$ im ψ in this case. □

3 Normal congruences

For any congruence ρ on S, we have the restriction $\rho|_E$ of ρ on E which is called *the trace* of ρ, denoted by tr ρ. Clearly tr ρ is a congruence on E. Further, if $e, f \in E$ with $e\,\rho\,f$ and $a \in S$, then $(e\,a, f\,a) \in \rho$ and $(a\,e, a\,f) \in \rho$. If ρ is admissible, then by Lemma 2.2, we get

$$(e\,a)^*\,\rho\,(f\,a)^* \text{ and } (a\,e)^\dagger\,\rho\,(a\,f)^\dagger.$$

Accordingly, a congruence π on E is said to be *normal* if for any $e, f \in E$ and $a \in S$;

$$e\ \pi\ f \text{ implies} (e\,a)^*\,\pi\ (f\ a)^* \text{ and } (a\,e)^\dagger\,\pi\ (a\ f)^\dagger.$$

We give in this section; expressions for the minimum and the maximum admissible congruence on S whose restriction to E is a given normal congruence π on E. Before we begin with the expression for the former, we start with the following Lemma.

Lemma 3.1 *If π is a normal congruence on E, then for any elements a, b in S, the following two statements are equivalent:*

(1) $a^* \pi\, b^*,\ a\,e = b\,e$ *for some* $e \in E, e\,\pi\, a^*$*;*
(2) $a^\dagger \pi\, b^\dagger,\ f\,a = f\,b$ *for some* $f \in E,\ f\,\pi\, a^\dagger$.

Proof Let $a, b \in S$ and suppose (1) holds, then as S is ample;

$$a\,e = b\,e \text{ implies } (a\,e)^\dagger a = (b\,e)^\dagger b \quad \text{and} \quad (a\,e)^\dagger = (b\,e)^\dagger.$$

From the normality of π,

$$e\,\pi\, a^* \text{ implies } (a\,e)^\dagger\,\pi\,(a\,a^*)^\dagger, \text{ that is, } (a\,e)^\dagger\,\pi\, a^\dagger \text{ and}$$
$$e\,\pi\, b^* \text{ implies } (b\,e)^\dagger\pi\,(b\,b^*)^\dagger, \text{ that is, } (b\,e)^\dagger\pi\, b^\dagger.$$

As $(a\,e)^\dagger = (b\,e)^\dagger$, then $a^\dagger\,\pi\, b^\dagger$ and (2) holds. The argument for (2) implies (1) is similar. □

We use one of the statements of Lemma 3.1 to define the required congruence as the following Theorem demonstrates:

Theorem 3.2 *For any normal congruence* π *on* E *, the relation:*

$$\sigma_\pi = \{(a, b) \in S \times S : a^*\pi\, b^*,\ a\,e = b\,e \textit{ for some } e \in E,\ e\,\pi\, a^*\}$$

is the minimum congruence on S *whose restriction to* E *is* π*. Further,* σ_π *is an admissible congruence.*

Proof Clearly σ_π is an equivalence relation. Let $a, b, c \in S$ such that $(a, b) \in \sigma_\pi$, then in particular $a^*\pi\, b^*,\ a\,e = b\,e$ for some $e \in E,\ e\,\pi\, a^*$and, $c\,a\,e = c\,b\,e$.
Thus $(c\,a)^*\,e = (c\,b)^*\,e$, $(c\,a)^*\,e\,\pi\,(c\,a)^*\,a^*$, $(c\,a)^*\,a^* = (c\,a)^*$, $(c\,b)^*\,e\,\pi\,(c\,b)^*\,b^*$, $(c\,b)^*\,b^* = (c\,b)^*$.
Therefore, $(c\,a)^*\,\pi\,(c\,b)^*$ and $(c\,a)\,(c\,a)^*e = (c\,b)\,(c\,b)^*e = (c\,b)\,(c\,a)^*e$, where $(c\,a)^*e\,\pi\,(c\,a)^*$. Hence, $(c\,a, c\,b) \in \sigma_\pi$.
On the other side,

$$a\,e = b\,e \implies a\,e\,c = b\,e\,c \implies a\,c\,(e\,c)^* = b\,c\,(e\,c)^*.$$

By the normality of π,

$$e\,\pi\, a^* \text{ implies } (e\,c)^*\pi\,(a^*\,c)^*;\ (a^*\,c)^* = (a\,c)^*, \text{ so that } (e\,c)^*\pi\,(a\,c)^*.$$

Similarly, $(e\,c)^*\pi\,(b\,c)^*$ and therefore ;$(a\,c)^*\pi(b\,c)^*$. As $a\,c\,(e\,c)^* = b\,c\,(e\,c)^*$ and $(e\,c)^*\pi(a\,c)^*$, we obtain $(a\,c, b\,c) \in \sigma_\pi$. Hence σ_π is congruence.

It is easy to see that $e\,\pi\, f$ if and only if $(e, f) \in \sigma_\pi$ and $\operatorname{tr} \sigma_\pi = \pi$. Now suppose τ is a congruence on S such that $\operatorname{tr} \tau = \pi$, and $(a, b) \in \sigma_\pi$ for some $a, b \in S$; then—in particular—we have $a^*\pi\, b^*$ and $a\,e = b\,e$ for some $e \in E,\ e\,\pi\, a^*$ and thus $(a^*, e) \in \tau,\ (b^*, e) \in \tau$. Therefore,

$$\begin{aligned} a\,\tau &= a\,a^*\tau = a\,\tau a^*\tau = a\,\tau e\,\tau = (a\,e)\,\tau = (b\,e)\,\tau \\ &= b\,\tau e\,\tau = b\,\tau b^*\,\tau = b\,b^*\,\tau = b\,\tau; \end{aligned}$$

that is, $(a, b) \in \tau$. Hence $\sigma_\pi \subseteq \tau$ and σ_π is the minimum congruence on S whose restriction to E is π. To prove that σ_π is admissible.

Let $a \in S$, $s, t \in S^1$ such that $(a\,s, a\,t) \in \sigma_\pi$. Then

$$(a\,s)^* \pi (a\,t)^* \text{ and } a\,s\,e = a\,t\,e \text{ for some } e \in E,\ (a\,s)^* \pi\, e.$$

That is, $(a^* s)^* \pi\ (a^* t)^*$ and $a^* s\,e = a^* t\,e$, $e\,\pi (a^* s)^*$. Therefore $(a^* s, a^* t) \in \sigma_\pi$.

On the other side, suppose $(s\,a, t\,a) \in \sigma_\pi$; that is,

$$(s\,a)^\dagger \pi (t\,a)^\dagger,\ f\,s\,a = f\,t\,a, \text{ and } f\,\pi\,(s\,a)^\dagger \text{ for some } f \in E \text{ (by Lemma 3.1)}$$

Notice that $f\,s\,a = f\,t\,a$, implies $f\,s\,a^\dagger = f\,t\,a^\dagger$ where $f\,\pi\,(s\,a^\dagger)^\dagger$, $f\,\pi\,(t\,a^\dagger)^\dagger$. Therefore,

$$(s\,a^\dagger)^\dagger\,\pi\,(t\,a^\dagger)^\dagger,\ f\,s\,a^\dagger = f\,t\,a^\dagger,\ f\,\pi\,(s\,a^\dagger)^\dagger.$$

Again by Lemma 3.1 we have, $(s\,a^\dagger, t\,a^\dagger) \in \sigma_\pi$. Hence σ_π is an admissible congruence and we have σ_π as required. □

Lemma 3.1 offers an alternative expression for σ_π of Theorem 3.2 which can be stated in the following corollary:

Corollary 3.3 *The congruence σ_π of Theorem 3.2 has also the following form:*

$$\sigma_\pi = \{(a, b) \in S \times S : a^\dagger \pi\, b^\dagger,\ f\,a = f\,b;\ \textit{where}\ f\,\pi\,a^\dagger,\ f \in E\}.$$

If the congruence ρ is admissible on S, then, as we recall, $\pi = \rho|_E$ is a normal congruence on E and σ_π of Theorem 3.2 (and Corollary 3.3) coincides with ρ_{min} described in [[13], Proposition 9.2].

Let the congruence π be normal on E. As σ_π is the minimum admissible congruence on S whose trace is π, we are in a position to generalize Lemma 3.1 (1) of [23] as follows:

Proposition 3.4 *Let ρ be an admissible congruence on S whose trace is π. Then S/ρ is an idempotent-separating homomorphic image of S/σ_π.*

Proof The mapping $\phi : S/\sigma_\pi \longrightarrow S/\rho$ defined by $(s\,\sigma_\pi)\,\phi = s\,\rho$ is a homomorphism of S/σ_π onto S/ρ. From [6], we conclude that,

$$E(S/\sigma_\pi) = \{e\,\sigma_\pi : e \in E\}.$$

Let $e\,\sigma_\pi$, $f\,\sigma_\pi$ be two idempotents in S/σ_π $(e, f \in E)$

$$\begin{aligned}(e\,\sigma_\pi)\,\phi = (f\,\sigma_\pi)\,\phi &\Longrightarrow e\,\rho = f\,\rho \\ &\Longrightarrow (e, f) \in \rho \\ &\Longrightarrow (e, f) \in \pi \quad (\mathrm{tr}\,\rho = \pi) \\ &\Longrightarrow e\,\sigma_\pi = f\,\sigma_\pi\end{aligned}$$

Therefore, ϕ is idempotent-separating. □

To turn to the second objective of this section, let π be a normal congruence on E. Define μ_π on S by the following rule:

$$(a,\, b) \in \mu_\pi \ \text{ if and only if } \ (e\,a)^*\pi\,(e\,b)^* \ \text{ and } \ (a\,e)^\dagger\pi\,(b\,e)^\dagger \ \text{ for any } \ e \in E.$$

The following Lemma gives an alternative description of μ_π.

Lemma 3.5 *Let π be a normal congruence on E. Then for any elements $a,\, b$ of S, the following statements are equivalent:*

(i) $(a,\, b) \in \mu_\pi$.
(ii) $(e\,a)^*\,\pi\,(f\,b)^*$ *and* $(a\,e)^\dagger\,\pi\,(b\,f)^\dagger$ *for any* $e,\, f \in E$ *with* $e\,\pi\,f$.
(iii) $(a\,\sigma_\pi,\, b\,\sigma_\pi) \in \mu(S/\sigma_\pi)$.

Proof (i) $\Longleftrightarrow$ (ii) For any $b \in S$, $e,\, f \in E$ with $e\,\pi\,f$ we have $(e\,b)^*\,\pi\,(f\,b)^*$. If $(a,\, b) \in \mu_\pi$, then $(e\,a)^*\,\pi(e\,b)^*$ so that $(e\,a)^*\,\pi(f\,b)^*$. Similarly, $(a\,e)^\dagger\pi(b\,f)^\dagger$. Hence (ii) follows from (i).
It is clear that (i) is an immediate consequence of (ii).
(i) $\Longleftrightarrow$ (iii) For any $a,\, b \in S$, we have;

$$\begin{aligned}
(a,\, b) \in \mu_\pi &\Longleftrightarrow (e\,a)^*\,\pi\,(e\,b)^* \text{ and } (a\,e)^\dagger\pi\,(b\,e)^\dagger;\text{ for all } e \in E\\
&\Longleftrightarrow (e\,a)^*\sigma_\pi = (e\,b)^*\sigma_\pi \text{ and } (a\,e)^\dagger\sigma_\pi = (b\,e)^\dagger\sigma_\pi;\ \text{ for all } e \in E\\
&\Longleftrightarrow (e\sigma_\pi\, a\sigma_\pi)^* = (e\,\sigma_\pi\, b\sigma_\pi)^* \text{ and}\\
&\qquad (a\sigma_\pi\, e\sigma_\pi)^\dagger = (b\sigma_\pi\, e\sigma_\pi)^\dagger;\ \text{ for all } e \in E\ \ (\sigma_\pi \text{ is admissible})\\
&\Longleftrightarrow (a\sigma_\pi,\, b\sigma_\pi) \in \mu(S/\sigma_\pi).
\end{aligned}$$

□

Now the second main result of the section follows;

Theorem 3.6 *The relation μ_π is the maximum admissible congruence on S whose restriction to E is π.*

Proof It is clear that μ_π is an equivalence relation. Let $a,\, b,\, c \in S$ with $(a,\, b) \in \mu_\pi$ and $e \in E$. Then $(e\,a)^*\,\pi\,(e\,b)^*$ and by the normality of π, it follows that:

$$((e\,a)^*c)^*\,\pi\,((e\,b)^*c)^*,\ \text{ that is; }\ (e\,a\,c)^*\,\pi\,(e\,b\,c)^*.$$

Since $(e\,c)^\dagger \in E$, we have, $(a\,(c\,e)^\dagger)^\dagger\,\pi\,(b\,(c\,e)^\dagger)^\dagger$, that is; $(a\,c\,e)^\dagger\,\pi\,(b\,c\,e)^\dagger$.
Therefore, $(a\,c,\, b\,c) \in \mu_\pi$. Similarly, $(c\,a,\, c\,b) \in \mu_\pi$.
Hence, μ_π is congruence.
It is obvious that $\pi \subseteq \mu_\pi$. Let $f, g \in E$ with $f\,\mu_\pi\,g$. Then for any $e \in E$, $ef\pi eg$.
Take in turn $e = f$ and $e = g$ to get $f\,\pi\,f\,g$ and $g\,f\,\pi\,g$.
As $f\,g = g\,f$. So $f\,\pi\,g$. Thus $\mathrm{tr}\,\mu_\pi = \pi$.

To prove that μ_π is admissible, let $a \in S$ and $s,\ t \in S^1$. Then;

$$\begin{aligned}(a\,s,\ a\,t) \in \mu_\pi &\Longrightarrow\ (a\,s\sigma_\pi,\ a\,t\sigma_\pi) \in \mu(S/\sigma_\pi) \ \text{(Lemma 3.5)}\\ &\Longrightarrow\ (a^*\,s\sigma_\pi,\ a^*\,t\sigma_\pi) \in \mu(S/\sigma_\pi)\\ &\qquad\quad (\mu(S/\sigma_\pi),\ \text{and } \sigma_\pi \text{ are admissble})\\ &\Longrightarrow\ (a^*\,s,\ a^*\,t) \in \mu_\pi. \qquad\qquad \text{(Lemma 3.5)}\end{aligned}$$

Similarly, $(s\,a,\ t\,a) \in \mu_\pi$ implies that $(s\,a^\dagger,\ t\,a^\dagger) \in \mu_\pi$.
Therefore, μ_π is admissible.

It remains to prove that μ_π contains any admissible congruence on S whose restriction to E is π. Let ρ be an admissble congruence on S such that $\rho|_E = \pi$ and $(a,\ b) \in \rho$ for some $a,\ b \in S$. Then for any $e \in E$, $(e\,a,\ e\,b) \in \rho$ and $(a\,e,\ b\,e) \in \rho$. In particular, we have, (Lemma 2.2)

$$((e\,a)^*,\ (e\,b)^*) \in \rho,\ ((a\,e)^\dagger,\ (b\,e)^\dagger) \in \rho.$$

Thus

$$(e\,a)^*\,\pi\,(e\,b)^*,\ (a\,e)^\dagger\,\pi\,(b\,e)^\dagger.$$

and $(a,\ b) \in \mu_\pi$.
Hence the result holds. □

4 Congruences with the same trace

The objective of this section is to characterize the relationship between two admissible congruences, ρ and τ on S having the same trace. The proof of the main result is obtained by adopting Petrich's proof [21] of the corresponding result for inverse semigroups.

It follows from the results of Sect. 3, that for any admissible congruence ρ on S; $tr\,\rho$ is a normal congruence on E and $\sigma_{\mathrm{tr}\,\rho}$, $\mu_{\mathrm{tr}\,\rho}$ are respectively the minimum and the maximum admissible congruence on S such that:

$$\mathrm{tr}\ \sigma_{\mathrm{tr}\ \rho} = \mathrm{tr}\ \rho = \mathrm{tr}\ \mu_{\mathrm{tr}\ \rho}$$

where

$$\sigma_{\mathrm{tr}\ \rho} = \{(a,\ b) \in S \times S;\ a^*\rho\,b^*,\ a\,e = b\,e \text{ for some } e \in a^*\rho \cap E\}.$$

or equivalently

$$\sigma_{\mathrm{tr}\ \rho} = \{(a,\ b) \in S \times S;\ a^\dagger\rho\,b^\dagger,\ f\,a = f\,b \text{ for some } f \in a^\dagger\rho \cap E\}.$$

and

$$\mu_{\mathrm{tr}\ \rho} = \{(a,\ b) \in S \times S;\ (e\,a)^*\rho\,(e\,b)^*,\ \text{and } (a\,e)^\dagger\rho\,(b\,e)^\dagger \text{ for all } e \in E\}.$$

We may denote $\sigma_{\mathrm{tr}\ \rho}$ and $\mu_{tr\rho}$ by σ_ρ and μ_ρ respectively.
From Theorems 3.2 and 3.6 we conclude the following Corollary:

Corollary 4.1 *For any admissible congruence ρ on S,*

$$\sigma_\rho \subseteq \rho \subseteq \mu_\rho \text{ and } \mathrm{tr}\ \sigma_\rho = \mathrm{tr}\ \rho = \mathrm{tr}\ \mu_\rho.$$

If i and ω are respectively, the equality and the universal relations on S, then clearly, $\mu_i = \mu$, the maximum congruence contained in $\mathcal{H}^*$. That is, to say (Theorem 3.6), μ is the maximum admissible congruence whose trace is the identity relation on E. Therefore, μ is the maximum idempotent-separating admissible congruence on S, and $\sigma_\omega = \sigma$, the minimum cancellative congruence on S.

Recall that for any congruences ρ and τ on S with $\tau \subseteq \rho$, the congruence ρ/τ is defined on S/τ by the rule:

$$a\,\tau\ \ \rho/\tau\ \ b\,\tau \quad \text{if and only if} \quad a\,\rho\,b;\ \ (a,\ b \in S) \quad \text{(see [2], or [17])}.$$

Theorem 4.2 *The following statements concerning admissible congruences ρ and τ on the ample semigroup S are equivalent:*

(i) $\mathrm{tr}\,\rho = \mathrm{tr}\,\tau$.
(ii) $\rho \subseteq \mu_\tau$; $\mu_\tau/\rho = \mu(S/\rho)$.
(iii) $a\,\rho\ \ \mu(S/\rho)\ \ b\,\rho \iff a\,\tau\ \ \mu(S/\tau)\ \ b\,\tau,\ (a,\ b \in S)$.
(iv) $a\,\rho\ \ \mathcal{H}^*(S/\rho)\ \ b\,\rho \iff a\,\tau\ \ \mathcal{H}^*(S/\tau)\ \ b\,\tau,\ (a,\ b \in S)$.
(v) $\rho \cap \tau|_{e\,\rho}$ *and* $\rho \cap \tau|_{e\,\tau}$ *are cancellative congruences,* $(e \in E)$.
(vi) $\rho/\rho \cap \tau$ *and* $\tau/\rho \cap \tau$ *are congruences contained in* $\mathcal{H}^*(S/\rho \cap \tau)$.

Proof (i) $\Longrightarrow$ (ii) First note that $\mu_\rho = \mu_\tau$ so that $\rho \subseteq \mu_\tau$.
For any $a,\ b \in S$ we have,

$$\begin{aligned} a\,\rho\ \mu_\tau/\rho\ b\,\rho &\iff a\,\rho\ \mu_\rho/\rho\ b\,\rho \\ &\iff a\,\mu_\rho\,b \\ &\iff (e\,a)^*\,\rho\,(e\,b)^* \text{ and } (a\,e)^\dagger\,\rho\,(b\,e)^\dagger; \quad \text{for any } e \in E \\ &\iff (a\,\rho,\ b\,\rho) \in \mu(S/\rho). \end{aligned}$$

(ii) $\Longrightarrow$ (i) Observe that $\mathrm{tr}\ \rho \subseteq \mathrm{tr}\ \mu_\tau \subseteq \mathrm{tr}\ \tau$. Further, for any $e,\ f \in E$, we have,

$$e\,\tau\,f \implies e\,\mu_\tau\,f \implies e\,\rho\,\mu_\tau/\rho\ f\,\rho \implies e\,\rho\,\mu(S/\rho)\,f\,\rho \implies e\,\rho = f\,\rho$$
$$\implies e\,\rho\,f.$$

and thus also $\mathrm{tr}\ \tau \subseteq \mathrm{tr}\ \rho$.

(i) $\Longrightarrow$ (iii) For any $a,\ b \in S$ we have,

$$\begin{aligned} a\,\rho\ \mu(S/\rho)\ b\,\rho &\iff (e\,a)^*\rho = (e\,b)^*\rho,\ (a\,e)^\dagger\,\rho = (b\,e)^\dagger\rho; \quad \text{for any } e \in E \\ &\iff (e\,a)^*\tau = (e\,b)^*\tau,\ (a\,e)^\dagger\,\tau = (b\,e)^\dagger\tau; \quad \text{for any } e \in E \\ &\iff a\,\tau\ \mu(S/\tau)\ b\,\tau. \end{aligned}$$

(iii) $\Longrightarrow$ (i) For any $e,\ f \in E$, we obtain;

$$\begin{aligned} e\,\rho\,f &\iff e\,\rho = f\,\rho \\ &\iff e\,\rho\ \mu(S/\rho)\ f\,\rho \\ &\iff e\,\tau\ \mu(S/\tau)\ f\,\tau \\ &\iff e\,\tau = f\,\tau \\ &\iff e\,\tau\,f. \end{aligned}$$

(i) $\Longrightarrow$ (iv) Let $a,\ b \in S$ and assume $a\,\rho\ \mathcal{H}^*(S/\rho)\ b\,\rho$.
Thus $a^*\rho = b^*\rho$ and $a^\dagger\rho = b^\dagger\rho$.
The hypothesis implies $a^*\tau = b^*\tau$ and $a^\dagger\tau = b^\dagger\tau$, which evidently imply $a\,\tau\ \mathcal{H}^*(S/\tau)\ b\,\tau$.
By symmetry, $a\,\tau\ \mathcal{H}^*(S/\tau)\ b\,\tau$ implies $a\,\rho\ \mathcal{H}^*(S/\rho)\ b\,\rho$.
(iv) $\Longrightarrow$ (i) Let $e,\ f \in E$ and assume $e\,\rho\,f$, then $e\,\rho\ \mathcal{H}^*(S/\rho)\ f\,\rho$ so that by hypothesis, $e\,\tau\ \mathcal{H}^*(S/\tau)\ f\,\tau$ and hence $e\,\tau\,f$. Symmetrically, $e\,\tau\,f$ implies $e\,\rho\,f$.
(i) $\Longrightarrow$ (v) Observe that $e\,\rho$ is an ample semigroup ($e \in E$). Let $a,\ b,\ c \in e\,\rho$.
If $(a\,b,\ a\,c) \in \rho \cap \tau$, and since $\rho,\ \tau$ are admissible congruences, then $(a^*b,\ a^*c) \in \rho \cap \tau$. Notice that $a^*,\ b^\dagger \in e\,\rho$ and so $(b^\dagger,\ a^*) \in \rho \cap \tau$ (by the hypothesis), then $(b,\ a^*b) \in \rho\cap\tau$. Also, $(a^*,\ c^\dagger) \in \rho\cap\tau$ so that $(a^*c,\ c) \in \rho\cap\tau$. Hence $(b,\ c) \in \rho\cap\tau$.
Similarly, $(b\,a,\ c\,a) \in \rho \cap \tau$ implies $(b,\ c) \in \rho \cap \tau$.
Therefore, $\rho \cap \tau|_{e\,\rho}$ is cancellative congruence.
Similarly; $e\,\tau$ is an ample semigroup ($e \in E$), and $\rho \cap \tau|_{e\,\tau}$ is cancellative congruence.

(v) $\Longrightarrow$ (i) Let $g,\ h \in e\,\rho \cap E$. As $g.g\,h = g.h$ where $g,\ g\,h,\ h \in e\,\rho$
so $(g.\,g\,h,\ g\,h) \in \rho \cap \tau$, then by the hypothesis, $(g\,h,\ h) \in \rho \cap \tau$.
Symmetrically, $(h\,g,\ g) \in \rho \cap \tau$.
But $gh = hg$, then $(g,\ h) \in \rho \cap \tau$.

In particular, for any $e,\ f \in E$.

$$e\,\rho\,f \implies f \in e\,\rho \implies (e,\ f) \in \rho \cap \tau \implies (e,\ f) \in \tau \implies e\,\tau\,f.$$

Similarly $e\,\tau\,f \implies e\,\rho\,f$.

(i) $\Longrightarrow$ (vi) For any $a,\ b \in S$, we have,

$$\begin{aligned} a(\rho \cap \tau)\ \rho/_{\rho\cap\tau}\ b(\rho \cap \tau) &\Longrightarrow a\,\rho\, b \Longrightarrow a^*\rho\, b^* \text{ and } a^{\dagger}\rho\, b^{\dagger} \\ &\Longrightarrow a^*(\rho \cap \tau)\ \rho/_{\rho\cap\tau}\ b^*(\rho \cap \tau) \text{ and} \\ &\quad\ a^{\dagger}(\rho \cap \tau)\ \rho/_{\rho\cap\tau}\ b^{\dagger}(\rho\cap\tau) \\ &\Longrightarrow a\,(\rho \cap \tau)\ \mathcal{L}^*(S/_{\rho\cap\tau})\ b\,(\rho \cap \tau) \text{ and} \\ &\quad\ a\,(\rho \cap \tau)\ \mathcal{R}^*(S/_{\rho\cap\tau})\ b\,(\rho \cap \tau) \\ &\Longrightarrow a\,(\rho \cap \tau)\ \mathcal{H}^*(S/_{\rho\cap\tau})\ b\,(\rho \cap \tau). \end{aligned}$$

Therefore, $\rho/_{\rho\cap\tau} \subseteq \mathcal{H}^*(S/_{\rho\cap\tau})$. Similarly, $\tau/_{\rho\cap\tau} \subseteq \mathcal{H}^*(S/_{\rho\cap\tau})$.
(vi) $\Longrightarrow$ (i) For any $e,\ f \in E$, we have;

$$\begin{aligned} e\,\rho\, f &\Longrightarrow e(\rho\cap\tau)\ \rho/_{\rho\cap\tau}\ f(\rho\cap\tau) \\ &\Longrightarrow e\,(\rho\cap\tau)\ \mathcal{H}^*(S/_{\rho\cap\tau})\ f\,(\rho\cap\tau) \\ &\Longrightarrow e\,(\rho\cap\tau) = f\,(\rho\cap\tau) \\ &\Longrightarrow e\,\tau\, f \end{aligned}$$

and similarly, $e\,\tau\, f$ implies $e\,\rho\, f$. □

Corollary 4.3 *An admissible congruence ρ on S is equal to μ_ρ if and only if S/ρ is fundamental.*

Proof Using the equivalence of (i) and (ii) of Theorem 4.2, we obtain;

$$\rho = \mu_\rho \iff \mu_\rho/\rho = i \iff \mu(S/\rho) = i \iff S/\rho \text{ is fundamental .}$$

□

Corollary 4.4 *If ρ is an admissible congruence on S, then for any $e \in E$, $\sigma_\rho|_{e\,\rho}$ is the minimum cancellative congruence on $e\,\rho$.*

Proof Clearly, $\sigma_\rho \subseteq \rho$ so that $\rho\cap\sigma_\rho = \sigma_\rho$. Since $\operatorname{tr}\rho = \operatorname{tr}\sigma_\rho$, then by the equivalence of (i) and (v) in Theorem 4.2; $\sigma_\rho|_{e\,\rho}$ is a cancellative congruence on $e\,\rho$ for any $e \in E$. Since $e\,\rho$ is an ample semigroup, then the minimum cancellative congruence on $e\,\rho$ can be defined by the rule:

$$a\,\sigma\, b \text{ if and only if } a\, f = b\, f \text{ for some } f \in e\rho \cap E.$$

For any $a,\ b \in e\,\rho$, we have $a^*\rho = e\,\rho$, and $\sigma_\rho\, b \iff a\, f = b\, f$ for some $f \in a^*\rho \cap E$. Therefore, $a\,\sigma_\rho\, b \iff a\, f = b\, f$ for some $f \in e\rho \cap E$. Thus, $\sigma_\rho|_{e\,\rho} \subseteq \sigma$. But $\sigma_\rho|_{e\,\rho}$ itself is a cancellative congruence on $e\rho$. Hence, $\sigma_\rho|_{e\,\rho} = \sigma$ on $e\,\rho$. □

Proposition 4.5 *Let ρ and τ be two admissible congruences on S such that $\tau \subseteq \rho$ and for every $e \in E$; $\tau|_{e\rho}$ is the minimum cancellative congruence on $e\,\rho$. Then $e\,\tau = e\,\sigma_\rho$ for any $e \in E$.*

Proof Let $e, f \in E$ such that $e \rho f$. Then in particular, $e, f \in e\rho$ and $e.ef = f.ef$, $ef \in e\rho \cap E$. Thus $e \tau f$ and $\operatorname{tr} \rho \subseteq \operatorname{tr} \tau$. By the hypothesis $\operatorname{tr} \tau \subseteq \operatorname{tr} \rho$. Therefore, $\operatorname{tr} \tau = \operatorname{tr} \rho$ so that $\sigma_\rho \subseteq \tau$ and $e \sigma_\rho \subseteq e \tau$ for any $e \in E$.

Conversely, for any $a, \in S, e \in E$, since $\tau \subseteq \rho$ and ρ is an admissible congruence, we have $a \in e \tau$ implies $a \in e \rho$ and $e \rho a^*$, $ef \rho (af)^*$. As $\tau|_{e\rho} = \sigma$ is the minimum cancellative congruence on $e \rho$, so also $a \in e \tau$ implies $a f = e f$ for some $f \in e \rho \cap E$, that is, whenever $a \in e \tau$, we have $a^* \rho e$, $a f = e f$ for some $f \in e \rho \cap E$. Thus we obtain $a \in e \sigma_\rho$. Therefore, $e \tau \subseteq e \sigma_\rho$. Hence $e \tau = e \sigma_\rho$. □

Recall that for any congruence ρ on S, *the kernel* of ρ is denoted by $\ker \rho$, thus

$$\ker \rho = \{a \in S : (e, a) \in \rho \text{ for some } e \in E\}.$$

Directly, from Lemma 2.2 we have:

Proposition 4.6 *If ρ is admissible congruence on S and $a \in \ker \rho$, then $(a^\dagger, a^*) \in \operatorname{tr} \rho$.*

Proposition 4.7 *For any admissible congruence ρ on S and any $e \in E$, we have $e \rho = e \mu_\rho \cap \ker \rho$.*

Proof Let $a \in S$ such that $a \in e \mu_\rho \cap \ker \rho$. Then $a \mu_\rho e$ and $a \rho f$ for some $f \in E$ so that $a^* \mu_\rho e$ and $a^* \rho f$. We have that $\operatorname{tr} \rho = \operatorname{tr} \mu_\rho$, then $a^* \mu_\rho f$ and thus $e \mu_\rho f$. Therefore, $e \rho f$ and $a \rho e$, that is, $a \in e \rho$.

Conversely, if $a \in e \rho$, then $a \in \ker \rho$, and since $\rho \subseteq \mu_\rho$, $(a, e) \in \mu_\rho$, that is, $a \in e \mu_\rho$. Hence $a \in e \mu_\rho \cap \ker \rho$. □

5 The kernels of σ_ρ and μ_ρ

It follows from Sect. 2, that the congruences σ and μ are respectively, the minimum cancellative congruence on S and the maximum congruence contained in $\mathcal{H}^*$ on S. From Sect. 4, σ is σ_ω and μ is μ_i where ω and i are respectively, the universal and the equality relations on S, that is, σ is the minimum admissible congruence on S whose trace in the universal relation on E and μ is the maximum admissible congruence on S whose trace is the equality relation on E. In this section we determine the kernels of σ_ρ and μ_ρ, where these congruences as defined in Sect. 4 for any admissible congruence ρ on S. The kernels of σ and μ will follow. We begin with the kernel of σ_ρ.

Proposition 5.1 *For any admissible congruence ρ on S, the following subsets of S are equal:*

(1) $K_1 = \ker \sigma_\rho$.
(2) $K_2 = \{a \in S : ae = e \text{ for some } e \in a^* \rho \cap E\}$.
(3) $K_3 = \{a \in S : ea = e \text{ for some } e \in a^\dagger \rho \cap E\}$.

Proof For any $a \in S$; we notice that;

$$\begin{aligned} a \in K_1 &\Longrightarrow a\,\sigma_\rho\, f \ \text{ for some } \ f \in E \\ &\Longrightarrow a^*\rho\, f,\ ae = fe \ \text{ for some } \ e \in a^*\rho \cap E \\ &\Longrightarrow aef = ef,\ ef \in a^*\rho \cap E \\ &\Longrightarrow a \in K_2. \end{aligned}$$

Therefore, $K_1 \subseteq K_2$. Suppose $a \in K_2$, that is, $a \in S$ and $ae = e$ for some $e \in a^*\rho \cap E$. As $ae = (ae)^\dagger a$, then $ea = e$ and $(ae, a) \in \rho$, so that $(e, a) \in \rho$ and, in particular. $(e, a^\dagger) \in \rho$. Therefore, $a \in K_3$. Thus $K_2 \subseteq K_3$. Now, let $a \in K_3$, that is, $a \in S$; $ea = e$ for some $e \in a^\dagger\rho \cap E$. It follows as $e\,\rho\, a^\dagger$ so $e = ea\,\rho\, a^\dagger a$. Then as ρ preserves $*$ (Lemma 2.2), $a^*\rho\, e^* = e$. But $(a^*)^\dagger = a^*$ and $a^*\rho\, e\,\rho\, a^\dagger$ so we get:

$$a^\dagger\rho(a^*)^\dagger,\ ea = ea^* \text{ where } e\,\rho\, a^\dagger,\ e \in E.$$

By Corollary 3.3 $(a, a^*) \in \sigma_\rho$ and $a \in K_1$. Therefore, $K_3 \subseteq K_1$. Hence the result holds. □

As a direct consequence of proposition 5.1 we have the following Corollary:

Corollary 5.2 *For any normal congruence π on E, the following subsets of S are equal:*

(1) $\ker \sigma_\pi$.
(2) $\{a \in S :\ ae = e\ \textit{for some}\ e \in E,\ e\,\pi\, a^*\}$.
(3) $\{a \in S :\ ea = e\ \textit{for some}\ e \in E,\ e\,\pi\, a^\dagger\}$.

The determination of the kernel of μ_ρ follows:

Proposition 5.3 *Let ρ be an admissible congruence on S. Then the following subsets of S are equal:*

(1) $K_1 = \ker \mu_\rho$.
(2) $K_2 = \{a \in S :\ ea\ \rho\ ae,\ \textit{for all}\ e \in E\}$.
(3) $K_3 = \{a \in S :\ (ea)^*\ \rho\ ea^*,\ \textit{for all}\ e \in E\}$.
(4) $K_4 = \{a \in S :\ (ae)^\dagger\ \rho\ a^\dagger e,\ \textit{for all}\ e \in E\}$.

Proof To prove $K_1 = K_2$, let $a \in K_1$, then $a\,\mu_\rho\, f$ for some $f \in E$. In particular, $(ea)^*\ \rho\ ef$, for any $e \in E$ which implies $a\,(ea)^*\ \rho\ aef$. Since S is ample, $ea\,\rho\ aef$. Further, as μ_ρ is admissible congruence, $a^*\,\mu_\rho\, f$ and $a^*\ \rho\ f$, that is, $a\ \rho\ af$. Therefore; $ae\ \rho\ afe$, $afe = aef$ and $aef\ \rho\ ea$. Hence $ae\ \rho\ ea$ and $a \in K_2$; $K_1 \subseteq K_2$.

Assume that $a \in K_2$, that is, $a \in S$ and $ae\ \rho\ ea$ for any $e \in E$. Take in turn $e = a^*$ and $e = a^\dagger$ to get $a\,\rho\ a^*a$ and $aa^\dagger\ \rho\ a$ so that

$$a^\dagger\ \rho\ a^*a^\dagger \text{ and } a^*a^\dagger\ \rho\ a^*$$

Thus $a^*\ \rho\ a^\dagger$.

For any $e \in E$, $a\,e\ \rho\ e\,a$ implies $(a\,e)^* \ \rho\ (e\,a)^*$ and $(a\,e)^\dagger\ \rho\ (e\,a)^\dagger$.
Notice that

$$(a\,e)^* \ = \ a^* e = (e\,a^*)^*$$
$$(e\,a)^\dagger = e\,a^\dagger,\ e\,a^\dagger\ \rho\ e\,a^*,\ e\,a^* = (a^* e)^\dagger.$$

Therefore, $a\,e\,\rho\ e\,a$ implies $(e\,a)^*\,\rho\ (e\,a^*)^*$ and $(a\,e)^\dagger\ \rho\ (a^*\,e)^\dagger$ so that, $a\,\mu_\rho\,a^*$ and $a \in K_1$, $K_2 \subseteq K_1$.
Hence $K_1 = K_2$.

To prove $K_2 = K_3$, let $a \in K_2$, that is, $a \in S$ such that $ea\ \rho\ ae$ for any $e \in E$. Then $(e\,a)^*\rho\,(a\,e)^*$. But, $(a\,e)^* = a^* e = e\,a^*$.
Hence $(e\,a)^*\ \rho\ e\,a^*$ and $a \in K_3$. Thus $K_2 \subseteq K_3$.

Let $a \in K_3$, then $a \in S$, $(e\,a)^*\ \rho\ e\,a^*$ for any $e \in E$, so that

$$a\,(e\,a)^*\ \rho\ a\,e\,a^*.$$

Notice that

$$e\,a = a\,(e\,a)^*,\ e\,a\ \rho\ a\,e \text{ and } a \in K_2,\ K_3 \subseteq K_2.$$

Hence $K_2 = K_3$.

To prove $K_2 = K_4$, let $a \in K_2$, that is, $a \in S$, $ea\ \rho\ ae$ for any $e \in E$. Then $(e\,a)^\dagger\ \rho\ (a\,e)^\dagger$ and $(e\,a)^\dagger = e\,a^\dagger = a^\dagger e$. Therefore; $(a\,e)^\dagger\ \rho\ a^\dagger e$, $a \in K_4$ and $K_2 \subseteq K_4$. For any element $a \in K_4$, that is, $a \in S$, $(a\,e)^\dagger\ \rho\ a^\dagger e$ for any $e \in E$, we have:

$$(a\,e)^\dagger\,a\ \rho\ a^\dagger e\,a \text{ and } a\,e\ \rho\ e\,a,\ a \in K_2.$$

Therefore; $K_4 \subseteq K_2$ and thus $K_2 = K_4$. Hence the result. □

The following Corollary follows as proposition 5.3:

Corollary 5.4 *For any normal congruence π on E. The following subsets of S are equal:*

(1) $\ker\ \mu_\pi$.
(2) $\{a \in S :\ (e\,a)^*\,\pi\ e\,a^*,\ \ for\ any\ e \in E\}$.
(3) $\{a \in S :\ (a\,e)^\dagger\,\pi\ a^\dagger e,\ \ for\ any\ e \in E\}$.

Recall that $\sigma_\omega = \sigma$ and $\mu_i = \mu$ on S, The following Corollary is direct application of Corollary 5.2, Proposition 5.3 and Corollary 5.4.

Corollary 5.5 (a) $\ker\ \sigma = \{a \in S :\ a\,e = e,\ for\ some\ e \in E\}$.
(b) *The following subsets of S are equal*

(1) $\ker\ \mu$.
(2) $\{a \in S :\ a\,e = e\,a,\ \ for\ any\ \ e \in E\}$.
(3) $\{a \in S :\ (e\,a)^* = e\,a^*,\ \ for\ any\ \ e \in E\}$.

(4) $\{a \in S : (a\,e)^{\dagger} = a^{\dagger}e, \ \textit{for any} \ e \in E\}$.

A natural partial order $\leq$ on S can be defined for any $a, \ b \in S$ by the rule:

$$a \ \leq \ b \ \text{ if and only if } \ a = b\,e \text{ for some } \ e \in E$$

or equivalently

$$a \ \leq \ b \text{ if and only if } \ a = b\,a^{*}.$$

We refer the reader to [18] for its properties.

Corollary 5.5 shows that the kernel of σ is the closure E_{ω} of E in S relative to the natural partial order $\leq$ on S and that the kernel of μ is the centralizer $E\xi$ of E in S. Consequently we have,

Corollary 5.6 *For any $x \in S$, if $x \in E\xi$, then*

$$x^{*} = x^{\dagger} \quad \textit{and} \quad x\,\mathcal{H}^{*}x^{\dagger}.$$

Proof For any $x \in S$, if $x \in E\,\xi$, then $e\,x = x\,e$ for all $e \in E$. In particular,

$$\begin{aligned} & x = x^{\dagger}x = x\,x^{\dagger} \ \text{ and thus } \ x^{*} = x^{*}\,x^{\dagger} \\ \text{and} \qquad & x = x\,x^{*} = x^{*}x \ \text{ and thus } \ x^{\dagger} = x^{*}\,x^{\dagger}. \\ \text{Hence} \quad & x^{\dagger} = x^{*} = e \ \text{ and } \ x\,\mathcal{H}^{*}e. \end{aligned}$$

□

6 Congruences with the same kernel

Green [14] introduces the concept of normal subsemigroups in the class of inverse semigroups. We extend this concept to the class of ample semigroups and define a *normal subsemigroup* of the ample semigroup S to be a full subsemigroup N of S which satisfies the following two conditions:

(1) for any $x, \ y \in S, \ n \in N, \ x\,y \in N$ together imply $x\,n\,y \in N$ and
(2) for any $x, \ y \in S, \ n \in N, \ x\,n\,y \in N$ together imply; $x\,n^{\dagger}y \in N$, $x\,n^{*}y \in N$.

As the semilattice E of S is a full subsemigroup satisfying condition (2) and if S is commutative, condition (1) holds for any subsemigroup of S, then if S is commutative, E is normal. However, there exist full subsemigroups of commutative ample monoids which do not satisfy condition (2). Recall that the set of numbers; $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are commutative ample monoids under the usual multiplication, so easy to see the set of non-negative integers is a normal subsemigroup of $\mathbb{Z}$ and $\mathbb{Q}$ is a normal subsemigroup of $\mathbb{R}$. While the set of real numbers represented by the interval $[-1, 1]$ is a full subsemigroup of $\mathbb{R}$ which is not normal as well as $\mathbb{Z}$ is not normal subsemigroup of $\mathbb{Q}$ (condition (2) is not satisfied). The last statement indicates that condition (1) is not sufficient for N to be normal in S. This is in contrast to the inverse case, for, if S is an inverse semigroup and N is a full inverse subsemigroup of S which satisfies condition (1), then for any $x,\ y \in S,\ n \in N$ such that $x\,n\,y \in N$, as $n^{-1} \in N$, we get - by (1) - $x\,n\,n^{-1}\,y \in N$ and $x\,n^{-1}\,n\,y \in N$. Therefore, condition (2) holds for N in S.

For more justification to the definition, we notice the condition (1) reduces to that of normal subgroups when both N and S are groups. In such case it is clear that for any $x,\ y \in S,\ n \in N$;

$$x\,y \in N \ \text{ implies } \ x\,n\,y \in N \ \text{ if and only if } \ N \ \text{ is normal.}$$

It follows that every non-normal subgroup does not satisfy condition (1). But there exist non-normal subgroups in which condition (2) holds. For example, consider the following two sets of matrices:

$$S = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x,\ y \in \mathbb{R},\ xy \neq 0 \right\} \text{ and } T = \left\{ \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} : x,\ y \in \mathbb{R},\ xy \neq 0 \right\}.$$

Put $G = S \cup T$. G is a group (so it is an ample monoid) under the matrix multiplication where $H = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} : 0 \neq x \in \mathbb{R} \right\}$ is a subgroup. Since for any $\alpha = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$ in T, $\alpha^{-1} = \begin{bmatrix} 0 & b^{-1} \\ a^{-1} & 0 \end{bmatrix}$ and if we take $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}$ in H provided that $x \neq 1$ we find $\alpha\,\gamma\,\alpha^{-1} \notin H$. Then H is not normal. Let $\alpha,\ \beta \in G,\ \gamma \in H$ $\left(\gamma^{\dagger} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \gamma^{*}\right)$.
Notice that:
when $\alpha,\ \beta \in S$ and $\alpha\,\gamma\,\beta \in H$, then $\alpha\,\beta \in H$,
when $\alpha \in S$ and $\beta \in T$, then $\alpha\,\gamma\,\beta \notin H$,
when $\alpha \in T$ and $\beta \in S$, then $\alpha\,\gamma\,\beta \notin H$,
when $\alpha,\ \beta \in T$, then $\alpha\,\gamma\,\beta \in H$ if $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $a\,d = 1$, where $\alpha = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}$, and in this case $\alpha\,\beta \in H$. Therefore, whenever $\alpha,\ \beta \in G$ and $\gamma \in H$ such that $\alpha\,\gamma\,\beta \in H$, we get $\alpha\,\beta \in H$. Hence H satisfies condition (2).
In conclusion, conditions (1) and (2) are independent.
Similar verification of [14] shows that this definition is reasonable. Moreover, it is a direct extension of the cited one. Further, it is based on the following observation:

Proposition 6.1 *The kernel of any admissible congruence γ on S is a normal subsemigroup of S.*

Proof It is evident that ker γ is a full subsemigroup of S. Let $x,\ y$ be in S, and n be in ker γ. Then $(n,\ e) \in \gamma$ for some $e \in E$. Thus $(x\,n\,y,\ x\,e\,y) \in \gamma$. if $x\,y \in \ker\,\gamma$, then

$$x\,e\,y = x\,y\,(e\,y)^{*} \in \ker\,\gamma \ \text{ and } \ x\,n\,y \in \ker\,\gamma.$$

Further, since γ is admissible, $(n, e) \in \gamma$, then $(n^*, e) \in \gamma$ and $(n, n^*) \in \gamma$. Thus we have $(x\,n\,y,\ x\,n^*y) \in \gamma$ and, clearly, $x\,n\,y \in \ker\gamma \implies x\,n^*y \in \ker\gamma$. Similarly, $x\,n\,y \in \ker\gamma$ implies that $x\,n^{\dagger}y \in \ker\gamma$. Hence the result holds. □

For any subset N of a semigroup S, the well known *syntactic congruence* η_N of N on S is defined as follows:

$$\eta_N = \{(a,\ b) \in S \times S;\ \text{for any } x,\ y \in S^1;\ x\,a\,y \in N \iff x\,b\,y \in N\}.$$

It is well known that η_N is a congruence and it is the maximum congruence for which N is union of η_N-classes. If $E(S) \subseteq N$, it follows immediately that $\ker\eta_N \subseteq N$; the other inclusion will follow if S is ample and N is normal. So we have the maximum congruence on S whose kernel is N. However, we will argue in a direct way to prove the result in the following Proposition:

Proposition 6.2 *Let N be a normal subsemigroup of the ample semigroup S. Then the relation η_N is the maximum congruence on S whose kernel is N.*

Proof Clearly η_N is congruence. Let $a \in S$.

$$\begin{aligned} a \in \ker\eta_N &\implies (a,\ e) \in \eta_N \quad \text{for some } \ e \in E \\ &\implies x\,a\,y \in N \ \text{ if and only if } \ x\,e\,y \in N\ (x,\ y \in S^1) \end{aligned}$$

Since $a^{\dagger}e\,a^* \in E \subseteq N$, then $a^{\dagger}a\,a^* \in N$ and $a \in N$. On the other hand, let $n \in N$. For any $x,\ y \in S^1$,

$$\begin{aligned} x\,n\,y \in N &\implies x\,n^*y \in N \quad (N \ \text{is normal}) \\ &\implies x\,n\,n^*y \in N \\ &\implies x\,n\,y \in N. \end{aligned}$$

Therefore, $(n,\ n^*) \in \eta_N$ and $n \in \ker\eta_N$. Hence $\ker\eta_N = N$. Finally, let ρ be a congruence on S whose kernel is N and for $a,\ b \in S$, let $(a,\ b) \in \rho$, then for any $x,\ y \in S^1$; $(x\,a\,y,\ x\,b\,y) \in \rho$. Therefore,

$$x\,a\,y \in \ker\rho \iff x\,b\,y \in \ker\rho.$$

$$\text{So that} \qquad x\,a\,y \in N \iff x\,b\,y \in N \text{ and } (a,\ b) \in \eta_N.$$

Hence the result holds. □

The following example shows that the syntactic congruence is not in general admissible congruence on an ample semigroup.

Example 6.3 Consider the ample monoid of integers $\mathbb{Z}$ with its normal subsemigroup $N = \{-1,\ 0,\ 1\}$. For syntactic congruence η_N of N in $\mathbb{Z}$ we can see that for any $a, b \in \mathbb{Z}\backslash N$;

$$x\,a\,y \in N \iff x = 0 \text{ or } y = 0 \iff x\,b\,y \in N;\ \ x,\ y \in \mathbb{Z}$$

Therefore, $(a, b) \in \eta_N$ for any $a, b \in \mathbb{Z}\backslash N$.
But $(1, a) \notin \eta_N$ if $a \in \mathbb{Z}\backslash N$ and $(0, x) \notin \eta_N$ for any $x \neq 0$ in $\mathbb{Z}$
The equivalence classes of η_N are:

$$0\,\eta_N = \{0\},\ 1\,\eta_N = \{-1, 1\},\ a\,\eta_N = \mathbb{Z}\backslash N\ (a \in \mathbb{Z}\backslash N)$$

Let F be three element band $\{u, i, e\}$ where u is its zero element and i is its identity element. Let ψ be the map from $\mathbb{Z}/\eta_N$ onto F define by:

$$(0\,\eta_N)\psi = u, (1\,\eta_N)\psi = i, (a\,\eta_N)\psi = e.$$

It is routine matter to check that ψ is a semigroup isomorphism. Notice that for any $a \in \mathbb{Z}\backslash N$, we have $(1, a) \in \mathcal{L}^*(\mathbb{Z})$. But $(1\,\eta_N)\psi = i$, $(a\,\eta_N)\psi = e$ where $(i, e) \notin \mathcal{L}(F)$, so that $(1\,\eta_N, a\,\eta_N) \notin \mathcal{L}^*(\mathbb{Z}/\eta_N)$.
Hence, η_N is not admissible.

Let S be the ample semigroup and N be a normal subsemigroup of S. To give an expression for a congruence whose kernel is N and is contained in any admissible congruence on S whose kernel is N, we will depend on the following Lemma.

Lemma 6.4 *Let N be a normal subsemigroup of S and τ_N be the relation on S defined by:*

$$\tau_N = \{(x\,n_1\,y,\ x\,n_2\,y) : x, y \in S^1; n_1, n_2 \in N; n_1^\dagger = n_2^\dagger\}.$$

Then

(1) *τ_N is reflexive, symmetric and compatible relation on S.*
(2) $N = \{a \in S : (a, e) \in \tau_N$ *for some* $e \in E\}$.
(3) *τ_N is contained in any admissible congruence on S whose kernel is N.*

Proof Clearly, (1) holds; To establish (2), let $a \in N$. As; $a = a^\dagger a\, a^*, a^\dagger a^* = a^\dagger a^\dagger a^*$, then $(a, a^\dagger a^*) \in \tau_N$ and $(a, e) \in \tau_N$ for some $e \in E$.
Conversely, suppose, there exists $f \in E$ such that $(a, f) \in \tau_N$ $(a \in S)$, then;

$$a = x\,n_1\,y,\ f = x\,n_2\,y; \quad n_1^\dagger = n_2^\dagger; \quad n_1, n_2 \in N$$

As $f \in N$ $(E \subseteq N)$,

$$\begin{aligned} f \in N &\implies x\,n_2\,y \in N \implies x\,n_2^\dagger y \in N \ \ (N \text{ is normal}) \\ &\implies x\,n_1^\dagger n_1 y \in N \implies x\,n_1\,y \in N \implies a \in N, \end{aligned}$$

and (2) holds.
If ρ is an admissible congruence on S whose kernel is N, then for any $a, b \in S$; $(a, b) \in \tau_N \implies a = x\,n_1 y,\ b = x\,n_2 y$ where $x, y \in S$, $n_1, n_2 \in N$, $n_1^\dagger = n_2^\dagger$. since $N = \ker \rho$, then there exists $e, f \in E$ such that; $(n_1, e) \in \rho$, $(n_2, f) \in \rho$. by the admissibility of ρ we get; $(n_1^\dagger, e) \in \rho$ and $(n_2^\dagger, f) \in \rho$ so that $(e, f) \in \rho$ and $(x\,e\,y, x\,f\,y) \in \rho$. But also; $(x\,n_1\,y, x\,e\,y) \in \rho$, $(x\,n_2\,y, x\,f\,y) \in \rho$. Hence $(a, b) \in \rho$ and (3) holds. □

Proposition 6.5 *Let λ_N be the transitive closure of τ_N ($\lambda_N = \tau_N^t$). Then λ_N is a congruence on S whose kernel is N and it is contained in any admissible congruence on S whose kernel is N.*

Proof It is evident that λ_N is congruence on S. If $n \in N$, then as in the proof of Lemma 6.4, $(n, e) \in \tau_N$ and thus $(n, e) \in \lambda_N$ for some $e \in E$, and $N \subseteq \ker \lambda_N$. On the other hand, if $a \in \ker \lambda_N$, then $(a, f) \in \lambda_N$ for some $f \in E$, and there exists; $a_1, a_2, \ldots, a_n \in S$ such that
$(a, a_1) \in \tau_N, (a_1, a_2) \in \tau_N, \ldots, (a_{n-1}, a_n) \in \tau_N, (a_n, f) \in \tau_N$. Notice that $(a_n, f) \in \tau_N \implies a_n = x n_1 y,\ f = x n_2 y;\quad n_1^\dagger = n_2^\dagger;\quad n_1, n_2 \in N, f \in N$

$$x n_2 y \in N \implies x n_2^\dagger y \in N \implies x n_1^\dagger n_1 y \in N \implies x n_1 y \in N \implies a_n \in N$$

Similarly, we get all the elements; $a_{n-1}, a_{n-2}, \ldots, a_1, a \in N$ and $\ker \lambda_N \subseteq N$. Hence $\ker \lambda_N = N$.

If ρ is an admissible congruence on S whose kernel is N, then by Lemma 6.4, $\tau_N \subseteq \rho$. Hence $\lambda_N \subseteq \rho$. □

Combine Propositions 6.2 and 6.5 to get:

Corollary 6.6 *If ρ is an admissible congruence on S and $N = \ker \rho$, then $\lambda_N \subseteq \rho \subseteq \eta_N$ and $\ker \lambda_N = \ker \rho = \ker \eta_N$.*

The proof of the following Proposition is omitted since it is similar to the one of Lemma 3.1(2) of [23].

Proposition 6.7 *Let ρ be an admissible congruence on S whose kernel is N. If λ_N is admissible on S, then S/ρ is idempotent-pure homomorphic image of S/λ_N.*

7 Congruence pairs

We extend in this section the concept of congruence pairs directly from inverse semigroups (see [14] or [21]) to ample semigourps and investigate to what extend this idea is useful in the study of congruences on ample semigroups. Certainly, there is a congruence pair associated with each admissible congruence ρ on the ample semigroup S, namely ($\operatorname{tr} \rho$, $\ker \rho$). We give an example to show that it is possible to have two distinct admissible congruences on an ample semigroup associated with the same congruence pair. Indeed we use in our example; cancellative congruences on a cancellative monoid. However, given a congruence pair, there is a congruence associated with that pair which contains (is contained in) any admissible congruence associated with it.

We facilitate the necessary and sufficient conditions of the existence of a congruence ρ on an inverse semigroup S' whose kernel is a given normal subsemigroup of S' and whose trace is a given normal congruence on $E(S')$ ([14], Proposition 3.9) to consider a definition of a congruence pair as follows:

Definition 7.1 Let N be a normal subsemigroup of the ample semigroup S and π be a normal congruence on E; $(\pi, N)$is a congruence pair for S if:

(1) *for any* $n \in N$; $n^{\dagger}\pi\, n^{*}$.
(2) *for any* $x,\ y \in S$, *and any* $e,\ f \in E$; $x\,e\,y \in N$ *and* $e\,\pi\, f$ *together imply* $x\, f\, y \in N$.

To justify our assertion, we state the following Lemma:

Lemma 7.2 *If* ρ *is an admissible congruence on* S*, then* $(\text{tr } \rho,\ \ker\, \rho)$ *is a congruence pair for* S.

Proof It follows from Sect. 3 that tr ρ is a normal congruence on E and from Proposition 6.1; ker ρ is a normal subsemigroup of S. In order to prove that $(\text{tr } \rho,\ \ker\, \rho)$ are congruence pair, let $n \in \ker\, \rho$. Then by Proposition 4.6, $(n^{\dagger},\ n^{*}) \in \text{tr } \rho$. Now let $x,\ y \in S$, $e,\ f \in E$. It is clear that,

$$(e,\ f) \in \text{tr } \rho \implies (e,\ f) \in \rho \implies (x\,e\,y,\ x\,f\,y) \in \rho.$$

Therefore,

$$x\,e\,y \in \ker\, \rho \ \text{ if and only if } \ x\,f\,y \in \ker\, \rho.$$

Hence $(\text{tr } \rho, \ker\, \rho)$ is a congruence pair for S. □

Let N be a normal subsemigroup of S, and π be a normal congruence on E such that $(\pi,\ N)$ is a congruence pair for S. We use μ_π of Theorem 3.6 and η_N of Proposition 6.2 to establish a congruence on S associated with the congruence pair $(\pi,\ N)$.

Proposition 7.3 *The relation* $\mu_\pi \cap \eta_N$ *is a congruence on* S *associated with the congruence pair* $(\pi,\ N)$.

Proof Put $\rho = \mu_\pi \cap \eta_N$. It is obvious that ρ is a congruence on S, and

$$\ker\, \rho \subseteq \ker\, \eta_N, \quad \ker\, \eta_N = N.$$

Let $n \in N$. Then for any $e \in E$ $(E \subseteq N)$, $e\,n \in N$, $(e\,n)^{\dagger}\pi\,(e\,n)^{*}$.
Notice that;$(e\,n^{*})^{*} = e\,n^{*}$, $e\,n^{*}\,\pi\, e\,n^{\dagger}$, $e\,n^{\dagger} = (e\,n)^{\dagger}$, $(e\,n)^{\dagger}\,\pi(e\,n)^{*}$.
Similarly; $(n\,e)^{\dagger}\,\pi(n\,e)^{*}$, $(n\,e)^{*} = n^{*}e$, $n^{*}e = (n^{*}e)^{\dagger}$ Therefore; $(n,\ n^{*}) \in \mu_\pi$. Recall from the proof of Proposition 6.2 that also $(n,\ n^{*}) \in \eta_N$. Hence $(n,\ n^{*}) \in \rho$ and $n \in \ker\, \rho$; that is, $N \subseteq \ker\, \rho$, and, $\ker\, \rho = N$.

To show that tr $\rho = \pi$, let $e,\ f \in E$ with $e\,\pi\, f$. Then from the definition of the congruence pair $(\pi,\ N)$, we have $x\,e\,y \in N$ if and only if $x\,f\,y \in N$; for all $x,\ y \in S$. Therefore $(e,\ f) \in \eta_N$. But also $(e,\ f) \in \mu_\pi$. Thus $(e,\ f) \in \rho$ and $\pi \subseteq \text{tr } \rho$. Conversely, since $\rho \subseteq \mu_\pi$, then for any $e,\ f \in E$, $(e,\ f) \in \rho$ implies $(e,\ f) \in \mu_\pi$ and thus $e\,\pi\, f$. Hence tr $\rho = \pi$ and the proof is complete. □

If ρ is an admissible congruence on S such that $\ker\, \rho = N$, tr $\rho = \pi$, then by Theorem 3.6, $\rho \subseteq \mu_\pi$ and by Proposition 6.2, $\rho \subseteq \eta_N$ (ρ need not to be admissible in this case). Therefore, $\rho \subseteq \mu_\pi \cap \eta_N$ and we have:

Corollary 7.4 *If ρ is an admissible congruence on S such that* $\ker\rho = N$, $\operatorname{tr}\rho = \pi$, *then $\rho \subseteq \mu_\pi \cap \eta_N$.*

To establish an analogue of Proposition 7.3, let N be a normal subsemigroup of S and π be a normal congruence on E such that (π, N) is a congruence pair for S. We recall σ_π of Theorem 3.2 and λ_N of Proposition 6.5 to form $\sigma_\pi \vee \lambda_N$.

Proposition 7.5 *The relation $\sigma_\pi \vee \lambda_N$ is a congruence on S associated with the congruence pair (π, N).*

Proof Put $\alpha = \sigma_\pi \vee \lambda_N$. It is well known (see [2] or [17]) that

$$\alpha = \bigcap\{\rho : \rho \text{ is a congruence on } S, \sigma_\pi \subseteq \rho, \lambda_N \subseteq \rho\}$$

and for any $a, b \in S$,

$$(a, b) \in \alpha \text{ if and only if there exist } a_1, a_2, \ldots, a_n \in S \text{ such that}$$
$$(a, a_1) \in \lambda_N, (a_1, a_2) \in \sigma_\pi, (a_2, a_3) \in \lambda_N, \ldots, (a_{n-1}, a_n) \in \sigma_\pi, (a_n, b) \in \lambda_N.$$

and α is a congruence on S, that is, the minimum congruence containing both σ_π and λ_N.

Let $a \in \ker\alpha$. Then there exists $e \in E$, $(a, e) \in \alpha$ so that for

$$a_1, a_2, \ldots, a_n \in S$$
$$(a, a_1) \in \lambda_N, (a_1, a_2) \in \sigma_\pi, (a_2, a_3) \in \lambda_N, \ldots, (a_{n-1}, a_n) \in \sigma_\pi, (a_n, e) \in \lambda_N.$$

As $(a_n, e) \in \lambda_N$; by Proposition 6.5, $a_n \in N$, $(a_{n-1}, a_n) \in \sigma_\pi$.
By definition of σ_π (Theorem 3.2)

$$a^*_{n-1} \, \pi \, a^*_n, \quad a_{n-1}\, e = a_n\, e \text{ for some } e \in E, \quad e\, \pi\, a^*_{n-1}.$$

As N is normal, $a_n \in N$, then $a_n\, e \in N$ and thus $a_{n-1}\, e \in N$, that is

$$a_{n-1}\; e\; a^*_{n-1} \in N, \quad e\, \pi\, a^*_{n-1}.$$

By the definition of congruence pair, we get:

$$a_{n-1} = a_{n-1}\; a^*_{n-1}\; a^*_{n-1} \in N$$

so that $a_{n-1} \in N$, $(a_{n-2}, a_{n-1}) \in \lambda_N$.
Then by the same argument as in the proof of Proposition 6.5, we conclude that $a_{n-2} \in N$. The present step is $a_{n-2} \in N$; $(a_{n-3}, a_{n-2}) \in \sigma_\pi$. So as the argument of the above, we get $a_{n-3} \in N$.

The process will continue until we reach $a \in N$. Hence ker $\alpha \subseteq N$. On the other hand,

$$\begin{aligned} n \in N &\implies n \in \ker \lambda_N. \ \text{(Proposition 6.5)} \\ &\implies (n, e) \in \lambda_N \quad \text{for some } e \in E \\ &\implies (n, e) \in \alpha \quad \text{for some } e \in E \\ &\implies n \in \ker \alpha \end{aligned}$$

Therefore; $N \subseteq \ker \alpha$. Hence $\ker \alpha = N$.
To show that tr $\alpha = \pi$, let $e, f \in E$.
Notice that,

$$(e, f) \in \pi \implies (e, f) \in \sigma_\pi \implies (e, f) \in \alpha$$

and $\pi \subseteq \operatorname{tr} \alpha$.

For the other inclusion, let $e, f \in E$ such that $(e, f) \in \operatorname{tr} \alpha$, that is, $(e, f) \in \alpha$. Then for some elements $a_1, a_2, \ldots, a_n \in S$; $(e, a_1) \in \lambda_N$, $(a_1, a_2) \in \sigma_\pi$, $(a_2, a_3) \in \lambda_N, \ldots, (a_{n-1}, a_n) \in \sigma_\pi$, $(a_n, f) \in \lambda_N$.
But, $(e, a_1) \in \lambda_N$ implies $(e, b_1) \in \tau_N$, $(b_1, b_2) \in \tau_N, \ldots, (b_{k-1}, b_k) \in \tau_N$, $(b_k, a_1) \in \tau_N$. where $b_1, b_2, \ldots, b_n \in S$ and τ_N of Lemma 6.4. Notice that, $(e, b_1) \in \tau_N$ implies for some $x, y \in S$, $n_1, n_2 \in N$;

$$e = x n_1 y, \ b_1 = x n_2 y; \quad n_1^\dagger = n_2^\dagger$$

Since for any $n \in N$, $n^* \pi n^\dagger$, and then for any $x, y \in S$, $n y^\dagger \in N$ and $(n y^\dagger)^\dagger \pi (n y^\dagger)^*$;

$$(n y)^\dagger = (n y^\dagger)^\dagger, \quad \text{thus } (n y)^\dagger \pi (n y^\dagger)^*$$

By the normality of π,

$$(x (n y)^\dagger)^\dagger \pi (x (n y^\dagger)^*)^\dagger.$$

That is, $(x n y)^\dagger \pi (x (n y^\dagger)^*)^\dagger$. In particular, we have

$$(x n_1 y)^\dagger \pi (x (n_1 y^\dagger)^*)^\dagger \quad \text{and} \quad (x n_2 y)^\dagger \pi (x (n_2 y^\dagger)^*)^\dagger.$$

Since $n_1^* \pi n_1^\dagger$, $n_1^\dagger = n_2^\dagger$ and $n_2^\dagger \pi n_2^*$. Then $n_1^* \pi n_2^*$ and $n_1^* y^\dagger \pi n_2^* y^\dagger$ so that $(n_1 y^\dagger)^* \pi (n_2 y^\dagger)^*$. Again, by normality of π, $(x (n_1 y^\dagger)^*)^\dagger \pi (x (n_2 y^\dagger)^*)^\dagger$. Therefore, $(x n_1 y)^\dagger \pi (x n_2 y)^\dagger$ and $e \pi b_1^\dagger$. Also,

$$(b_1, b_2) \in \tau_N, \quad b_1 = x_1 n_3 y_1 \text{ and } b_2 = x_1 n_4 y_1 \text{ where } n_3^\dagger = n_4^\dagger.$$

By similar argument to the above, we get $b_1^\dagger \,\pi\, b_2^\dagger$ and so, $b_2^\dagger \,\pi\, b_3^\dagger, \ldots, b_k^\dagger \,\pi\, a_1^\dagger$. Hence $e \,\pi\, a_1^\dagger$. As $(a_1, a_2) \in \sigma_\pi$, then $a_1^\dagger \,\pi\, a_2^\dagger$ and $(a_2, a_3) \in \lambda_N$. By the same procedure as before, we get $a_2^\dagger \,\pi\, a_3^\dagger$. Thus $a_3^\dagger \,\pi\, a_4^\dagger,\ a_4^\dagger \,\pi\, a_5^\dagger, \ldots, a_{n-1}^\dagger \,\pi\, a_n^\dagger,\ a_n^\dagger \,\pi\, f$. Therefore; $e \,\pi\, f$ and $\operatorname{tr} \alpha \subseteq \pi$. Hence $\operatorname{tr} \alpha = \pi$. And we conclude that, α is a congruence associated with the congruence pair (π, N).

Let ρ be an admissible congruence on S. If N is the kernel of ρ and π is the trace of ρ. Then by Theorem 3.2, $\sigma_\pi \subseteq \rho$ (ρ need not to be admissible in this case) and by Proposition 6.5 $\lambda_N \subseteq \rho$. Hence $\sigma_\pi \vee \lambda_N \subseteq \rho$.

Corollary 7.6 *If ρ is an admissible congruence on S such that;* $\ker \rho = N$, $\operatorname{tr} \rho = \pi$, *then $\sigma_\pi \vee \lambda_N \subseteq \rho$.*

Considering Lemma 7.2, we may combine Propositions 7.3 and 7.5 and Corollaries 7.4 and 7.6 to have the following result:

Theorem 7.7 *If ρ is an admissible congruence on S whose kernel is N and trace is π, then $\mu_\pi \cap \eta_N$ and $\sigma_\pi \vee \lambda_N$ are congruences associated with the congruence pair (π, N). Moreover;*

$$\sigma_\pi \vee \lambda_N \subseteq \rho \subseteq \mu_\pi \cap \eta_N.$$

Theorem 7.7 can be compared with ([14] Theorem 3.8) in the case of inverse semigroups where we have the equality of the two congruences, $\sigma_\pi \vee \lambda_N$ and $\mu_\pi \cap \eta_N$ and thus a characterization of the congruence associated with a congruence pair (π, N). We did not hope to have the equality of the two congruences in the case of ample semigroups. The congruences constructed in Propositions 7.3 and 7.5 may not be admissible congruences. Moreover, even if those congruences which are associated with a given congruence pair were admissible congruences, we can not hope for a characterization of admissible congruences in this way. Distinct admissible congruences may have the same trace and kernel and be associated with the same congruence pair. The next example where the congruences involved are cancellative congruences on a cancellative monoid demonstrates this fact:

Example 7.8 Let $M = \{a, b\}^*$ be the free monoid on the elements a and b. Let π be the universal relation and $N = \{1\}$. Consider the congruence pair (π, N) for M. Let $T = \{c\}^*$ be the free monoid on the element c. Let $\phi : M \to T$ be the admissible homomorphism determined by; $a\phi = c = b\phi$ and put $\rho_1 = \phi \circ \phi^{-1}$. Then ρ_1 is admissible congruence, $\operatorname{tr} \rho_1 = \pi$ and $\ker \rho_1 = N$. Let $\psi : M \to T$ be the admissible homomorphism determined by; $a\psi = c,\ b\psi = c^2$ and put $\rho_2 = \psi \circ \psi^{-1}$. Then ρ_2 is admissible congruence, $\operatorname{tr} \rho_2 = \pi$ and $\ker \rho_2 = N$. So we can have two different admissible congruences on M associated with the same congruence pair (π, N). □

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