

Subtraction Menger algebras

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Abstract We give an abstract characterization of algebras of partial functions from A^n to A endowed with the operations of the Menger superposition and the set-theoretic difference of functions as subsets of A^{n+1} .

Keywords Menger algebra · Algebra of multiplace functions · Subtraction algebra

1. Let A^n be the n -th Cartesian power of a set A . Any partial mapping from A^n into A is called a *partial n -place function*. The set of all such mappings is denoted by $\mathcal{F}(A^n, A)$. On $\mathcal{F}(A^n, A)$ we define the *Menger superposition* (composition) of n -place functions $O: (f, g_1, \dots, g_n) \mapsto f[g_1 \dots g_n]$ as follows:

$$(\bar{a}, c) \in f[g_1 \dots g_n] \iff (\exists \bar{b})((\bar{a}, b_1) \in g_1 \wedge \dots \wedge (\bar{a}, b_n) \in g_n \wedge (\bar{b}, c) \in f) \quad (1)$$

for all $\bar{a} \in A^n$, $\bar{b} = (b_1, \dots, b_n) \in A^n$, $c \in A$.

Each subalgebra (Φ, O) , where $\Phi \subset \mathcal{F}(A^n, A)$, of the algebra $(\mathcal{F}(A^n, A), O)$ is a Menger algebra of rank n in the sense of [2–4, 8]. Menger algebras of partial n -place functions are partially ordered by the set-theoretic inclusion, i.e., such algebras can be considered as algebras of the form (Φ, O, \subset) . The first abstract characterization of such algebras was given in [9]. Later, in [10, 11] there have been found abstract characterizations of Menger algebras of n -place functions closed with respect to the

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set-theoretic intersection and union of functions, i.e., Menger algebras of the form (Φ, O, \cap) , (Φ, O, \cup) and (Φ, O, \cap, \cup) .

As is well known, the set-theoretic inclusion \subset and the operations \cap, \cup can be expressed via the set-theoretic difference (subtraction) in the following way:

$$A \subset B \iff A \setminus B = \emptyset, \quad A \cap B = A \setminus (A \setminus B),$$

$$A \cup B = C \setminus ((C \setminus A) \cap (C \setminus B)),$$

where A, B, C are arbitrary sets such that $A \subset C$ and $B \subset C$.

Thus it makes sense to examine sets of functions closed with respect to the subtraction of functions. Such sets of functions are called *difference semigroups*, while their abstract analogs are called *subtraction semigroups*. Some properties of subtraction semigroups can be found in [1]. The investigation of difference semigroups was initiated by Schein [7].

Below we present a generalization of Schein’s results to the case of Menger algebras of n -place functions, i.e., to the case of algebras $(\Phi, O, \setminus, \emptyset)$, where $\Phi \subset \mathcal{F}(A^n, A)$, $\emptyset \in \Phi$. Such algebras will be called *difference Menger algebras*.

2. A Menger algebra of rank n is a non-empty set G with one $(n + 1)$ -ary operation $o(x, y_1, \dots, y_n) = x[y_1 \dots y_n]$ satisfying the identity:

$$x[y_1 \dots y_n][z_1 \dots z_n] = x[y_1[z_1 \dots z_n] \dots y_n[z_1 \dots z_n]]. \tag{2}$$

A Menger algebra of rank 1 is a semigroup. A Menger algebra (G, o) of rank n is called *unitary* if it contains *selectors*, i.e., elements $e_1, \dots, e_n \in G$ such that $x[e_1 \dots e_n] = x$ and $e_i[x_1 \dots x_n] = x_i$ for all $x, x_1, \dots, x_n \in G, i = 1, \dots, n$. One can prove (see [2, 3]) that every Menger algebra (G, o) of rank n can be isomorphically embedded into a unitary Menger algebra (G^*, o^*) of the same rank with selectors $e_1, \dots, e_n \notin G$ such that $G \cup \{e_1, \dots, e_n\}$ is a generating set of (G^*, o^*) .

Let (G, o) be a Menger algebra of rank n . Consider the alphabet $G \cup \{[,], x\}$, where the symbols $[,], x$ do not belong to G , and construct the set $T_n(G)$ of *polynomials* over this alphabet by the following rules:

- (a) $x \in T_n(G)$;
- (b) if $i \in \{1, \dots, n\}, a, b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n \in G, t \in T_n(G)$, then $a[b_1 \dots b_{i-1} t b_{i+1} \dots b_n] \in T_n(G)$;
- (c) $T_n(G)$ contains those and only those polynomials which are constructed by (a) and (b).

A binary relation $\rho \subset G \times G$, where (G, o) is a Menger algebra of rank n , is

- *stable* if for all $x, y, x_i, y_i \in G, i = 1, \dots, n$

$$(x, y), (x_1, y_1), \dots, (x_n, y_n) \in \rho \implies (x[x_1 \dots x_n], y[y_1 \dots y_n]) \in \rho;$$

- *l-regular*, if for any $x, y, z_i \in G, i = 1, \dots, n$

$$(x, y) \in \rho \implies (x[z_1 \dots z_n], y[z_1 \dots z_n]) \in \rho;$$

- *v-regular*, if for all $x_i, y_i, z \in G, i = 1, \dots, n$

$$(x_1, y_1), \dots, (x_n, y_n) \in \rho \longrightarrow (z[x_1 \dots x_n], z[y_1 \dots y_n]) \in \rho;$$

- *i-regular* ($1 \leq i \leq n$), if for all $u, x, y \in G, \bar{w} \in G^n$

$$(x, y) \in \rho \longrightarrow (u[\bar{w}|_i x], u[\bar{w}|_i y]) \in \rho;$$

- *weakly steady* if for all $x, y, z \in G, t_1, t_2 \in T_n(G)$

$$(x, y), (z, t_1(x)), (z, t_2(y)) \in \rho \longrightarrow (z, t_2(x)) \in \rho,$$

where $\bar{w} = (w_1, \dots, w_n)$ and $u[\bar{w}|_i x] = u[w_1 \dots w_{i-1} x w_{i+1} \dots w_n]$. It is clear that a quasiorder¹ on a Menger algebra is *v-regular* if and only if it is *i-regular* for every $i = 1, \dots, n$. A quasiorder is stable if and only if it is both *v-regular* and *l-regular*.

A subset H of a Menger algebra (G, o) is called

- *stable* if

$$g, g_1, \dots, g_n \in H \longrightarrow g[g_1 \dots g_n] \in H;$$

- an *l-ideal*, if for all $x, h_1, \dots, h_n \in G$

$$(h_1, \dots, h_n) \in G^n \setminus (G \setminus H)^n \longrightarrow x[h_1 \dots h_n] \in H;$$

- an *i-ideal* ($1 \leq i \leq n$), if for all $h, u \in G, \bar{w} \in G^n$

$$h \in H \longrightarrow u[\bar{w}|_i h] \in H.$$

Clearly, H is an *l-ideal* if and only if it is an *i-ideal* for every $i = 1, \dots, n$.

Definition 1 An algebra $(G, -, 0)$ of type $(2, 0)$ is called a *subtraction algebra* if it satisfies the following identities:

$$x - (y - x) = x, \tag{3}$$

$$x - (x - y) = y - (y - x), \tag{4}$$

$$(x - y) - z = (x - z) - y, \tag{5}$$

$$0 - 0 = 0. \tag{6}$$

Proposition 1 (Abbott [1]) *Every subtraction algebra satisfies the identity*

$$0 = x - x. \tag{7}$$

Proof Below we give a short proof of this identity:

¹Recall that a *quasiorder* is a reflexive and transitive binary relation.

$$\begin{aligned}
 0 &\stackrel{(3)}{=} 0 - ((0 - (x - x)) - 0) \stackrel{(5)}{=} 0 - ((0 - 0) - (x - x)) \stackrel{(6)}{=} 0 - (0 - (x - x)) \\
 &\stackrel{(4)}{=} (x - x) - ((x - x) - 0) \stackrel{(5)}{=} (x - x) - ((x - 0) - x) \\
 &\stackrel{(5)}{=} (x - ((x - 0) - x)) - x \stackrel{(3)}{=} x - x,
 \end{aligned}$$

as required. □

From (7), by using (3), we obtain the following two identities:

$$x - 0 = x, \quad 0 - x = 0. \tag{8}$$

Similarly, from (4), (5), (7) and (8) we can deduce the identities

$$((x - y) - (x - z)) - (z - y) = 0, \tag{9}$$

$$(x - (x - y)) - y = 0. \tag{10}$$

Thus, subtraction algebras are implicative BCK-algebras (cf. [5, 6]).

Definition 2 An algebra $(G, o, -, 0)$ of type $(n + 1, 2, 0)$ is called a *subtraction Menger algebra* of rank n , if (G, o) is a Menger algebra of rank n , $(G, -, 0)$ is a subtraction algebra and the conditions

$$(x - y)[z_1 \dots z_n] = x[z_1 \dots z_n] - y[z_1 \dots z_n], \tag{11}$$

$$u[\bar{w}|_i (x - (x - y))] = u[\bar{w}|_i x] - u[\bar{w}|_i (x - y)], \tag{12}$$

$$x - y = 0 \wedge z - t_1(x) = 0 \wedge z - t_2(y) = 0 \implies z - t_2(x) = 0 \tag{13}$$

hold for all $x, y, z, u, z_1, \dots, z_n \in G, \bar{w} \in G^n, i = 1, \dots, n$ and $t_1, t_2 \in T_n(G)$.

By putting $n = 1$ in the above definition we obtain the notion of a *weak subtraction semigroup*² studied by Schein (cf. [7]). Such semigroups are isomorphic to some subtraction semigroups of the form (Φ, o, \setminus) .

3. Now we can present the first result of our paper.

Theorem 1 *Each difference Menger algebra of n -place functions is a subtraction Menger algebra of rank n .*

Proof Let $(\Phi, O, \setminus, \emptyset)$ be a difference Menger algebra of n -place functions defined on A . Since, as it is proved in [2], the superposition O satisfies (2), the algebra (Φ, O) is a Menger algebra of rank n . From the results proved in [1] it follows that the operation \setminus satisfies (3), (4) and (5). Hence $(\Phi, \setminus, \emptyset)$ is a subtraction algebra. Thus,

²A weak subtraction semigroup $(S, \cdot, -)$ is a semigroup (S, \cdot) satisfying the identities (3), (4), (5), $x(y - z) = xy - xz$ and $(x - (x - y))z = xz - (x - y)z$.

$(\Phi, O, \setminus, \emptyset)$ will be a subtraction Menger algebra if (11), (12) and (13) will be satisfied.

To verify (11) observe that for each $(\bar{a}, c) \in (f \setminus g)[h_1 \dots h_n]$, where $f, g, h_1, \dots, h_n \in \Phi, \bar{a} \in A^n, c \in A$ there exists $\bar{b} = (b_1, \dots, b_n) \in A^n$ such that $(\bar{b}, c) \in f \setminus g$ and $(\bar{a}, b_i) \in h_i$ for each $i = 1, \dots, n$. Consequently, $(\bar{b}, c) \in f$ and $(\bar{b}, c) \notin g$. Thus, $(\bar{a}, c) \in f[h_1 \dots h_n]$. If $(\bar{a}, c) \in g[h_1 \dots h_n]$, then there exists $\bar{d} = (d_1, \dots, d_n) \in A^n$ such that $(\bar{d}, c) \in g$ and $(\bar{a}, d_i) \in h_i$ for every $i = 1, \dots, n$. Since h_1, \dots, h_n are functions, we obtain $b_i = d_i$ for all $i = 1, \dots, n$. Thus $\bar{b} = \bar{d}$. Therefore $(\bar{b}, c) \in g$, which is impossible. Hence $(\bar{a}, c) \notin g[h_1 \dots h_n]$. This means that $(\bar{a}, c) \in f[h_1 \dots h_n] \setminus g[h_1 \dots h_n]$. So, the following implication

$$(\bar{a}, c) \in (f \setminus g)[h_1 \dots h_n] \longrightarrow (\bar{a}, c) \in f[h_1 \dots h_n] \setminus g[h_1 \dots h_n]$$

is valid for any $\bar{a} \in A^n, c \in A$, i.e., $(f \setminus g)[h_1 \dots h_n] \subset f[h_1 \dots h_n] \setminus g[h_1 \dots h_n]$.

Conversely, let $(\bar{a}, c) \in f[h_1 \dots h_n] \setminus g[h_1 \dots h_n]$. Then $(\bar{a}, c) \in f[h_1 \dots h_n]$ and $(\bar{a}, c) \notin g[h_1 \dots h_n]$. Thus, there exists $\bar{b} = (b_1, \dots, b_n) \in A^n$ such that $(\bar{b}, c) \in f, (\bar{b}, c) \notin g$ and $(\bar{a}, b_i) \in h_i$ for each $i = 1, \dots, n$. Hence, $(\bar{b}, c) \in f \setminus g$ and $(\bar{a}, c) \in (f \setminus g)[h_1 \dots h_n]$. So,

$$(\bar{a}, c) \in f[h_1 \dots h_n] \setminus g[h_1 \dots h_n] \longrightarrow (\bar{a}, c) \in (f \setminus g)[h_1 \dots h_n]$$

for any $\bar{a} \in A^n, c \in A$, i.e., $f[h_1 \dots h_n] \setminus g[h_1 \dots h_n] \subset (f \setminus g)[h_1 \dots h_n]$. Thus,

$$(f \setminus g)[h_1 \dots h_n] = f[h_1 \dots h_n] \setminus g[h_1 \dots h_n],$$

which proves (11).

Now, let $(\bar{a}, c) \in u[\bar{\omega}|_i(f \setminus (f \setminus g))] = u[\bar{\omega}|_i(f \cap g)]$, where $f, g, u \in \Phi, \bar{\omega} \in \Phi^n, \bar{a} \in A^n, c \in A$. Then there exists $\bar{b} = (b_1, \dots, b_n) \in A^n$ such that $(\bar{a}, b_i) \in f \cap g, (\bar{a}, b_j) \in \omega_j, j \in \{1, \dots, n\} \setminus \{i\}$ and $(\bar{b}, c) \in u$. Since $(\bar{a}, b_i) \in f \cap g$ implies $(\bar{a}, b_i) \notin f \setminus g$, we have $(\bar{a}, c) \in u[\bar{\omega}|_i f]$ and $(\bar{a}, c) \notin u[\bar{\omega}|_i(f \setminus g)]$. Therefore $(\bar{a}, c) \in u[\bar{\omega}|_i f] \setminus u[\bar{\omega}|_i(f \setminus g)]$. Thus, we have shown that for any $\bar{a} \in A^n, c \in A$ holds the implication

$$(\bar{a}, c) \in u[\bar{\omega}|_i(f \setminus (f \setminus g))] \longrightarrow (\bar{a}, c) \in u[\bar{\omega}|_i f] \setminus u[\bar{\omega}|_i(f \setminus g)],$$

which is equivalent to the inclusion $u[\bar{\omega}|_i(f \setminus (f \setminus g))] \subset u[\bar{\omega}|_i f] \setminus u[\bar{\omega}|_i(f \setminus g)]$.

Conversely, let $(\bar{a}, c) \in u[\bar{\omega}|_i f] \setminus u[\bar{\omega}|_i(f \setminus g)]$. Then $(\bar{a}, c) \in u[\bar{\omega}|_i f]$ and $(\bar{a}, c) \notin u[\bar{\omega}|_i(f \setminus g)]$. The first of these two conditions means that there exists $\bar{b} = (b_1, \dots, b_n) \in A^n$ such that $(\bar{a}, b_i) \in f, (\bar{a}, b_j) \in \omega_j$ for each $j \in \{1, \dots, n\} \setminus \{i\}$ and $(\bar{b}, c) \in u$. It is easy to see that the second condition $(\bar{a}, c) \notin u[\bar{\omega}|_i(f \setminus g)]$ is equivalent to the implication

$$(\forall \bar{d}) \left((\bar{a}, d_i) \in f \wedge \bigwedge_{j=1, j \neq i}^n (\bar{a}, d_j) \in \omega_j \wedge (\bar{d}, c) \in u \longrightarrow (\bar{a}, d_i) \in g \right), \tag{14}$$

where $\bar{d} = (d_1, \dots, d_n) \in A^n$. From this implication for $\bar{d} = \bar{b}$, we obtain

$$(\bar{a}, b_i) \in f \wedge \bigwedge_{j=1, j \neq i}^n (\bar{a}, b_j) \in \omega_j \wedge (\bar{b}, c) \in u \longrightarrow (\bar{a}, b_i) \in g,$$

which gives $(\bar{a}, b_i) \in g$. Therefore $(\bar{a}, b_i) \in f \cap g = f \setminus (f \setminus g)$. This means that $(\bar{a}, c) \in u[\bar{\omega}|_i(f \setminus (f \setminus g))]$. So, the implication

$$(\bar{a}, c) \in u[\bar{\omega}|_i f] \setminus u[\bar{\omega}|_i(f \setminus g)] \longrightarrow (\bar{a}, c) \in u[\bar{\omega}|_i(f \setminus (f \setminus g))]$$

is valid for all $\bar{a} \in A^n, c \in A$. Hence $u[\bar{\omega}|_i f] \setminus u[\bar{\omega}|_i(f \setminus g)] \subset u[\bar{\omega}|_i(f \setminus (f \setminus g))]$. Thus

$$u[\bar{\omega}|_i(f \setminus (f \setminus g))] = u[\bar{\omega}|_i f] \setminus u[\bar{\omega}|_i(f \setminus g)].$$

This proves (12).

To prove (13) suppose that for some $f, g, h \in \Phi$ and $t_1, t_2 \in T_n(\Phi)$ we have $f \setminus g = \emptyset, h \setminus t_1(f) = \emptyset$ and $h \setminus t_2(g) = \emptyset$. Then $f \subset g, h \subset t_1(f)$ and $h \subset t_2(g)$. Hence $f = g \circ \Delta_{pr_1 f}$ and $pr_1 h \subset pr_1 f$, where $pr_1 f$ denotes the domain of f and $\Delta_{pr_1 f}$ is the identity binary relation on $pr_1 f$.

From the inclusion $h \subset t_2(g)$ we obtain

$$h = h \circ \Delta_{pr_1 f} \subset t_2(g) \circ \Delta_{pr_1 f} = t_2(g \circ \Delta_{pr_1 f}) = t_2(f),$$

which means that (13) is also satisfied. This completes the proof that $(\Phi, O, \setminus, \emptyset)$ is a subtraction Menger algebra of rank n . □

To prove the converse statement, we should first consider a number of properties of subtraction Menger algebras of rank n , introduce some definitions and prove a few auxiliary propositions.

4. Let $(G, o, -, 0)$ be a subtraction Menger algebra of rank n .

Proposition 2 *In every subtraction Menger algebra of rank n we have*

$$0[x_1 \dots x_n] = 0, \quad x[x_1 \dots x_{i-1} 0 x_{i+1} \dots x_n] = 0$$

for all $x, x_1, \dots, x_n \in G, i = 1, \dots, n$.

Proof Indeed, using (7) and (11) we obtain

$$0[x_1 \dots x_n] = (0 - 0)[x_1 \dots x_n] = 0[x_1 \dots x_n] - 0[x_1 \dots x_n] = 0.$$

Similarly, applying (12) and (7) we get

$$u[\bar{w}|_i 0] = u[\bar{w}|_i(0 - (0 - 0))] = u[\bar{w}|_i 0] - u[\bar{w}|_i(0 - 0)] = u[\bar{w}|_i 0] - u[\bar{w}|_i 0] = 0,$$

which was to show. □

Let ω be a binary relation defined on $(G, o, -, 0)$ in the following way:

$$\omega = \{(x, y) \in G \times G \mid x - y = 0\}.$$

Using (7), (8) and (9) it is easy to see that this is an order, i.e., a reflexive, transitive and antisymmetric relation. In connection with this fact we will sometimes write $x \leq y$ instead of $(x, y) \in \omega$. Using this notation it is not difficult to verify that

$$0 \leq x, \quad x - y \leq x, \tag{15}$$

$$x \leq y \iff x - (x - y) = x, \tag{16}$$

$$x \leq y \implies x - z \leq y - z, \tag{17}$$

$$x \leq y \implies z - y \leq z - x, \tag{18}$$

$$x \leq y \wedge u \leq v \implies x - v \leq y - u \tag{19}$$

holds for all $x, y, z, u, v \in G$.

Moreover, in a subtraction algebra the following two identities

$$(x - y) - y = x - y, \tag{107}$$

$$(x - y) - z = (x - z) - (y - z) \tag{108}$$

are valid (cf. [1, 5, 6]).

Proposition 3 *The relation ω on the algebra $(G, o, -, 0)$ is stable and weakly steady.*

Proof Let $x \leq y$ for some $x, y \in G$. Then $x - y = 0$ and

$$(x - y)[z_1 \dots z_n] = 0[z_1 \dots z_n] = (0 - 0)[z_1 \dots z_n] = 0[z_1 \dots z_n] - 0[z_1 \dots z_n] = 0$$

for all $z_1, \dots, z_n \in G$. This, by (11), implies

$$x[z_1 \dots z_n] - y[z_1 \dots z_n] = 0,$$

i.e., $x[z_1 \dots z_n] \leq y[z_1 \dots z_n]$. Thus, ω is l -regular.

Moreover, from $x \leq y$, using (8), we obtain $x - (x - y) = x$, which together with (4), gives $y - (y - x) = x$. Consequently, for any $u \in G, \bar{w} \in G^n$ we have $u[\bar{w}|_i(y - (y - x))] = u[\bar{w}|_i x]$. This and (11) give $u[\bar{w}|_i y] - u[\bar{w}|_i(y - x)] = u[\bar{w}|_i x]$. Hence, according to (15), we obtain $u[\bar{w}|_i x] \leq u[\bar{w}|_i y]$. Thus, ω is i -regular for every $i = 1, \dots, n$. Since ω is a quasiorder, this means that ω is v -regular. But ω also is l -regular, hence it is stable.

It is clear that ω is weakly steady if and only if it satisfies (13).³ □

Proposition 4 *The axiom (12) is equivalent to each of the following conditions:*

$$x \leq y \implies u[\bar{w}|_i(y - x)] = u[\bar{w}|_i y] - u[\bar{w}|_i x], \tag{122}$$

$$x \leq y \implies t(y - x) = t(y) - t(x), \tag{123}$$

³In the case of semigroups the fact that ω is weakly steady can be deduced directly from the axioms of a weak subtraction semigroup (cf. [7]).

$$t(x - (x - y)) = t(x) - t(x - y) \tag{24}$$

for all $x, y, u \in G, \bar{w} \in G^n, i = 1, \dots, n, t \in T_n(G)$.

Proof (12) \rightarrow (22). Suppose that the condition (12) is satisfied and $x \leq y$ for some $x, y \in G$. Then, according to (16), we have $x - (x - y) = x$. Hence, by (4), we obtain $y - (y - x) = x$. Thus, $y - x = y - (y - (y - x))$, which, in view of (12), gives $u[\bar{w}|_i(y - x)] = u[\bar{w}|_i(y - (y - (y - x)))] = u[\bar{w}|_i y] - u[\bar{w}|_i(y - (y - x))] = u[\bar{w}|_i y] - u[\bar{w}|_i x]$. This means that (12) implies (22).

(22) \rightarrow (23). From (22) it follows that for $x \leq y$ and all polynomials $t \in T_n(G)$ of the form $t(x) = u[\bar{w}|_i x]$ the condition (23) is satisfied. To prove that (23) is satisfied by an arbitrary polynomial from $T_n(G)$ suppose that it is satisfied by some $t' \in T_n(G)$. Since the relation ω is stable on the algebra $(G, o, -, 0)$, from $x \leq y$ it follows $t'(x) \leq t'(y)$, which in view of (22), implies

$$u[\bar{w}|_i(t'(y) - t'(x))] = u[\bar{w}|_i t'(y)] - u[\bar{w}|_i t'(x)].$$

But according to the assumption on t' for $x \leq y$ we have $t'(y) - t'(x) = t'(y - x)$, so the above equation can be written as

$$u[\bar{w}|_i t'(y - x)] = u[\bar{w}|_i t'(y)] - u[\bar{w}|_i t'(x)].$$

Thus, (23) is satisfied by polynomials of the form $t(x) = u[\bar{w}|_i t'(x)]$.

From the construction of $T_n(G)$ it follows that (23) is satisfied by all polynomials $t \in T_n(G)$. Therefore (22) implies (23).

(23) \rightarrow (24). Since, by (15), $x - y \leq x$ holds for all $x, y \in G$, from (23) it follows $t(x - (x - y)) = t(x) - t(x - y)$ for any polynomial $t \in T_n(G)$. Thus, (23) implies (24).

(24) \rightarrow (12). By putting $t(x) = u[\bar{w}|_i x]$ we obtain (12). □

On a subtraction Menger algebra $(G, o, -, 0)$ of rank n we can define a binary operation \wedge by putting:

$$x \wedge y \stackrel{\text{def}}{=} x - (x - y). \tag{25}$$

By using this operation the conditions (11), (16), (24) can be written in a more useful form:

$$u[\bar{w}|_i(x \wedge y)] = u[\bar{w}|_i x] - u[\bar{w}|_i(x - y)], \tag{26}$$

$$x \leq y \iff x \wedge y = x, \tag{27}$$

$$t(x \wedge y) = t(x) - t(x - y), \tag{28}$$

where $x, y, u \in G, \bar{w} \in G^n, i = 1, \dots, n, t \in T_n(G)$. Moreover, from (11) and (25), we can deduce the identity:

$$(x \wedge y)[z_1 \dots z_n] = x[z_1 \dots z_n] \wedge y[z_1 \dots z_n]. \tag{29}$$

The algebra (G, \wedge) is a lower semilattice. Directly from the conditions (3)–(10) we obtain (cf. [1]) the following properties:

$$x \leq y \wedge x \leq z \longrightarrow x \leq y \wedge z, \tag{30}$$

$$x \leq y \longrightarrow x \wedge z \leq y \wedge z, \tag{31}$$

$$x \wedge y = 0 \longrightarrow x - y = x, \tag{32}$$

$$(x - y) \wedge y = 0, \tag{33}$$

$$x \wedge (y - z) = (x \wedge y) - (x \wedge z), \tag{34}$$

$$x - y = x - (x \wedge y), \tag{35}$$

$$(x \wedge y) - (y - z) = x \wedge y \wedge z, \tag{36}$$

$$(x \wedge y) - z = (x - z) \wedge (y - z), \tag{37}$$

$$(x \wedge y) - z = (x - z) \wedge y \tag{38}$$

for all $x, y, z \in G$.

Proposition 5 *In a subtraction Menger algebra $(G, o, -, 0)$ of rank n the following conditions*

$$t(x - y) = t(x) - t(x \wedge y), \tag{39}$$

$$t(x) - t(y) \leq t(x - y) \tag{40}$$

are valid for each $t \in T_n(G)$ and $x, y \in G$.

Proof From (35) we obtain $t(x - y) = t(x - (x \wedge y))$ for every $t \in T_n(G)$. (25) and (15) imply $x \wedge y \leq x$, which together with (23) gives $t(x - (x \wedge y)) = t(x) - t(x \wedge y)$. Hence, $t(x - y) = t(x) - t(x \wedge y)$. This proves (39).

Since $x \wedge y \leq y$, the stability of ω implies $t(x \wedge y) \leq t(y)$ for every $t \in T_n(G)$. From this, by applying (15) and (18), we obtain $t(x) - t(y) \leq t(x) - t(x \wedge y) = t(x - y)$, which proves (40). \square

By $[0, a]$ we denote the *initial segment* of the algebra $(G, -, 0)$, i.e., the set of all $x \in G$ such that $0 \leq x \leq a$. According to [7], on any $[0, a]$ we can define a binary operation Υ by putting:

$$x \Upsilon y \stackrel{\text{def}}{=} a - ((a - x) \wedge (a - y)) \tag{41}$$

for all $x, y \in [0, a]$. It is not difficult to see that this operation is idempotent and commutative, and 0 is its neutral element, i.e., $x \Upsilon x = x, x \Upsilon y = y \Upsilon x, x \Upsilon 0 = x$ for all $x, y \in [0, a]$.

Proposition 6 *For any $x, y \in [0, b] \subset [0, a]$, where $a, b \in G$, we have*

$$b - ((b - x) \wedge (b - y)) = a - ((a - x) \wedge (a - y)). \tag{42}$$

Proof Note first that $b = b \wedge a$ because $b \leq a$. Moreover, from $x \leq b$ and $y \leq b$, according to (18), we obtain $a - b \leq a - x$ and $a - b \leq a - y$. This together with (30) gives $a - b \leq (a - x) \wedge (a - y)$. Thus, $(a - b) - ((a - x) \wedge (a - y)) = 0$.

By (15) we have $b - ((a - x) \wedge (a - y)) \leq b$, which implies

$$b \wedge (b - ((a - x) \wedge (a - y))) = b - ((a - x) \wedge (a - y)). \tag{43}$$

Obviously $b = b \wedge b = b \wedge a, x = b \wedge x, y = b \wedge y$. Therefore:⁴

$$\begin{aligned} & b - ((b - x) \wedge (b - y)) \\ &= b \wedge b - ((b \wedge a - b \wedge x) \wedge (b \wedge a - b \wedge y)) \\ &\stackrel{(34)}{=} b \wedge b - (b \wedge (a - x) \wedge b \wedge (a - y)) = b \wedge b - b \wedge ((a - x) \wedge (a - y)) \\ &\stackrel{(34)}{=} b \wedge (b - ((a - x) \wedge (a - y))) \stackrel{(42)}{=} b - ((a - x) \wedge (a - y)) \\ &= a \wedge b - ((a - x) \wedge (a - y)) \stackrel{(25)}{=} (a - (a - b)) - ((a - x) \wedge (a - y)) \\ &\stackrel{(21)}{=} (a - ((a - x) \wedge (a - y))) - ((a - b) - ((a - x) \wedge (a - y))) \\ &= (a - ((a - x) \wedge (a - y))) - 0 \stackrel{(8)}{=} a - ((a - x) \wedge (a - y)), \end{aligned}$$

which completes the proof. □

Corollary 1 *The condition (42) is valid for all $x, y \in [0, a] \cap [0, b]$.*

Proof Since $[0, a] \cap [0, b] = [0, a \wedge b] \subset [0, a] \cup [0, b]$, by Proposition 6, for all $x, y \in [0, a] \cap [0, b]$ we have:

$$\begin{aligned} a - ((a - x) \wedge (a - y)) &= a \wedge b - ((a \wedge b - x) \wedge (a \wedge b - y)), \\ b - ((b - x) \wedge (b - y)) &= a \wedge b - ((a \wedge b - x) \wedge (a \wedge b - y)). \end{aligned}$$

This implies (42). □

From the above corollary it follows that the value of $x \vee y$, if it exists, does not depend on the choice of the interval $[0, a]$ containing the elements x and y . In [1] it is proved that for $x, y, z \in [0, a]$ we have:

$$x \wedge (x \vee y) = x, \tag{44}$$

$$x \vee (x \wedge y) = x, \tag{45}$$

$$(x \vee y) \vee z = x \vee (y \vee z), \tag{46}$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \tag{47}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \tag{48}$$

⁴To reduce the number of brackets we will write $x \wedge y - z$ instead of $(x \wedge y) - z$.

$$(x \Upsilon y) - z = (x - z) \Upsilon (y - z), \tag{49}$$

$$x \leq z \wedge y \leq z \longrightarrow x \Upsilon y \leq z, \tag{50}$$

$$y \leq x \longrightarrow x = (x - y) \Upsilon y, \tag{51}$$

$$x = (x \Upsilon y) - (y - x), \tag{52}$$

$$x = (x \wedge y) \Upsilon (x - y). \tag{53}$$

From (44) it follows $x \leq x \Upsilon y$.

Proposition 7 *If for some $x, y \in G$ there exists $x \Upsilon y$, then for all $u \in G, \bar{z}, \bar{w} \in G^n, i = 1, \dots, n$ there are also elements $x[\bar{z}] \Upsilon y[\bar{z}]$ and $u[\bar{w}|_i x] \Upsilon u[\bar{w}|_i y]$, and the following identities are satisfied:*

$$(x \Upsilon y)[\bar{z}] = x[\bar{z}] \Upsilon y[\bar{z}], \tag{54}$$

$$u[\bar{w}|_i(x \Upsilon y)] = u[\bar{w}|_i x] \Upsilon u[\bar{w}|_i y]. \tag{55}$$

Proof Suppose that the element $x \Upsilon y$ exists. Then $x \leq a$ and $y \leq a$ for some $a \in G$, which, by the l -regularity of the relation ω , implies $x[\bar{z}] \leq a[\bar{z}]$ and $y[\bar{z}] \leq a[\bar{z}]$ for any $\bar{z} \in G^n$. This means that $x[\bar{z}] \Upsilon y[\bar{z}]$ exists and

$$\begin{aligned} (x \Upsilon y)[\bar{z}] &\stackrel{(41)}{=} (a - ((a - x) \wedge (a - y)))[\bar{z}] \stackrel{(11)}{=} a[\bar{z}] - ((a - x) \wedge (a - y))[\bar{z}] \\ &\stackrel{(29)}{=} a[\bar{z}] - ((a - x)[\bar{z}] \wedge (a - y)[\bar{z}]) \\ &\stackrel{(11)}{=} a[\bar{z}] - ((a[\bar{z}] - x[\bar{z}]) \wedge (a[\bar{z}] - y[\bar{z}])) \stackrel{(41)}{=} x[\bar{z}] \Upsilon y[\bar{z}]. \end{aligned}$$

This proves (54).

Further, from $x \leq a, y \leq a$ and the i -regularity of ω we obtain $u[\bar{w}|_i x] \leq u[\bar{w}|_i a]$ and $u[\bar{w}|_i y] \leq u[\bar{w}|_i a]$. Hence, the element $u[\bar{w}|_i x] \Upsilon u[\bar{w}|_i y]$ exists. Since $x \leq x \Upsilon y$ and $y \leq x \Upsilon y$, we also have $u[\bar{w}|_i x] \leq u[\bar{w}|_i(x \Upsilon y)]$ and $u[\bar{w}|_i y] \leq u[\bar{w}|_i(x \Upsilon y)]$, which, according to (50), gives

$$u[\bar{w}|_i x] \Upsilon u[\bar{w}|_i y] \leq u[\bar{w}|_i(x \Upsilon y)]. \tag{56}$$

On the other side, the existence of $u[\bar{w}|_i x] \Upsilon u[\bar{w}|_i y]$ implies

$$u[\bar{w}|_i x] \leq u[\bar{w}|_i x] \Upsilon u[\bar{w}|_i y] \quad \text{and} \quad u[\bar{w}|_i y] \leq u[\bar{w}|_i x] \Upsilon u[\bar{w}|_i y].$$

Moreover,

$$u[\bar{w}|_i(x \Upsilon y)] - u[\bar{w}|_i(y - x)] \stackrel{(40)}{\leq} u[\bar{w}|_i((x \Upsilon y) - (y - x))] \stackrel{(52)}{=} u[\bar{w}|_i x].$$

Consequently,

$$u[\bar{w}|_i(x \Upsilon y)] - u[\bar{w}|_i(y - x)] \leq u[\bar{w}|_i x] \Upsilon u[\bar{w}|_i y]. \tag{57}$$

But $y - x \leq y$, so, $u[\bar{w}|_i(y - x)] \leq u[\bar{w}|_i y]$ and

$$u[\bar{w}|_i(y - x)] \leq u[\bar{w}|_i x] \vee u[\bar{w}|_i y].$$

This and (57) guarantee the existence of the element

$$(u[\bar{w}|_i(x \vee y)] - u[\bar{w}|_i(y - x)]) \vee u[\bar{w}|_i(y - x)]$$

such that

$$(u[\bar{w}|_i(x \vee y)] - u[\bar{w}|_i(y - x)]) \vee u[\bar{w}|_i(y - x)] \leq u[\bar{w}|_i x] \vee u[\bar{w}|_i y]. \tag{58}$$

Since $u[\bar{w}|_i(y - x)] \leq u[\bar{w}|_i y] \leq u[\bar{w}|_i(x \vee y)]$, the last inequality and (51) imply

$$(u[\bar{w}|_i(x \vee y)] - u[\bar{w}|_i(y - x)]) \vee u[\bar{w}|_i(y - x)] = u[\bar{w}|_i(x \vee y)],$$

which together with (58) gives

$$u[\bar{w}|_i(x \vee y)] \leq u[\bar{w}|_i x] \vee u[\bar{w}|_i y].$$

Comparing this inequality with (56) we obtain (55). □

Corollary 2 *If for some $x, y \in G$ an element $x \vee y$ exists, then for any polynomial $t \in T_n(G)$ an element $t(x) \vee t(y)$ also exists and $t(x \vee y) = t(x) \vee t(y)$.*

Proposition 8 *For all $x, y \in G$ and all polynomials $t_1, t_2 \in T_n(G)$ we have:*

$$t_1(x \wedge y) \wedge t_2(x - y) = 0.$$

Proof Let $t_1(x \wedge y) \wedge t_2(x - y) = h$. Obviously $h \leq t_1(x \wedge y)$ and $h \leq t_2(x - y)$. Since $t_2(x - y) \leq t_2(x)$, we have $h \leq t_2(x)$. Thus, $x \wedge y \leq x$, $h \leq t_1(x \wedge y)$ and $h \leq t_2(x)$. This, in view of Proposition 3 and (13), gives $h \leq t_2(x \wedge y)$. Consequently,

$$h \leq t_2(x - y) \wedge t_2(x \wedge y). \tag{59}$$

Further,

$$\begin{aligned} t_2(x - y) - t_2(x \wedge y) &\stackrel{(39)}{=} (t_2(x) - t_2(x \wedge y)) - t_2(x \wedge y) \\ &\stackrel{(20)}{=} t_2(x) - t_2(x \wedge y) \stackrel{(39)}{=} t_2(x - y). \end{aligned}$$

Therefore,

$$\begin{aligned} t_2(x - y) \wedge t_2(x \wedge y) &\stackrel{(25)}{=} t_2(x - y) - (t_2(x - y) - t_2(x \wedge y)) \\ &= t_2(x - y) - t_2(x - y) = 0, \end{aligned}$$

which together with (59) implies $h \leq 0$. Hence $h = 0$. This completes the proof. □

Proposition 9 For all $x, y, z, g \in G$ and all polynomials $t_1, t_2 \in T_n(G)$ the following conditions are valid:

$$t_1(x \wedge y) \wedge t_2(y) = t_1(x \wedge y) \wedge t_2(x \wedge y), \tag{60}$$

$$t_1(x \wedge y \wedge z) \wedge t_2(y) \leq t_1(x \wedge y) \wedge t_2(y \wedge z), \tag{61}$$

$$g \leq t_1(x \wedge y) \wedge g \leq t_2(y \wedge z) \longrightarrow g \leq t_2(x \wedge y \wedge z). \tag{62}$$

Proof To prove (60) observe first that for $z = t_1(x \wedge y) \wedge t_2(y)$ we have $z \leq t_1(x \wedge y)$ and $z \leq t_2(y)$. Since the relation ω is weakly steady and $x \wedge y \leq y$, from the above we conclude $z \leq t_2(x \wedge y)$, i.e., $t_1(x \wedge y) \wedge t_2(y) \leq t_2(x \wedge y)$. This, by (31), implies $t_1(x \wedge y) \wedge t_2(y) \leq t_1(x \wedge y) \wedge t_2(x \wedge y)$.

On the other side, the stability of ω and $x \wedge y \leq y$ imply $t_2(x \wedge y) \leq t_2(y)$ for every $t_2 \in T_n(G)$. Hence, $t_1(x \wedge y) \wedge t_2(x \wedge y) \leq t_1(x \wedge y) \wedge t_2(y)$ by (31). This completes the proof of (60).

Further: $t_1(x \wedge y \wedge z) \wedge t_2(y) = t_1((x \wedge z) \wedge y) \wedge t_2(y) \stackrel{(60)}{=} t_1((x \wedge z) \wedge y) \wedge t_2((x \wedge z) \wedge y) \leq t_1(x \wedge y) \wedge t_2(y \wedge z)$ proves (61).

Finally, let $g \leq t_1(x \wedge y)$ and $g \leq t_2(y \wedge z)$. Then

$$\begin{aligned} g &\leq t_1(x \wedge y) \wedge t_2(y \wedge z) \\ &\stackrel{(28)}{=} t_1(x \wedge y) \wedge (t_2(y) - t_2(y - z)) \\ &\stackrel{(34)}{=} (t_1(x \wedge y) \wedge t_2(y)) - (t_1(x \wedge y) \wedge t_2(y - z)) \\ &\stackrel{(60)}{=} (t_1(x \wedge y) \wedge t_2(x \wedge y)) - (t_1(x \wedge y) \wedge t_2(y - z)) \\ &\stackrel{(34)}{=} t_1(x \wedge y) \wedge (t_2(x \wedge y) - t_2(y - z)) \leq t_2(x \wedge y) - t_2(y - z) \\ &\stackrel{(40)}{\leq} t_2((x \wedge y) - (y - z)) \stackrel{(36)}{=} t_2(x \wedge y \wedge z). \end{aligned}$$

This proves (62) and completes the proof of our proposition. □

Corollary 3 For all $x, y, z \in G$ and all polynomials $t_1, t_2 \in T_n(G)$ we have:

$$t_1(x \wedge y \wedge z) \wedge t_2(y) = t_1(x \wedge y) \wedge t_2(y \wedge z). \tag{63}$$

Proof We have $t_1(x \wedge y) \wedge t_2(y \wedge z) \leq t_1(x \wedge y)$ and $t_1(x \wedge y) \wedge t_2(y \wedge z) \leq t_2(y \wedge z)$, so by (62) we obtain $t_1(x \wedge y) \wedge t_2(y \wedge z) \leq t_1(x \wedge y \wedge z)$. Considering now that $t_1(x \wedge y) \wedge t_2(y \wedge z) \leq t_2(y \wedge z) \leq t_2(y)$, by (30), we get $t_1(x \wedge y) \wedge t_2(y \wedge z) \leq t_1(x \wedge y \wedge z) \wedge t_2(y)$. Taking now into account the condition (61) we obtain (63). □

5. Let $(G, o, -, 0)$ be a subtraction Menger algebra of rank n .

Definition 3 By a *determining pair* of a subtraction Menger algebra $(G, o, -, 0)$ of rank n we mean an ordered pair (ε^*, W) , where ε is a v -regular equivalence relation

defined on (G, o) , $\varepsilon^* = \varepsilon \cup \{(e_1, e_1), \dots, (e_n, e_n)\}$, e_1, \dots, e_n are the selectors of the unitary extension (G^*, o^*) of (G, o) and W is the empty set or an l -ideal of (G, o) which is an ε -class.

Definition 4 A non-empty subset F of a subtraction Menger algebra $(G, o, -, 0)$ of rank n is called a *filter* if:

- (1) $0 \notin F$;
- (2) $x \in F \wedge x \leq y \implies y \in F$;
- (3) $x \in F \wedge y \in F \implies x \wedge y \in F$

for all $x, y \in G$.

If $a, b \in G$ and $a \not\leq b$, then $[a] = \{x \in G \mid a \leq x\}$ is a filter with $a \in [a]$ and $b \notin [a]$. By Zorn’s Lemma the collection of filters which contain an element a , but do not contain an element b , has a maximal element which is denoted by $F_{a,b}$. Using this filter we define the following three sets:

$$\begin{aligned}
 W_{a,b} &= \{x \in G \mid (\forall t \in T_n(G)) t(x) \notin F_{a,b}\}, \\
 \varepsilon_{a,b} &= \{(x, y) \in G \times G \mid x \wedge y \notin W_{a,b} \vee x, y \in W_{a,b}\}, \\
 \varepsilon_{a,b}^* &= \varepsilon_{a,b} \cup \{(e_1, e_1), \dots, (e_n, e_n)\}.
 \end{aligned}$$

Proposition 10 For any $a, b \in G$, the pair $(\varepsilon_{a,b}^*, W_{a,b})$ is the determining pair of the algebra $(G, o, -, 0)$.

Proof First we show that $\varepsilon_{a,b}$ is an equivalence relation on G . It is clear that this relation is reflexive and symmetric. To prove its transitivity let $(x, y), (y, z) \in \varepsilon_{a,b}$. We have four possibilities:

- (a) $x \wedge y \notin W_{a,b} \wedge y \wedge z \notin W_{a,b}$,
- (b) $x \wedge y \notin W_{a,b} \wedge y, z \in W_{a,b}$,
- (c) $x, y \in W_{a,b} \wedge y \wedge z \notin W_{a,b}$,
- (d) $x, y \in W_{a,b} \wedge y, z \in W_{a,b}$.

In the case (a) we have $t_1(x \wedge y), t_2(y \wedge z) \in F_{a,b}$ for some $t_1, t_2 \in T_n(G)$. Since $F_{a,b}$ is a filter, then, obviously, $t_1(x \wedge y) \wedge t_2(y \wedge z) \in F_{a,b}$. This, according to (63), implies $t_1(x \wedge y \wedge z) \wedge t_2(y) \in F_{a,b}$. But $t_1(x \wedge y \wedge z) \wedge t_2(y) \leq t_1(x \wedge z)$, hence also $t_1(x \wedge z) \in F_{a,b}$, i.e., $x \wedge z \notin W_{a,b}$. Thus, $(x, z) \in \varepsilon_{a,b}$.

In the case (b) from $x \wedge y \notin W_{a,b}$ it follows $t(x \wedge y) \in F_{a,b}$ for some polynomial $t \in T_n(G)$. But $x \wedge y \leq y$, and consequently $t(x \wedge y) \leq t(y)$. Thus $t(y) \in F_{a,b}$, i.e., $y \notin W_{a,b}$, which is a contradiction. Hence the case (b) is impossible. Analogously we can show that also the case (c) is impossible. The case (d) is obvious, because in this case $x, z \in W_{a,b}$ which means that $(x, z) \in \varepsilon_{a,b}$. This completes the proof that $\varepsilon_{a,b}$ is transitive.

Moreover, if $x \in W_{a,b}$, then $t(x) \notin F_{a,b}$ for every $t \in T_n(G)$. In particular, for all $t(x) = t'(u[\bar{w}|_i x]) \in T_n(G)$ we have $t'(u[\bar{w}|_i x]) \notin F_{a,b}$. Thus, $u[\bar{w}|_i x] \in W_{a,b}$ for every $i = 1, \dots, n$. Hence, $W_{a,b}$ is an i -ideal of (G, o) , and consequently, an l -ideal. It is clear that $W_{a,b}$ is an $\varepsilon_{a,b}$ -class.

Next, we prove that the relation $\varepsilon_{a,b}$ is v -regular. Let $x \equiv y(\varepsilon_{a,b})$. Then $x \wedge y \notin W_{a,b}$ or $x, y \in W_{a,b}$. In the case $x, y \in W_{a,b}$ we obtain $u[\bar{w}|_i x], u[\bar{w}|_i y] \in W_{a,b}$ because $W_{a,b}$ is an l -ideal of (G, o) . Thus, $u[\bar{w}|_i x] \equiv u[\bar{w}|_i y](\varepsilon_{a,b})$. In the case $x \wedge y \notin W_{a,b}$ elements $u[\bar{w}|_i x], u[\bar{w}|_i y]$ belong or not belong to $W_{a,b}$ simultaneously. Indeed, if $u[\bar{w}|_i x], u[\bar{w}|_i y] \in W_{a,b}$, then obviously $u[\bar{w}|_i x] \equiv u[\bar{w}|_i y](\varepsilon_{a,b})$. Now, if $u[\bar{w}|_i x] \notin W_{a,b}$, then $t(u[\bar{w}|_i x]) \in F_{a,b}$ for some $t \in T_n(G)$. Since $x \wedge y \notin W_{a,b}$, then also $t_1(x \wedge y) \in F_{a,b}$ for some $t_1 \in T_n(G)$. Thus $t_1(x \wedge y) \wedge t(u[\bar{w}|_i x]) \in F_{a,b}$, which, by (60), implies $t_1(x \wedge y) \wedge t(u[\bar{w}|_i(x \wedge y)]) \in F_{a,b}$. But $t_1(x \wedge y) \wedge t(u[\bar{w}|_i(x \wedge y)]) \leq t(u[\bar{w}|_i y])$, hence $t(u[\bar{w}|_i y]) \in F_{a,b}$, i.e., $u[\bar{w}|_i y] \notin W_{a,b}$. So, we have shown that $x \wedge y \notin W_{a,b}$ and $u[\bar{w}|_i x] \notin W_{a,b}$ imply $u[\bar{w}|_i y] \notin W_{a,b}$. Similarly we can show that $x \wedge y \notin W_{a,b}$ and $u[\bar{w}|_i y] \notin W_{a,b}$ imply $u[\bar{w}|_i x] \notin W_{a,b}$. Therefore, we have proved that in the case $x \wedge y \notin W_{a,b}$ elements $u[\bar{w}|_i x], u[\bar{w}|_i y]$ belong or not belong to $W_{a,b}$ simultaneously.

So, if for $x \wedge y \notin W_{a,b}$ we have $u[\bar{w}|_i x], u[\bar{w}|_i y] \in W_{a,b}$, then clearly $u[\bar{w}|_i x] \equiv u[\bar{w}|_i y](\varepsilon_{a,b})$. Therefore assume that $u[\bar{w}|_i x] \notin W_{a,b}$ (hence $u[\bar{w}|_i y] \notin W_{a,b}$). Thus, $x \wedge y \notin W_{a,b}, u[\bar{w}|_i x] \notin W_{a,b}$, i.e., $t(x \wedge y) \in F_{a,b}, t_1(u[\bar{w}|_i x]) \in F_{a,b}$ for some $t, t_1 \in T_n(G)$. Hence, $t(y \wedge x \wedge y) \wedge t_1(u[\bar{w}|_i x]) \in F_{a,b}$. From this, according to (63), we obtain $t(y \wedge x) \wedge t_1(u[\bar{w}|_i(x \wedge y)]) \in F_{a,b}$. This implies $t_1(u[\bar{w}|_i(x \wedge y)]) \in F_{a,b}$. Since $u[\bar{w}|_i(x \wedge y)] \leq u[\bar{w}|_i x]$ and $u[\bar{w}|_i(x \wedge y)] \leq u[\bar{w}|_i y]$, we have $u[\bar{w}|_i(x \wedge y)] \leq u[\bar{w}|_i x] \wedge u[\bar{w}|_i y]$, which, by the stability of ω gives $t_1(u[\bar{w}|_i(x \wedge y)]) \leq t_1(u[\bar{w}|_i x] \wedge u[\bar{w}|_i y])$. Consequently, $t_1(u[\bar{w}|_i x] \wedge u[\bar{w}|_i y]) \in F_{a,b}$, so $u[\bar{w}|_i x] \wedge u[\bar{w}|_i y] \notin W_{a,b}$, i.e., $u[\bar{w}|_i x] \equiv u[\bar{w}|_i y](\varepsilon_{a,b})$. In this way we have proved that the relation $\varepsilon_{a,b}$ is i -regular for every $i = 1, \dots, n$. Thus it is v -regular. \square

Proposition 11 *All equivalence classes of $\varepsilon_{a,b}$, except of $W_{a,b}$, are filters.*

Proof Indeed, let $H \neq W_{a,b}$ be an arbitrary class of $\varepsilon_{a,b}$. If $x \in H$ and $x \leq y$, then $x \wedge y = x \notin W_{a,b}$, consequently, $(x, y) \in \varepsilon_{a,b}$. Hence, $y \in H$. Further, let $x, y \in H$, then $(x, y) \in \varepsilon_{a,b}$. Thus $x \wedge y \notin W_{a,b}$, i.e., $t(x \wedge y) \in F_{a,b}$ for some $t \in T_n(G)$. But $x \wedge y = x \wedge (x \wedge y)$, hence, $t(x \wedge (x \wedge y)) \in F_{a,b}$ and $x \wedge (x \wedge y) \notin W_{a,b}$. So $x \equiv x \wedge y(\varepsilon_{a,b})$. This implies $x \wedge y \in H$. Thus, we have shown that H is a filter. \square

Proposition 12 *If $x \Upsilon y$ exists for some $x, y \in W_{a,b}$, then $x \Upsilon y \in W_{a,b}$.*

Proof Let $x \Upsilon y$ exists for some $x, y \in W_{a,b}$. If $x \Upsilon y \notin W_{a,b}$, then $t(x \Upsilon y) \in F_{a,b}$ for some $t \in T_n(G)$, and, according to Corollary 2, $t(x \Upsilon y) = t(x) \Upsilon t(y)$. If $t(x) \notin F_{a,b}$, then $F_{a,b}$ is a proper subset of the set

$$U = \{u \in G \mid (\exists z \in F_{a,b}) z \wedge t(x) \leq u\}$$

because $t(x) \in U$.

We show that U is a filter. $0 \notin U$ because, by (15), we have $0 \leq z \wedge t(x)$ for any $z \in F_{a,b}$. Let $s \in U$ and $s \leq r$. Then $z \wedge t(x) \leq s$ for some $z \in F_{a,b}$. Consequently, $z \wedge t(x) \leq r$, so $r \in U$. Now let $s \in U$ and $r \in U$, i.e., $z_1 \wedge t(x) \leq s$ and $z_2 \wedge t(x) \leq r$ for some $z_1, z_2 \in F_{a,b}$. Since $F_{a,b}$ is a filter, we have $z_1 \wedge z_2 \in F_{a,b}$. Hence, $(z_1 \wedge z_2) \wedge t(x) \leq s \wedge r$, which implies $s \wedge r \in U$. Thus U is a filter. But by assumption $F_{a,b} \subset U$ is a maximal filter, which does not contain b , so $b \in U$. Consequently,

$z_1 \wedge t(x) \leq b$ for some $z_1 \in F_{a,b}$. Similarly, if $t(y) \notin F_{a,b}$, then $z_2 \wedge t(y) \leq b$ for some $z_2 \in F_{a,b}$. This implies $z \wedge t(x) \leq b$ and $z \wedge t(y) \leq b$ for $z = z_1 \wedge z_2$. Hence $(z \wedge t(x)) \vee (z \wedge t(y))$ exists and

$$(z \wedge t(x)) \vee (z \wedge t(y)) = z \wedge (t(x) \vee t(y)) = z \wedge t(x \vee y) \in F_{a,b}$$

by (47). But by (50) we have $(z \wedge t(x)) \vee (z \wedge t(y)) \leq b$, so $z \wedge t(x \vee y) \leq b$. Since $z \wedge t(x \vee y) \in F_{a,b}$, then, obviously, $b \in F_{a,b}$, which is impossible. So, $t(x) \in F_{a,b}$ or $t(y) \in F_{a,b}$, hence $x \notin W_{a,b}$ or $y \notin W_{a,b}$, contrary to the assumption that $x, y \in W_{a,b}$. Thus, the assumption that $x \vee y \notin W_{a,b}$ is incorrect. Therefore $x \vee y \in W_{a,b}$. \square

6. Each homomorphism of a Menger algebra (G, o) of rank n into a Menger algebra $(\mathcal{F}(A^n, A), O)$ is called a *representation by n -place functions*. Thus, $P : G \rightarrow \mathcal{F}(A^n, A)$ is a representation, if

$$P(x[y_1 \dots y_n]) = P(x)[P(y_1) \dots P(y_n)]$$

for all $x, y_1, \dots, y_n \in G$. A representation which is an isomorphism is called *faithful* (cf. [2–4, 8]). A representation P of (G, o) is a representation of $(G, o, -, 0)$ if

$$P(x - y) = P(x) \setminus P(y) \quad \text{and} \quad P(0) = \emptyset$$

for all $x, y \in G$.

Let $(P_i)_{i \in I}$ be the family of representations of a subtraction Menger algebra $(G, o, -, 0)$ of rank n by n -place functions defined on pairwise disjoint sets $(A_i)_{i \in I}$. By the *sum* of the family $(P_i)_{i \in I}$ we mean the map $P : g \mapsto P(g)$, denoted by $\sum_{i \in I} P_i$, where $P(g)$ is an n -place function on $A = \bigcup_{i \in I} A_i$ defined by $P(g) = \bigcup_{i \in I} P_i(g)$. It is clear (cf. [2, 3]) that P is a representation of $(G, o, -, 0)$.

Similarly as in [2, 3] with each determining pair (ε^*, W) we can associate the so-called *simplest representation* $P_{(\varepsilon^*, W)}$ of (G, o) which assigns to each element $g \in G$ the n -place function $P_{(\varepsilon^*, W)}(g)$ defined on $\mathcal{H} = \mathcal{H}_0 \cup \{\{e_1\}, \dots, \{e_n\}\}$, where \mathcal{H}_0 is the set of all ε -classes of G different from W such that

$$(H_1, \dots, H_n, H) \in P_{(\varepsilon, W)}(g) \iff g[H_1 \dots H_n] \subset H,$$

for $(H_1, \dots, H_n) \in \mathcal{H}_0^n \cup \{\{e_1\}, \dots, \{e_n\}\}$ and $H \in \mathcal{H}$.

Theorem 2 *Each subtraction Menger algebra of rank n is isomorphic to some difference Menger algebra of n -place functions.*

Proof Let $(G, o, -, 0)$ be a subtraction Menger algebra of rank n . Then the sum

$$P = \sum_{a, b \in G, a \not\leq b} P_{(\varepsilon_{a,b}^*, W_{a,b})}$$

of the family $(P_{(\varepsilon_{a,b}^*, W_{a,b})})_{a, b \in G, a \not\leq b}$ of simplest representations of (G, o) is a representation of (G, o) .

Now we show that P is a representation of $(G, o, -, 0)$. Let \mathcal{H}_0 be the set of all $\varepsilon_{a,b}$ -classes of G different from $W_{a,b}$. Consider $H_1, \dots, H_n, H \in \mathcal{H}$, where $\mathcal{H} = \mathcal{H}_0 \cup \{\{e_1\}, \dots, \{e_n\}\}$, such that $(H_1, \dots, H_n, H) \in P_{(\varepsilon_{a,b}, W_{a,b})}^*(g_1 - g_2)$ for some $g_1, g_2 \in G$. Then, obviously, $(g_1 - g_2)[H_1 \dots H_n] \subset H \neq W_{a,b}$. Thus $(g_1 - g_2)[\bar{x}] \in H$ for each $\bar{x} \in H_1 \times \dots \times H_n$, which, by (11), gives $g_1[\bar{x}] - g_2[\bar{x}] \in H$. But $g_1[\bar{x}] - g_2[\bar{x}] \leq g_1[\bar{x}]$ and H is a filter (Proposition 11), hence $g_1[\bar{x}] \in H$. Thus $(g_1[\bar{x}] - g_2[\bar{x}]) \wedge g_2[\bar{x}] = 0$, by (33). Consequently, $(g_1[\bar{x}] - g_2[\bar{x}]) \wedge g_2[\bar{x}] \in W_{a,b}$, because the other $\varepsilon_{a,b}$ -classes as filters do not contain 0. This means that $g_1[\bar{x}] - g_2[\bar{x}] \not\equiv g_2[\bar{x}]_{(\varepsilon_{a,b})}$. Hence, $g_2[\bar{x}] \notin H$. Therefore $g_1[H_1 \dots H_n] \subset H$ and $g_2[h_1 \dots H_n] \cap H = \emptyset$, which implies

$$(H_1, \dots, H_n, H) \in P_{(\varepsilon_{a,b}, W_{a,b})}^*(g_1) \setminus P_{(\varepsilon_{a,b}, W_{a,b})}^*(g_2).$$

In this way, we have proved the inclusion

$$P_{(\varepsilon_{a,b}, W_{a,b})}^*(g_1 - g_2) \subset P_{(\varepsilon_{a,b}, W_{a,b})}^*(g_1) \setminus P_{(\varepsilon_{a,b}, W_{a,b})}^*(g_2). \tag{64}$$

To show the reverse inclusion let

$$(H_1, \dots, H_n, H) \in P_{(\varepsilon_{a,b}, W_{a,b})}^*(g_1) \setminus P_{(\varepsilon_{a,b}, W_{a,b})}^*(g_2).$$

Then $(H_1, \dots, H_n, H) \in P_{(\varepsilon_{a,b}, W_{a,b})}^*(g_1)$ and $(H_1, \dots, H_n, H) \notin P_{(\varepsilon_{a,b}, W_{a,b})}^*(g_2)$, i.e., $g_1[H_1 \dots H_n] \subset H$ and $g_2[H_1 \dots H_n] \cap H = \emptyset$. Thus $g_1[\bar{x}] \in H$ and $g_2[\bar{x}] \notin H$ for all $\bar{x} \in H_1 \times \dots \times H_n$. Since from $g_1[\bar{x}] \wedge g_2[\bar{x}] \notin W_{a,b}$, it follows $g_1[\bar{x}] \equiv g_2[\bar{x}]_{(\varepsilon_{a,b})}$ and $g_2[\bar{x}] \in H$, which is a contradiction, we conclude that $g_1[\bar{x}] \wedge g_2[\bar{x}] \in W_{a,b}$.

If $g_1[\bar{x}] - g_2[\bar{x}] \in W_{a,b}$, then, by (53) and Proposition 12, we obtain $g_1[\bar{x}] = (g_1[\bar{x}] \wedge g_2[\bar{x}]) \vee (g_1[\bar{x}] - g_2[\bar{x}]) \in W_{a,b}$. Consequently, $g_1[\bar{x}] \in W_{a,b}$, which is impossible because $g_1[\bar{x}] \in H$. Thus, $(g_1[\bar{x}] - g_2[\bar{x}]) \wedge g_1[\bar{x}] = g_1[\bar{x}] - g_2[\bar{x}] \notin W_{a,b}$. Hence, $g_1[\bar{x}] - g_2[\bar{x}] \equiv g_1[\bar{x}]_{(\varepsilon_{a,b})}$. This implies $(g_1 - g_2)[\bar{x}] = g_1[\bar{x}] - g_2[\bar{x}] \in H$. Therefore, $(g_1 - g_2)[H_1 \dots H_n] \subset H$, i.e., $(H_1, \dots, H_n, H) \in P_{(\varepsilon_{a,b}, W_{a,b})}^*(g_1 - g_2)$. So, we have proved

$$P_{(\varepsilon_{a,b}, W_{a,b})}^*(g_1) \setminus P_{(\varepsilon_{a,b}, W_{a,b})}^*(g_2) \subset P_{(\varepsilon_{a,b}, W_{a,b})}^*(g_1 - g_2).$$

This together with (64) proves

$$P_{(\varepsilon_{a,b}, W_{a,b})}^*(g_1 - g_2) = P_{(\varepsilon_{a,b}, W_{a,b})}^*(g_1) \setminus P_{(\varepsilon_{a,b}, W_{a,b})}^*(g_2),$$

which means that $P(g_1 - g_2) = P(g_1) \setminus P(g_2)$ for $g_1, g_2 \in G$. Further, $P(0) = P(0 - 0) = P(0) \setminus P(0) = \emptyset$. So, P is a representation of $(G, o, -, 0)$ by n -place functions.

We show that this representation is faithful. Let $P(g_1) = P(g_2)$ for some $g_1, g_2 \in G$. If $g_1 \neq g_2$, then both inequalities $g_1 \leq g_2$ and $g_2 \leq g_1$ at the same time are impossible. Suppose that $g_1 \not\leq g_2$. Then $g_1 \in F_{g_1, g_2}$ and, consequently,

$$(\{e_1\}, \dots, \{e_n\}, F_{g_1, g_2}) \in P_{(\varepsilon_{g_1, g_2}, W_{g_1, g_2})}^*(g_2).$$

Since $P_{(\varepsilon_{g_1, g_2}^*, W_{g_1, g_2})}(g_1) = P_{(\varepsilon_{g_1, g_2}^*, W_{g_1, g_2})}(g_2)$, then, obviously,

$$(\{e_1\}, \dots, \{e_n\}, F_{g_1, g_2}) \in P_{(\varepsilon_{g_1, g_2}^*, W_{g_1, g_2})}(g_2).$$

Thus $\{g_2\} = g_2[\{e_1\} \dots \{e_n\}] \subset F_{g_1, g_2}$, hence $g_2 \in F_{g_1, g_2}$. This is a contradiction because F_{g_1, g_2} is a filter containing g_1 but not containing g_2 . The case $g_2 \not\leq g_1$ is analogous. So, the supposition $g_1 \neq g_2$ is not true. Hence $g_1 = g_2$ and P is a faithful representation. The theorem is proved. \square

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References

1. Abbott, J.C.: Sets, Lattices and Boolean Algebras. Allyn & Bacon, Boston (1969)
2. Dudek, W.A., Trokhimenko, V.S.: Menger Algebras of Multiplace Functions. Centrul Ed. USM, Chişinău (2006) (Russian)
3. Dudek, W.A., Trokhimenko, V.S.: Algebras of Multiplace Functions. De Gruyter/Versita, Berlin/London (2010)
4. Dudek, W.A., Trokhimenko, V.S.: Menger algebras of n -place functions. In: Proc. Intern. Confer. Algebra 2010. Advances in Algebraic Structures, pp. 198–218. World Scientific, Singapore (2011)
5. Iséki, K.: On implicative BCK-algebras. Math. Semin. Notes, Kobe Univ. **4**, 9–10 (1976)
6. Kim, Y.H., Kim, H.S.: Subtraction algebras and BCK-algebras. Math. Bohem. **1**, 21–24 (2003)
7. Schein, B.M.: Difference semigroups. Commun. Algebra **20**, 2153–2169 (1992)
8. Schein, B.M., Trokhimenko, V.S.: Algebras of multiplace functions. Semigroup Forum **17**, 1–64 (1979)
9. Trokhimenko, V.S.: Ordered algebras of multiplace functions. Izv. Vysš. Učebn. Zaved., Mat. **104**, 90–98 (1971) (Russian)
10. Trokhimenko, V.S.: A characterization of \mathcal{P} -algebras of multiplace functions. Sib. Math. J. **16**, 461–470 (1976). Translation from Sibirsk. Mat. Zh. **16**, 599–611 (1975)
11. Trokhimenko, V.S., Schein, B.M.: Compatible algebras of multiplace functions. Teor. Polugrupp Prilozh. **4**, 89–98 (1978) (Izdat. Saratov. Gos. Univ.) (Russian)