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On right chain ordered semigroups

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Abstract A right chain ordered semigroup is an ordered semigroup whose right ideals form a chain. In this paper we study the ideal theory of right chain ordered semigroups in terms of prime ideals, completely prime ideals and prime segments, extending to these semigroups results on right chain semigroups proved in Ferrero et al. (J Algebra 292:574–584, 2005).

1 Introduction and preliminaries

Problems studied in this paper have their roots in the theory of chain rings. Recall that a ring R with unity is said to be a *right* (respectively *left*) *chain ring* if its right (respectively left) ideals form a chain, i.e., are totally ordered by set inclusion. If R

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is a right and left chain ring, then R is called a *chain ring*. These rings are natural generalizations of commutative valuation rings to the noncommutative case and they have been extensively studied in many papers.

In 1976, Brungs and Törner proved in [6, Theorem 3.6] that a semi-invariant chain ring is invariant provided it satisfies d.c.c. for prime ideals. This result indicated the importance of the structure of the lattice of prime ideals of a chain ring. In [1, Theorem 3.5], Bessenrodt, Brungs and Törner noted that in a right chain ring, a prime ideal which is not completely prime is always pairing with a unique completely prime ideal, and this result drew attention to the structure of the lattice of completely prime ideals of a right chain ring. As noted in [7], essential for the understanding of the ideal theory of right chain rings R is the understanding of the ideals between two neighbouring completely prime ideals $P_1 \supset P_2$ of R; such a pair (P_1, P_2) is called a *prime segment* of R. In [7, Theorem 2.2], Brungs and Törner proved that a prime segment of a right chain ring falls in exactly one of three classes: it is either archimedean, or simple, or exceptional. An analogous classification for prime segments of Dubrovin valuation rings was obtained by Brungs et al. [2], and for so called semiprime segments of any ring by Törner and the third author in [13]. Further results on prime segments can be found in, e.g., [3–5,12].

A natural generalization of right chain rings are *right chain semigroups*, i.e., semigroups with unity whose right ideals form a chain. Examples of right chain semigroups include the cones of left ordered groups and the multiplicative semigroups of right chain rings. In [8] Brungs and Törner extended the ideal theory of right chain rings in terms of prime ideals, completely prime ideals and prime segments to right cones, that is to right chain semigroups with a left cancellation law. In particular, in [8, Theorem 1.14] Brungs and Törner classified prime segments of a right cone as either archimedean, or simple, or exceptional. An analogous classification of prime segments $P_1 \supset P_2$ for right P_1-comparable semigroups was obtained by Halimi in [11, Theorem 4.8]. In [9], Ferrero, Sant'Ana, and the third author generalized the ideal theory of right cones to right chain semigroups, but in this case it was necessary to add to the three known types of prime segments (archimedean, simple, exceptional) another one, which was named "supplementary".

In this paper we do a next step, namely we introduce right chain ordered semigroups and extend to them the ideal theory of right chain semigroups developed in [9]. Below in this section we explain why the new class of semigroups is a generalization of right chain semigroups.

Recall (see, e.g., [10]) that an *ordered semigroup* $(S, \cdot, \leq)$ is a semigroup $(S, \cdot)$ together with a partial order $\leq$ that is compatible with the semigroup operation, i.e., for any $x, y, z \in S$ we have

$$x \leq y \Rightarrow xz \leq yz \text{ and } zx \leq zy.$$

For nonempty subsets A, B of S we define

$$(A] = \{s \in S : s \leq a \text{ for some } a \in A\} \text{ and } AB = \{ab : a \in A, b \in B\}.$$

A nonempty subset I of an ordered semigroup $(S, \cdot, \leq)$ is called a *right* (respectively, *left*) *ideal* of S if it satisfies the following conditions:

(1) $IS \subseteq I$ (respectively, $SI \subseteq I$);
(2) $I = (I]$, that is, for any $s \in S$ and $a \in I$, $s \leq a$ implies $s \in I$.

If I is both a left and a right ideal of S, then I is called a *two-sided ideal* of S, or simply an *ideal* of S.

Definition 1.1 An ordered semigroup $(S, \cdot, \leq)$ is called a *right chain ordered semigroup* if the right ideals of S form a chain, i.e., for any right ideals I, J of S we have $I \subseteq J$ or $J \subseteq I$. Left chain semigroups are defined analogously, and S is a *chain ordered semigroup* if it is a right and left chain ordered semigroup.

Note that any semigroup $(S, \cdot)$ is an ordered semigroup with respect to the trivial order $\leq$ on S (i.e., the order defined by $x \leq y \Leftrightarrow x = y$). Furthermore, if $\leq$ is the trivial order on S, then a subset $A \subseteq S$ is a right ideal of the ordered semigroup $(S, \cdot, \leq)$ if and only if A is a right ideal of the semigroup $(S, \cdot)$. Hence right chain semigroups are exactly right chain ordered semigroups with respect to the trivial order. Therefore, the notion of a right chain ordered semigroup generalizes the notion of a right chain semigroup. It is also obvious that any general result on right chain ordered semigroups $(S, \cdot, \leq)$, when applied to the trivial order $\leq$ on S, gives its counterpart for the right chain semigroup $(S, \cdot)$.

An element e of an ordered semigroup $(S, \cdot, \leq)$ is called an *identity element* of S if $ex = x = xe$ for any $x \in S$. An element 0 of S is called a *zero element* of S if $0x = 0 = 0x$ for any $x \in S$. In this paper we assume that each ordered semigroup is with identity element e, and with zero element 0, and $e \neq 0$.

The following example shows that a right chain ordered semigroup need not be a right chain semigroup.

Example 1.2 The set $T = \{a, b\}$ with the multiplication $xy = x$ for any $x, y \in T$ is a semigroup (without zero and identity). Let S be the semigroup obtained from T by adjoining zero and unity elements, i.e., $S = \{0, e, a, b\}$ and the multiplication in S is defined as follows:

$$xy = \begin{cases} x \text{ if } x \neq e \text{ and } y \neq 0, \\ y \text{ if } x = e \text{ or } y = 0. \end{cases}$$

Then $\{0, a\}$ and $\{0, b\}$ are incomparable right ideals of the semigroup $(S, \cdot)$ and thus S is not a right chain semigroup. On the other hand, S (with the above multiplication) is an ordered semigroup with respect to the order

$$x \leq y \Leftrightarrow x = y \text{ or } (x, y) = (a, b),$$

and the only right ideals of the ordered semigroup $(S, \cdot, \leq)$ are

$$\{0\} \subset \{0, a\} \subset \{0, a, b\} \subset S.$$

Hence $(S, \cdot, \leq)$ is a right chain ordered semigroup.

The paper is organized as follows. In Sect. 2 we study relationships between prime, semiprime, completely prime, and completely semiprime right ideals of an ordered semigroup $(S, \cdot, \leq)$, and we provide some methods for constructing such right ideals. In Sect. 3 we show that if furthermore $(S, \cdot, \leq)$ is a right chain ordered semigroup, then for any right ideal of S, being semiprime is equivalent to being prime, and being completely semiprime is the same as being completely prime. Moreover, in Sect. 3 we use powers of an ideal and powers of an element to construct completely prime right ideals. In Sect. 4 we focus on prime segments of right chain ordered semigroups, proving that any prime segment falls into one of four categories: it has to be archimedean, or simple, or exceptional, or supplementary.

In the paper the symbol $\subset$ denotes proper inclusion of sets. The set of positive integers is denoted by $\mathbb{N}$.

2 Prime, semiprime, completely prime and completely semiprime ideals of ordered semigroups

Let $(S, \cdot, \leq)$ be an ordered semigroup and let A and B be nonempty subsets of S. Recall that AB denotes the set of all products ab, where $a \in A$ and $b \in B$. If $s \in S$, then we write sA (respectively, As) instead of $\{s\}A$ (respectively, $A\{s\}$). If $n \in \mathbb{N}$, then A^n denotes the set of all products $a_1 a_2 \cdots a_n$, where $a_1, a_2, \ldots, a_n \in A$. The symbol $(A]$ denotes the set of all elements $s \in S$ such that $s \leq a$ for some $a \in A$. In the following proposition we record some basic properties of these sets. We will freely use this proposition in the paper; its easy proof is left to the reader.

Proposition 2.1 *Let $(S, \cdot, \leq)$ be an ordered semigroup.*

(1) *For any nonempty subsets A, B of S, the following hold:*
  (a) $A \subseteq (A]$ *and* $((A]] = (A]$,
  (b) *If $A \subseteq B$, then $(A] \subseteq (B]$,*
  (c) $((A](B]] = ((A]B] = (A(B]] = (AB]$. *Consequently,* $(A](B] \subseteq (AB]$, $(A]B \subseteq (AB]$, *and* $A(B] \subseteq (AB]$,
  (d) $((A^m]^n] = (A^{mn}]$ *for any $m, n \in \mathbb{N}$,*
  (e) *If $x, y \in S$ and $x \leq y$, then $(xA] \subseteq (yA]$ and $(Ax] \subseteq (Ay]$.*
(2) *If $\{A_k\}_{k \in K}$ is a family of nonempty subsets of S, then $(\bigcup_{k \in K} A_k] = \bigcup_{k \in K}(A_k]$ and $(\bigcap_{k \in K} A_k] \subseteq \bigcap_{k \in K}(A_k]$.*

It is easy to see that if I and J are right (respectively, two-sided) ideals of S, then $(IJ]$ is a right (respectively, two-sided) ideal of S. Furthermore, directly from Proposition 2.1(2) we obtain the following corollary.

Corollary 2.2 *If $\{I_k\}_{k \in K}$ is a family of right (resp. left, two-sided) ideals of an ordered semigroup $(S, \cdot, \leq)$, then $\bigcup_{k \in K} I_k$ and $\bigcap_{k \in K} I_k$ are right (resp. left, two-sided) ideals of S.*

Let $(S, \cdot, \leq)$ be an ordered semigroup. A right ideal I of S is said to be

– *prime* if I is proper and for any right ideals A, B of S, $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$;

- *completely prime* if I is proper and for any elements a, b of S, $ab \in I$ implies $a \in I$ or $b \in I$;
- *semiprime* if I is proper and for any right ideal A of S, $A^2 \subseteq I$ implies $A \subseteq I$;
- *completely semiprime* if I is proper and for any element a of S, $a^2 \in I$ implies $a \in I$.

From the above definitions we obtain immediately the following implication chart for the considered types of right ideals:

$$\begin{array}{ccc} \text{prime} & \Rightarrow & \text{semiprime} \\ \Uparrow & & \Uparrow \\ \text{completely prime} & \Rightarrow & \text{completely semiprime} \end{array} \qquad (2.1)$$

Below we prove another interrelation between considered types of ideals.

Proposition 2.3 *Let I be an ideal of an ordered semigroup $(S, \cdot, \leq)$. Then I is completely prime if and only if I is prime and completely semiprime.*

Proof Obviously if I is completely prime, then I is prime and completely semiprime. To prove the converse implication, assume that I is prime and completely semiprime. To show that I is completely prime, consider any $a, b \in S$ such that $ab \in I$. Then

$$(bSa]^2 = (bSa](bSa] \subseteq (bSabSa] \subseteq (I] = I,$$

and since I is completely semiprime, we obtain $(bSa] \subseteq I$ and thus $bSa \subseteq I$. Hence

$$(bS](aS] \subseteq (bSaS] \subseteq (I] = I, \qquad (2.2)$$

and since I is prime, it follows from (2.2) that $(bS] \subseteq I$ or $(aS] \subseteq I$. Thus $b \in I$ or $a \in I$, which shows that I is completely prime. $\qquad \square$

The following concept will be useful in constructing semiprime ideals of an ordered semigroup $(S, \cdot, \leq)$. For any proper right ideal A of S we define the *Hoehnke ideal of S associated with A* to be the set

$$H_A(S) = \{h \in S : s \notin (shS] \; for \; all \; s \in S \backslash A\}.$$

If the order $\leq$ is trivial and A is an ideal of S, then the Hoehnke ideal coincides with the set $H_A(S) = \{h \in S : s \notin shS \; for \; all \; s \in S \backslash A\}$, which was defined and studied in [9] (for information why Hoehnke's name appears in this context, the interested reader is referred to [9]). Below we extend [9, Proposition 2] to ordered semigroups, showing in particular that indeed $H_A(S)$ is an ideal of S.

Theorem 2.4 *Let A be a proper right ideal of an ordered semigroup $(S, \cdot, \leq)$. Then*

(1) $H_A(S)$ *is a semiprime ideal of S.*
(2) *For any right ideal I of S, $I \subseteq H_A(S)$ if and only if $s \notin (sI]$ for all $s \in S \backslash A$.*
(3) *If A is an ideal of S, then $A \subseteq H_A(S)$.*

Proof We show first that $H_A(S)$ is an ideal of S. Since $0 \in A$, $(s0S] = (0] \subseteq (A] = A$ for any $s \in S$. Thus $s \notin (s0S]$ for all $s \in S \backslash A$, which shows that $0 \in H_A(S)$. Hence the set $H_A(S)$ is nonempty. To show that $H_A(S)$ is closed under multiplication (from both the left and the right) by elements of S, suppose for a contradiction that $t_1 h t_2 \notin H_A(S)$ for some $h \in H_A(S)$ and $t_1, t_2 \in S$. Then there exists $s \in S \backslash A$ such that

$$s \in (st_1 h t_2 S]. \qquad (2.3)$$

From (2.3) we get $st_1 \in (st_1 h t_2 S]S \subseteq (st_1 h t_2 S] \subseteq (st_1 h S]$, and since $h \in H_A(S)$, it follows that $st_1 \in A$. Thus $(st_1 h t_2 S] \subseteq (A] = A$, so (2.3) implies $s \in A$, a contradiction. Hence $SH_A(S)S \subseteq H_A(S)$. To complete the proof that $H_A(S)$ is an ideal of S, it suffices to show that $(H_A(S)] \subseteq H_A(S)$. For this, consider any $y \in (H_A(S)]$. Then $y \leq h$ for some $h \in H_A(S)$. Since for any $s \in S \backslash A$ we have $sy \leq sh$, we get $(syS] \subseteq (shS]$. Now $h \in H_A(S)$ implies $s \notin (shS]$, so also $s \notin (syS]$, and $y \in H_A(S)$ follows. Hence $(H_A(S)] \subseteq H_A(S)$, as desired.

Before showing that the ideal $H_A(S)$ is semiprime, we prove for any right ideal I of S the equivalence stated in (2). We proceed by contraposition. If $I \nsubseteq H_A(S)$, then there exist $i \in I$ and $s \in S \backslash A$ such that $s \in (siS]$, and $s \in (sI]$ follows, which proves the implication "$\Leftarrow$" in (2). To prove the opposite implication, suppose that $s \in (sI]$ for some $s \in S \backslash A$. Then for some $i \in I$ we have $s \leq si = sie \in siS$, and thus $s \in (siS]$, which implies that $i \notin H_A(S)$. Hence $I \nsubseteq H_A(S)$ and the proof of (2) is complete.

To establish (1), it remains to show that the ideal $H_A(S)$ is semiprime. Since $e \notin H_A(S)$, the ideal $H_A(S)$ is proper. Let I be a right ideal of S such that $I^2 \subseteq H_A(S)$. If $I \nsubseteq H_A(S)$, then by (2) there exists $s \in S \backslash A$ such that $s \in (sI]$. Hence $sI \subseteq (sI]I \subseteq (sI^2]$, so $(sI] \subseteq (sI^2]$, and thus

$$s \in (sI] \subseteq (sI^2] \subseteq (sH_A(S)],$$

which is a contradiction by (2). Thus $I \subseteq H_A(S)$ and therefore $H_A(S)$ is a semiprime ideal of S. The prove of (1) is complete.

To prove (3), assume that A is an ideal of S. Then $(sA] \subseteq (A] = A$ for any $s \in S$, and thus $s \notin (sA]$ for all $s \in S \backslash A$. Hence (2) implies $A \subseteq H_A(S)$, as desired. $\qquad \square$

For any proper right ideal A of an ordered semigroup $(S, \cdot, \leq)$ we define the *associated prime right ideal of A* to be the set

$$P_r(A) = \{p \in S : sp \in A \ for \ some \ s \in S \backslash A\}.$$

This concept is an analogue of the notion introduced in [9, Definition 12]. Below we extend [9, Lemma 13(i)] to ordered semigroups.

Proposition 2.5 *Let A be a proper right ideal of an ordered semigroup $(S, \cdot, \leq)$. Then $P_r(A)$ is a completely prime right ideal of S containing A.*

Proof By assumption, A is a proper right ideal of S. Hence $e \notin A$, and since for any $a \in A$ we have $ea = a \in A$, it follows that $A \subseteq P_r(A)$. Now we show that $P_r(A)$ is a

right ideal of S. For this, let us consider any $s \in S$ and $p \in P_r(A)$. Since $p \in P_r(A)$, for some $x \in S \backslash A$ we have $xp \in A$, and since A is a right ideal of S, we obtain $x(ps) = (xp)s \in A$ and thus $ps \in P_r(A)$. Hence $P_r(A)S \subseteq P_r(A)$. To complete the proof that $P_r(A)$ is a right ideal of S, it suffices to show that $(P_r(A)] \subseteq P_r(A)$. For this, consider any $q \in (P_r(A)]$. Then there exists $p \in S$ such that $q \leq p$ and $xp \in A$ for some $x \in S \backslash A$. Since $xq \leq xp$ and $xp \in A$, it follows that $xq \in (A] = A$ and thus $q \in P_r(A)$, as desired. Since obviously $e \notin P_r(A)$, it follows that $P_r(A)$ is a proper right ideal of S. To show that this right ideal is completely prime, consider any $a, b \in S$ with $ab \in P_r(A)$. Then there exists $x \in S \backslash A$ such that $xab \in A$. If $xa \in A$, then we have $a \in P_r(A)$. Otherwise $xa \in S \backslash A$, and since $(xa)b = xab \in A$, $b \in P_r(A)$ follows. □

3 Prime and completely prime ideals of right chain ordered semigroups

The following result shows that for any right chain ordered semigroup the horizontal implications on the chart (2.1) are in fact equivalences. The result is a generalization of [9, Lemma 8].

Proposition 3.1 *If $(S, \cdot, \leq)$ is a right chain ordered semigroup, then*

(1) *A right ideal I of S is semiprime if and only if I is prime.*
(2) *An ideal I of S is completely semiprime if and only if I is completely prime.*

Proof (1) Assume that I is a semiprime right ideal of S. Let A, B be right ideals of S such that $AB \subseteq I$. Since S is a right chain ordered semigroup, we must have $A \subseteq B$ or $B \subseteq A$. If $A \subseteq B$, then $A^2 \subseteq AB \subseteq I$ and $A \subseteq I$ follows. Similarly $B \subseteq A$ implies $B \subseteq I$. Thus I is prime. The converse statement is obvious.

(2) If an ideal I is completely semiprime, then (1) implies that I is prime, and thus I is completely prime by Proposition 2.3. The converse statement is clear. □

Let A be an ideal of an ordered semigroup $(S, \cdot, \leq)$. We adopt from [9] the following two useful notions. An ideal I of S is said to be *A-nilpotent* if $I^n \subseteq A$ for some $n \in \mathbb{N}$. An element t of S is said to be *A-nilpotent* if $t^n \in A$ for some $n \in \mathbb{N}$.

The following result extends [9, Proposition 9] to right chain ordered semigroups.

Proposition 3.2 *Let A be a proper ideal of a right chain ordered semigroup $(S, \cdot, \leq)$.*

(1) *If I is an ideal of S such that $I \subseteq H_A(S)$ and I is not A-nilpotent, then $\bigcap_{n \in \mathbb{N}}(I^n]$ is a completely prime ideal of S.*
(2) *If $t \in S$ is such that $t \in H_A(S)$ and t is not A-nilpotent, then $\bigcap_{n \in \mathbb{N}}(t^n S]$ is a prime right ideal of S.*

Proof (1) Assume that I is an ideal of S such that $I \subseteq H_A(S)$ and I is not A-nilpotent. Since $H_A(S)$ is a proper ideal of S by Theorem 2.4(1), so is I and thus Corollary 2.2 implies that $\bigcap_{n \in \mathbb{N}}(I^n]$ is a proper ideal of S. By Proposition 3.1(2), to complete the proof of (1), it suffices to show that for any $a \in S$, $a^2 \in \bigcap_{n \in \mathbb{N}}(I^n]$ implies $a \in \bigcap_{n \in \mathbb{N}}(I^n]$. For a contradiction, assume that there exists $a \in S$ such that

$a^2 \in \bigcap_{n\in\mathbb{N}}(I^n]$ but $a \notin \bigcap_{n\in\mathbb{N}}(I^n]$. Then $a \notin (I^m]$ for some $m \in \mathbb{N}$. Since S is a right chain ordered semigroup, we must have $(I^m] \subseteq (aS]$. Hence

$$
\begin{aligned}
a^2 \in (I^{2m+1}] &= (I^m I^{m+1}] \subseteq ((aS]I^{m+1}] \\
&\subseteq (aI^{m+1}] \subseteq (aI^m I] \subseteq (a(aS]I] \subseteq (a^2 I],
\end{aligned} \tag{3.1}
$$

and Theorem 2.4(2) implies $a^2 \in A$. Thus from (3.1) we obtain

$$
I^{2m+1} \subseteq (I^{2m+1}] \subseteq (a^2 I] \subseteq (A] = A,
$$

so I is A-nilpotent. This contradiction completes the proof of (1).

(2) Assume $t \in H_A(S)$ and t is not A-nilpotent. Since $t \in H_A(S)$, it follows from Theorem 2.4(1) that $\bigcap_{n\in\mathbb{N}}(t^n S] \subseteq H_A(S)$ and thus Corollary 2.2 implies that $\bigcap_{n\in\mathbb{N}}(t^n S]$ is a proper right ideal of S. By Proposition 3.1(1), to complete the proof of (2), it is enough to show for any right ideal J of S that $J^2 \subseteq \bigcap_{n\in\mathbb{N}}(t^n S]$ implies $J \subseteq \bigcap_{n\in\mathbb{N}}(t^n S]$. Suppose for a contradiction that $J^2 \subseteq \bigcap_{n\in\mathbb{N}}(t^n S]$ but $J \not\subseteq \bigcap_{n\in\mathbb{N}}(t^n S]$. Then $J \not\subseteq (t^m S]$ for some $m \in \mathbb{N}$, and since S is a right chain ordered semigroup, we have $(t^m S] \subseteq J$. Hence

$$
t^{2m} \in (t^m S](t^m S] \subseteq J^2 \subseteq (t^{2m+1} S] = (t^{2m} t S] \subseteq (t^{2m}(t S]]
$$

and thus $t^{2m} \in A$ by Theorem 2.4(2). But this is a contradiction, since t is not A-nilpotent.   □

Example 10 in [9] shows that in Proposition 3.2 the assumptions $I \subseteq H_A(S)$ in part (1) and $t \in H_A(S)$ in part (2) are both necessary.

The following corollary generalizes [9, Corollary 11].

Corollary 3.3 *Let I be an ideal of a right chain ordered semigroup $(S, \cdot, \leq)$ such that $(I^n] \neq (I^{n+1}]$ for any $n \in \mathbb{N}$. Then $\bigcap_{n\in\mathbb{N}}(I^n]$ is a completely prime ideal of S.*

Proof By Corollary 2.2, $A = \bigcap_{n\in\mathbb{N}}(I^n]$ is an ideal of S. By Proposition 3.2(1), to prove that the ideal A is completely prime it suffices to show that the ideal A is proper, $I \subseteq H_A(S)$ and I is not A-nilpotent. If $A = S$, then for any $n \in \mathbb{N}$ we have $(I^n] = S$, hence $(I^n] = (I^{n+1}]$, and this contradiction shows that the ideal A is proper. Let $s \in S \backslash A$. Then $s \notin (I^m]$ for some $m \in \mathbb{N}$. If $s \in (sI]$, then $sI] \subseteq ((sI]I] \subseteq (sI^2]$, so $s \in (sI^2]$ and continuing this way we obtain $s \in (sI^m] \subseteq (I^m]$, a contradiction. Hence $s \notin (sI]$, and thus $I \subseteq H_A(S)$ by Theorem 2.4(2). To show that I is not A-nilpotent, suppose for a contradiction that $I^k \subseteq A$ for some $k \in \mathbb{N}$. Then $(I^k] \subseteq (A] = A$, and thus $(I^k] \subseteq A \subseteq (I^{k+1}] \subseteq (I^k]$, which implies $(I^k] = (I^{k+1}]$, a contradiction. Thus I is not A-nilpotent.   □

Later on we will need the following generalization of [9, Lemma 13(ii)].

Proposition 3.4 *Let A be a prime right ideal of a right chain ordered semigroup $(S, \cdot, \leq)$. Then for any ideal I of S we have $I \subseteq A$ or $P_r(A) \subseteq I$.*

Proof Let I be an ideal of S such that $P_r(A) \nsubseteq I$. Then there exists $p \in P_r(A) \backslash I$. Since $p \in P_r(A)$, for some $x \in S \backslash A$ we have $xp \in A$, and since S is a right chain ordered semigroup and $p \notin I$, it follows that $I \subseteq (pS]$. Hence

$$(xS]I \subseteq (xSI] \subseteq (xI] \subseteq (x(pS]] \subseteq (xpS] \subseteq (AS] \subseteq (A] = A,$$

and since A is prime and $x \notin A$, it follows that $I \subseteq A$. □

The following lemma generalizes [9, Lemma 16].

Lemma 3.5 *If A is a proper ideal of a right chain ordered semigroup $(S, \cdot, \leq)$ such that $A = (A^2]$, then $A = (s^n A]$ for any $s \in S \backslash A$ and $n \in \mathbb{N}$.*

Proof Let $s \in S \backslash A$. Since S is a right chain ordered semigroup, we have $A \subseteq (sS]$ and thus

$$A = (A^2] = (AA] \subseteq ((sS]A] \subseteq (sA] \subseteq (A] = A.$$

Hence $A = (sA]$. Suppose that for some $n \in \mathbb{N}$ we have already proved that $A = (s^n A]$. Then

$$A = (s^n A] = (s^n (sA]] = (s^n sA] = (s^{n+1} A].$$

Thus the result follows by induction. □

The following two notions are obvious analogues of the concepts defined in [9, p. 580].

Definition 3.6 Let $(S, \cdot, \leq)$ be an ordered semigroup. An ideal Q of S is called an *exceptional prime ideal* of S if Q is prime but not completely prime. If $I \subset J$ are ideals of S such that there are no further ideals properly between I and J, then we say that J is *minimal over I*.

We close this section with the following generalization of [9, Lemmas 15 and 17].

Proposition 3.7 *Let $(S, \cdot, \leq)$ be a right chain ordered semigroup, and let Q be an exceptional prime ideal of S. Then there exists a unique ideal D of S such that $Q \subset D$ and D is minimal over Q. Furthermore, $D = (D^2]$ and there exists an element $a \in D \backslash Q$ such that $Q \subset \bigcap_{n \in \mathbb{N}} (a^n S]$. In particular, there exist elements in $D \backslash Q$ that are not Q-nilpotent.*

Proof Let D denote the intersection of all ideals I of S such that $Q \subset I$. Proposition 3.4 implies that for any such an ideal I we have $P_r(Q) \subseteq I$ and thus $P_r(Q) \subseteq D$. By Proposition 2.5, $P_r(Q)$ is a completely prime right ideal of S containing Q, and since Q is an exceptional prime ideal, it follows that $Q \subset P_r(Q)$. Hence $Q \subset D$ and now the definition of D and Corollary 2.2 imply that the ideal D is minimal over Q, and obviously D is a unique ideal of S with this property. If $D \neq (D^2]$, then $(D^2] \subset D$, and the minimality of D over Q implies $(D^2] \subseteq Q$. Hence $D^2 \subseteq (D^2] \subseteq Q$, and

since Q is prime, we get $D \subseteq Q$. This contradiction shows that $D = (D^2]$. We now prove that there exists $a \in D \backslash Q$ such that $Q \subset \bigcap_{n \in \mathbb{N}} (a^n S]$. Set

$$C = \{c \in S : \bigcap_{n \in \mathbb{N}} (c^n D] \subseteq Q\}.$$

Note that $Q \subseteq C$ and thus the set C is nonempty. We claim that $C \subseteq D$. Indeed, if $s \in S \backslash D$, then Lemma 3.5 implies that $\bigcap_{n \in \mathbb{N}} (s^n D] = D$, and since $D \nsubseteq Q$, it follows that $s \notin C$, which proves our claim.

Since Q is an exceptional prime ideal of S, by Proposition 3.1(2) for some $b \in S \backslash Q$ we have $b^2 \in Q$. If $b \in (CbD]$, then $b \in (cbD]$ for some $c \in C$. Hence $(bD] \subseteq (cbD]$, and thus

$$(bD] \subseteq (cbD] \subseteq (c(bD]] \subseteq (c(cbD]] \subseteq (c^2(bD]] \subseteq (c^2(cbD]] \subseteq (c^3(bD]] \subseteq \cdots,$$

and continuing this way, we obtain for any $n \in \mathbb{N}$ that $(bD] \subseteq (c^n bD] \subseteq (c^n D]$. Hence, since $c \in C$, we obtain

$$(bS]D \subseteq (bSD] \subseteq (bD] \subseteq \bigcap_{n \in \mathbb{N}} (c^n D] \subseteq Q,$$

and since Q is prime, $b \in Q$ or $D \subseteq Q$, a contradiction. Hence $b \notin (CbD]$, and since S is a right chain ordered semigroup, we must have $(CbD] \subseteq (bS]$. If $C = D$, then $(DbD] \subseteq (bS]$ and thus

$$((bS]D]^2 \subseteq (bD]^2 = (bD](bD] \subseteq (bDbD] = (b(DbD]] \subseteq (b(bS]] \subseteq (b^2 S] \subseteq Q.$$

Since Q is prime and $((bS]D]^2 \subseteq Q$, we get $(bS]D \subseteq Q$, so $b \in Q$ or $D \subseteq Q$, a contradiction. Hence we must have $C \subset D$. To complete the proof, take any $a \in D \backslash C$. Then $\bigcap_{n \in \mathbb{N}} (a^n D] \nsubseteq Q$, and since S is a right chain ordered semigroup, we obtain $Q \subset \bigcap_{n \in \mathbb{N}} (a^n D] \subseteq \bigcap_{n \in \mathbb{N}} (a^n S]$. $\qquad\square$

4 Prime segments of right chain ordered semigroups

Following [9], we define a *prime segment* of a right chain ordered semigroup $(S, \cdot, \leq)$ to be a pair $P_2 \subset P_1$ of completely prime ideals of S such that no further completely prime ideal of S exists between P_2 and P_1. In the following theorem we extend to right chain ordered semigroups the classification of prime segments of right chain semigroups given in [9, Theorem 18].

Theorem 4.1 *Let $(S, \cdot, \leq)$ be a right chain ordered semigroup, and let $P_2 \subset P_1$ be a prime segment of S. Then one of the following possibilities occurs:*

(a) *There are no further ideals of S between P_2 and P_1; the prime segment is called* simple *in this case.*

(b) *For every $a \in P_1 \backslash P_2$ there exists an ideal $I \subseteq P_1$ of S such that $a \in I$ and $\bigcap_{n \in \mathbb{N}}(I^n] = P_2$; the prime segment is called* archimedean *in this case.*

(c) *There exists a prime ideal Q of S with $P_2 \subset Q \subset P_1$; the prime segment is called* exceptional *in this case.*

(d) *There exists an ideal D of S such that $P_2 \subset D \subset P_1$ and D is minimal over P_2; the prime segment is called* supplementary *in this case.*

Possibilities (a), (b), (c) *are mutually exclusive, and possibilities* (a), (b), (d) *are mutually exclusive.*

Proof Assume that the prime segment $P_2 \subset P_1$ is not simple, i.e., case (a) does not hold. Then there exists an ideal I of S such that $P_2 \subset I \subset P_1$. If $P_1 \nsubseteq H_I(S)$, then since S is a right chain ordered semigroup, we must have $H_I(S) \subset P_1$, and by combining Theorem 2.4(1,3) with Proposition 3.1(1) we can see that $H_I(S)$ is a prime ideal of S lying properly between P_2 and P_1. Thus the prime segment $P_2 \subset P_1$ is exceptional in this case. Hence to the end of the proof we assume that

$$\text{there exists an ideal } I \text{ of } S \text{ with } P_2 \subset I \subset P_1 \text{ and for any such an ideal } I$$
$$\text{we have } P_1 \subseteq H_I(S). \tag{4.1}$$

Let us first consider the case where the prime segment $P_2 \subset P_1$ contains an ideal I of S such that $(I^m] = (I^{m+1}]$ for some $m \in \mathbb{N}$. Then

$$(I^{m+1}] = (I^m I] = ((I^m]I] = ((I^{m+1}]I] = (I^{m+2}]$$

and thus $(I^m] = (I^{m+1}] = (I^{m+2}]$. Continuing this way we obtain

$$(I^m] = (I^{m+k}] \text{ for any } k \in \mathbb{N}.$$

Thus for the ideal $D = (I^m]$ and any $n \in \mathbb{N}$ we have $D = (I^m] = (I^{mn}] = ((I^m]^n] = (D^n]$ and $D \subseteq I \subset P_1$. If we would have $D = (I^m] \subseteq P_2$, then since P_2 is completely prime, we would get $I \subseteq P_2 \subset I$, a contradiction. Hence, since S is a right chain ordered semigroup, we must have $P_2 \subset D$. We show that furthermore D is minimal over P_2. If not, then there exists an ideal A of S such that $P_2 \subset A \subset D$. Then $P_2 \subset A \subset P_1$ and by (4.1) we have $D \subset P_1 \subseteq H_A(S)$. Hence by Proposition 3.2(1), $\bigcap_{n \in \mathbb{N}}(D^n] = D$ is a completely prime ideal of S, which however is impossible, since $P_2 \subset P_1$ is a prime segment. Hence D is minimal over P_2 and thus the prime segment $P_2 \subset P_1$ is supplementary in this case.

We are left with the case where there exists an ideal I of S such that $P_2 \subset I \subset P_1$ and for any such an ideal I we have $(I^n] \neq (I^{n+1}]$ for all $n \in \mathbb{N}$. Let $\mathcal{I}$ be the set of all ideals I of S such that $P_2 \subset I \subset P_1$. Since S is a right chain ordered semigroup and the ideal P_2 is completely prime, for any $I \in \mathcal{I}$ and $n \in \mathbb{N}$ we have $P_2 \subseteq (I^n]$, and thus $P_2 \subseteq \bigcap_{n \in \mathbb{N}}(I^n] \subset P_1$. Since by Corollary 3.3 the ideal $\bigcap_{n \in \mathbb{N}}(I^n]$ is completely prime, it follows that

$$\bigcap_{n \in \mathbb{N}} (I^n] = P_2 \ for \ any \ I \in \mathcal{I}. \tag{4.2}$$

Let $Q = \bigcup \{I : I \in \mathcal{I}\}$. By Corollary 2.2, Q is an ideal of S. If $Q = P_1$, then (4.2) implies that the prime segment $P_2 \subset P_1$ is archimedean. Assume that $Q \neq P_1$. Then $Q \subset P_1$. We consider two cases:

Case1: $(P_1^2] \subset P_1$. Then $P_2 \subset (P_1^2] \subset P_1$ and thus $(P_1^2] \in \mathcal{I}$. Hence $\bigcap_{n \in \mathbb{N}} (P_1^{2n}] = \bigcap_{n \in \mathbb{N}} ((P_1^2]^n] = P_2$ by (4.2) and thus

$$P_2 \subseteq \bigcap_{n \in \mathbb{N}} (P_1^n] \subseteq \bigcap_{n \in \mathbb{N}} (P_1^{2n}] \subseteq P_2.$$

Therefore $P_2 = \bigcap_{n \in \mathbb{N}} (P_1^n]$ and the prime segment $P_2 \subset P_1$ is archimedean in this case.

Case2: $(P_1^2] = P_1$. We show that the ideal Q is prime in this case. By Proposition 3.1(1), it suffices to show that Q is semiprime. For this, consider any right ideal A of S such that $A^2 \subseteq Q$. Then $A^2 \subseteq P_1$, and since P_1 is completely prime, $A \subseteq P_1$ follows. If $A = P_1$, then $P_1 = (P_1^2] = (A^2] \subseteq (Q] = Q \subset P_1$, a contradiction. Hence $A \subset P_1$ and thus $A \subseteq Q$ by the definition of Q. Therefore Q is prime and the prime segment $P_2 \subset P_1$ is exceptional in this case.

It is easy to see that possibilities (a), (b), (c) are mutually exclusive. It is also clear that (a) and (d) are mutually exclusive. To complete the proof, assume that the possibility (d) occurs and D is minimal over P_2. Then $D = (D^2]$, and Lemma 3.5 implies that for any $a \in P_1 \backslash D$ we have $P_2 \subset D = \bigcap_{n \in \mathbb{N}} (a^n D] \subseteq \bigcap_{n \in \mathbb{N}} (a^n S]$. Hence (b) and (d) cannot happen simultaneously. □

Example 19 from [9] shows that possibilities (c) and (d) of Theorem 4.1 can occur simultaneously.

We close the paper with the following characterization of archimedean prime segments of right chain ordered semigroups. The result is a generalization of [9, Corollary 20].

Corollary 4.2 *Let $P_2 \subset P_1$ be a prime segment of a right chain ordered semigroup $(S, \cdot, \leq)$. Then the following conditions are equivalent.*

(i) *The prime segment $P_2 \subset P_1$ is archimedean.*
(ii) *For any $a \in P_1 \backslash P_2$, $\bigcap_{n \in \mathbb{N}} (a^n S] = P_2$.*
(iii) *For any $a \in P_1 \backslash P_2$, $(P_1 a S] \subset (a S]$.*

Proof (i) $\Rightarrow$ (ii) follows directly from the definition of an archimedean prime segment.

(ii) $\Rightarrow$ (iii) Let $a \in P_1 \backslash P_2$. Suppose that $(a S] \subseteq (P_1 a S]$. Then $a \leq pas$ for some $p \in P_1$ and $s \in S$. If $p \in P_2$, then $a \in (P_2 a S] \subseteq (P_2] = P_2$, a contradiction. Hence $p \in P_1 \backslash P_2$. Furthermore, $a \leq pas$ implies

$$a \leq pas \leq p(pas)s = p^2 a s^2 \leq p^2 (pas) s^2 = p^3 a s^3 \leq p^3 (pas) s^3 = p^4 a s^4 \leq \ldots$$

and thus for any $n \in \mathbb{N}$ we have $a \leq p^n a s^n$, and $a \in (p^n S]$ follows. Hence by (ii), $a \in \bigcap_{n \in \mathbb{N}} (p^n S] = P_2$, which is a contradiction. Thus $(a S] \nsubseteq (P_1 a S]$, and since S is a right chain ordered semigroup, we obtain $(P_1 a S] \subset (a S]$, as desired.

(iii) $\Rightarrow$ (i) Assume (iii). Then for any $a \in P_1 \backslash P_2$ we have $P_2 \subset (P_1 a S] \subset (a S] \subseteq P_1$ and thus the prime segment $P_2 \subset P_1$ is not simple. Suppose the prime segment $P_2 \subset P_1$ is exceptional, i.e., there exists a prime ideal Q of S such that $P_2 \subset Q \subset P_1$. Then by Proposition 3.7 there exists an ideal D of S which is minimal over Q. This however is impossible, since (iii) implies that for any $a \in D \backslash Q$ we have $Q \subset (P_1 a S] \subset (a S] \subseteq D$. Finally, suppose the prime segment $P_2 \subset P_1$ is supplementary. Then there exists an ideal D' of S such that $P_2 \subset D' \subset P_1$ and D' is minimal over P_2. Then by (iii), for any $a \in D' \backslash P_2$ we have $P_2 \subseteq (P_1 a S] \subset (a S] \subseteq D'$, a contradiction. Hence the prime segment $P_2 \subset P_1$ is neither simple, nor exceptional, nor supplementary, and thus by Theorem 4.1 it must be archimedean. $\square$

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