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Yet another proof of the density in energy of Lipschitz functions

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Abstract. We provide a new, short proof of the density in energy of Lipschitz functions into the metric Sobolev space defined by using plans with barycenter (and thus, a fortiori, into the Newtonian–Sobolev space). Our result covers first-order Sobolev spaces of exponent $p \in (1, \infty)$, defined over a complete separable metric space endowed with a boundedly-finite Borel measure. Our proof is based on a completely smooth analysis: first we reduce the problem to the Banach space setting, where we consider smooth functions instead of Lipschitz ones, then we rely on classical tools in convex analysis and on the superposition principle for normal 1-currents. Along the way, we obtain a new proof of the density in energy of smooth cylindrical functions in Sobolev spaces defined over a separable Banach space endowed with a finite Borel measure.

1. Introduction

1.1. General overview

Sobolev calculus on metric measure spaces has been a field of intense research activity for almost three decades. In this paper, we focus on two approaches: the Sobolev space $H^{1,p}(\mathrm{X}, \mu)$ obtained via relaxation and the Sobolev space $W^{1,p}(\mathrm{X}, \mu)$ defined using plans with barycenter. More specifically, fix a metric measure space $(\mathrm{X}, \mathsf{d}, \mu)$, i.e. (X, d) is a complete separable metric space endowed with a boundedly-finite Borel measure $\mu \geq 0$, and $p \in (1, \infty)$. Then:

- By $H^{1,p}(\mathrm{X}, \mu)$ we mean the Sobolev space via relaxation of Lipschitz functions, which was introduced by Ambrosio–Gigli–Savaré [3] as a variant of Cheeger's approach [6]; see Definition 2.4. We denote by $|Df|_H$ the minimal relaxed slope of $f \in H^{1,p}(\mathrm{X}, \mu)$.
- By $W^{1,p}(\mathrm{X}, \mu)$ we mean the Sobolev space defined using plans with barycenter, which was introduced by Savaré [23] after a series of works by Ambrosio, Di Marino, Gigli, and Savaré [1–3]; see Definition 2.8. The notion of plan with barycenter we consider is essentially taken from [23]. We denote by $|Df|_W$ the minimal weak upper gradient of $f \in W^{1,p}(\mathrm{X}, \mu)$, while $\mathcal{B}_q(\mathrm{X}, \mu)$ is the space

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of plans $\boldsymbol{\pi}$ with barycenter $\mathsf{Bar}(\boldsymbol{\pi})$ in $L^q(\mu)$, where q stands for the conjugate exponent of p; see Definition 2.6.

In this paper, we provide a new proof of the equivalence between $H^{1,p}(\mathrm{X},\mu)$ and $W^{1,p}(\mathrm{X},\mu)$, i.e.

$$H^{1,p}(\mathrm{X},\mu) = W^{1,p}(\mathrm{X},\mu), \qquad |Df|_W = |Df|_H \ \text{ for every } f \in W^{1,p}(\mathrm{X},\mu). \tag{1.1}$$

It is worth underlining that we do not make any additional assumptions on $(\mathrm{X},\mathsf{d},\mu)$. In particular, we are not assuming that μ is doubling nor the validity of a Poincaré inequality; in the doubling-Poincaré framework, the equivalence was proved in [6,24]. We can rephrase (1.1) as follows:

Lipschitz functions are *dense in energy* in $W^{1,p}(\mathrm{X},\mu)$.

The result stated in (1.1) or variants of it were obtained earlier in the literature:

- The first proof was obtained by Ambrosio–Gigli–Savaré in [2], where another class of plans (called *test plans*) was used; see Remark 3.6 for a comparison with the notion of $W^{1,p}(\mathrm{X},\mu)$ we consider in this paper. The proof in [2] is based on the metric Hopf–Lax semigroup.
- Savaré proved in [23] that (1.1) holds by using the von Neumann min-max principle and two representations of the dual Cheeger energy. This approach is the closest to ours.
- Eriksson-Bique proved in [10] a variant of (1.1) via a more direct approximation technique. On the one hand, the result in [10] is (a priori) weaker, since it shows the identification between $H^{1,p}(\mathrm{X},\mu)$ and the Newtonian–Sobolev space $N^{1,p}(\mathrm{X},\mu)$; see the relative discussion in Sect. 1.3. On the other hand, [10] covers also the case of the exponent $p = 1$.

Compared to the previous arguments, the novelty of our proof of (1.1) is that it relies on a purely smooth analysis. More precisely, since we can embed (X,d) isometrically into a Banach space, we can reduce the problem to the case where $\mathrm{X} = \mathbb{B}$ itself is a Banach space. In this framework, we argue by using only smooth functions, their Fréchet differentials, classical tools in convex analysis, and normal 1-currents. Neither Lipschitz functions nor other metric tools are actually needed.

1.2. The proof strategy

Up to a localisation argument and a Kuratowski embedding, we can reduce ourselves to addressing the problem in the case where μ is finite and $\mathrm{X} = \mathbb{B}$ is a separable Banach space. We then consider the algebra $\mathrm{Cyl}(\mathbb{B})$ of *cylindrical functions* (Definition 2.2). The advantage of working with cylindrical functions is that they are both smooth (of class C^∞) and strongly dense in $L^p(\mu)$. We define the space $H^{1,p}_{cyl}(\mathbb{B},\mu)$ in analogy with $H^{1,p}(\mathbb{B},\mu)$, but using cylindrical functions in the relaxation procedure instead of Lipschitz functions. We denote the corresponding minimal relaxed slope by $|Df|_{H,cyl}$. Therefore, the new goal is to prove that

$$H^{1,p}_{cyl}(\mathbb{B},\mu) = W^{1,p}(\mathbb{B},\mu), \qquad |Df|_W = |Df|_{H,cyl} \text{ for every } f \in W^{1,p}(\mathbb{B},\mu). \tag{1.2}$$

It is easy to show that $H^{1,p}_{cyl}(\mathbb{B},\mu) \subseteq H^{1,p}(\mathbb{B},\mu) \subseteq W^{1,p}(\mathbb{B},\mu)$ and $|Df|_W \leq |Df|_H \leq |Df|_{H,cyl}$ for every $f \in H^{1,p}_{cyl}(\mathbb{B},\mu)$. It follows that (1.2) implies (1.1). To prove (1.2), it suffices to check that

$$W^{1,p}(\mathbb{B},\mu) \subseteq H^{1,p}_{cyl}(\mathbb{B},\mu),$$
$$\||Df|_{H,cyl}\|_{L^p(\mu)} \leq \||Df|_W\|_{L^p(\mu)} \text{ for all } f \in W^{1,p}(\mathbb{B},\mu). \tag{1.3}$$

In order to prove (1.3), we apply well-known results in convex analysis about Fenchel conjugates; see (the proof of) Theorem 3.3. Our arguments are strongly inspired by some ideas contained in Bouchitté–Buttazzo–Seppecher's paper [5], where Sobolev spaces on weighted Euclidean spaces were introduced. Roughly speaking, we consider the densely-defined unbounded linear operator $\mathrm{d}\colon L^p(\mu) \to L^p(\mu;\mathbb{B}^*)$ with domain $D(\mathrm{d}) = \mathrm{Cyl}(\mathbb{B})$, which assigns to each function $f \in \mathrm{Cyl}(\mathbb{B})$ the μ-a.e. equivalence class $\mathrm{d}f$ of its Fréchet differential. To prove the property (1.3) amounts to showing that $\mathrm{sc}^-\mathcal{F}(f) \leq \frac{1}{p}\||Df|_W\|^p_{L^p(\mu)}$ for every $f \in W^{1,p}(\mathbb{B},\mu)$, where $\mathrm{sc}^-\mathcal{F}$ denotes the weak lower semicontinuous envelope of the functional $\mathcal{F}\colon L^p(\mu) \to [0,+\infty]$ given by

$$\mathcal{F}(f) := \frac{1}{p}\int \|\mathrm{d}_x f\|^p_{\mathbb{B}^*}\,\mathrm{d}\mu(x) \quad \text{for every } f \in \mathrm{Cyl}(\mathbb{B})$$

and $\mathcal{F}(f) := +\infty$ otherwise (some extra care is needed when μ is not fully supported). In order to achieve this goal, we need to prove the following statement: given any $L \in D(\mathrm{d}^*)$, there exists a plan $\boldsymbol{\pi} \in \mathcal{B}_q(\mathbb{B},\mu)$ such that $\partial\boldsymbol{\pi} = (\mathrm{d}^*L)\mu$ (see (2.1)) and $\|\mathsf{Bar}(\boldsymbol{\pi})\|_{L^q(\mu)} \leq \|L\|_{L^p(\mu;\mathbb{B}^*)^*}$. Here, we denote by $\mathrm{d}^*\colon L^p(\mu;\mathbb{B}^*)^* \to L^q(\mu)$ the adjoint operator of d. This is the content of Proposition 3.1, whose proof is based on Smirnov's superposition principle for normal 1-currents [25].

1.3. Some additional comments

With (1.2), we recover a result by Savaré [23], which states that cylindrical functions are dense in energy in $W^{1,p}(\mathbb{B},\mu)$; see also the paper [12]. Whereas Savaré obtains (1.2) as a consequence of the density in energy of Lipschitz functions, our proof goes in the opposite direction: we prove directly (1.2), then we obtain (1.1) as a corollary.

Sobolev spaces over certain classes of *weighted* Banach spaces (i.e. Banach spaces equipped with an arbitrary Borel measure) have been thoroughly investigated in several articles. Weighted Euclidean spaces were studied e.g. in [5,8,14,16,17,27], weighted Hilbert spaces (more generally, weighted locally-$\mathsf{CAT}(\kappa)$ spaces) in [7], and weighted reflexive Banach spaces in [12,21,23,26].

We now consider the Newtonian–Sobolev space $N^{1,p}(\mathrm{X},\mu)$ introduced by Shanmugalingam [24] (see also [15]) and the associated notion of minimal weak upper gradient $|Df|_N$. It is not too difficult to show that $H^{1,p}(\mathrm{X},\mu) \subseteq N^{1,p}(\mathrm{X},\mu) \subseteq W^{1,p}(\mathrm{X},\mu)$ and $|Df|_W \leq |Df|_N \leq |Df|_H$ for every $f \in H^{1,p}(\mathrm{X},\mu)$. Therefore, it follows directly from (1.1) that $H^{1,p}(\mathrm{X},\mu) = N^{1,p}(\mathrm{X},\mu)$ and that $|Df|_N = |Df|_H$ for every $f \in N^{1,p}(\mathrm{X},\mu)$, in other words that Lipschitz functions are dense in energy in the Newtonian–Sobolev space; cf. with Remark 3.5.

We also mention that it seems that our proof strategy cannot be used to prove the identification between the spaces $H^{1,1}$ and $W^{1,1}$. Nevertheless, we do believe that it can be adapted to show the equivalence of the different notions of functions of bounded variation, as well as to study various notions of Sobolev spaces of exponent $p = \infty$. These questions will be addressed in future works.

2. Preliminaries

Given $p \in [1,\infty)$, we tacitly denote by $q := \frac{p}{p-1} \in (1,\infty]$ its conjugate exponent, and vice versa.

2.1. Metric and measure spaces

Given metric spaces $(\mathrm{X},\mathsf{d}_\mathrm{X})$, $(\mathrm{Y},\mathsf{d}_\mathrm{Y})$, we denote by $C(\mathrm{X};\mathrm{Y})$ the space of continuous maps from X to Y. We endow its subset $C_b(\mathrm{X};\mathrm{Y})$ consisting of bounded elements with the distance $\mathsf{d}_{C_b(\mathrm{X};\mathrm{Y})}(\varphi,\psi) := \sup_{x\in\mathrm{X}} \mathsf{d}_\mathrm{Y}(\varphi(x),\psi(x))$. If $\mathrm{Y} = \mathbb{B}$ is a Banach space, $C_b(\mathrm{X};\mathbb{B})$ is a vector space and $\mathsf{d}_{C_b(\mathrm{X};\mathbb{B})}$ is induced by the supremum norm $\|\cdot\|_{C_b(\mathrm{X};\mathbb{B})}$. We denote by $\mathfrak{M}(\mathrm{X})$ the set of (finite) signed Borel measures on X and $\mathfrak{M}_+(\mathrm{X}) := \{\mu \in \mathfrak{M}(\mathrm{X}) : \mu \geq 0\}$. For any $\mu \in \mathfrak{M}(\mathrm{X})$, we denote by $\mu^+, \mu^- \in \mathfrak{M}_+(\mathrm{X})$ the *positive part* and the *negative part* of μ, respectively. Recall that $\mu = \mu^+ - \mu^-$. The total variation measure of $\mu \in \mathfrak{M}(\mathrm{X})$ is defined as $|\mu| := \mu^+ + \mu^- \in \mathfrak{M}_+(\mathrm{X})$. We endow $\mathfrak{M}(\mathrm{X})$ with the *weak topology*, i.e. with the coarsest topology such that $\mathfrak{M}(\mathrm{X}) \ni \mu \mapsto \int f\,\mathrm{d}\mu$ is a continuous function for every $f \in C_b(\mathrm{X}) := C_b(\mathrm{X};\mathbb{R})$. We denote by $\mathrm{LIP}(\mathrm{X};\mathrm{Y}) \subseteq C(\mathrm{X};\mathrm{Y})$ the space of all Lipschitz maps from X to Y, and by $\mathrm{Lip}(\varphi)$ the Lipschitz constant of a map $\varphi \in \mathrm{LIP}(\mathrm{X};\mathrm{Y})$. Notice that $\mathrm{LIP}_b(\mathrm{X};\mathrm{Y}) := \mathrm{LIP}(\mathrm{X};\mathrm{Y}) \cap C_b(\mathrm{X};\mathrm{Y})$ is a Borel subset of $C_b(\mathrm{X};\mathrm{Y})$, since it can be written as $\bigcup_{n\in\mathbb{N}}\{\varphi \in \mathrm{LIP}_b(\mathrm{X};\mathrm{Y}) : \mathrm{Lip}(\varphi) \leq n\}$ and each set $\{\varphi \in \mathrm{LIP}_b(\mathrm{X};\mathrm{Y}) : \mathrm{Lip}(\varphi) \leq n\}$ is closed in $C_b(\mathrm{X};\mathrm{Y})$. Moreover, we define $\mathrm{LIP}(\mathrm{X}) := \mathrm{LIP}(\mathrm{X};\mathbb{R})$ and $\mathrm{LIP}_b(\mathrm{X}) := \mathrm{LIP}_b(\mathrm{X};\mathbb{R})$. We denote by $C_{bs}(\mathrm{X})$ the space of all functions $f \in C(\mathrm{X})$ whose support $\mathrm{spt}(f)$ is bounded and we define $\mathrm{LIP}_{bs}(\mathrm{X}) := \mathrm{LIP}(\mathrm{X}) \cap C_{bs}(\mathrm{X})$. The *asymptotic slope* $\mathrm{lip}_a(f)\colon \mathrm{X} \to [0,+\infty)$ of a function $f \in \mathrm{LIP}(\mathrm{X})$ is given by

$$\mathrm{lip}_a(f)(x) := \inf_{r>0} \mathrm{Lip}(f|_{B_r(x)}) \quad \text{for every } x \in \mathrm{X}.$$

Let us now focus on the space $C([0,1];\mathrm{X})$ of curves. The *evaluation maps* $\mathrm{e}_\pm\colon C([0,1];\mathrm{X}) \to \mathrm{X}$ are the 1-Lipschitz maps given by $\mathrm{e}_+(\gamma) := \gamma_1$

and $\mathrm{e}_-(\gamma) := \gamma_0$. When $\gamma \in \mathrm{LIP}([0,1];\mathrm{X})$, the *metric speed* $|\dot{\gamma}_t| := \lim_{h\to 0} \mathsf{d}_\mathrm{X}(\gamma_{t+h},\gamma_t)/|h|$ exists for $\mathcal{L}_1$-a.e. $t \in [0,1]$, where $\mathcal{L}_1$ stands for the restriction of the one-dimensional Lebesgue measure to $[0,1]$. The *length* of γ is defined as $\ell(\gamma) := \int_0^1 |\dot{\gamma}_t|\,\mathrm{d}t$. We say that γ is *of constant speed* if $|\dot{\gamma}|$ is $\mathcal{L}_1$-a.e. constant, so that $|\dot{\gamma}_t| = \ell(\gamma)$ for $\mathcal{L}_1$-a.e. $t \in [0,1]$. The length functional $\ell\colon \mathrm{LIP}([0,1];\mathrm{X}) \to [0,+\infty)$ is lower semicontinuous, since it holds that

$$\ell(\gamma) = \sup \sum_{i=1}^{n} \mathsf{d}_\mathrm{X}(\gamma_{t_i},\gamma_{t_{i-1}}) \quad \text{for every } \gamma \in \mathrm{LIP}([0,1];\mathrm{X}),$$

where the supremum is taken among all partitions $0 = t_0 < t_1 < \ldots < t_n = 1$ of the interval $[0,1]$. By a *plan* on X we mean any measure $\boldsymbol{\pi} \in \mathfrak{M}_+(C([0,1];\mathrm{X}))$ that is concentrated on $\mathrm{LIP}([0,1];\mathrm{X})$. We denote by $\Pi(\mathrm{X})$ the set of all plans on X. We define the *boundary* of a plan $\boldsymbol{\pi} \in \Pi(\mathrm{X})$ as

$$\partial\boldsymbol{\pi} := (\mathrm{e}_+)_\#\boldsymbol{\pi} - (\mathrm{e}_-)_\#\boldsymbol{\pi} \in \mathfrak{M}(\mathrm{X}). \tag{2.1}$$

Moreover, we define the Borel measure $\|\boldsymbol{\pi}\| \geq 0$ on X as $\|\boldsymbol{\pi}\| := \int \ell(\gamma)\,\gamma_\#\mathcal{L}_1\,\mathrm{d}\boldsymbol{\pi}(\gamma)$.

One can also readily prove that, given any function $f \in C_b(\mathrm{X})$ such that $f \geq 0$, it holds that

$$\Pi(\mathrm{X}) \ni \boldsymbol{\pi} \mapsto \iint_0^1 f(\gamma_t)|\dot{\gamma}_t|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma) \quad \text{is weakly lower semicontinuous.} \tag{2.2}$$

Indeed, by exploiting the continuity of f, one can check that for any $\gamma \in \mathrm{LIP}([0,1];\mathrm{X})$ it holds

$$\int_0^1 f(\gamma_t)|\dot{\gamma}_t|\,\mathrm{d}t = \sup \sum_{i=1}^{n} \min_{t\in[t_{i-1},t_i]} f(\gamma_t)\,\mathsf{d}_\mathrm{X}(\gamma_{t_i},\gamma_{t_{i-1}}),$$

where the supremum is taken among all partitions $0 = t_0 < t_1 < \ldots < t_n = 1$ of the interval $[0,1]$. This shows that $\mathrm{LIP}([0,1];\mathrm{X}) \ni \gamma \mapsto F(\gamma) := \int_0^1 f(\gamma_t)|\dot{\gamma}_t|\,\mathrm{d}t$ is lower semicontinuous. Hence, we can find an increasing sequence of bounded continuous functions $F_j\colon \mathrm{LIP}([0,1];\mathrm{X}) \to [0,+\infty)$ such that $F(\gamma) = \lim_j F_j(\gamma)$ for all $\gamma \in \mathrm{LIP}([0,1];\mathrm{X})$. The monotone convergence theorem then gives

$$\iint_0^1 f(\gamma_t)|\dot{\gamma}_t|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma) = \int F\,\mathrm{d}\boldsymbol{\pi} = \sup_{j\in\mathbb{N}} \int F_j\,\mathrm{d}\boldsymbol{\pi} \quad \text{for every } \boldsymbol{\pi} \in \Pi(\mathrm{X}),$$

whence (2.2) follows since each functional $\Pi(\mathrm{X}) \ni \boldsymbol{\pi} \mapsto \int F_j\,\mathrm{d}\boldsymbol{\pi}$ is weakly continuous.

2.2. *Banach spaces and* 1*-currents*

Let $\mathbb{B}$ be a Banach space. For any function $f \in C^\infty(\mathbb{B})$, we denote by $\mathrm{d}f \in C^\infty(\mathbb{B};\mathbb{B}^*)$ its Fréchet differential $x \mapsto \mathrm{d}_x f$, where $\mathbb{B}^*$ is the dual of $\mathbb{B}$. We define $C_b^\infty(\mathbb{B};\mathbb{B}^*) := C^\infty(\mathbb{B};\mathbb{B}^*) \cap C_b(\mathbb{B};\mathbb{B}^*)$. If $\mathbb{V}$ is a finite-dimensional Banach space, then we also consider the space $C_c^\infty(\mathbb{V};\mathbb{V}^*)$ of all those $\omega \in C_b^\infty(\mathbb{V};\mathbb{V}^*)$ having compact support. The space of 1-*currents* in $\mathbb{V}$ is defined as the dual $\mathbf{M}_1(\mathbb{V})$ of the normed space $(C_c^\infty(\mathbb{V};\mathbb{V}^*), \|\cdot\|_{C_b(\mathbb{V};\mathbb{V}^*)})$. When $\mathbb{V}$ is a Euclidean space, these are the 1-currents in the sense of Federer–Fleming [11]. The elements of $\mathbf{M}_1(\mathbb{V})$ can be identified with the $\mathbb{V}$-valued Borel measures on $\mathbb{V}$, thus we can consider the total variation measure $\|T\| \in \mathfrak{M}_+(\mathbb{V})$ of every $T \in \mathbf{M}_1(\mathbb{V})$. Given any $T \in \mathbf{M}_1(\mathbb{V})$, we define

$$\partial T(f) := T(\mathrm{d}f) \quad \text{for every } f \in C_c^\infty(\mathbb{V}).$$

When the resulting operator $\partial T : C_c^\infty(\mathbb{V}) \to \mathbb{R}$ – which is called the *boundary* of T – belongs to the dual of $(C_c^\infty(\mathbb{V}), \|\cdot\|_{C_b(\mathbb{V})})$, we say that T is a *normal* 1*-current*. We denote by $\mathbf{N}_1(\mathbb{V})$ the space of all normal 1-currents in $\mathbb{V}$. The boundary ∂T of each $T \in \mathbf{N}_1(\mathbb{V})$ can be identified with a (finite) signed Borel measure on $\mathbb{V}$. It also holds that the elements $T \in \mathbf{N}_1(\mathbb{V})$ can be identified with those $\mathbb{V}$-valued Borel measures on $\mathbb{V}$ whose distributional divergence is a finite Borel measure (which coincides with $-\partial T$). A *subcurrent* of $T \in \mathbf{M}_1(\mathbb{V})$ is a current $S \in \mathbf{M}_1(\mathbb{V})$ such that $\|S\| + \|T - S\| = \|T\|$. By a *cycle* of T we mean a subcurrent $C \in \mathbf{N}_1(\mathbb{V})$ of T such that $\partial C = 0$. We say that T is *acyclic* if its unique cycle is the null current. Then the following result holds (see e.g. [18, Proposition 3.8]): for any $T \in \mathbf{M}_1(\mathbb{V})$, there exists a cycle C of T such that $T - C$ is acyclic. The following result states that acyclic normal 1-currents are superpositions of curves:

Theorem 2.1. (Superposition principle) *Let $\mathbb{V}$ be a finite-dimensional Banach space. Then for every acyclic current $T \in \mathbf{N}_1(\mathbb{V})$ there exists $\boldsymbol{\pi} \in \mathfrak{M}_+(C([0,1];\mathbb{V}))$ concentrated on non-constant Lipschitz curves of constant speed such that $(\mathrm{e}_+)_\# \boldsymbol{\pi} = (\partial T)^+$, $(\mathrm{e}_-)_\# \boldsymbol{\pi} = (\partial T)^-$, and $\|T\| = \|\boldsymbol{\pi}\|$.*

Proof. Since all norms on a finite-dimensional vector space are equivalent and the Euclidean norm is strictly convex, one can deduce the statement from Smirnov's results in [25]. Alternatively, one can argue as follows: since the normal metric 1-currents (in the sense of Ambrosio–Kirchheim [4]) on $\mathbb{V}$ can be identified with those $\mathbb{V}$-valued Borel measures on $\mathbb{V}$ whose distributional divergence is a finite Borel measure (see [20, Lemma A.3]) – and thus they can be identified also with the elements of $\mathbf{N}_1(\mathbb{V})$ – the statement follows from [18, Lemma 5.4] (see also [18, Theorem 5.1]). □

We will focus on a distinguished class of smooth functions: the algebra of cylindrical functions.

Definition 2.2. (Cylindrical function) Let $\mathbb{B}$ be a Banach space. Then we say that $f : \mathbb{B} \to \mathbb{R}$ is a *cylindrical function* if $f = g \circ p$ for some finite-dimensional Banach space $\mathbb{V}$, some $g \in C_c^\infty(\mathbb{V})$, and some bounded linear map $p : \mathbb{B} \to \mathbb{V}$. We denote by $\mathrm{Cyl}(\mathbb{B})$ the space of cylindrical functions.

It holds that $f \in \mathrm{LIP}_b(\mathbb{B})$ and $\mathrm{d}f \in C_b^\infty(\mathbb{B};\mathbb{B}^*)$ for every $f \in \mathrm{Cyl}(\mathbb{B})$. Moreover, it holds that

$$\mathrm{lip}_a(f)(x) = \|\mathrm{d}_x f\|_{\mathbb{B}^*} \quad \text{for every } f \in \mathrm{Cyl}(\mathbb{B}) \text{ and } x \in \mathbb{B}.$$

Given Banach spaces $\mathbb{B}$, $\mathbb{V}$ with $\mathbb{V}$ finite-dimensional and a linear 1-Lipschitz operator $p\colon \mathbb{B} \to \mathbb{V}$, we define the *pullback* operator $p^*\colon C_c^\infty(\mathbb{V};\mathbb{V}^*) \to C_b^\infty(\mathbb{B};\mathbb{B}^*)$ as follows: given any $\omega \in C_c^\infty(\mathbb{V};\mathbb{V}^*)$,

$$(p^*\omega)(x) := p^{\mathrm{adj}} \circ ((\omega \circ p)(x)) \in \mathbb{B}^* \quad \text{for every } x \in \mathbb{B}, \tag{2.3}$$

where $p^{\mathrm{adj}}\colon \mathbb{V}^* \to \mathbb{B}^*$ stands for the adjoint of p, which is a linear 1-Lipschitz operator. Hence,

$$\|(p^*\omega)(x)\|_{\mathbb{B}^*} \leq \|\omega(p(x))\|_{\mathbb{V}^*} \quad \text{for every } x \in \mathbb{B}, \tag{2.4}$$

thus in particular $\|p^*\omega\|_{C_b(\mathbb{B};\mathbb{B}^*)} \leq \|\omega\|_{C_b(\mathbb{V};\mathbb{V}^*)}$. Notice that $g \circ p \in \mathrm{Cyl}(\mathbb{B})$ for every $g \in C_c^\infty(\mathbb{V})$, and that $p^*(\mathrm{d}g) = \mathrm{d}(g \circ p)$ thanks to the chain rule for Fréchet differentials. Given any $\mu \in \mathfrak{M}_+(\mathbb{B})$ and $p \in [1,\infty)$, the μ-a.e. equivalence class $[p^*\omega]_\mu$ of $p^*\omega$ belongs to the *Lebesgue–Bochner space* $L^p(\mu;\mathbb{B}^*)$, which consists of all $L^p(\mu)$-integrable maps from $\mathbb{B}$ to $\mathbb{B}^*$ in the sense of Bochner [9]. Notice that $(C_c^\infty(\mathbb{V};\mathbb{V}^*), \|\cdot\|_{C_b(\mathbb{V};\mathbb{V}^*)}) \ni \omega \mapsto [p^*\omega]_\mu \in L^p(\mu;\mathbb{B}^*)$ is linear $\mu(\mathbb{B})$-Lipschitz by (2.4).

2.3. Metric Sobolev spaces

By a *metric measure space* $(\mathrm{X},\mathsf{d},\mu)$ we mean a complete and separable metric space (X,d) together with a boundedly-finite Borel measure $\mu \geq 0$ on X, where "boundedly-finite" means that $\mu(B) < +\infty$ whenever $B \subseteq \mathrm{X}$ is a bounded Borel set. Given any exponent $p \in [1,\infty]$, we denote by $(L^p(\mu), \|\cdot\|_{L^p(\mu)})$ the p-Lebesgue space on $(\mathrm{X},\mathsf{d},\mu)$. For any measurable function $f\colon \mathrm{X} \to \mathbb{R}$, we denote by $[f]_\mu$ its equivalence class up to μ-a.e. equality. If $\tilde\mu$ is a boundedly-finite Borel measure on X such that $\mu \leq \tilde\mu$, then we denote by $\mathrm{ext}_{\tilde\mu}\colon L^p(\mu) \to L^p(\tilde\mu)$ the unique map satisfying $[\mathrm{ext}_{\tilde\mu}(f)]_\mu = f$ and $\mathrm{ext}_{\tilde\mu}(f) = 0$ $\tilde\mu$-a.e. on $\left\{\frac{\mathrm{d}\mu}{\mathrm{d}\tilde\mu} = 0\right\}$ for every $f \in L^p(\mu)$.

Remark 2.3. Let $(\mathrm{X},\mathsf{d},\mu)$ be a metric measure space with $\mathrm{spt}(\mu) = \mathrm{X}$ and let $p \in [1,\infty]$. Then

$$\left\{g \in C(\mathrm{X}) : [g]_\mu \in L^p(\mu)\right\} \ni f \mapsto [f]_\mu \in L^p(\mu) \quad \text{is injective.}$$

Indeed, if two continuous functions agree μ-a.e. on X, then they agree everywhere on $\mathrm{spt}(\mu)$. □

2.3.1. Sobolev spaces via relaxation The first notion of metric Sobolev space we recall is based on a relaxation procedure. The next definition, taken from [2], is a variant of Cheeger's one [6].

Definition 2.4. (Sobolev space via relaxation of Lipschitz functions) Let $(\mathrm{X}, \mathsf{d}, \mu)$ be a metric measure space and $p \in (1, \infty)$. We define the *Cheeger energy* functional $\mathrm{Ch}\colon L^p(\mu) \to [0, +\infty]$ as

$$\mathrm{Ch}(f) := \inf\left\{ \varliminf_n \frac{1}{p} \int \mathrm{lip}_a(f_n)^p \, \mathrm{d}\mu \;\middle|\; (f_n)_n \subseteq \mathrm{LIP}_{bs}(\mathrm{X}),\ [f_n]_\mu \rightharpoonup f \text{ weakly in } L^p(\mu) \right\}.$$

Then we define the space $H^{1,p}(\mathrm{X}, \mu)$ as $H^{1,p}(\mathrm{X}, \mu) := \{f \in L^p(\mu) : \mathrm{Ch}(f) < +\infty\}$.

Given two functions $f \in L^p(\mu)$ and $g \in L^p(\mu)^+$, we say that g is a *p-relaxed slope* of f if there exist a sequence $(f_n)_n \subseteq \mathrm{LIP}_{bs}(\mathrm{X})$ and a function $g' \in L^p(\mu)^+$ with $g' \leq g$ such that $[f_n]_\mu \rightharpoonup f$ and $[\mathrm{lip}_a(f_n)]_\mu \rightharpoonup g'$ weakly in $L^p(\mu)$. It can be readily checked that $H^{1,p}(\mathrm{X}, \mu)$ coincides with the set of $L^p(\mu)$-functions having a p-relaxed slope. Moreover, the set of all p-relaxed slopes of a given function $f \in H^{1,p}(\mathrm{X}, \mu)$ is a closed sublattice of $L^p(\mu)^+$, whose μ-a.e. minimal element $|Df|_H \in L^p(\mu)^+$ is called the *minimal p-relaxed slope* of f. We also have $\mathrm{Ch}(f) = \frac{1}{p} \int |Df|_H^p \, \mathrm{d}\mu$.

On a weighted Banach space, one can give a similar definition using cylindrical functions instead:

Definition 2.5. (Sobolev space via relaxation of cylindrical functions) Let $\mathbb{B}$ be a separable Banach space and $\mu \in \mathfrak{M}_+(\mathbb{B})$. Let $p \in (1, \infty)$ be a given exponent. We define the *cylindrical Cheeger energy* functional $\mathrm{Ch}_{cyl}\colon L^p(\mu) \to [0, +\infty]$ as

$$\mathrm{Ch}_{cyl}(f) := \inf\left\{ \varliminf_n \frac{1}{p} \int \|\mathrm{d}_x f_n\|_{\mathbb{B}^*}^p \, \mathrm{d}\mu \;\middle|\; (f_n)_n \subseteq \mathrm{Cyl}(\mathbb{B}),\ [f_n]_\mu \rightharpoonup f \text{ weakly in } L^p(\mu) \right\}.$$

Then we define the space $H^{1,p}_{cyl}(\mathbb{B}, \mu)$ as $H^{1,p}_{cyl}(\mathbb{B}, \mu) := \{f \in L^p(\mu) : \mathrm{Ch}_{cyl}(f) < +\infty\}$.

Similarly as for the space $H^{1,p}$, each function $f \in H^{1,p}_{cyl}(\mathbb{B}, \mu)$ is associated with a *minimal cylindrical p-relaxed slope* $|Df|_{H,cyl} \in L^p(\mu)^+$. It also holds that $\mathrm{Ch}_{cyl}(f) = \frac{1}{p} \int |Df|_{H,cyl}^p \, \mathrm{d}\mu$.

Definition 2.5 is a particular instance of the notion of metric Sobolev space via relaxation introduced in [23], because $\mathrm{Cyl}(\mathbb{B})$ is a unital separating subalgebra of $\mathrm{LIP}_b(\mathbb{B})$ [23, Example 2.1.19]. In particular, we know from [23, Lemma 2.1.27] that $[\mathrm{Cyl}(\mathbb{B})]_\mu$ is dense in $L^p(\mu)$. Moreover, the inclusion $\mathrm{Cyl}(\mathbb{B}) \subseteq \mathrm{LIP}_b(\mathbb{B})$, a standard cut-off argument, and the pointwise minimality properties of minimal relaxed slopes (see [23, Lemma 3.1.11]) ensure that $H^{1,p}_{cyl}(\mathbb{B}, \mu) \subseteq H^{1,p}(\mathbb{B}, \mu)$ and

$$|Df|_H \leq |Df|_{H,cyl} \quad \text{for every } f \in H^{1,p}_{cyl}(\mathbb{B}, \mu). \tag{2.5}$$

2.3.2. Sobolev spaces via plans The next notion was introduced in [23, Definition 5.1.1] after [1].

Definition 2.6. (Plan with barycenter) Let $(\mathrm{X}, \mathsf{d}, \mu)$ be a metric measure space and $q \in (1, \infty]$. We define $\mathcal{B}_q(\mathrm{X}, \mu)$ as the set of all $\boldsymbol{\pi} \in \mathfrak{M}_+(C([0,1]; \mathrm{X}))$ concentrated on $\mathrm{LIP}([0,1]; \mathrm{X})$ such that:

i) $\boldsymbol{\pi}$ has *barycenter* in $L^q(\mu)$, i.e. there exists a (unique) function $\mathsf{Bar}(\boldsymbol{\pi}) \in L^q(\mu)$ such that

$$\int f\, \mathsf{Bar}(\boldsymbol{\pi})\, \mathrm{d}\mu = \iint_0^1 f(\gamma_t)|\dot{\gamma}_t|\, \mathrm{d}t\, \mathrm{d}\boldsymbol{\pi}(\gamma) \quad \text{for every } f \in C_{bs}(\mathrm{X}).$$

ii) It holds that $(\mathrm{e}_\pm)_\# \boldsymbol{\pi} \ll \mu$ and $\frac{\mathrm{d}(\mathrm{e}_\pm)_\# \boldsymbol{\pi}}{\mathrm{d}\mu} \in L^q(\mu)$.

In Sect. 3, we will need to reduce our study to the case of fully-supported reference measures (due to technical reasons that will be discussed in the first paragraph of Sect. 3). The following auxiliary result about plans will be helpful in this direction.

Lemma 2.7. *Let* $(\mathrm{X}, \mathsf{d}, \mu)$ *be a metric measure space such that* $S := \mathrm{spt}(\mu) \neq \mathrm{X}$. *Let* $C \subseteq \mathrm{X} \setminus S$ *be a countable set. Let* $\tilde{\mu} \geq 0$ *be a boundedly-finite Borel measure on* X *concentrated on* $S \cup C$ *such that* $\tilde{\mu}|_S = \mu$. *Fix any* $q \in (1, \infty]$ *and* $\tilde{\boldsymbol{\pi}} \in \mathcal{B}_q(\mathrm{X}, \tilde{\mu})$. *Then* $\tilde{\boldsymbol{\pi}}$*-a.e. curve* γ *is either contained in* S *or constant. In particular,* $\boldsymbol{\pi} := \tilde{\boldsymbol{\pi}}|_{\mathrm{LIP}([0,1];S)} \in \mathcal{B}_q(\mathrm{X}, \mu)$, $\mathrm{ext}_{\tilde{\mu}}(\mathsf{Bar}(\boldsymbol{\pi})) = \mathsf{Bar}(\tilde{\boldsymbol{\pi}})$, *and* $\partial \boldsymbol{\pi} = \partial \tilde{\boldsymbol{\pi}}$.

Proof. Let Γ_{const} be the set of constant curves in X and define $\Gamma_S := \mathrm{LIP}([0,1]; S)$. We claim that

$$\Gamma := \big\{\gamma \in \mathrm{LIP}([0,1]; \mathrm{X}) \,\big|\, \gamma_t \in S \cup C \text{ for } \mathcal{L}_1\text{-a.e. } t \in \{|\dot{\gamma}| > 0\}\big\} \subseteq \Gamma_S \cup \Gamma_{\mathrm{const}}. \tag{2.6}$$

Let us prove (2.6). Fix any $\gamma \in \Gamma \setminus \Gamma_{\mathrm{const}}$. We aim to show that $\gamma([0,1]) \cap C = \varnothing$. We argue by contradiction: suppose $\gamma_a = x$ for some $a \in [0,1]$ and $x \in C$. Up to replacing γ with $t \mapsto \gamma_{-t}$, we can assume $a < 1$, and we can find $b \in (a, 1]$ such that $r := \mathsf{d}(\gamma_b, x) > 0$ and $\gamma([a,b]) \subseteq \mathrm{X} \setminus S$. Since $N := \{\mathsf{d}(y, x) : y \in C\}$ is countable and $[a,b] \ni t \mapsto f(t) := \mathsf{d}(\gamma_t, x)$ is Lipschitz, we deduce that $f' = 0$ holds $\mathcal{L}_1$-a.e. on $f^{-1}(N)$. Moreover, $f^{-1}(\mathbb{R} \setminus N) \subseteq \gamma^{-1}(\mathrm{X} \setminus (S \cup C))$ and thus $|\dot{\gamma}| = 0$ holds $\mathcal{L}_1$-a.e. on $f^{-1}(\mathbb{R} \setminus N)$. All in all, it follows that

$$\begin{aligned} 0 < r = \mathsf{d}(\gamma_b, \gamma_a) = f(b) - f(a) &= \int_a^b f'(t)\, \mathrm{d}t = \int_{f^{-1}(\mathbb{R}\setminus N)} f'(t)\, \mathrm{d}t \\ &\leq \int_{f^{-1}(\mathbb{R}\setminus N)} |\dot{\gamma}_t|\, \mathrm{d}t = 0, \end{aligned}$$

which leads to a contradiction. Therefore, we proved that $\gamma([0,1]) \cap C = \varnothing$. Now consider the Lipschitz function $[0,1] \ni t \mapsto g(t) := \mathsf{d}(\gamma_t, S)$. Given that $\gamma \in \Gamma$, we have that $g = 0$ holds $\mathcal{L}_1$-a.e. on $\{|\dot{\gamma}| > 0\}$, thus $g' = 0$ holds $\mathcal{L}_1$-a.e. on $\{|\dot{\gamma}| > 0\}$. Since $|g'| \leq |\dot{\gamma}|$ holds $\mathcal{L}_1$-a.e. on $[0,1]$, we conclude that $g' = 0$ holds $\mathcal{L}_1$-a.e.

on [0, 1] and thus g is constant. Being γ non-constant, we know that $\mathcal{L}_1(\{|\dot\gamma| > 0\}) > 0$, so that $g = 0$ on [0, 1]. This means that $\gamma \in \Gamma_S$, so that (2.6) is proved. Finally, observe that $\iint_0^1 \mathbb{1}_{X\setminus(S\cup C)}(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\tilde{\boldsymbol{\pi}}(\gamma) = \int_{X\setminus(S\cup C)} \mathsf{Bar}(\tilde{\boldsymbol{\pi}})\,\mathrm{d}\tilde\mu = 0$, whence it follows that for $\tilde{\boldsymbol{\pi}}$-a.e. γ it holds that $\mathbb{1}_{X\setminus(S\cup C)}(\gamma_t)|\dot\gamma_t| = 0$ for $\mathcal{L}_1$-a.e. $t \in [0, 1]$. In particular, we deduce that $\tilde{\boldsymbol{\pi}}(\mathrm{LIP}([0,1];X) \setminus \Gamma) = 0$. Taking into account also (2.6), we have that the first part of the statement is proved. The last part of the statement then easily follows. □

The following definition of Sobolev space via plans is taken from [23, Definition 5.1.4]. Similar notions were previously introduced in [1–3], see Remark 3.6 for a quick comparison.

Definition 2.8. (Sobolev space via plans) Let (X, d, μ) be a metric measure space and $p \in (1, \infty)$. Then we declare that $f \in L^p(\mu)$ belongs to $W^{1,p}(X, \mu)$ if there exists $G \in L^p(\mu)^+$ such that

$$\int f\,\mathrm{d}\partial\boldsymbol{\pi} \le \int G\,\mathsf{Bar}(\boldsymbol{\pi})\,\mathrm{d}\mu \quad \text{for every } \boldsymbol{\pi} \in \mathcal{B}_q(X, \mu).$$

The μ-a.e. minimal such G is called the *minimal p-weak upper gradient* $|Df|_W \in L^p(\mu)^+$ of f.

One can readily deduce from the definitions that $H^{1,p}(X, \mu) \subseteq W^{1,p}(X, \mu)$ and that

$$|Df|_W \le |Df|_H \quad \text{for every } f \in H^{1,p}(X, \mu). \tag{2.7}$$

Indeed, for every $f \in H^{1,p}(X, \mu)$ there exists a sequence $(f_n)_n \subseteq \mathrm{LIP}_{bs}(X)$ such that $[f_n]_\mu \rightharpoonup f$ and $[\mathrm{lip}_a(f_n)]_\mu \rightharpoonup |Df|_H$ weakly in $L^p(\mu)$. Therefore, for any given $\boldsymbol{\pi} \in \mathcal{B}_q(X, \mu)$ we can let $n \to \infty$ in

$$\begin{aligned}\int f_n\,\mathrm{d}\partial\boldsymbol{\pi} &= \iint_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f_n(\gamma_t)\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma) \le \iint_0^1 \mathrm{lip}_a(f_n)(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\ &= \int \mathrm{lip}_a(f_n)\,\mathsf{Bar}(\boldsymbol{\pi})\,\mathrm{d}\mu,\end{aligned}$$

thus obtaining that $\int f\,\mathrm{d}\partial\boldsymbol{\pi} \le \int |Df|_H\,\mathsf{Bar}(\boldsymbol{\pi})\,\mathrm{d}\mu$. This gives $f \in W^{1,p}(X, \mu)$ and $|Df|_W \le |Df|_H$.

Remark 2.9. Let (X, d_X, μ_X), (Y, d_Y, μ_Y) be metric measure spaces. Let $S := \mathrm{spt}(\mu_X)$. We call $\phi\colon S \to Y$ a *short map* if it is 1-Lipschitz and $\phi_\#\mu_X \le \mu_Y$. Define $\Phi\colon \mathrm{LIP}([0,1];S) \to \mathrm{LIP}([0,1];Y)$ as $\Phi(\gamma)_t := \phi(\gamma_t)$ for every $\gamma \in \mathrm{LIP}([0,1];S)$ and $t \in [0,1]$. Then the following claim can be readily checked: given any $q \in (1, \infty)$ and $\boldsymbol{\pi} \in \mathcal{B}_q(S, \mu_X)$, it holds that $\Phi_\#\boldsymbol{\pi} \in \mathcal{B}_q(Y, \mu_Y)$ and

$$\mathsf{Bar}(\Phi_\#\boldsymbol{\pi}) \le \frac{\mathrm{d}\phi_\#(\mathsf{Bar}(\boldsymbol{\pi})\mu_X)}{\mathrm{d}\mu_Y}, \qquad \partial(\Phi_\#\boldsymbol{\pi}) = \phi_\#(\partial\boldsymbol{\pi}). \tag{2.8}$$

Moreover, the map ϕ induces via pre-composition a 1-Lipschitz linear map $\phi^*\colon L^p(\mu_Y) \to L^p(\mu_X)$. Using (2.8), one can easily show that $\phi^* W^{1,p}(Y, \mu_Y) \subseteq$

$W^{1,p}(\mathrm{X},\mu_{\mathrm{X}})$ and $|D(\phi^* f)|_W \leq \phi^*|Df|_W$ for every $f \in W^{1,p}(\mathrm{Y},\mu_{\mathrm{Y}})$. It can also be readily checked that – assuming in addition that ϕ can be extended to a 1-Lipschitz map from X to Y – it holds that $\phi^* H^{1,p}(\mathrm{Y},\mu_{\mathrm{Y}}) \subseteq H^{1,p}(\mathrm{X},\mu_{\mathrm{X}})$ and that $|D(\phi^* f)|_H \leq \phi^*|Df|_H$ for every $f \in H^{1,p}(\mathrm{Y},\mu_{\mathrm{Y}})$. □

The following technical statement, which will allow us to reduce the study of the Sobolev space $W^{1,p}$ to the case of fully-supported reference measures, is a direct consequence of Lemma 2.7.

Corollary 2.10. *Let* $(\mathrm{X},\mathsf{d},\mu)$ *be a metric measure space and let* $p \in (1,\infty)$. *Let* $C \subseteq \mathrm{X} \setminus \mathrm{spt}(\mu)$ *be a countable set. Let* $\tilde{\mu} \geq 0$ *be a boundedly-finite Borel measure on* X *concentrated on* $\mathrm{spt}(\mu) \cup C$ *such that* $\tilde{\mu}|_{\mathrm{spt}(\mu)} = \mu$. *Then it holds that* $W^{1,p}(\mathrm{X},\tilde{\mu}) = \left\{ f \in L^p(\tilde{\mu}) : [f]_\mu \in W^{1,p}(\mathrm{X},\mu) \right\}$ *and*

$$|Df|_W = \mathrm{ext}_{\tilde{\mu}}(|D[f]_\mu|_W) \quad \textit{for every } f \in W^{1,p}(\mathrm{X},\tilde{\mu}).$$

Proof. On the one hand, taking $\phi := \mathrm{id}_{\mathrm{X}} \colon (\mathrm{X},\tilde{\mu}) \to (\mathrm{X},\mu)$ in Remark 2.9 we get $[f]_\mu \in W^{1,p}(\mathrm{X},\mu)$ and $|D[f]_\mu|_W \leq [|Df|_W]_\mu$ for all $f \in W^{1,p}(\mathrm{X},\tilde{\mu})$. Conversely, if $f \in L^p(\tilde{\mu})$ and $[f]_\mu \in W^{1,p}(\mathrm{X},\mu)$, then for every $\tilde{\boldsymbol{\pi}} \in \mathcal{B}_q(\mathrm{X},\tilde{\mu})$ we deduce from Lemma 2.7 that $\boldsymbol{\pi} := \tilde{\boldsymbol{\pi}}|_{\mathrm{LIP}([0,1];\mathrm{spt}(\mu))} \in \mathcal{B}_q(\mathrm{X},\mu)$ and

$$\begin{aligned}\int f \,\mathrm{d}\partial\tilde{\boldsymbol{\pi}} &= \int [f]_\mu \,\mathrm{d}\partial\boldsymbol{\pi} \leq \int |D[f]_\mu|_W \,\mathsf{Bar}(\boldsymbol{\pi})\,\mathrm{d}\mu \\ &= \int \mathrm{ext}_{\tilde{\mu}}\big(|D[f]_\mu|_W\big)\mathsf{Bar}(\tilde{\boldsymbol{\pi}})\,\mathrm{d}\tilde{\mu},\end{aligned}$$

which implies that $f \in W^{1,p}(\mathrm{X},\tilde{\mu})$ and $|Df|_W \leq \mathrm{ext}_{\tilde{\mu}}(|D[f]_\mu|_W)$. The statement follows. □

Remark 2.11. Let $(\mathrm{X},\mathsf{d},\mu)$ be a metric measure space and $p \in (1,\infty)$. Fix any $\bar{x} \in \mathrm{X}$ and define $\Omega_n := B_n(\bar{x})$ for every $n \in \mathbb{N}$. Let $f \in L^p(\mu)$ be such that $f_n \in H^{1,p}(\mathrm{X},\mu|_{\Omega_n})$ for every $n \in \mathbb{N}$ and $s := \sup_n \int |Df_n|_H^p \,\mathrm{d}\mu|_{\Omega_n} < +\infty$, where $f_n := [f]_{\mu|_{\Omega_n}}$. Then it holds that $f \in H^{1,p}(\mathrm{X},\mu)$ and $\int |Df|_H^p \,\mathrm{d}\mu = s$. This property can be proved by combining the locality of minimal relaxed slopes with a cut-off argument, see e.g. [6, Proposition 2.17]. □

3. Main results

Let $\mathbb{B}$ be a separable Banach space, $\mu \in \mathfrak{M}_+(\mathbb{B})$ a measure satisfying $\mathrm{spt}(\mu) = \mathbb{B}$, and $p \in [1,\infty)$. In view of Remark 2.3, we can identify $\mathrm{Cyl}(\mathbb{B})$ with a subspace of $L^p(\mu)$, thus the Fréchet differential induces an unbounded linear operator $\mathrm{d} \colon L^p(\mu) \to L^p(\mu;\mathbb{B}^*)$ with domain $D(\mathrm{d}) = \mathrm{Cyl}(\mathbb{B})$. It is worth highlighting that the well-posedness of d is guaranteed by the assumption that $\mathrm{spt}(\mu) = \mathbb{B}$; without such assumption, the well-posedness might fail (e.g. if μ is Dirac delta). Since the operator $\mathrm{d} \colon L^p(\mu) \to L^p(\mu;\mathbb{B}^*)$ is densely defined, its adjoint operator $\mathrm{d}^* \colon L^p(\mu;\mathbb{B}^*)^* \to L^q(\mu)$ is well-posed.

Proposition 3.1. *Let $\mathbb{B}$ be a separable Banach space. Let $\mu \in \mathfrak{M}_+(\mathbb{B})$ be such that* $\mathrm{spt}(\mu) = \mathbb{B}$. *Let $p \in [1, \infty)$ and $L \in D(\mathrm{d}^*)$ be given. Then there exists a plan $\boldsymbol{\pi} \in \mathcal{B}_q(\mathbb{B}, \mu)$ such that*

$$\partial \boldsymbol{\pi} = (\mathrm{d}^* L)\mu, \qquad \|\mathsf{Bar}(\boldsymbol{\pi})\|_{L^q(\mu)} \leq \|L\|_{L^p(\mu;\mathbb{B}^*)^*}.$$

Proof. By [13, Proposition 1.2.13], there exists a unique $L^\infty(\mu)$-linear map $\ell\colon L^p(\mu;\mathbb{B}^*) \to L^1(\mu)$ such that $L(\omega) = \int \ell(\omega)\,\mathrm{d}\mu$ and $|\ell(\omega)| \leq |L|\|\omega(\cdot)\|_{\mathbb{B}^*}$ in the μ-a.e. sense for every $\omega \in L^p(\mu;\mathbb{B}^*)$, for some $|L| \in L^q(\mu)^+$ satisfying $\||L|\|_{L^q(\mu)} = \|L\|_{L^p(\mu;\mathbb{B}^*)^*}$. Since $\mathbb{B}$ can be embedded linearly and isometrically into ℓ^∞ via a Kuratowski embedding and ℓ^∞ has the metric approximation property (see e.g. [18, Lemma 5.7]), we have that $\mathbb{B}$ is the subspace of a Banach space $\tilde{\mathbb{B}}$ having the metric approximation property. Given that μ is concentrated on a σ-compact set, we can find a sequence $(\tilde{p}_n)_n$ of finite-rank 1-Lipschitz linear operators $\tilde{p}_n\colon \tilde{\mathbb{B}} \to \tilde{\mathbb{B}}$ such that $\lim_n \|\tilde{p}_n(x) - x\|_{\tilde{\mathbb{B}}} = 0$ holds for μ-a.e. $x \in \tilde{\mathbb{B}}$. Now let us fix a separable closed subspace $\hat{\mathbb{B}}$ of $\tilde{\mathbb{B}}$ containing $\mathbb{B} \cup \bigcup_{n\in\mathbb{N}} \tilde{p}_n(\mathbb{B})$. For any $n \in \mathbb{N}$, we denote by $\mathbb{V}_n$ the finite-dimensional Banach space $\tilde{p}_n(\mathbb{B}) \subseteq \hat{\mathbb{B}}$ and we define the operator $p_n\colon \mathbb{B} \to \mathbb{V}_n$ as $p_n := \tilde{p}_n|_{\mathbb{B}}$. We define the 1-current $T_n \in \mathbf{M}_1(\mathbb{V}_n)$ as $T_n(\omega) := L([p_n^*\omega]_\mu)$ for every $\omega \in C_c^\infty(\mathbb{V}_n;\mathbb{V}_n^*)$, where $p_n^*\omega$ is given by (2.3). We claim that $T_n \in \mathbf{N}_1(\mathbb{V}_n)$ and

$$\|T_n\| \leq (p_n)_\#(|L|\mu), \qquad \partial T_n = (p_n)_\#((\mathrm{d}^* L)\mu). \tag{3.1}$$

To prove the first property in (3.1), fix any open set $\Omega \subseteq \mathbb{V}_n$ and an element $\omega \in C_c^\infty(\mathbb{V}_n;\mathbb{V}_n^*)$ satisfying $\mathrm{spt}(\omega) \subseteq \Omega$ and $\|\omega(x)\|_{\mathbb{V}_n^*} \leq 1$ for every $x \in \mathbb{V}_n$. Recalling (2.4), we can estimate

$$\begin{aligned}|T_n(\omega)| &\leq \int |\ell([p_n^*\omega]_\mu)|\,\mathrm{d}\mu \leq \int_{p_n^{-1}(\Omega)} |L|(x)\|\omega(p_n(x))\|_{\mathbb{V}_n^*}\,\mathrm{d}\mu(x)\\ &\leq (p_n)_\#(|L|\mu)(\Omega),\end{aligned}$$

whence it follows that $\|T_n\|(\Omega) \leq (p_n)_\#(|L|\mu)(\Omega)$ and thus $\|T_n\| \leq (p_n)_\#(|L|\mu)$. To prove the second property in (3.1), notice that for every given function $f \in C_c^\infty(\mathbb{V}_n)$ we can compute

$$\begin{aligned}\partial T_n(f) = T_n(\mathrm{d}f) = L([p_n^*\mathrm{d}f]_\mu) = L(\mathrm{d}(f \circ p_n)) &= \int f \circ p_n\,\mathrm{d}^* L\,\mathrm{d}\mu\\ &= \int f\,\mathrm{d}(p_n)_\#((\mathrm{d}^* L)\mu),\end{aligned}$$

whence it follows that T_n is normal and $\partial T_n = (p_n)_\#((\mathrm{d}^* L)\mu)$. All in all, the claim (3.1) is proved. Since $(p_n)_\#(|L|\mu) \rightharpoonup |L|\mu$ and $(p_n)_\#(|\mathrm{d}^* L|\mu) \rightharpoonup |\mathrm{d}^* L|\mu$ weakly in $\mathfrak{M}(\hat{\mathbb{B}})$ by the dominated convergence theorem, Prokhorov's theorem gives that $(\|T_n\|)_n, (|\partial T_n|)_n \subseteq \mathfrak{M}_+(\hat{\mathbb{B}})$ are tight sequences.

For any $n \in \mathbb{N}$, we choose a cycle C_n of T_n such that $\tilde{T}_n := T_n - C_n \in \mathbf{N}_1(\mathbb{V}_n)$ is acyclic. Using Theorem 2.1, we obtain a plan $\boldsymbol{\pi}_n \in \mathfrak{M}_+(C([0,1];\hat{\mathbb{B}}))$, concentrated on the set Γ of non-constant Lipschitz curves in $\mathbb{V}_n$ of constant speed, such that $\|\tilde{T}_n\| = \|\boldsymbol{\pi}_n\|$ and $(\mathrm{e}_\pm)_\#\boldsymbol{\pi}_n = (\partial\tilde{T}_n)^\pm$. Now we follow the proof of [18, Lemma

4.11]. Since $\|\boldsymbol{\pi}_n\| \leq \|T_n\|$ and $(\mathrm{e}_\pm)_\# \boldsymbol{\pi}_n \leq |\partial T_n|$ for every $n \in \mathbb{N}$, the sequences $(\|\boldsymbol{\pi}_n\|)_n, ((\mathrm{e}_\pm)_\# \boldsymbol{\pi}_n)_n \subseteq \mathfrak{M}_+(\hat{\mathbb{B}})$ are tight, thus we can find compact subsets $(K_j)_j$ of $\hat{\mathbb{B}}$ such that $\|\boldsymbol{\pi}_n\|(\hat{\mathbb{B}} \setminus K_j) \leq 4^{-j}$ and $((\mathrm{e}_+)_\# \boldsymbol{\pi}_n)(\hat{\mathbb{B}} \setminus K_j) \leq 2^{-j}$ for all $j, n \in \mathbb{N}$. We also define

$$\Gamma_j := \{\gamma \in \Gamma \mid \ell(\gamma) \leq 2^j\} \cap \mathrm{e}_+^{-1}(K_j) \cap \bigcap_{k>j} \tilde{\Gamma}_k,$$

where $\tilde{\Gamma}_k := \{\gamma \in \Gamma : \mathcal{L}_1(\gamma^{-1}(\hat{\mathbb{B}} \backslash K_k)) \leq 2^{-k}/\ell(\gamma)\}$. Since $2^{-k}\boldsymbol{\pi}_n(\Gamma \backslash \tilde{\Gamma}_k) \leq \|\boldsymbol{\pi}_n\|(\hat{\mathbb{B}} \backslash K_k) \leq 4^{-k}$ and $2^j \boldsymbol{\pi}_n(\{\gamma \in \Gamma : \ell(\gamma) > 2^j\}) \leq \|\boldsymbol{\pi}_n\|(\hat{\mathbb{B}}) \leq \int |L| \,\mathrm{d}\mu =: m$ for every $n \in \mathbb{N}$, we deduce that

$$\begin{aligned} \boldsymbol{\pi}_n(\Gamma \setminus \Gamma_j) &\leq \frac{m}{2^j} + ((\mathrm{e}_+)_\# \boldsymbol{\pi}_n)(\hat{\mathbb{B}} \backslash K_j) + \sum_{k>j} \boldsymbol{\pi}_n(\Gamma \setminus \tilde{\Gamma}_k) \\ &\leq \frac{m}{2^j} + \frac{1}{2^j} + \sum_{k>j} \frac{1}{2^k} = \frac{m+2}{2^j} \end{aligned} \tag{3.2}$$

for all $j, n \in \mathbb{N}$. We now show that Γ_j is a precompact subset of $C([0, 1]; \hat{\mathbb{B}})$. Fix $(\gamma^i)_i \subseteq \Gamma_j$. Then:

- Suppose $\lim_i \ell(\gamma^i) = 0$. Since $((\gamma^i)_1)_i \subseteq K_j$ and K_j is compact, $x := \lim_i (\gamma^i)_1 \in K_j$ exists, up to subsequence. Hence, $(\gamma^i)_i$ converges uniformly to the curve constantly in x.
- Suppose $\overline{\lim}_i \, \ell(\gamma^i) > 0$, so that $c := \inf_i \ell(\gamma^i) > 0$ up to subsequence. Since $\mathrm{Lip}(\gamma^i) \leq 2^j$ and $\mathcal{L}_1((\gamma^i)^{-1}(\hat{\mathbb{B}} \backslash K_k)) \leq 2^{-k}/c$ for every $i \in \mathbb{N}$ and $k > j$, the sequence $(\gamma^i)_i$ has a uniformly converging subsequence by the Arzelà–Ascoli theorem [18, Proposition 2.1].

All in all, we proved that each Γ_j is precompact, thus (3.2) implies that $(\boldsymbol{\pi}_n)_n \subseteq \mathfrak{M}_+(C([0, 1]; \hat{\mathbb{B}}))$ is a tight sequence. Since $\boldsymbol{\pi}_n(C([0, 1]; \hat{\mathbb{B}})) \leq |\partial T_n|(\hat{\mathbb{B}}) \leq \int |\mathrm{d}^* L| \,\mathrm{d}\mu$ for every $n \in \mathbb{N}$, we know from Prokhorov's theorem that $\boldsymbol{\pi}_n \rightharpoonup \hat{\boldsymbol{\pi}}$ for some $\hat{\boldsymbol{\pi}} \in \mathfrak{M}_+(C([0, 1]; \hat{\mathbb{B}}))$, up to subsequence. Note that

$$\begin{aligned} &\left| \iint_0^1 f(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}t \,\mathrm{d}\hat{\boldsymbol{\pi}}(\gamma) \right| \\ &\leq \underline{\lim}_n \iint_0^1 |f|(\gamma_t)\|\dot{\gamma}_t\|_{\mathbb{V}_n} \,\mathrm{d}t \,\mathrm{d}\boldsymbol{\pi}_n(\gamma) = \underline{\lim}_n \int |f| \,\mathrm{d}\|\boldsymbol{\pi}_n\| \\ &\leq \lim_n \int |f| \circ p_n \, |L| \,\mathrm{d}\mu = \int |f||L| \,\mathrm{d}\mu \leq \|f\|_{L^p(\mu)} \|L\|_{L^p(\mu;\mathbb{B}^*)^*} \end{aligned}$$

for every $f \in C_{bs}(\hat{\mathbb{B}})$ thanks to (2.2), (3.1), and the dominated convergence theorem. This implies that $\hat{\boldsymbol{\pi}}$ has barycenter in $L^q(\mu)$ and $\|\mathsf{Bar}(\hat{\boldsymbol{\pi}})\|_{L^q(\mu)} \leq \|L\|_{L^p(\mu;\mathbb{B}^*)^*}$. Given that $(\mathrm{e}_\pm)_\# \boldsymbol{\pi}_n \rightharpoonup (\mathrm{e}_\pm)_\# \hat{\boldsymbol{\pi}}$ and $(\mathrm{e}_\pm)_\# \boldsymbol{\pi}_n \leq (p_n)_\#(|\mathrm{d}^* L|\mu) \rightharpoonup |\mathrm{d}^* L|\mu$, we have that $(\mathrm{e}_\pm)_\# \hat{\boldsymbol{\pi}} \ll \mu$ and $\frac{\mathrm{d}(\mathrm{e}_\pm)_\# \hat{\boldsymbol{\pi}}}{\mathrm{d}\mu} \leq |\mathrm{d}^* L| \in L^q(\mu)$. Therefore, it holds $\hat{\boldsymbol{\pi}} \in \mathcal{B}_q(\hat{\mathbb{B}}, \mu)$. Also, $\partial \boldsymbol{\pi}_n = (\mathrm{e}_+)_\# \boldsymbol{\pi}_n - (\mathrm{e}_-)_\# \boldsymbol{\pi}_n \rightharpoonup (\mathrm{e}_+)_\# \hat{\boldsymbol{\pi}} - (\mathrm{e}_-)_\# \hat{\boldsymbol{\pi}} = \partial \hat{\boldsymbol{\pi}}$ and $\partial \boldsymbol{\pi}_n = \partial \tilde{T}_n = \partial T_n = (p_n)_\#((\mathrm{d}^* L)\mu) \rightharpoonup (\mathrm{d}^* L)\mu$, so that $\partial \hat{\boldsymbol{\pi}} = (\mathrm{d}^* L)\mu$.

Thanks to the last part of Lemma 2.7, we then conclude that $\boldsymbol{\pi} := \hat{\boldsymbol{\pi}}|_{\mathrm{LIP}([0,1];\mathbb{B})} \in \mathcal{B}_q(\mathbb{B}, \mu)$ verifies the statement. □

Remark 3.2. Alternatively, in the proof of Proposition 3.1 we could have used metric 1-currents in the sense of Ambrosio–Kirchheim [4] and Paolini–Stepanov's metric version of the superposition principle [18,19]. We opted for the proof we presented for two reasons: first, the current T we want to associate to L is defined on cylindrical functions, but it is not obvious how to extend it to Lipschitz functions; second, extending T to Lipschitz functions is de facto unnecessary. □

With Proposition 3.1 at disposal, we prove the equivalence result for weighted Banach spaces.

Theorem 3.3. (Equivalence of Sobolev spaces on weighted Banach spaces) *Let $\mathbb{B}$ be a separable Banach space, $\mu \in \mathfrak{M}_+(\mathbb{B})$, and $p \in (1, \infty)$. Then it holds that $H^{1,p}_{cyl}(\mathbb{B}, \mu) = W^{1,p}(\mathbb{B}, \mu)$ and*

$$|Df|_W = |Df|_{H,cyl} \quad \textit{for every } f \in W^{1,p}(\mathbb{B}, \mu).$$

Proof. If $\mathrm{spt}(\mu) = \mathbb{B}$, define $\tilde{\mu} := \mu$. Otherwise, fix any dense sequence $(x_n)_n$ in $\mathbb{B} \setminus \mathrm{spt}(\mu)$ and call $\tilde{\mu} := \mu + \sum_n 2^{-n}\delta_{x_n} \in \mathfrak{M}_+(\mathbb{B})$. By Remark 2.3, it makes sense to define $\mathcal{F} \colon L^p(\tilde{\mu}) \to [0, +\infty]$ as

$$\mathcal{F}(f) := \frac{1}{p} \int \|\mathrm{d}_x f\|^p_{\mathbb{B}^*} \, \mathrm{d}\tilde{\mu}(x) \quad \text{if } f \in \mathrm{Cyl}(\mathbb{B}) \subseteq L^p(\tilde{\mu})$$

and $\mathcal{F}(f) := +\infty$ otherwise. Consider its Fenchel conjugate and its double Fenchel conjugate, i.e.

$$\mathcal{F}^*(g) := \sup_{\tilde{h} \in L^p(\tilde{\mu})} \int g\tilde{h} \, \mathrm{d}\tilde{\mu} - \mathcal{F}(\tilde{h}), \qquad \mathcal{F}^{**}(f) := \sup_{\tilde{g} \in L^q(\tilde{\mu})} \int f\tilde{g} \, \mathrm{d}\tilde{\mu} - \mathcal{F}^*(\tilde{g})$$

for every $g \in L^q(\tilde{\mu})$ and $f \in L^p(\tilde{\mu})$. Since $\mathcal{F}$ is convex, we know e.g. from [22, Theorem 5] that $\mathcal{F}^{**}$ coincides with the weak lower semicontinuous envelope $\mathrm{sc}^-\mathcal{F} \colon L^p(\tilde{\mu}) \to [0, +\infty]$ of $\mathcal{F}$. Moreover, letting $\phi \colon L^p(\tilde{\mu}; \mathbb{B}^*) \to [0, +\infty)$ be given by $\phi := \frac{1}{p}\|\cdot\|^p_{L^p(\tilde{\mu};\mathbb{B}^*)}$, we have that $\mathcal{F} = \phi \circ \mathrm{d}$ and thus

$$\mathcal{F}^*(g) = \inf\left\{\frac{1}{q}\|L\|^q_{L^p(\tilde{\mu};\mathbb{B}^*)^*} \;\middle|\; L \in D(\mathrm{d}^*),\ \mathrm{d}^*L = g\right\} \quad \text{for every } g \in L^q(\tilde{\mu})$$

thanks to [5, Theorem 5.1]. Indeed, ϕ is convex and continuous, and its Fenchel conjugate is given by $\phi^* = \frac{1}{q}\|\cdot\|^q_{L^p(\tilde{\mu};\mathbb{B}^*)^*}$. Combining this with Proposition 3.1

and Young's inequality, we obtain

$$\begin{aligned}
\mathrm{Ch}_{cyl}(\tilde f) = \mathrm{sc}^-\mathcal{F}(\tilde f) &= \sup_{g\in L^q(\tilde\mu)} \left(\int \tilde f g \,\mathrm{d}\tilde\mu - \inf_{\substack{L\in D(\mathrm{d}^*):\\ \mathrm{d}^* L = g}} \frac{1}{q}\|L\|^q_{L^p(\tilde\mu;\mathbb{B}^*)^*} \right) \\
&\le \sup_{g\in L^q(\tilde\mu)} \left(\int \tilde f g \,\mathrm{d}\tilde\mu - \inf_{\substack{\boldsymbol{\pi}\in\mathcal{B}_q(\mathbb{B},\tilde\mu):\\ \partial\boldsymbol{\pi} = g\tilde\mu}} \frac{1}{q}\|\mathsf{Bar}(\boldsymbol{\pi})\|^q_{L^q(\tilde\mu)} \right) \\
&= \sup_{\boldsymbol{\pi}\in\mathcal{B}_q(\mathbb{B},\tilde\mu)} \left(\int \tilde f \,\mathrm{d}\partial\boldsymbol{\pi} - \frac{1}{q}\|\mathsf{Bar}(\boldsymbol{\pi})\|^q_{L^q(\tilde\mu)} \right) \\
&\le \sup_{\boldsymbol{\pi}\in\mathcal{B}_q(\mathbb{B},\tilde\mu)} \left(\int |D\tilde f|_W \,\mathsf{Bar}(\boldsymbol{\pi}) \,\mathrm{d}\tilde\mu - \frac{1}{q}\|\mathsf{Bar}(\boldsymbol{\pi})\|^q_{L^q(\tilde\mu)} \right) \\
&\le \frac{1}{p}\int |D\tilde f|^p_W \,\mathrm{d}\tilde\mu \quad \text{for every } \tilde f \in W^{1,p}(\mathbb{B},\tilde\mu).
\end{aligned}$$

This gives that $W^{1,p}(\mathbb{B},\tilde\mu) \subseteq H^{1,p}_{cyl}(\mathbb{B},\tilde\mu)$ and $\int |D\tilde f|^p_{H,cyl}\,\mathrm{d}\tilde\mu \le \int |D\tilde f|^p_W\,\mathrm{d}\tilde\mu$ for all $\tilde f \in W^{1,p}(\mathbb{B},\tilde\mu)$.

Now fix any function $f \in W^{1,p}(\mathbb{B},\mu)$ and define $\tilde f := \mathrm{ext}_{\tilde\mu}(f) \in L^p(\tilde\mu)$. The first part of the proof, Corollary 2.10, and Remark 2.9 ensure that $\tilde f \in W^{1,p}(\mathbb{B},\tilde\mu)$, as well as $f \in H^{1,p}_{cyl}(\mathbb{B},\mu)$ and

$$\int |Df|^p_{H,cyl}\,\mathrm{d}\mu \le \int |D\tilde f|^p_{H,cyl}\,\mathrm{d}\tilde\mu \le \int |D\tilde f|^p_W\,\mathrm{d}\tilde\mu = \int |Df|^p_W\,\mathrm{d}\mu.$$

Since $|Df|_W \le |Df|_{H,cyl}$ by (2.5), (2.7) and $H^{1,p}_{cyl}(\mathbb{B},\mu) \subseteq W^{1,p}(\mathbb{B},\mu)$, the statement follows. □

The equivalence result for arbitrary metric measure spaces easily follows:

Theorem 3.4. (Equivalence of metric Sobolev spaces) *Let* $(\mathrm{X},\mathsf{d},\mu)$ *be a metric measure space and* $p \in (1,\infty)$. *Then it holds that* $H^{1,p}(\mathrm{X},\mu) = W^{1,p}(\mathrm{X},\mu)$ *and*

$$|Df|_W = |Df|_H \quad \textit{for every } f \in W^{1,p}(\mathrm{X},\mu).$$

Proof. In view of (2.7), it suffices to show that $f \in H^{1,p}(\mathrm{X},\mu)$ and $\int |Df|^p_H\,\mathrm{d}\mu \le \int |Df|^p_W\,\mathrm{d}\mu$ for every fixed $f \in W^{1,p}(\mathrm{X},\mu)$. Fix a linear isometric embedding $\iota\colon \mathrm{X} \hookrightarrow \mathbb{B}$ into some separable Banach space $\mathbb{B}$ and call $\tilde\mu := \iota_\#\mu$. Define $\tilde\Omega_n := B_n(0) \subseteq \mathbb{B}$ and $\Omega_n := \iota^{-1}(\tilde\Omega_n)$ for every $n \in \mathbb{N}$. Both $\iota_n := \iota|_{\Omega_n}\colon (\mathrm{spt}(\mu_n),\mu_n) \to (\mathbb{B},\tilde\mu_n)$ and $\phi_n := (\iota|_{\mathrm{spt}(\mu_n)})^{-1}\colon (\mathrm{spt}(\tilde\mu_n),\tilde\mu_n) \to (\mathrm{X},\mu_n)$ are short maps (in the sense of Remark 2.9), where we set $\mu_n := \mu|_{\Omega_n}$ and $\tilde\mu_n := \tilde\mu|_{\tilde\Omega_n} = \iota_\#\mu_n$. Hence:

- $f_n := [f]_{\mu_n} \in W^{1,p}(\mathrm{X},\mu_n)$ and $\int |Df_n|^p_W\,\mathrm{d}\mu_n \le \int |Df|^p_W\,\mathrm{d}\mu$ thanks to Remark 2.9.
- $\tilde f_n := \phi_n^* f_n \in W^{1,p}(\mathbb{B},\tilde\mu_n)$ and $\int |D\tilde f_n|^p_W\,\mathrm{d}\tilde\mu_n \le \int |Df_n|^p_W\,\mathrm{d}\mu_n$ again by Remark 2.9.
- $\tilde f_n \in H^{1,p}(\mathbb{B},\tilde\mu_n)$ and $\int |D\tilde f_n|^p_H\,\mathrm{d}\tilde\mu_n = \int |D\tilde f_n|^p_W\,\mathrm{d}\tilde\mu_n$ by Theorem 3.3, (2.5), and (2.7).

- $f_n = \iota_n^* \tilde{f}_n \in H^{1,p}(\mathrm{X}, \mu_n)$ and $\int |Df_n|_H^p \, \mathrm{d}\mu_n \leq \int |D\tilde{f}_n|_H^p \, \mathrm{d}\tilde{\mu}_n$ thanks to the last part of Remark 2.9, which can be applied since ι_n is a restriction of the 1-Lipschitz map ι.

All in all, we proved that $f_n \in H^{1,p}(\mathrm{X}, \mu_n)$ and $\int |Df_n|_H^p \, \mathrm{d}\mu_n \leq \int |Df|_W^p \, \mathrm{d}\mu$ for all $n \in \mathbb{N}$. By Remark 2.11, we conclude that $f \in H^{1,p}(\mathrm{X}, \mu)$ and $\int |Df|_H^p \, \mathrm{d}\mu = \sup_n \int |Df_n|_H^p \, \mathrm{d}\mu_n \leq \int |Df|_W^p \, \mathrm{d}\mu$. □

To conclude, we briefly comment on the equivalence with other notions of metric Sobolev space. First, it follows from our results that Cheeger's original definition of Sobolev space in [6] (defined in terms of the relaxation of *upper gradients*) is equivalent to $H^{1,p}(\mathrm{X}, \mu) = W^{1,p}(\mathrm{X}, \mu)$. Furthermore:

Remark 3.5. (Equivalence with Newtonian–Sobolev spaces) Let $(\mathrm{X}, \mathsf{d}, \mu)$ be a metric measure space and $p \in (1, \infty)$. We denote by $N^{1,p}(\mathrm{X}, \mu)$ the *Newtonian–Sobolev space*, in the sense of [24]. It is known that $H^{1,p}(\mathrm{X}, \mu) \subseteq N^{1,p}(\mathrm{X}, \mu) \subseteq W^{1,p}(\mathrm{X}, \mu)$ and that $|Df|_W \leq |Df|_N \leq |Df|_H$ for every $f \in H^{1,p}(\mathrm{X}, \mu)$, where $|Df|_N$ denotes the minimal p-weak upper gradient in the sense of Newtonian–Sobolev spaces; see [15] for the first inclusion and [1,23] for the second. Consequently, Theorem 3.4 implies that $N^{1,p}(\mathrm{X}, \mu) = H^{1,p}(\mathrm{X}, \mu)$ and $|Df|_N = |Df|_H$ for every $f \in N^{1,p}(\mathrm{X}, \mu)$. □

Remark 3.6. (Equivalence with Sobolev spaces via test plans) In the paper [2], $W^{1,p}(\mathrm{X}, \mu)$ was defined in a different way, in terms of *test plans*. Nevertheless, it follows from the proof arguments of [1, Theorems 8.5 and 9.4] that the notion in [2] coincides with ours. We omit the details. □

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no Conflict of interest.

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References

[1] Ambrosio, L., Di Marino, S., Savaré, G.: On the duality between p-modulus and probability measures. J. Eur. Math. Soc. **17**(8), 1817–1853 (2015)
[2] Ambrosio, L., Gigli, N., Savaré, G.: Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces. Rev. Mat. Iberoam. **29**(3), 969–996 (2013)
[3] Ambrosio, L., Gigli, N., Savaré, G.: Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below. Invent. Math. **195**(2), 289–391 (2014)
[4] Ambrosio, L., Kirchheim, B.: Currents in metric spaces. Acta Math. **185**(1), 1–80 (2000)
[5] Bouchitté, G., Buttazzo, G., Seppecher, P.: Energies with respect to a measure and applications to low dimensional structures. Calc. Var. Partial Differ. Equ. **5**, 37–54 (1997)
[6] Cheeger, J.: Differentiability of Lipschitz functions on metric measure spaces. Geom. Funct. Anal. **9**(3), 428–517 (1999)
[7] Di Marino, S., Gigli, N., Pasqualetto, E., Soultanis, E.: Infinitesimal Hilbertianity of locally CAT(κ)-spaces. J. Geom. Anal. **31**(8), 7621–7685 (2021)
[8] Di Marino, S., Lučić, D., Pasqualetto, E.: A short proof of the infinitesimal Hilbertianity of the weighted Euclidean space. C. R. Math. Acad. Sci. Paris **358**(7), 817–825 (2020)
[9] Diestel, J., Uhl, Jr, J.J.: Vector Measures. American Mathematical Society, Providence, R.I. Mathematical Surveys, No. 15 (1977)
[10] Eriksson-Bique, S.: Density of Lipschitz functions in energy. Calc. Var. Partial Differ. Equ. **62**(2), (2023)
[11] Federer, H., Fleming, W.H.: Normal and integral currents. Ann. Math. **72**(3), 458–520 (1960)
[12] Fornasier, M., Savaré, G., Sodini, G.E.: Density of subalgebras of Lipschitz functions in metric Sobolev spaces and applications to Wasserstein Sobolev spaces. J. Funct. Anal. **285**(11), 110153 (2023)
[13] Gigli, N.: Nonsmooth differential geometry—an approach tailored for spaces with Ricci curvature bounded from below. Mem. Am. Math. Soc. **251**(1196):v+161 (2018)
[14] Gigli, N., Pasqualetto, E.: Behaviour of the reference measure on RCD spaces under charts. Commun. Anal. Geom. **29**(6), 1391–1414 (2021)
[15] Heinonen, J., Koskela, P., Shanmugalingam, N., Tyson, J.T.: Sobolev Spaces on Metric Measure Spaces. New Mathematical Monographs, vol. 27. Cambridge University Press, Cambridge (2015)
[16] Louet, J.: Some results on Sobolev spaces with respect to a measure and applications to a new transport problem. J. Math. Sci. (N.Y.) **196**, 152–164 (2014)
[17] Lučić, D., Pasqualetto, E., Rajala, T.: Characterisation of upper gradients on the weighted Euclidean space and applications. Ann. Mat. Pura Appl. **4**(200), 2473–2513 (2021)
[18] Paolini, E., Stepanov, E.: Decomposition of acyclic normal currents in a metric space. J. Funct. Anal. **263**(11), 3358–3390 (2012)
[19] Paolini, E., Stepanov, E.: Structure of metric cycles and normal one-dimensional currents. J. Funct. Anal. **264**(6), 1269–1295 (2013)
[20] Paolini, E., Stepanov, E.: Flows of measures generated by vector fields. Proc. R. Soc. Edinb. Sect. A **148**(4), 773–818 (2018)

[21] Pasqualetto, E., Rajala, T.: Vector calculus on weighted reflexive Banach spaces. Preprint (2023). arXiv:2306.03684
[22] Rockafellar, R.T.: Conjugate Duality and Optimization. Society for Industrial and Applied Mathematics, Philadelphia (1974)
[23] Savaré, G.: Sobolev spaces in extended metric-measure spaces. In: New Trends on Analysis and Geometry in Metric Spaces, pp. 117–276. Springer International Publishing, (2022)
[24] Shanmugalingam, N.: Newtonian spaces: an extension of Sobolev spaces to metric measure spaces. Rev. Mat. Iberoam. **16**(2), 243–279 (2000)
[25] Smirnov, S.K.: Decomposition of solenoidal vector charges into elementary solenoids, and the structure of normal one-dimensional flows. Algebra i Analiz **5**(4), 206–238 (1993)
[26] Sodini, G.E.: The general class of Wasserstein Sobolev spaces: density of cylinder functions, reflexivity, uniform convexity and Clarkson's inequalities. Calc. Var. Partial Differ. Equ. **62**(7), 1–41 (2023)
[27] Zhikov, V.V.: On an extension of the method of two-scale convergence and its applications. Sb. Math. **191**, 973–1014 (2000)