The variety of commutative additively and multiplicatively idempotent semirings

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Abstract The variety $\mathcal{Z}$ of commutative additively and multiplicatively idempotent semirings is studied. We prove that $\mathcal{Z}$ is generated by a single subdirectly irreducible three-element semiring and it has a canonical form for its terms. Hence, $\mathcal{Z}$ is locally finite despite the fact that it is residually large. The word problem in $\mathcal{Z}$ is solvable.

Keywords Semiring · Commutative · Additively idempotent · Multiplicatively idempotent · Variety · Locally finite · Residually large · Word problem

The concept of semiring is surprisingly successful in applications both in mathematics and computer science as it was shown by Golan in [6] and Kuich and Salomaa in [8]. In this paper we concentrate on the variety $\mathcal{C}$ of commutative multiplicatively idempotent semirings which was investigated by the authors already in [2–4] and in the subvariety of this variety satisfying $x + x \approx x$. The concept of a Boolean semiring was introduced by Guzmán in [7]. It is a commutative multiplicatively idempotent semiring satisfying

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the identity $1 + x + x \approx 1$. The variety $\mathcal{B}$ of Boolean semirings contains just two subdirectly irreducible members, namely the two-element (distributive) lattice and the two-element Boolean ring. On the contrary, $\mathcal{C}$ has a proper class of subdirectly irreducible members, i. e. it is residually large, as it was proved by the authors in [2]. The natural question arises if by adding another elementary identity to the identities defining $\mathcal{C}$ the number of subdirectly irreducible members can be reduced. In our previous paper we studied members of $\mathcal{C}$ satisfying $x + y + xyz \approx x + y$. In the present paper we investigate the subvariety $\mathcal{Z}$ of $\mathcal{C}$ determined by $x + x \approx x$ and make frequent use of the identity $x + y + xy \approx x + y$. That this identity indeed holds in $\mathcal{Z}$ can be seen as follows:

$$x + y + xy \approx x^2 + xy + yx + y^2 \approx (x + y)^2 \approx x + y.$$

We show that $\mathcal{Z}$ is locally finite and we provide an explicit upper bound for the cardinalities of finite free algebras in $\mathcal{Z}$. Our main result is to prove that $\mathcal{Z}$ is generated by a single three-element subdirectly irreducible semiring. Moreover, it turns out that there are single semirings of arbitrary cardinality generating $\mathcal{Z}$.

For the reader's convenience, all the concepts concerning semirings are included. What concerns concepts from Universal Algebra necessary for understanding the paper, the reader is referred to [1].

We start with the definition of a semiring in the sense of the monograph [6] by Golan.

Definition 1 A *semiring* is an algebra $\mathbf{S} = (S, +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ such that

- $(S, +, 0)$ is a commutative monoid.
- $(S, \cdot, 1)$ is a monoid.
- The operation $\cdot$ is distributive with respect to $+$.
- $x0 = 0x = 0$ for all $x \in S$

$\mathbf{S}$ is called *commutative* if $\cdot$ is commutative, *additively idempotent* if $+$ is idempotent and *multiplicatively idempotent* if $\cdot$ is idempotent. Let $\mathcal{Z}$ denote the variety of commutative additively and multiplicatively idempotent semirings.

The corresponding semiring of type $(2, 2)$ has been called a $\cdot$ -distributive bisemilattice by Romanowska and a bisemilattice by Pastijn and Zhao. McKenzie and Romanowska [9] showed that the variety of all bisemilattices has exactly 5 subvarieties. Based on this result, Ghosh et al. [5] and Pastijn [10] have shown that the variety of all semirings with type $(2, 2)$ whose additive semigroup is a semilattice and multiplicative semigroup is a band has exactly 78 subvarieties. In this process, the identity $x + y + xy \approx x + y$ plays a role.

Every bounded distributive lattice $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ is a semiring in $\mathcal{Z}$. On the contrary, non-trivial Boolean rings with unit 1 do not belong to $\mathcal{Z}$ since in every non-trivial Boolean ring with 1 we have $1 + 1 + 1 \cdot 1 = 1 \neq 0 = 1 + 1$. In the following we present an example of a semiring in $\mathcal{Z}$ which is not a lattice since it does not satisfy the absorption law.

Example 2 The algebra $\mathbf{S}_3 := (S_3, +, \cdot, 0, 1)$ with $S_3 = \{0, a, 1\}$,

$$\begin{array}{c|ccc} + & 0 & a & 1 \\ \hline 0 & 0 & a & 1 \\ a & a & a & a \\ 1 & 1 & a & 1 \end{array} \quad \text{and} \quad \begin{array}{c|ccc} \cdot & 0 & a & 1 \\ \hline 0 & 0 & 0 & 0 \\ a & 0 & a & a \\ 1 & 0 & a & 1 \end{array}$$

belongs to $\mathcal{Z}$. Because of $1(1 + a) = a \neq 1$, $\mathbf{S}_3$ is not a lattice. Let us remark that $\mathbf{S}_3$ satisfies the so-called weak-absorption law $x(x + y) \approx x + xy$.

It is easy to see that $\mathbf{S}_3$ is the unique semiring in $\mathcal{Z}$ with cardinality less than 4 which is not a lattice.

Now we repeat the definitions of certain members of $\mathcal{Z}$ introduced in [2]:

Definition 3 For every integer $n > 1$ let $\mathbf{S}_n$ be the semiring with universe $S_n = \{1, \dots, n\}$ and operations defined as follows:

$$x + y := \begin{cases} \max(x, y) & \text{for } x \text{ and } y \text{ are odd} \\ y & \text{for } x \text{ is odd and } y \text{ is even} \\ x & \text{for } x \text{ is even and } y \text{ is odd} \\ \min(x, y) & \text{for } x \text{ and } y \text{ are even} \end{cases}$$

and

$$xy := \min(x, y)$$

($x, y \in S_n$). Here the operation symbols 0 and 1 are interpreted as 1 and n , respectively. Moreover, for every infinite cardinal k let $\mathbf{C}_k = (C_k, \leq, 0, 1)$ be a bounded chain of cardinality k and $\mathbf{U}_k$ be the semiring with universe $U_k := C_k \times \{1, 2\}$ and operations defined as follows:

$$(x, i) + (y, j) := \begin{cases} (\max(x, y), 1) & \text{for } (i, j) = (1, 1) \\ (y, 2) & \text{for } (i, j) = (1, 2) \\ (x, 2) & \text{for } (i, j) = (2, 1) \\ (\min(x, y), 2) & \text{for } (i, j) = (2, 2) \end{cases}$$

and

$$(x, i)(y, j) := \begin{cases} (x, i) & \text{for } x < y \\ (x, \min(i, j)) & \text{for } x = y \\ (y, j) & \text{for } x > y \end{cases}$$

((x, i), (y, j) $\in U_k$). Here the operation symbols 0 and 1 are interpreted as (0, 1) and (1, 2), respectively.

It is easy to see that the semiring $\mathbf{S}_3$ of Example 2 is isomorphic to the semiring $\mathbf{S}_3$ of Definition 3.

It was proved in [2] that $\mathbf{S}_3$ is subdirectly irreducible. Moreover, in this paper it was shown that for every integer $n > 1$ the semiring $\mathbf{S}_n$ is a subdirectly irreducible member of $\mathcal{Z}$ of cardinality n and that for every infinite cardinal k the semiring $\mathbf{U}_k$ is a subdirectly irreducible member of $\mathcal{Z}$ of cardinality k . Hence $\mathcal{Z}$ is residually large.

We are going to derive a canonical form of terms in $\mathcal{Z}$.

In the following let n denote an arbitrary, but fixed non-negative integer and put $N := \{1, \dots, n\}$.

Lemma 4 Every term $t(x_1, \dots, x_n)$ in $\mathcal{Z}$ can be written in the form

$$t_A(x_1, \dots, x_n) = \sum_{I \in A} \prod_{s \in I} x_s \text{ with } A \subseteq 2^N \quad (1)$$

where the empty sum is defined as 0 and the empty product as 1.

Proof Let W denote the set of all sums of products of elements of $\{x_1, \dots, x_n\}$. Obviously, the elements of W are terms in $\mathcal{Z}$. It is easy to see that $0, 1 \in W$ and that the sum of two elements of W again belongs to W . Using the distributive law one can see that also the product of two elements of W again belongs to W . Idempotency of addition and multiplication shows that the elements of W can be written in the form (1). $\square$

Corollary 5 There exist only finitely many different terms in $\mathcal{Z}$ of fixed finite arity which means that $\mathcal{Z}$ is locally finite. The number of different n -ary terms in $\mathcal{Z}$ is at most 2^{2^n} .

In fact, one can show that $\mathcal{Z}$ is locally finite by another method. Indeed, if $\mathbf{S}$ is a semiring whose additive semigroup is a semilattice, then $\mathbf{S}$ is locally finite if and only if $(S, \cdot)$ is locally finite. It is well-known that every semilattice is locally finite.

Within $\mathcal{Z}$ we can write terms in a more economic way.

Definition 6 A subset A of 2^N is called *reduced* if there does not exist an integer $k > 2$ and pairwise different elements $I_1, \dots, I_k$ of A with $I_1 \cup \dots \cup I_{k-1} = I_k$. Let $\mathbb{A}$ denote the set of all reduced subsets of 2^N . The term t_A is called *reduced* if $A \in \mathbb{A}$.

At first, we show that the identity $x + y + xy \approx x + y$ can be extended to arbitrary products of variables as follows:

Lemma 7 We have

$$\mathcal{Z} \models \prod_{i \in I} x_i + \prod_{i \in J} x_i \approx \prod_{i \in I} x_i + \prod_{i \in J} x_i + \prod_{i \in I \cup J} x_i$$

for all $I, J \subseteq N$.

Proof

$$\mathcal{Z} \models \prod_{i \in I} x_i + \prod_{i \in J} x_i \approx \prod_{i \in I} x_i + \prod_{i \in J} x_i + \prod_{i \in I} x_i \prod_{i \in J} x_i \approx \prod_{i \in I} x_i + \prod_{i \in J} x_i + \prod_{i \in I \cup J} x_i$$

$\square$

In the following we are going to show that every term in $\mathcal{Z}$ can be reduced.

Theorem 8 *To every $A \subseteq 2^N$ there exists some $B \in \mathbb{A}$ with $\mathcal{Z} \models t_A \approx t_B$.*

Proof Let $A \subseteq 2^N$. If $A \in \mathbb{A}$ then we are done. Hence assume $A \notin \mathbb{A}$. Then there exist an integer $k > 2$ and pairwise different elements $I_1, \dots, I_k$ of A with $I_1 \cup \dots \cup I_{k-1} = I_k$. Then by Lemma 7 we have

$$\begin{aligned} \mathcal{Z} \models \prod_{i \in I_1} x_i + \dots + \prod_{i \in I_k} x_i &\approx \prod_{i \in I_1} x_i + \prod_{i \in I_2} x_i + \prod_{i \in I_1 \cup I_2} x_i + \prod_{i \in I_3} x_i + \dots + \prod_{i \in I_k} x_i \approx \\ &\approx \prod_{i \in I_1} x_i + \prod_{i \in I_2} x_i + \prod_{i \in I_1 \cup I_2} x_i + \prod_{i \in I_3} x_i + \prod_{i \in I_1 \cup I_2 \cup I_3} x_i + \prod_{i \in I_4} x_i + \dots + \prod_{i \in I_k} x_i \approx \dots \approx \\ &\approx \prod_{i \in I_1} x_i + \prod_{i \in I_2} x_i + \prod_{i \in I_1 \cup I_2} x_i + \prod_{i \in I_3} x_i + \dots + \prod_{i \in I_1 \cup \dots \cup I_{k-2}} x_i + \prod_{i \in I_{k-1}} x_i + \prod_{i \in I_k} x_i \approx \\ &\approx \prod_{i \in I_1} x_i + \prod_{i \in I_2} x_i + \prod_{i \in I_1 \cup I_2} x_i + \prod_{i \in I_3} x_i + \dots + \prod_{i \in I_1 \cup \dots \cup I_{k-2}} x_i + \prod_{i \in I_{k-1}} x_i \approx \dots \approx \\ &\approx \prod_{i \in I_1} x_i + \prod_{i \in I_2} x_i + \prod_{i \in I_1 \cup I_2} x_i + \prod_{i \in I_3} x_i + \prod_{i \in I_1 \cup I_2 \cup I_3} x_i + \prod_{i \in I_4} x_i + \dots + \prod_{i \in I_{k-1}} x_i \approx \\ &\approx \prod_{i \in I_1} x_i + \prod_{i \in I_2} x_i + \prod_{i \in I_1 \cup I_2} x_i + \prod_{i \in I_3} x_i + \dots + \prod_{i \in I_{k-1}} x_i \approx \prod_{i \in I_1} x_i + \dots + \prod_{i \in I_{k-1}} x_i \end{aligned}$$

and hence

$$\mathcal{Z} \models t_A \approx t_{A \setminus \{I_k\}}.$$

Either $A \setminus \{I_k\} \in \mathbb{A}$ or again one element of $A \setminus \{I_k\}$ can be cancelled. After a finite number of steps one finally ends up with some $B \in \mathbb{A}$ satisfying $\mathcal{Z} \models t_A \approx t_B$ (every subset of 2^N having less than three elements automatically belongs to $\mathbb{A}$). $\square$

Corollary 9 *For $n = 0, 1, 2$ resp. 3 there are exactly 2, 4, 14 resp. 122 reduced n -ary terms in $\mathcal{Z}$.*

Proof The proof is evident. $\square$

Now we state and prove our main theorem saying that $\mathcal{Z}$ is generated by $\mathbf{S}_3$.

Theorem 10 *If $A, B \in \mathbb{A}$ then $\mathbf{S}_3 \models t_A \approx t_B$ if and only if $A = B$.*

Proof Assume $t_A \approx t_B$ and $A \neq B$. Without loss of generality $A \setminus B \neq \emptyset$. Let $I \in A \setminus B$. Since $I = \emptyset$ would imply $1 = t_A(0, \dots, 0) = t_B(0, \dots, 0) = 0$ we have $I \neq \emptyset$. Let $i \in I$. Put

$$a_j := \begin{cases} 0 & \text{if } j \in N \setminus I \\ a & \text{if } j = i \\ 1 & \text{if } j \in I \setminus \{i\} \end{cases}$$

Then $t_A(a_1, \dots, a_n) = a$. Hence we have $t_B(a_1, \dots, a_n) = a$. From this we conclude that there exists some $J_i \in B$ with $i \in J_i \subseteq I$. Since $I \notin B$ we have $J_i \subsetneq I$. Now define

$$b_j := \begin{cases} 0 & \text{if } j \in N \setminus J_i \\ a & \text{if } j = i \\ 1 & \text{if } j \in J_i \setminus \{i\} \end{cases}$$

Then $t_B(b_1, \dots, b_n) = a$. Hence $t_A(b_1, \dots, b_n) = a$ which shows that there exists some $K_i \in A$ with $i \in K_i \subseteq J_i$. Therefore $i \in K_i \subsetneq I$. Obviously,

$$I = \bigcup_{i \in I} K_i.$$

Let $K_1, \dots, K_s$ denote the pairwise distinct elements of $\{K_i \mid i \in I\}$. Then $s > 1$, $K_1, \dots, K_s, I$ are pairwise distinct and $K_1 \cup \dots \cup K_s = I$. This contradicts $A \in \mathbb{A}$. Hence $A = B$. $\square$

Remark 11 It follows from Theorem 10 that B in Theorem 8 is uniquely determined by A . So we could write $B = f(A)$ in Theorem 8. For $A, B \subseteq 2^N$ we then have $\mathcal{Z} \models t_A \approx t_B$ if and only if $f(A) = f(B)$. Thus, the word problem in $\mathcal{Z}$ is solvable.

Remark 12 From Corollary 9 and Theorem 10 we conclude that the free algebras in $\mathcal{Z}$ with 0, 1, 2 resp. 3 free generators have 2, 4, 14 resp. 122 elements.

From Theorem 10 we finally obtain

Theorem 13 $\mathcal{Z}$ is generated by $\mathbf{S}_3$. Moreover, $\mathcal{Z}$ is generated by every $\mathbf{S}_n$ for each integer $n > 2$ and by every $\mathbf{U}_k$ for each infinite cardinal k .

Proof Let $A, B \subseteq 2^N$ and assume $\mathbf{S}_3 \models t_A \approx t_B$. According to Theorem 8 there exist $C, D \in \mathbb{A}$ with $\mathcal{Z} \models t_C \approx t_A$ and $\mathcal{Z} \models t_D \approx t_B$. We conclude $\mathbf{S}_3 \models t_C \approx t_D$. According to Theorem 10 we obtain $C = D$ and hence $\mathcal{Z} \models t_A \approx t_C = t_D \approx t_B$. Therefore $\mathbf{S}_3$ generates $\mathcal{Z}$. If $n > 2$ is an integer then $\mathbf{S}_n$ has a subalgebra being isomorphic to $\mathbf{S}_3$, namely the subalgebra with base set $\{1, 2, n\}$. Hence, every identity holding in $\mathbf{S}_n$ also holds in $\mathcal{Z}$. Analogously, if k is an infinite cardinal then $\mathbf{U}_k$ has a subalgebra being isomorphic to $\mathbf{S}_3$, namely the subalgebra with base set $\{(0, 1), (0, 2), (1, 2)\}$. Hence, every identity holding in $\mathbf{U}_k$ also holds in $\mathcal{Z}$. This completes the proof of the theorem. $\square$

Remark 14 As mentioned in Remark 12 and Corollary 9, the free algebra $\mathbf{F}_{\mathcal{Z}}(x)$ with one free generator x in $\mathcal{Z}$ consists just of the four elements $0, 1, x, 1 + x$. It is easy to see that the equivalence relation Θ with classes $\{0\}, \{x, 1 + x\}, \{1\}$ is a congruence on $\mathbf{F}_{\mathcal{Z}}(x)$ and $(\mathbf{F}_{\mathcal{Z}}(x))/\Theta \cong \mathbf{S}_3$. This shows that also $\mathbf{F}_{\mathcal{Z}}(x)$ generates $\mathcal{Z}$.

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