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OBITUARY

John Macintosh Howie: work and legacy

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John McIntosh Howie (1936 - 2011)

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John M. Howie (1936–2011) was one of the founding fathers of semigroup theory as we know it today. His active research life spanned almost half a century. During this time he published around 80 research articles, on topics such as amalgamated products, transformation semigroups, generators, products of idempotents and others. Generations of future researchers in semigroup theory learned the basics of their trade from the two editions of John's seminal monograph [56,67]. John was a much loved and respected teacher, and his mastery is reflected in the undergraduate texts on automata [66], real analysis [68], complex analysis [69] and Galois theory [70]. John worked with a large number of collaborators, and supervised twelve Ph.D. students, including the present authors. In this article we will attempt to give the barest of sketches of John's life and the main themes running through John's research work, and to indicate how it has shaped research in semigroups to the present day.

John was born in Chryston, near Glasgow in Scotland, where his father was a Church of Scotland minister. In 1937 the family moved to Keith in North Scotland. John received secondary and higher education in Aberdeen, graduating in 1958. He was immediately appointed assistant at the University of Aberdeen, and in 1959 he enrolled for D.Phil. at Balliol College, University of Oxford. His official supervisor was G. Higman, but John ended up interacting with G. Preston who was in Oxford at the time. Higman and Preston obviously influenced the key components of John's early research— amalgamation and semigroups. In 1960 John married Dorothy, who was to remain his lifelong companion, and with whom he would have two daughters. In 1961 he was awarded D.Phil. from Oxford, returned to Scotland, to positions in Glasgow (1961), and then in the newly founded University of Stirling (1967). In 1970 John was appointed Regius Professor of Mathematics at the University of St Andrews—a Crown appointment at the oldest University in Scotland, and third oldest in the English speaking world. John held this position until retirement in 1997, and remained in St Andrews as an Emeritus Professor until his death in 2011.

St Andrews years witnessed the zenith of John's career. He opened several new research areas, published the first edition of his celebrated monograph, hosted a series of distinguished visitors, and started supervising Ph.D. students. John was a long-standing Head of Pure Mathematics in St Andrews, and also served as the Dean of Science (1976–79). He was a member of the London Mathematical Society, where he held several positions of responsibility, most notably that of Vice-President (1986–88, 1990–92); John also served as the President of the Edinburgh Mathematical Society (1973–74). John was elected to the Royal Society of Edinburgh in 1971, and awarded their Keith Prize. He was also editor for several international mathematical journals: Semigroup Forum (1976–1998), Executive Editor of Semigroup Forum (1990–94), Proceedings of the Royal Society of Edinburgh Series A (1988–91, 1997–2000), Communications in Algebra (1994–99), and Portugaliae Mathematica (1987–2001). John was a very respected figure in the educational circles in Scotland. In 1990–92 he chaired a committee set up to review the final years in Scottish secondary schools; the committee's report *Upper Secondary Education in Scotland*, proposed radical changes to the Scottish school curriculum, but the proposals were not implemented. In 1993 John was honoured with a Commander of the British Empire 'for services to education'.

In the course of his professional life John had the opportunity to travel a great deal. His early research visits to the United States—6 weeks at Pennsylvania State University in 1963 and ten months at Tulane University in 1964–65—were certainly formative. But it is fair to say that of all the foreign countries it was Portugal, which he visited on numerous occasions between 1980 and 2009, that John felt the strongest attachment to. It is therefore particularly fitting that five of his Ph.D. students were Portuguese. The final visit in 2009 was to participate in the workshop in memory of another Scottish giant of semigroup theory, Douglas Munn, with whom John maintained a close professional relationship and personal friendship. A workshop in memory of John himself was also held in Lisbon in May 2012.

In addition to dedication to his family and mathematics, John had many other interests and hobbies. The most absorbing of these was certainly music. John was an accomplished organist and singer. In fact, his first trip to Portugal in 1980 was with the St Andrews Renaissance Group, an occasion that John loved to recount. For many years John was the Church Organist and Choir Master at the Hope Park Parish Church in St Andrews. He was also a member of St Andrews Chorus with whom he had many public appearances, and he gave a series of solo Lunchtime Concerts.

A slightly less known hobby of John's was gardening. As Dorothy recalls, for many years John kept the household supplied with fresh fruit and vegetables. Just as, one might say, for many years he fed all of us around him with his mathematical knowledge and ideas.

1 Amalgamation

John's early research work was devoted to the problems of amalgamation in semigroups, the topic of his D.Phil. dissertation in Oxford under the supervision of Graham Higman and Gordon Preston. The dissertation was entitled "Some problems in the theory of semigroups", and its main findings were published in [47].

An *amalgam* $[U; S_i; \phi_i\ (i \in I)]$ consists of a pairwise disjoint family of semigroups U, $S_i\ (i \in I)$, and a family of embeddings $\phi_i : U \to S_i\ (i \in I)$. The 'common subsemigroup' U is called the *core* of the amalgam. The amalgam is *embeddable* into a semigroup T if there exist monomorphisms $\lambda : U \to T$ and $\lambda_i : S_i \to T$ such that we have

(A1) $\phi_i \lambda_i = \lambda$ for all $i \in I$;
(A2) $S_i \lambda_i \cap S_j \lambda_j = U\lambda$ for all distinct $i, j \in I$.

Intuitively, this means that all S_i embed into T, and any two such embeddings intersect *precisely* in the designated copy of U.

At the time John was working on his D.Phil. dissertation, it was known that every group amalgam embeds into a group, specifically the group amalgamated free product $\prod_U^* S_i$, and the importance of these free products was beginning to emerge through the work of mathematicians such as A.G. Kurosh, B.H. Neumann and G. Higman. It was also known that semigroup amalgams do not always embed [89], and that this depends on whether or not they embed into the *semigroup amalgamated free product* defined by the presentation

$$\prod_U^* S_i = \langle A_i \mid R_i,\ u\phi_i = u\phi_j\ (u \in U,\ i, j \in I)\rangle,$$

where $\langle A_i \mid R_i\rangle$ are presentations for S_i $(i \in I)$. John's oeuvre on amalgamation was devoted to understanding the intricacies of when and why semigroup amalgams do embed, and was, due to the above observation, predominantly word-combinatorial in nature.

In [47], John generalises the emergent notion of unitary subsemigroups: a subsemigroup $U \leq S$ is *unitary* if it satisfies:

$$(\forall s \in S)(\forall u \in U)(su \in U \text{ or } us \in U \Rightarrow s \in U).$$

John defines an *almost unitary* subsemigroup to be one for which there exist two mappings $\lambda, \rho : S \to S$, λ acting on the left, ρ on the right, satisfying the following properties:

- $\lambda^2 = \lambda$, $\rho^2 = \rho$, $\lambda\rho = \rho\lambda$;
- $\lambda|_U = \rho|_U = 1_U$, the identity mapping;
- $\lambda(st) = (\lambda s)t$, $(st)\rho = s(t\rho)$, $s(\lambda t) = (s\rho)t$ for all $s, t \in S$;
- U is unitary in $\lambda S\rho$.

If U is unitary then it is almost unitary, by choosing λ and ρ to be the identity mappings. The main result of [47] asserts:

Theorem 1 ([47, Theorem 3.3]) *Let $\{S_i : i \in I\}$ be a family of semigroups, and suppose that there exists a semigroup U and a monomorphism $\phi_i : U \to S_i$ for each i in I. Suppose also that, for each i in I, the subsemigroup $U\phi_i$ is almost unitary in S_i. Then the embedding $[U; S_i; \phi_i\ (i \in I)]$ is possible.*

Subsequent papers developed these results further; for instance:

- a cancellative semigroup amalgam with a group core is embeddable into a cancellative semigroup [48];
- a commutative semigroup amalgam, where the core is regular or totally ordered by division, is embeddable [53];
- any amalgam with an inverse core is embeddable [55].

Following John's pioneering work, amalgamation has rightfully taken a position of importance within semigroup theory, with a steady stream of articles continuing to appear. A very good introduction into the subject is given in the last chapter of John's iconic monograph on semigroup theory [67].

John worked on a couple of topics related in various ways to amalgamation. In combinatorial group theory, HNN extensions are often investigated side by side with amalgamated free products. John explored this angle for semigroups in [49]. It was not until 1997 and the work of Yamamura [112] that HNN extensions were revisited, but since then there has been a steady interest in them, e.g. see [25,29,114]. In 1967 John and Isbell [72] applied Isbell's general theory of dominions [85] to semigroups. They proved a variant of Isbell's celebrated zigzag theorem [85] for commutative semigroups, and derived a number of corollaries for commutative and other semigroups. John's pioneering work on amalgamation and zig-zags is a precursor to the categorical theory of monoid actions, which developed rapidly in 1980s and 1990s; for an overview see [88]. John supervised Renshaw's [103] Ph.D. studies on this topic, with the resulting thesis and several papers [104–106]. He returned to the topic of semigroup presentations once more with Ruškuc in [80], where they gave 'natural' presentations for a number of basic semigroup constructions. This generated considerable further work, see for example [3,26,87,92].

2 Products of idempotents

In 1966 John published a paper [51] on a topic completely unrelated to his work thus far, and outlined succinctly in the opening sentences: "The full transformation semigroup on a set X is defined to consist of all mappings of X into itself, the semigroup operation being composition of mappings. It is well-known [...] that this semigroup (which we shall denote by $\mathcal{T}_X$) is regular, and that the idempotents of $\mathcal{T}_X$ do not form a subsemigroup if X has more than two elements. The principal object of this paper is to identify the subsemigroup of $\mathcal{T}_X$ generated by the idempotents." The paper has two sections, dealing with situations where X is finite or infinite respectively, and each contains one main theorem.

Theorem 2 ([51, Theorem I]) *If X is a finite set, then the subsemigroup of $\mathcal{T}_X$ generated by the idempotents of non-zero defect is $\mathcal{T}_X \setminus \mathcal{S}_X$. In fact, every element of $\mathcal{T}_X \setminus \mathcal{S}_X$ is a product of idempotents of defect 1.*

Here $\mathcal{S}_X$ stands for the symmetric group on X, while the *defect* of a mapping is defined as the size of the complement of the image:

$$Z(\alpha) = X \backslash X\alpha, \ \operatorname{def}(\alpha) = |Z(\alpha)|.$$

In order to present the result for infinite X, John introduces two further parameters:

$$\begin{aligned} S(\alpha) &= \{x \in X : x\alpha \neq x\}, & \operatorname{sh}(\alpha) &= |S(\alpha)|, \\ C(\alpha) &= \bigcup \{t\alpha^{-1} : t \in X\alpha, |t\alpha^{-1}| \geqslant 2\}, & \operatorname{c}(\alpha) &= |C(\alpha)|, \end{aligned}$$

called the *shift* and the *collapse* respectively.

Theorem 3 ([51, Theorem III]) *Let $\mathcal{T}_X$ be the full transformation semigroup on an infinite set X. The subsemigroup of $\mathcal{T}_X$ generated by the idempotents of non-zero defect consists of all elements of finite shift and finite non-zero defect together with those elements α of infinite shift for which $|S(\alpha)| = |Z(\alpha)| = |C(\alpha)|$.*

From these two theorems it follows, as was shown also in [51], that any (finite) semigroup can be embedded into a (finite) regular idempotent generated semigroup.

This paper set the scene for what was to become a life-long fascination with generators and idempotents. Subsequent development can usefully be viewed under the following three strands, although it has to be remembered that in reality these strands are intertwined throughout John's work:

(I1) Given a semigroup S, identify the subsemigroup generated by the idempotents of S.
(I2) If S is idempotent-generated (or *semiband*, as John termed such semigroups), what is the smallest number of idempotents needed to generate it?
(I3) If S is a semiband, what is the minimal length of products of idempotents needed to generate the whole of S?

For a totally ordered set X, a map $\alpha \in \mathcal{T}(X)$ is said to be *order preserving* if $x \leqslant y$ implies $x\alpha \leqslant y\alpha$. In [54], the semigroup $\mathcal{O}_X$ of all such maps is considered. When $X = \{1 < 2 < \cdots < n\}$, we denote $\mathcal{O}_X$ by $\mathcal{O}_n$. It is proved that $\mathcal{O}_n$ is idempotent-generated (this was actually proved earlier by Aĭzenštat [1], unbeknownst to John) and $|\mathrm{O}_n|$ and $|E(\mathrm{O}_n)|$ are computed. These results have inspired similar questions in many other classes of transformation semigroups, and have often been cited, see [2,5,15,90,93,115]. In the infinite case the answer of course depends on the *order type* of X, and some progress was made in [54] for the case $X = \{1 < 2 < \dots\}$. But these results were significantly strengthened in a follow-on paper [74] joint with B.M. Schein. They consider an arbitrary well-ordered set (i.e. an ordinal), and obtain a complete description of the elements $\alpha \in \mathcal{O}_X$ that are products of idempotents. The same authors also consider another interesting semigroup of mappings in [76], namely the semigroup S of all endomorphisms of a finite Boolean algebra $(B, \vee, \wedge, ', 0, 1)$, and prove that there is a morphism from S onto $\mathcal{T}_A$, where A is the set of atoms of B, and an anti-morphism from $\mathrm{End}(B, \vee, 0)$ onto the semigroup $\mathcal{B}_A$ of all binary relations on A. As a consequence, they show that, as in $\mathcal{T}_X$, all singular endomorphisms are generated by the idempotents, and describe the elements generated by the idempotents in $\mathrm{End}(B, \vee, \wedge)$.

The results for the full transformation semigroup have their natural analogues for endomorphisms of vector spaces; this was established by Erdös [19]. In turn, they have common generalisations to the semigroup of all singular endomorphisms of a connected independence algebra of finite rank; this was first proved in [24] and later in [4].

Implicit in these early papers is the question of (idempotent) generation of completely 0-simple semigroups—after all, to generate $\mathrm{Sing}_n = \mathcal{T}_n \setminus \mathcal{S}_n$ by idempotents of rank $n-1$ one *must* generate the top $\mathcal{J}$-class too. This is the likely motivation for [57], where the subsemigroup $\langle E \rangle$ generated by the idempotents E of a completely 0-simple semigroup expressed as a Rees matrix semigroup $S = M^0[G; I, \Lambda; P)$ is described in a very neat way. John defines a bipartite graph on the set $I \dot{\cup} \Lambda$ by letting (i, λ) be an edge if and only if $p_{\lambda i} \neq 0$ in the structure matrix P. The connectivity relation on this graph is denoted by $\sim$. Furthermore, for each $i \in I$, $\lambda \in \Lambda$, a subset $V_{i,\lambda}$ of $G \times \{i\} \times \{\lambda\}$ is defined.

Theorem 4 ([57, Theorem 1]) *Let $S = M^0[G; I, \Lambda; P]$ be a completely 0-simple semigroup. Let E be the set of idempotents in S and $\langle E \rangle$ the subsemigroup of S generated by the idempotents. Then*

$$\langle E\rangle = \{(a, i, \lambda)\ :\ i \sim \lambda,\ \textit{and}\ a \in V_{i,\lambda}\} \cup \{0\}.$$

In the case of completely simple semigroups, the matrix P may be taken to be *normalised*, i.e. to have a row and a column consisting entirely of 1s, and then the following holds:

Theorem 5 ([57, Theorem 4]) *Let $S = M[G; I, \Lambda; P]$ be a completely simple semigroup in which P is normal. Then $\langle E\rangle = V \times I \times \Lambda$, where V is the subgroup of G generated by the entries of P. The semigroup S is idempotent-generated if and only if $V = G$.*

Recently, these results have been used in [41] to study a homological finiteness property in completely simple semigroups.

An interesting variation on the theme of idempotent generation in the setting of semigroups with zero is provided by considering the nilpotent elements as generators. A nonzero element a is *nilpotent* if it has a zero power and its *index* is the largest integer n such that a^n is not zero. In [65] John proves that any semigroup [regular, orthodox, inverse] embeds into a nilpotent generated [regular, orthodox, inverse] semigroup, the same happening for monoids that are [0-] simple, completely simple or bisimple. In [33] with Gomes, John considers the symmetric inverse semigroup $\mathcal{I}(X)$, consisting of all partial bijections on $X = \{1, \ldots, n\}$, and the subsemigroup $SP_n = \mathcal{I}(X)\backslash\mathcal{S}(X)$ of all proper partial bijections. They prove that SP_n is generated by its set of nilpotents if and only if n is even. Some citations of these results appear in [28,91,109]. In [77] with Marques-Smith, John presents another interesting example of a 2-nilpotent generated semigroup, i.e. a semigroup generated by the nilpotents of index 2, this time by considering special transformations on a set X whose cardinal is infinite and regular [102]. For X of infinite cardinality $\mathfrak{m}$, the *Baer–Levi* semigroup is defined by

$$B = \{\alpha \in \mathcal{I}(X) : |X\backslash X\alpha| = \mathfrak{m}\}.$$

In [75], again with Marques-Smith, John shows that the inverse subsemigroup of $\mathcal{I}(X)$ generated by $B^{-1}B$ is 2-nilpotent generated. A certain quotient of this semigroup is given as an example of a congruence-free 0-bisimple 2-nilpotent generated inverse semigroup. In [94] and [97] we find generalisations of these results, the first for independence algebras and so for vector spaces, and the second for certain linear versions of Baer–Levi semigroups.

3 Ranks

The question 2 above is most naturally viewed in the more general context of minimal generating sets. For a semigroup S, define its *rank* to be the smallest cardinality of a generating set of S:

$$\operatorname{rank}(S) = \min\{|A|\ :\ \langle A\rangle = S\}.$$

If we specialise to *idempotent* [*nilpotent*] *generating sets* we obtain the notion of the *idempotent* [*nilpotent*] *rank* denoted $\operatorname{irank}(S)$ [$\operatorname{nrank}(S)$]. It took some time to arrive at this general setting: the idempotent rank was first considered in the 1978 paper [58], the term idempotent rank was introduced in 1987 in a joint paper with Gomes [32], while the general rank and the notation were introduced in 1990 jointly with McFadden [79]. It should be pointed out that elsewhere in Algebra, and particularly in Group Theory, it is more common to use the notation $d(S)$ for the smallest number of generators.

The 1978 paper prefigures, in relative simplicity, the subsequent work, so a brief outline is appropriate. An idempotent ϵ of defect 1 in $\mathcal{T}_n$ fixes its image of size $n-1$, and maps the remaining point i to some $j \in \operatorname{im}(\epsilon)$; let us denote such an ϵ by $\binom{i}{j}$. For a set I of idempotents of defect 1, define a digraph $\Gamma(I)$ with vertices $\{1, \ldots, n\}$ and a directed edge $i \to j$ for every $\binom{i}{j} \in I$. We say that $\Gamma(I)$ is *strong* if it is strongly connected, and *complete* if for all distinct $i, j \in I$ at least one of $i \to j$ or $j \to i$ is an edge.

Theorem 6 ([58, Theorem 1]) *Let $X = \{1, 2, \ldots, n\}$ ($n \geq 3$) and let $S = \mathcal{T}(X) \setminus \mathcal{S}(X)$ be the semigroup of all singular mappings of X into itself. A set I of idempotents of defect* 1 *in S is a generating set if and only if the associated digraph* $\Gamma(I)$ *is strong and complete.*

As a consequence the following is derived:

Theorem 7 ([58, Theorem 2]) *If M is a minimal set of idempotent generators of the semigroup $S = \mathcal{T}(X) \setminus \mathcal{S}(X)$, where $|X| = n \geq 3$, then $|M| = n(n-1)/2$. The number of distinct sets M is*

$$2^{n-2} \prod_{r=2}^{n-2} (2^r - 1).$$

To this should be added [58, Theorem 2.1] where it is shown that $\operatorname{rank}(S) = \operatorname{irank}(S)$. The salient features emerging from this are:

- the rank of S is equal to the rank of the principal factor P corresponding to the unique maximal ideal of S;
- the computation of $\operatorname{rank}(P)$ depends on a certain graph-theoretic/combinatorial argument, and the rank turns out to be the number of $\mathcal{R}$-classes in P (which is greater than the number of $\mathcal{L}$-classes);
- the rank and the idempotent rank (of both S and P) are equal.

It is important to note that, certainly, none of these properties hold generally, they just happen to be true in this particular instance. But what is more remarkable is that they seem to hold in many more naturally occurring settings, as the subsequent work demonstrates. For instance, in [32] Gomes and John consider the singular part SP_n of the symmetric inverse semigroup $\mathcal{I}_n$, and prove that its rank is $n+1$, the same as the rank of the top principal factor. Of course, in this instance idempotent rank is undefined, as the only idempotent-generated inverse semigroups are the semilattices.

In [79] John and McFadden turn to the ideals of $\mathcal{T}_n$, namely

$$K(n,r) = \{\alpha \in \mathcal{T}_n \;:\; |\operatorname{im}(\alpha)| \le r\} \; (r = 1, \ldots, n-1).$$

They prove that $\operatorname{rank}(K(n,r)) = \operatorname{irank}(K(n,r)) = S(n,r)$, the Stirling number of the second kind, which is equal to the rank of the top principal factor, and to the number of $\mathcal{R}$-classes in this factor. Analogous results for the endomorphism semigroups of vector spaces and independence algebras were proved by other authors, see [7,14,24,38], and for order preserving transformations of finite chains by Gomes and John [34]. Variations on the theme of order preserving transformations of a finite chain have been studied by Fernandes and collaborators in a series of papers, see for example [20–22].

The question of rank of an abstract completely 0-simple semigroup was treated for the first time in [107] by the second author of the present article, who was John's Ph.D. student at the time. John devised the original question, provided several key ideas, and much overall support, so should have rightfully been a co-author of this article. The paper does not resolve the question fully, but gives the formula

$$\operatorname{rank}(\mathcal{M}^0[G; I, J; P]) = \max(|I|, |J|, \operatorname{rank}(G : H))$$

when the bipartite graph associated with P described in Sect. 2 is connected. Here H is a certain subgroup of G arising from the matrix P, and $\operatorname{rank}(G : H)$ denotes the *relative rank* of H inside G, the number of elements that need to be added to H to generate the whole of G. The general formula was established by Gray and Ruškuc [39], while Gray [40] provided a combinatorial framework for explaining the equality of the rank and idempotent rank.

John 'played' with and around the notions of rank for finite semigroups in several other pieces of work:

- [31] with Giraldes: semigroups of high rank, meaning $\operatorname{rank}(S) = |S|$ or $\operatorname{rank}(S) = |S| - 1$, and christened in a typically humorous fashion *royal* and *noble*;
- [81,82] with Ribeiro: relations between rank and various other numerical characteristics of a semigroup S to do with the existence of generating and/or independent sets;
- [33] with Gomes: nilpotent ranks in the singular part SP_n of $\mathcal{I}_n$;
- [6] with Ayık, Ayık, Ünlü (John's last research article): generating sets of Sing_n consisting of (m, r)*-path cycles*.

For infinite semigroups the notion of rank has only a limited interest, as many natural semigroups are not finitely generated (e.g. because they are uncountable). In this setting the notion of *relative rank*, which made its first appearance in the study of ranks of finite Rees matrix semigroups, has provided a perhaps surprisingly fertile ground for investigation. The first in a series of papers with Higgins and Ruškuc, and later Mitchell, appeared in 1998 [84]. There, the relative ranks of some distinguished subsemigroups of the full transformation semigroup $\mathcal{T}(X)$ on an infinite set X are considered. For instance, in the case of a symmetric group $\mathcal{S}_X$, we have

$$\operatorname{rank}(\mathcal{T}(X):\mathcal{S}(X))=2.$$

Furthermore, a new parameter $K(\alpha)$ associated with a mapping $\alpha \in \mathcal{T}(X)$ is introduced, called the *infinite contraction index*:

$$K(\alpha)=\{x\in X\ :\ |x\alpha^{-1}|=|X|\},\ k(\alpha)=|K(\alpha)|.$$

With this notation:

Theorem 8 ([84, Theorem 4.1]) *Let X be an infinite set of regular cardinality, and let $\mu,\nu\in\mathcal{T}_X$. Then the set $\mathcal{S}_X\cup\{\mu,\nu\}$ generates $\mathcal{T}_X$ if and only if one of the two mappings, say μ, is an injection of defect $|X|$, and the other is a surjection of infinite contraction index $|X|$.*

The key momentum for this area was generated when the authors (re)discovered an old result of Sierpiński [108]: *Every countable subset of $\mathcal{T}(X)$ (X infinite) is contained in a two-generated subsemigroup of $\mathcal{T}(X)$*. An immediate corollary of this is that the relative rank of any subsemigroup of $\mathcal{T}(X)$ is 0, 1, 2 or uncountable. In [44] it is proved that the semigroup $\mathcal{B}(X)$ of all binary relations on X (infinite) has the same properties. It should be remarked that the analogous properties for the symmetric group were proved by Galvin [27]. Relative ranks of several distinguished subsemigroups of $\mathcal{B}(X)$ are computed, e.g.

$$\operatorname{rank}(\mathcal{B}(X):\mathcal{T}(X))=\operatorname{rank}(\mathcal{B}(X):\mathcal{I}(X))=1,\ \operatorname{rank}(\mathcal{B}(X):\mathcal{S}(X))=2.$$

By way of contrast, the semigroup of all contractions on $\mathbb{N}$ (i.e. all mappings α satisfying $|x\alpha-y\alpha|\le|x-y|$ for all $x,y\in\mathbb{N}$) has an uncountable relative rank in $\mathcal{T}_{\mathbb{N}}$. The work begun in these two papers has been taken much further by Mitchell in collaboration with his Ph.D. student Péresse and others (see for example [12,46,99,100]), as well as by a number of other authors (e.g. [9,17,18,98]). Péresse's Ph.D. Thesis [101] gives a good introduction into this area.

As a spin-off from the relative ranks work, John, Higgins, and Ruškuc published some combinatorial insights into set products in $\mathcal{T}(X)$ in two articles. The factorization of $\mathcal{T}(X)=GE$, for X finite, where $G=\mathcal{S}(X)$ and E is the set of idempotents of $\mathcal{T}(X)$, is the starting point of [43]. The elements of $GE=EG$ in an arbitrary $\mathcal{T}(X)$ are described as the ones such that $c(\alpha)=\operatorname{def}(\alpha)$. More generally, for X finite, the subsets A of $\mathcal{T}(X)$ such that $\mathcal{T}(X)=AG$ or $\mathcal{T}(X)=GA$ are also described. For X infinite, $\mathcal{T}(X)\neq GE=EG=\langle G\cup E\rangle$. Various products are studied, for example SI, surjections times injections, that gives the $\mathcal{J}$-class of all transformations of rank $|X|$. In [45], the authors study further products in $\mathcal{T}(X)$, namely the product of an $\mathcal{L}$-($\mathcal{R}$-, $\mathcal{H}$-) class by E. Problems of this type appear in much wider contexts, see for example [13] regarding continuous maps on a metric space.

4 Lengths of products and depth

If a semigroup S is generated by a set A, then of course S can be written as $S=\bigcup_{i=1}^{\infty}A^i$. If this union is in fact finite, i.e. $S=\bigcup_{i=1}^{n}A^i$, then the smallest such n is called the *depth* of S (with respect to A) and is denoted by $\Delta(S, A)$. If the generating set is clear from the context (e.g. the set of all idempotents) we will write just $\Delta(S)$. The depth is clearly equal to the maximum length of elements of S with respect to A, where the length of an element s is the length of a shortest product of elements of A equal to s.

In his work John frequently returned to the questions of depth and length, predominantly in the context of idempotent generators in finite or infinite full transformation semigroups. In the case of T_n, the complete information is provided by John's elegant *gravity formula*:

Theorem 9 ([59, Theorem 3.1]) *Let S denote the semigroup of singular mappings from X into X, where X is the finite set $\{1, \dots, n\}$, and let E denote the set of idempotents of rank $n-1$ in S. For each α in S the least k for which $\alpha \in E^k$ is $k = g(\alpha)$, where $g(\alpha)$ is the gravity of α.*

Here, the *gravity* of α, is defined as

$$g(\alpha) = n + \operatorname{corb}(\alpha) - \operatorname{fix}(\alpha),$$

where $\operatorname{corb}(\alpha)$ is the number of *cyclic orbits* of α and fix (α) is the number of points fixed by α. The *orbits* are the equivalence classes of the relation ω on $X = \{1, \dots, n\}$ given as follows

$$x\omega y \Leftrightarrow x\alpha^l = y\alpha^m, \text{ for some } l, m \geqslant 0.$$

An orbit of the form $\{x, x\alpha, \dots, x\alpha^{r-1}\}$ with $r \geq 2$, for some $x \in X$, is called *cyclic*. From this the depth of Sing_n can be computed, and turns out to be to be $\left\lceil \frac{3(n-1)}{2} \right\rceil$, see [60].

Later, in [64], John considers the problem of writing an arbitrary $\alpha \in \operatorname{Sing}_n$ not just as a product of idempotents of rank $n-1$ but of arbitrary idempotents, showing that the least number $k(\alpha)$ of idempotents required satisfies $k(\alpha) \leqslant \operatorname{ran}(\alpha) + \operatorname{orb}(\alpha) - \operatorname{fix}(\alpha)$, where orb (α) is the number of non-singleton orbits of α. Clearly, $k(\alpha) \leqslant g(\alpha)$ and $g(\alpha) \leqslant n\left[\frac{1}{2}(\operatorname{ran}\alpha - 2)\right]$. A sharper upper bound for $k(\alpha)$ in terms of the orbits of α, and a lower bound $k(\alpha) \geq \operatorname{sh}(\alpha)/\operatorname{def}(\alpha)$ are also obtained. Precise characterisations of elements that can be written as products of 2 or 3 idempotents are given in a joint paper with Robertson and Schein [78], and this thread of research has been followed further by other authors, see for example [23]. Further results concerning the gravity and the decomposition of elements in Sing_n appear in [86].

Elements in Sing_n of minimum gravity 1 are simply idempotents of rank $n-1$; however, elements of maximum gravity are much more elaborate. With Lusk and McFadden [83], using computational tools to investigate examples, John calculated the number of elements α with maximum gravity. In particular, an interesting observation is made that when n is odd there are no elements with maximal gravity outside the group $\mathcal{H}$-classes. The question of embedding finite semigroups into semibands whilst minimising the depth was investigated with Giraldes [30].

Questions of depth for infinite transformation semigroups were first considered in [61]. Recall that by Theorem 3, for an infinite set X, the subsemigroup generated by the idempotents in $\mathcal{T}(X)$ can be written as

$$\langle E\rangle = F \cup (\bigcup_{\aleph_0 \leq \mathfrak{m} \leq |X|} Q_{\mathfrak{m}}),$$

where

$$F = \{\alpha \;:\; \mathrm{sh}(\alpha) < \aleph_0,\; 0 < \mathrm{def}(\alpha) < \aleph_0\},$$
$$Q_{\mathfrak{m}} = \{\alpha \;:\; \mathrm{sh}(\alpha) = \mathrm{def}(\alpha) = \mathrm{c}(\alpha) = \mathfrak{m}\}.$$

The elements of $Q_{\mathfrak{m}}$ are called *balanced mappings* of *weight* $\mathfrak{m}$.

Theorem 10 ([61, Theorem 3.2]) *Let X be an infinite set and let F be the subsemigroup of $\mathcal{T}(X)$ consisting of all elements with finite shift and non-zero defect. Then F is a regular idempotent-generated semigroup and $\Delta(F) = \infty$.*

Theorem 11 ([61, Theorem 3.7]) *Let X be an infinite set, let $\mathfrak{m}$ be a cardinal number such that $\aleph_0 \leq \mathfrak{m} \leq |X|$ and let $Q_{\mathfrak{m}}$ be the subsemigroup of $\mathcal{T}(X)$ consisting of all the balanced mappings of weight $\mathfrak{m}$. Then $Q_{\mathfrak{m}}$ is a regular idempotent-generated semigroup and $\Delta(Q_{\mathfrak{m}}) = 4$.*

This second theorem can be viewed as a distant intimation of what was to become known as *Bergman Property* in the 2000s, motivated by the result of Bergman [8] asserting that the infinite symmetric group $\mathcal{S}(X)$ has finite depth with respect to *any* generating set. This theme was brought back to semigroup theory by three of John's academic offspring in [95].

Meanwhile, questions of depth and lengths of products in infinite semigroups featured in further papers of John's with various collaborators. Given a set X with regular infinite cardinal $\mathfrak{m}$, the set of stable elements of $\mathcal{T}(X)$ is

$$S_{\mathfrak{m}} = \{\alpha \in \mathcal{T}(X) : \mathrm{ran}(\alpha) = \mathrm{def}(\alpha) = c(\alpha) = \mathfrak{m} \text{ and } \forall y \in \mathrm{im}\,\alpha,\, |y\alpha^{-1}| < \mathfrak{m}\}.$$

In [62], John shows that this is an idempotent-generated regular subsemigroup of $\mathcal{T}(X)$ of depth 4, and that it admits a quotient which is congruence-free (has no proper congruences), bisimple (has a unique $\mathcal{D}$-class), also idempotent-generated of depth 4, and contains a copy of every semigroup of cardinality less than $\mathfrak{m}$. In a similar vein, in [63], given another infinite cardinal $\mathfrak{n} < \mathfrak{m}$, John constructs another congruence-free bisimple semigroup, this time inverse, which is a homomorphic image of

$$\{\alpha \in \mathcal{I}(X) : \mathrm{sh}(\alpha) \leq \mathrm{def}(\alpha) = \mathrm{def}(\alpha^{-1}) = \mathfrak{n}\}.$$

Further related results include:

- with Schein [74]: a beautiful calculation of the length of any idempotent-generated endomorphism of an arbitrary well-ordered set (ordinal);
- with Gomes [35]: a description of the elements of the endomorphism monoid of a connected independence algebra that can be written as products of k singular idempotents, generalizing a result of Ballantine for vector spaces [7].

Finally, it is worth mentioning an attempt to bring the two strands of research represented by rank and depth closer together in a work with Cherubini and Piochi [11]. They define the *status* of a semigroup S to be the minimum value of $|A|\Delta(S, A)$ over all generating sets A for S. The authors then provide upper bounds for the status of certain groups, rectangular bands, monogenic semigroups and compute the exact value for the aperiodic Brandt semigroup.

5 Algebraic and structure theory

Although John's most enduring love was for transformations and generation, from time to time throughout his career he would venture into more classical, structural areas of semigroup theory.

In 1964–66, John published two papers [50,71] on the subject of congruences on an arbitrary semigroup. Contrary to what happens in group theory, for semigroups this is a non-trivial matter and continues to be an open field. The first paper deals with new descriptions of the maximum idempotent-separating congruence μ and the minimum group congruence σ on an inverse semigroup S, by means of the centralizer and the closure of $E(S)$ respectively. A recent citation appears in [42] in connection with the study of decompositions of semiheaps, a certain type of ternary algebra. In [71], jointly with Lallement, John discusses a collection of congruences on a regular semigroup, namely the least group congruence, the least band congruence, the least semilattice congruence, the least E-unitary band of groups congruence, and the least E-unitary Clifford congruence. Special attention is given to these congruences when S is orthodox and E-unitary. This is cited by various authors, e.g. [96,111]. Worth noting is an "innocent-looking" small lemma (2) in this paper, which tells how to obtain an inverse of a product of two idempotents in a regular semigroup, which is in fact a precursor of a now standard technique of finding inverses of products using the sandwich set [67, Theorem 2.5.4].

Given John's work on idempotent generation, it seems rather natural that his structural work would also be organised around idempotents. In [52] the bands for which the natural partial order is compatible with multiplication are characterised as strong semilattices of rectangular bands, this is in fact the case when every local subband is a semilattice. In [73] John and Schein consider *anti-uniform semilattices* E, i.e. semilattices for which no two principal ideals are isomorphic. They prove that a semilattice E is anti-uniform if and only if *every* inverse semigroup with set of idempotents E is a Clifford semigroup (union of groups). In an interesting convergence of themes, Yamamura [113] showed that these are the semilattices such that each locally full HNN extension (in a sense different from John's original) is a Clifford semigroup.

In [36], following a different line of research, Gomes and John prove an analogue of McAlister's P-theorem, which characterises E-unitary inverse semigroups, in terms of groups acting by order-automorphisms on partially ordered sets. The notion of E-unitarity never applies to inverse semigroups with zero. For this reason, the authors modify this notion to that of an E^*-unitary strongly categorical inverse semigroup. In this setting the role of the group is played by a *Brandt semigroup*, i.e. a completely 0-simple inverse semigroup, and the action is by partial maps. This description is extended to a wider class of E^*-unitary inverse semigroups in [10], while E^*-semigroups that are inverse have been subject of further study, see for example [16,110].

John and the first author continued their interest in structural results and treated an even more general situation in [37] where they considered semigroups with zero whose idempotents form a subsemigroup, i.e. *E-semigroups*, and which are E^*-dense and E^*-unitary as well as (strongly) categorical. Their structure is described in terms of a special *quiver* (a category without identities) acted upon by an inverse semigroup with zero that is *primitive*, i.e. in which every non zero idempotent is minimal with respect to the natural partial order. In this paper, E^*-unitary covers of E^*-dense categorical E-semigroups with zero are also constructed. In this volume we can find a paper by Fountain and Hayes on E^*-dense E-semigroups, their structure is discussed and strongly E^*-unitary covers are constructed.

6 Conclusion

As we have seen, John worked on a wide range of topics within semigroup theory, and his contributions have influenced the development of the field in many profound ways. However, the main qualities of John's work are impossible to convey in an article like this. Yet, it is sufficient to pick any single one of John's publications, to get a sense of what really mattered to him. John's writings reveal his instinctive commitment to communicating mathematics; uncompromisingly clear, and expressed with apparent lightness, they are invariably turned towards the reader as an open invitation to share in John's mathematical world. And pervading this is John's enduring love of mathematics and generosity in sharing his thoughts with collaborators and students. For those of us who have had the privilege to know John, learn from him and work with him, it is these qualities that will stay forever and constitute our memory of the great man.

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