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Zeta function of some Kummer Calabi-Yau 3-folds

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Abstract. We compute Hodge numbers and zeta function of a Kummer Calabi-Yau 3-folds introduced by M. Andreatta and J. Wiśniewski in [2] and investigated by M. Donten-Bury in [13].

1. Introduction

The classical Kummer construction involves an abelian surface A over $\mathbb{C}$ and an involution $i: A \to A$. The quotient $X := A/i$ is a normal surface having 16 rational singularities of type A_1. The minimal resolution of singularities $\widetilde{X}$ of X is a $K3$ surface.

Various ideas have been used to give generalizations of the Kummer construction. In [18] the author describes the generalized Kummer manifold us an algebraic variety X for which there exists an abelian variety A and a generically surjective rational map $A \to X$. S. Cynk and K. Hulek in [10] imitated the classical Kummer construction in the sense that they considered pairs of varieties X, Y with involutions i_X and i_Y. Then the authors derived a criterion for when the quotient $(X \times Y)/(i_X \times i_Y)$ admits a crepant resolution of singularities giving a Calabi-Yau manifold. In the paper [12] Cynk and M. Schütt using Cynk-Hulek generalised Kummer construction and the Weil restrictions discussed resulting Calabi-Yau manifolds over number fields. For $n = 3$, the construction was exhibited in [4] in the context of Calabi-Yau threefold with complex multiplication.

In [2] M. Andreatta and J. Wiśniewski gave another generalization, $\mathrm{Kum}_n(A, G)$, of the classical Kummer construction by resolving singularities of the quotient of a complex abelian variety A of dimension n by a finite integral matrix group G in $\mathrm{GL}_n(\mathbb{Z})$. Incidentally, holomorphic symplectic manifolds introduced by A. Beauville in [3] are examples of manifolds $\mathrm{Kum}_n(A, G)$ for some group G. Andreatta and Wiśniewski gave also a new method to compute the cohomology of the constructed Kummer variety. In the case of Beauville's holomorphic symplectic manifolds the cohomology has been computed by using another idea by L. Göttsche and W. Soergel in [15], B. Fantechi and L. Göttsche in [14]. In dimension

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3, under some restrictions on the resolution of singularities and the group action, the resulting variety is a Calabi-Yau threefold.

M. Donten-Bury in [13] investigated the Andreatta-Wiśniewski construction in the case of $A = E^3$ – a triple product of an elliptic curve E, together with an action of a finite subgroup of $\mathrm{SL}_3(\mathbb{Z})$ on E^3. The complete classification of finite subgroups of $\mathrm{SL}_3(\mathbb{Z})$ was given in [17], Donten-Bury classified these subgroups using different idea. Following Andreatta-Wiśniewski paper, the author gave detailed computations of the Poincaré polynomial of a crepant resolution $\mathrm{Kum}_3(E, G)$ of the quotient E^3/G.

The aim of this work is to give a formula for the Hodge numbers and local zeta function of $\mathrm{Kum}_3(E, G)$ using the Chen-Ruan orbifold cohomology theory [11] and the description of the Frobenius action on the orbifold cohomology [16]. The work is motivated by the limited range of examples of Calabi-Yau manifolds which are neither complete intersection in projective spaces nor hypersurfaces in toric space with explicit computed zeta functions.

Our main result is:

Theorem. *Hodge numbers and zeta functions of* $\mathrm{Kum}_3(E, G)$ *are given in tables from section 5.*

In [6], [7], [8] we successfully used that approach in order to compute Hodge numbers and zeta function of manifolds $(X_1 \times X_2 \times \ldots \times X_k)/G$, where X_i $(i = 1, 2, \ldots, k)$ are manifolds of a Calabi-Yau type (i.e. $K_{X_i} = \mathcal{O}_{X_i}$) and G is a finite group acting on a product variety $X_1 \times X_2 \times \ldots \times X_k$. All computed examples covered: the famous Borcea-Voisin Calabi-Yau threefolds ([5], [19]), n-dimensional Calabi-Yau manifold of a Borcea-Voisin type (see [9] for $n = 3$, and [8] for an arbitrary dimension n).

In Sects. 2 and 3 we briefly discuss Chen-Ruan cohomology theory and orbifold zeta function. In section 4 we give detailed computations of Hodge numbers and zeta functions of $\mathrm{Kum}_3(E, G)$, for some group $G \in \mathrm{SL}_3(\mathbb{Z})$ of order 24. In the Sect. 5 we give results of computations for all remaining groups.

2. Chen-Ruan cohomology

In [11] W. Chen and Y. Ruan introduced a new cohomology theory for an orbifold.

Definition 2.1. Let X be a projective variety and G be a finite group which acts on X. For a variety X/G define the *Chen-Ruan cohomology* by

$$H^{i,j}_{\mathrm{orb}}(X/G) := \bigoplus_{[g]\in \mathrm{Conj}(G)} \left(\bigoplus_{U\in\Lambda(g)} H^{i-\mathrm{age}(g),\ j-\mathrm{age}(g)}(U) \right)^{\mathrm{C}(g)}, \qquad (2.1)$$

where $\mathrm{Conj}(G)$ is the set of conjugacy classes of G (we choose a representative g of each conjugacy class), $\mathrm{C}(g)$ is the centralizer of g, $\Lambda(g)$ denotes the set of irreducible connected components of the set fixed by $g \in G$ and $\mathrm{age}(g)$ is the age of the matrix of linearised action of g near a point of U.

The dimension of $H^{i,j}_{\mathrm{orb}}(X/G)$ will be denoted by $h^{i,j}_{\mathrm{orb}}(X/G)$.

An important feature of the Chen-Ruan cohomology is the possibility of computing Hodge numbers of a crepant resolution of singularities of a quotient variety, without referring to an explicit construction of such a resolution i.e.

Theorem 2.2. *([20], Cor. 3.16) Let X and X' be complete varieties with Gorenstein quotient singularities. Suppose that there are proper birational morphisms $Z \to X$ and $Z \to X'$ such that $K_{Z/X} = K_{Z/X'}$. Then the orbifold cohomology groups of X and X' have the same Hodge structure.*

As a special case we have:

Theorem 2.3. *Let G be a finite group acting on an algebraic smooth variety X. If there exists a crepant resolution $\widetilde{X/G}$ of variety X/G, then the following equality holds*

$$h^{i,j}(\widetilde{X/G}) = h^{i,j}_{\mathrm{orb}}(X/G).$$

For a systematic exposition of the orbifold Chen-Ruan cohomology see [1].

3. The orbifold zeta function

Let q be a prime power. For a smooth, tame, Deligne-Mumford $\mathbb{F}_q$-stack $\mathcal{X}$ Rose in [16] defined *ℓ-adic Chen-Ruan orbifold cohomology* of $\mathcal{X} \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ denoted by $H^*_{\mathrm{CR}}\left(\mathcal{X} \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathbb{Q}_l\right)$ (Def. 3.1), introduced the *orbifold Frobenius morphisms* $\mathrm{Frob}_{\mathrm{orb}}$ on $H^*_{\mathrm{CR}}\left(\mathcal{X} \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathbb{Q}_l\right)$ (Prop. 1.1) and defined (Def. 6.1) the *orbifold zeta function* by

$$Z_{\mathrm{CR}}(\mathcal{X}, t) := \det\left(1 - \mathrm{Frob}_{\mathrm{orb}}\, t \mid H^*_{\mathrm{CR}}\left(\mathcal{X} \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathbb{Q}_l\right)\right).$$

It turns out that the orbifold zeta function of a crepant desingularization coincide with the usual zeta function i.e.

Theorem 3.1. *([16], Cor. 6.4) Let $\mathcal{X}$ be a proper, smooth, tame Deligne-Mumford stack satisfying the hard Lefschetz condition with trivial generic stabilizer. Suppose $\widetilde{X} \to X$ is a crepant resolution of the coarse moduli scheme X of $\mathcal{X}$, then*

$$Z_{H^*_{CR}}(\mathcal{X}, t) = Z_q(\widetilde{X}, t).$$

where $Z_q(\widetilde{X}, t)$ denotes the classical zeta function X.

As a special case we have:

Theorem 3.2. *Let G be a finite group acting on an algebraic smooth variety X. If there exists a crepant resolution $\widetilde{X/G}$ of variety X/G, then the following equality holds*

$$Z_q(\widetilde{X/G}, t) = Z_{H^*_{CR}}(X/G, t).$$

4. Quotients of E^3 by a finite subgroup of $\mathrm{SL}_3(\mathbb{Z})$

Let E be an elliptic curve, and let G be a finite subgroup of $\mathrm{SL}_3(\mathbb{Z})$. The action of G on E^3 is regarded as the action of G on the endomorphism ring of E^3. The quotient E^3/G admits a crepant resolution of singularities in the sense of Andreatta-Wiśniewski ([2]) denoted by $\mathrm{Kum}_3(E, G)$. If $H^{1,0}(E^3)^G = 0$, then $\mathrm{Kum}_3(E, G)$ is a Calabi-Yau threefold. The Calabi-Yau manifold $\mathrm{Kum}_3(E, G)$ is defined over any subfield of $\mathbb{C}$ containing the j-invariant of E. Note that all finite subgroups $\mathrm{SL}_3(\mathbb{Z})$ were classified in [17] as well as in [13] by using different idea.

Denote by τ, η a 2-torsion and 3-torsion point of E, respectively.

4.1. Hodge numbers

In this section we compute Hodge numbers of the variety $\mathrm{Kum}_3(E, G_{24.1})$, where

$$G_{24.1}: \quad \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle \simeq S_4$$

is a finite subgroup of $\mathrm{SL}_3(\mathbb{Z})$ of order 24 denoted by $G_{24.1}$. In the table below w collect all necessary data needed for computation of the Hodge numbers by using Chen-Ruan cohomology 2.1 i.e the fixed loci and the age of representative classes of conjugacy classes of $G_{24.1}$:

$$\mathrm{Conj}(G_{24.1}) = \{[g_0], [g_1], [g_2], [g_3], [g_4]\}$$

$g_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\mathrm{Fix}(g_0) = E^3$ $\mathrm{age}(g_0) = 0$ $C(g_0) = G_{24.1}$
$g_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$	$C(g_1) = \left\langle 1, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ $\mathrm{Fix}(g_1) = \bigcup_{\tau \in E[2]} \{(\tau, -x, x)\}_{x \in E} \simeq \bigcup_{\tau \in E[2]} E$ – four copies of E $\mathrm{age}(g_1) = 1$
$g_2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$	$C(g_2) = \langle g_2 \rangle \simeq \mathbb{Z}_3$ $\mathrm{Fix}(g_2) = \{(x, -x, x)\}_{x \in E} \simeq E$ – one copy of E $\mathrm{age}(g_2) = 1$
$g_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$C(g_3) = \langle g_3 \rangle \simeq \mathbb{Z}_4$ $\mathrm{Fix}(g_3) = \{(\tau, \tau, x)\}_{\tau \in E[2],\, x \in E} \simeq \bigcup_{\tau \in E[2]} E$ – four copies of E $\mathrm{age}(g_3) = 1$

$g_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$C(g_4) = \left\langle g_4, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \simeq D_8$ $\mathrm{Fix}(g_4) = \bigcup_{\tau_1 \in E[2]} \bigcup_{\tau_2 \in E[2]} \{(\tau_1, \tau_2, x)\}_{x \in E} \simeq \bigcup_{\tau_1 \in E[2]} \bigcup_{\tau_2 \in E[2]} E$ – sixteen copies of E $\mathrm{age}(g_4) = 1$

The fixed locus of the action of $g_0 = \mathrm{id}$. As $\mathrm{Fix}(g_0) = E^3$ we have to compute $H^{1,0}(E^3)^{G_{24.1}}$, $H^{1,1}(E^3)^{G_{24.1}}$, $H^{1,2}(E^3)^{G_{24.1}}$. Denote by $\mathrm{d}z_i$, $\mathrm{d}\bar{z_i}$ pullbacks of invariant form and its complex conjugation by the projection $E^3 \to E$ on i-th component. Then $\{\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}z_3\}$ is a basis of the vector space $H^{1,0}(E^3)$. The action of generators of $G_{24.1}$ on $H^{1,0}(E^3)$ is given by matrices

$$M := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Consequently the vector space $H^{1,0}(E^3)^{G_{24.1}}$ is given by the kernel of the 6×3 matrix

$$\left(\frac{M - \mathbb{1}_3}{N - \mathbb{1}_3} \right).$$

In this case

$$H^{1,0}(E^3)^{G_{24.1}} = \ker \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = (0).$$

Similarly the vector space $H^{1,1}(E^3)^{G_{24.1}}$ has basis $\mathrm{d}z_i \wedge \mathrm{d}\bar{z_j}$, hence the space $H^{1,1}(E^3)^{G_{24.1}}$ is the kernel of the 18×9 matrix

$$\left(\frac{M \otimes \bar{M} - \mathbb{1}_9}{N \otimes \bar{N} - \mathbb{1}_9} \right),$$

where $\otimes$ denotes the Kronecker product of matrices. In our case $H^{1,1}(E^3)^{G_{24.1}} \simeq \mathbb{C}$.

Finally $\mathrm{d}z_i \wedge \mathrm{d}\bar{z_j} \wedge \mathrm{d}\bar{z_k}$ forms a basis of $H^{1,2}(E^3)$, hence $H^{1,2}(E^3)^{G_{24.1}}$ is given by the kernel of the matrix

$$\left(\frac{M \otimes \bigwedge^2 \bar{M} - \mathbb{1}_9}{N \otimes \bigwedge^2 \bar{N} - \mathbb{1}_9} \right).$$

Consequently $H^{1,2}(E^3)^{G_{24.1}} \simeq \mathbb{C}$.

The fixed locus of the action of g_1. Since

$$\mathrm{Fix}(g_1) = \bigcup_{\tau\in E[2]} \{(\tau, -x, x)\}_{x\in E} \simeq \bigcup_{i\in\{1,2,3,4\}} E^{(i)} - \text{four copies of } E$$

and $C(g_1) = \langle g_1, g\rangle$ where

$$g := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

it follows that

- $g(E^{(i)}) = E^{(i)}$,
- g acts on $H^{0,0}(\mathrm{Fix}(g_1))$ as $\mathrm{diag}(1, 1, 1, 1)$,
- g acts on $H^{1,0}(\mathrm{Fix}(g_1))$ as $\mathrm{diag}(-1, -1, -1, -1)$.

Consequently

$$H^{0,0}(\mathrm{Fix}(g_1))^{C(g_1)} \simeq \mathbb{C}^4 \quad \text{and} \quad H^{1,0}(\mathrm{Fix}(g_1))^{C(g_1)} \simeq (0).$$

One can also see that

$$\mathrm{Fix}(g_1)/C(g_1) \simeq \text{four copies of } \mathbb{P}^1.$$

The fixed locus of the action of g_2. In that case

$$\mathrm{Fix}(g_2) = \{(x, -x, x)\}_{x\in E} \simeq E \quad \text{and} \quad C(g_2) = \langle g_2\rangle .$$

Therefore $\mathrm{Fix}(g_2)/C(g_2) \simeq E$, so

$$H^{0,0}(\mathrm{Fix}(g_2))^{C(g_2)} \simeq \mathbb{C} \quad \text{and} \quad H^{1,0}(\mathrm{Fix}(g_2))^{C(g_2)} \simeq \mathbb{C}.$$

The fixed locus of the action of g_3. In that case

$$\mathrm{Fix}(g_3) = \{(\tau, \tau, x)\}_{\tau\in E[2],\, x\in E} \simeq \bigcup_{i\in\{1,2,3,4\}} E^{(i)} - \text{four copies of } E$$

and $C(g_3) = \langle g_3\rangle$. Therefore

$$H^{0,0}(\mathrm{Fix}(g_3))^{C(g_3)} \simeq \mathbb{C}^4 \quad \text{and} \quad H^{1,0}(\mathrm{Fix}(g_3))^{C(g_3)} \simeq \mathbb{C}^4.$$

The fixed locus of the action of g_4. In that case

$$\mathrm{Fix}(g_4) = \bigcup_{\tau_1\in E[2]} \bigcup_{\tau_2\in E[2]} \{(\tau_1, \tau_2, x)\}_{x\in E} \simeq \bigcup_{i,j\in\{1,2,3,4\}} E^{(i,j)} - \text{sixteen copies of } E.$$

The action of

$$C(g_4) = \left\langle g_4, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\rangle \simeq D_8$$

on $H^{0,0}(\mathrm{Fix}(g_4))$ has the following matrix

$$(\mathbb{1})^4 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^6$$

while the action of $C(g_4)$ on $H^{1,0}(\mathrm{Fix}(g_4))$ has the matrix $-\mathbb{1}_{16}$. Therefore

$$H^{0,0}(\mathrm{Fix}(g_4))^{C(g_3)} \simeq \mathbb{C}^{10} \quad \text{and} \quad H^{1,0}(\mathrm{Fix}(g_4))^{C(g_3)} \simeq (0).$$

In total, the Calabi-Yau manifold X has the following Hodge numbers:

$$h^{1,1}(\mathrm{Kum}_3(E, G_{24.1})) = 1 + 4 + 1 + 4 + 10 = 20 \quad \text{and}$$
$$h^{2,1}(\mathrm{Kum}_3(E, G_{24.1})) = 1 + 0 + 1 + 4 + 0 = 6,$$

hence the Poincaré polynomial of $\mathrm{Kum}_3(E, G_{24.1})$ equals

$$t^6 + 20t^4 + 14t^3 + 20t^2 + 1,$$

which agrees with the result of [13].

4.2. *Zeta function*

Let q be a power of a prime $p \neq 2$ such that the curve E admits a good reduction E_q over $\mathbb{F}_q$. In the case of groups of order 6, 12, 24 we also assume that $p \neq 3$. These conditions imply that the Calabi-Yau 3-fold $X = \mathrm{Kum}_3(E_q, G)$ has a good reduction X_q over $\mathbb{F}_q$.

The zeta function of E_q is equal to

$$\frac{1 - a_q T + qT^2}{(1-T)(1-qT)},$$

where

$$a_q = q + 1 - \#E_q(\mathbb{F}_q) \quad \text{and} \quad |a_q| \leq 2\sqrt{q}.$$

Moreover trinomial $1 - a_q T + qT^2$ has two roots α_q and $\bar{\alpha_q}$ which are algebraic integers such that $|\alpha_q| = |\bar{\alpha_q}| = q^{\frac{3}{2}}$.

By [16] the orbifold zeta function is given by the product of zeta functions associated to fixed loci of actions of elements in $G_{24.1}$. The orbifold zeta function associated to $H^*(E_q^3)^{G_{24.1}}$ is a factor of zeta function of E_q^3 :

$$\frac{\left(1 - a_q T + qT^2\right)^3 \left(1 - (a_q^3 - 3a_q q)T + q^3T^2\right) \left(1 - a_q qT + q^3T^2\right)^9 \left(1 - a_q q^2 T + q^5 T^2\right)^3}{(1-T)\,(1-qT)^9 \left(1 - (a_q^2 - 2q)T + q^2T^2\right)^3 \left(1 - (a_q^2 q - 2q^2)T + q^4T^2\right)^3 \left(1 - q^2T\right)^9 \left(1 - q^3T\right)}.$$

The fixed locus of the action of $g_0 = \mathrm{id}$. The contribution to the zeta function coming from $H^0(X_q)$ and $H^6(X_q)$ obviously equals

$$\frac{1}{1-T} \quad \text{and} \quad \frac{1}{1-q^3T},$$

respectively.

By the orbifold formula (cf. Thm. 2.3) the Kummer 3-fold $\mathrm{Kum}_3(E, G)$ is Calabi-Yau exactly for 16 non-cyclic groups from [17] (see Table 2 in [13] for a complete list of subgroups and corresponding Poincaré polynomials).

Lemma 4.1. *Let E be an elliptic curve with a good reduction E_q over $\mathbb{F}_q$. Then for any group G (such that $\mathrm{Kum}_3(E, G)$ is a Calabi-Yau 3-fold) the group $H^2(E_q^3)^G$ is generated by divisors defined over $\mathbb{F}_q$.*

Consequently, the orbifold Frobenius polynomials for $H^2(X_q)$ and $H^4(X_q)$ equal

$$\frac{1}{(1-qT)^{\dim H^{1,1}(E^3)^G}} \quad \text{and} \quad \frac{1}{(1-q^2T)^{\dim H^{1,1}(E^3)^G}},$$

respectively.

Proof. The vector space $H^{1,1}(E_q^3)^G$ is generated by the classes of the following invariant divisors:

- $G_{24.1}$: $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_3 = 0\}$
- $G_{24.2}$: $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 + Z_2 + Z_3 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 + Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 + Z_3 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_3 = 0\}$
- $G_{24.3}$: $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 + Z_2 + Z_3 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 + Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 + Z_3 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_3 = 0\}$
- $G_{12.1}$: $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 + Z_3 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_3 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 = 0\}$

- $G_{12.2}$: $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_3 = 0\}$

- $G_{12.3}$: $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 - Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 - Z_3 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_3 - Z_1 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_3 = 0\}$

- $G_{12.4}$: $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 + Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 + Z_3 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 + Z_3 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 + Z_2 + Z_3 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_3 = 0\}$

- $G_{8.1}$: $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_3 = 0\}$

- $G_{8.2}$: $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 + Z_2 + Z_3 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 + Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 + Z_3 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_3 = 0\}$

- $G_{6.1}$: $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_3 - Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_3 = 0\}$

- $G_{6.2}$: $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_3 - Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_3 = 0\}$

- $G_{6.3}$: $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 + Z_2 + Z_3 = 0\}$

- $G_{4.1}$: $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_3 = 0\}$

- $G_{4.2}$: $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 + Z_3 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_3 = 0\}$

- $G_{4.3}$: $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_3 = 0\}$

- $G_{4.4}$: $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 - Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_1 + Z_3 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 - Z_3 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_2 = 0\}$,
  $\{(Z_1, Z_2, Z_3) \in E_q^3 \colon Z_3 = 0\}$

$\square$

Lemma 4.2. *The orbifold Frobenius polynomial for* $H^3_{ét}(E_q^3, \mathbb{Q}_\ell)^G$ *equals*

$$\left(1 - (a_q^3 - 3a_q q)T + q^3T^2\right)\left(1 - a_q qT + q^3T^2\right)^{\dim H^{2,1}(E^3,\mathbb{C})^G}.$$

Proof. The lemma is obvious when $a_q = 0$, so we can consider only ordinary case. Let $\omega, \eta \in H^1_{\mathrm{DR}}(E_q)$ such that

$$\mathrm{Frob}_q^*\,\omega = \alpha_q\omega \quad \text{and} \quad \mathrm{Frob}_q^*\,\eta = \bar{\alpha_q}\eta.$$

Denote by $\mathrm{d}z_i = \pi_i^*(\mathrm{d}z)$, $\omega_i = \pi_i^*(\omega)$, $\eta_i = \pi_i^*(\eta)$ where $\pi_i \colon E^3 \to E$ is projection on i-th coordinate ($i = 1, 2, 3$). As ω and η are linearly independent it follows

$$\mathrm{d}z = k\omega + \ell\eta \quad \text{and} \quad \mathrm{d}\bar{z} = m\omega + n\eta,$$

for some $k, \ell, m, n \in \mathbb{C}$. Now

$$\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3 = k^3\omega_1 \wedge \omega_2 \wedge \omega_3 + \ell^3\eta_1 \wedge \eta_2 \wedge \eta_3 + k^2\ell\xi + k\ell^2\psi,$$

where

$$\mathrm{Frob}_q^*(\xi) = q\alpha_q\xi, \qquad \mathrm{Frob}_q^*(\omega_1 \wedge \omega_2 \wedge \omega_3) = \alpha_q^3\omega_1 \wedge \omega_2 \wedge \omega_3$$

and

$$\mathrm{Frob}_q^*(\psi) = q\bar{\alpha_q}\psi, \qquad \mathrm{Frob}_q^*(\eta_1 \wedge \eta_2 \wedge \eta_3) = \bar{\alpha_q}^3\eta_1 \wedge \eta_2 \wedge \eta_3.$$

If all eigenvalues of Frob_q^* are equal $q\alpha_q$ and $q\bar{\alpha_q}$, then $k^3 = \ell^3 = 0$ as we consider the case $a_q \neq 0$. Consequently $\mathrm{d}z = 0$ which is impossible in the ordinary case.

□

From the above lemmas we immediately get the following corollary

Corollary 4.3. *The orbifold zeta function for* $H^3_{\acute{e}t}(E^3_q, \mathbb{Q}_\ell)^G$ *equals*

$$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^{\dim H^{2,1}(E^3,\mathbb{C})^G}}{(1-T)(1-qT)^{\dim H^{1,1}(E^3,\mathbb{C})^G}(1-q^2T)^{\dim H^{1,1}(E^3,\mathbb{C})^G}(1-q^3T)}.$$

For our main case (24.1) we have computed $\dim H^{1,1}(E^3,\mathbb{C})^{G_{24.1}} = \dim H^{2,1}(E^3,\mathbb{C})^{G_{24.1}} = 1$, so the orbifold zeta function $H^3_{\acute{e}t}(E^3_q, \mathbb{Q}_\ell)^G$ equals

$$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^{\dim H^{2,1}(E^3,\mathbb{C})^G}}{(1-T)(1-qT)^{\dim H^{1,1}(E^3,\mathbb{C})^G}(1-q^2T)^{\dim H^{1,1}(E^3,\mathbb{C})^G}(1-q^3T)}. \tag{4.1}$$

The fixed locus of the action of $g_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$

Since $\mathrm{Fix}(g_1)/C(g_1) \simeq$ four copies of indexed by 2-torsion points $E[2]$ the zeta function of $H^*(\mathrm{Fix}(g_1)/C(g_1))$ depends on the behaviour of torsion subgroups $E[2]$ under the reduction modulo q. Let $E_q[2] = \{a, b, c, d\}$ and denote by $P_{(a_1,a_2,\ldots,a_k)}$ the characteristic polynomial of the following matrix:

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}}_{a_1} \oplus \underbrace{\begin{pmatrix} 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}}_{a_2} \oplus \cdots \oplus \underbrace{\begin{pmatrix} 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}}_{a_k}$$

We have the following three cases:

- $(1, 1, 1, 1)$ All points a, b, c, d are defined over $\mathbb{F}_q$. Then the zeta function is equal to

$$P_{(1,1,1,1)}(T) = (1-T)^4.$$

- $(1, 1, 2)$ Points $a, b, c+d$ are defined over $\mathbb{F}_q$ but c, d are not. The action of the Frobenius has the following linearization

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and therefore the zeta function equals

$$P_{(1,1,2)}(T) = (1-T)^3(1+T).$$

- (1, 3) a is defined over $\mathbb{F}_q$ and b, c, d not. The action of the Frobenius has the following linearization

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and therefore the zeta function equals

$$P_{(1,3)}(T) = (1-T)^2(1+T+T^2).$$

From section 3 it follows that in the case (λ) we have the following formula

$$\begin{aligned} Z_q\left(H^*(\mathrm{Fix}(g_1)/C(g_1))\right) &= Z_q(H^*(\mathbb{P}^1))(qT) \otimes P_{(\lambda)}(T) \\ &= \frac{1}{\left((1-qT)(1-q^2T)\right) \otimes P_{(\lambda)}(T)}, \end{aligned} \tag{4.2}$$

for $(\lambda) \in \{(1,1,1,1), (1,1,2), (1,3)\}$.

The fixed locus of the action of $g_2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$

Since $\mathrm{Fix}(g_2)/C(g_2) \simeq E$ it follows that

$$Z_q\left(H^*(\mathrm{Fix}(g_2)/C(g_2))\right) = Z_q(H^*(E))(qT) = \frac{1 - a_q qT + q^3T^2}{(1-qT)(1-q^2T)}. \tag{4.3}$$

The fixed locus of the action of $g_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Since $C(g_3)$ is generated by g_3 then

$$\begin{aligned} Z_q\left(H^*(\mathrm{Fix}(g_3)/C(g_3))\right) &= Z_q(H^*(E))(qT) \otimes P_{(\lambda)}(T) \\ &= \frac{1 - a_q qT + q^3T^2}{(1-qT)(1-q^2T)} \otimes P_{(\lambda)}(T). \end{aligned} \tag{4.4}$$

The fixed locus of the action of $g_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

In the case of (1, 1, 1, 1) the action has the matrix $\mathbb{1}_{10}$ with characteristic polynomial equal to

$$P_{(1,1,1,1,1,1,1,1,1,1)}(T) = (1 - T)^{10}.$$

For (1,1,2) the Frobenius morphism has the following linearization:

$$(\mathbb{1})^4 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^3$$

with characteristic polynomial equal to

$$P_{(1,1,1,1,2,2,2)}(T) = (1 - T)^4(1 - T^2)^3.$$

Finally, for (1,3) the Frobenius morphism has the following linearization:

$$(\mathbb{1}^4 \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^2$$

with characteristic polynomial

$$P_{(1,1,1,1,3,3)}(T) = (1 - T)^6(1 + T + T^2)^2.$$

Therefore in the case (η) we get

$$\begin{aligned} Z_q\left(H^*(\mathrm{Fix}(g_4)/C(g_4))\right) &= Z_q(H^*(\mathbb{P}^1))(qT) \otimes P_{(\eta)}(T) \\ &= \frac{1}{\left((1 - qT)(1 - q^2T)\right) \otimes P_{(\eta)}(T)}, \end{aligned} \tag{4.5}$$

for $\eta \in \{(1, 1, 1, 1, 1, 1, 1, 1, 1, 1), (1, 1, 1, 1, 2, 2, 2), (1, 1, 1, 1, 3, 3)\}$.

Multiplying rational functions (4.1), (4.2), (4.3), (4.4), (4.5) we obtain the following

Theorem 4.4. *The zeta function of* $\mathrm{Kum}_3(E, G_{24.1})$ *equals*

Case:	$Z_q\ (\mathrm{Kum}_3(E, G_{24.1}))$
(1, 1, 1, 1)	$\dfrac{\left(1 - (a_q^3 - 3a_q q)T + q^3T^2\right)\left(1 - a_q qT + q^3T^2\right)^6}{(1 - T)(1 - qT)^{20}(1 - q^2T)^{20}(1 - q^3T)}$
(1, 1, 2)	$\dfrac{\left(1 - (a_q^3 - 3a_q q)T + q^3T^2\right)\left(1 - a_q qT + q^3T^2\right)^5\left(1 + a_q qT + q^3T^2\right)}{(1 - T)(1 - qT)^{12}(1 + qT)^5(1 - a_q qT + q^3T^2)^3(1 + q^2T)^5(1 - q^2T)^{12}(1 - q^3T)}$
(1, 3)	$\dfrac{\left(1 - (a_q^3 - 3a_q q)T + q^3T^2\right)\left(1 - a_q qT + q^3T^2\right)^4\left(1 + a_q qT - (q^3 - a_q^2q^2)T^2 + a_q q^4T^3 + q^6T^4\right)}{(1 - T)(1 - qT)^{12}(1 + qT + q^2T^2)^4(1 + q^2T + q^4T^2)^4(1 - q^2T)^{12}(1 - q^3T)}$

5. Examples

In the computation of zeta function we excluded powers of 2 and 3 because 2 is a prime of bad reduction of all considered Kummer manifolds while 3 is a prime of bad reduction of $\mathrm{Kum}_3(E, G)$ where G is a group of order 6, 12, 24.

In this section we collect result of computations for all remaining groups. The Frobenius morphism acts on $E[2]$ and $E[3]$ by a permutation and the shape of the zeta function depends on the cycle type of this permutation. In particular, if all elements in $E[2]$ and $E[3]$ are $\mathbb{F}_q$ rational then this action is trivial and hence the zeta function of X_q is given by the default (simplest) formula corresponding to the first row in the table.

5.1. Order 24

$$G_{24.2}\colon \left\langle \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle \simeq S_4$$

Hodge numbers:

$$h^{1,1}(\mathrm{Kum}_3(E, G_{24.2})) = 11 \quad \text{and} \quad h^{2,1}(\mathrm{Kum}_3(E, G_{24.2})) = 3$$

Zeta function:

Case:	$Z_q\ (\mathrm{Kum}_3(E, G_{24.2}))$
(1, 1, 1, 1)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^{11}(1-q^2T)^{11}(1-q^3T)}$
(1, 1, 2)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^{10}(1+qT)(1+q^2T)(1-q^2T)^{10}(1-q^3T)}$
(1, 3)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^7(1+qT+q^2T^2)^2(1+q^2T+q^4T^2)^2(1-q^2T)^7(1-q^3T)}$

$$G_{24.3}\colon \left\langle \begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle \simeq S_4$$

Hodge numbers:

$$h^{1,1}(\mathrm{Kum}_3(E, G_{24.3})) = 11 \quad \text{and} \quad h^{2,1}(\mathrm{Kum}_3(E, G_{24.3})) = 3$$

Zeta function:

Case:	$Z_q\ (\mathrm{Kum}_3(E, G_{24.3}))$
(1, 1, 1, 1)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^{11}(1-q^2T)^{11}(1-q^3T)}$
(1, 1, 2)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^{10}(1+qT)(1+q^2T)(1-q^2T)^{10}(1-q^3T)}$
(1, 3)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^7(1+qT+q^2T^2)^2(1+q^2T+q^4T^2)^2(1-q^2T)^7(1-q^3T)}$

5.2. Order 12

$$G_{12.1}\colon\ \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle \simeq D_6$$

Hodge numbers:

$$h^{1,1}(\mathrm{Kum}_3(E, G_{12.1})) = 21 \quad \text{and} \quad h^{2,1}(\mathrm{Kum}_3(E, G_{12.1})) = 9$$

Zeta function: We get 15 possibilities for the zeta function of $\mathrm{Kum}_3(E, G_{12.1})$:

$$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^4}{(1-T)(1-qT)^4\left(((1-qT)(1-q^2T))\otimes P_{(\eta)}\right)^3(1-q^2T)^4(1-q^3T)} \cdot \left(\left(\frac{1-a_qqT+q^3T^2}{(1-qT)(1-q^2T)}\right)\otimes P_{(\lambda)}\right),$$

where

$$((\lambda), (\eta)) \in \{(1,1,1,1,1), (1,1,1,2), (1,2,2), (1,1,3), (1,4)\} \times\{(1,1,1,1), (1,1,2), (1,3)\}$$

Case:	$Z_q\,(\mathrm{Kum}_3(E, G_{12.1}))$
(1, 1, 1, 1, 1) (1, 1, 1, 1)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^9}{(1-T)(1-qT)^{21}(1-q^2T)^{21}(1-q^3T)}$
(1, 1, 1, 1, 1) (1, 1, 2)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^9}{(1-T)(1-qT)^{18}(1+qT)^3(1+q^2T)^3(1-q^2T)^{18}(1-q^3T)}$
(1, 1, 1, 1, 1)(1, 3)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^9}{(1-T)(1-qT)^{15}(1+qT+q^2T^2)^3(1+q^2T+q^4T^2)^3(1-q^2T)^{15}(1-q^3T)}$
(1, 1, 1, 2)(1, 1, 1, 1)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1+a_qqT+q^3T^2\right)^5\left(1-a_qqT+q^3T^2\right)^4}{(1-T)(1-qT)^{20}(1+qT)(1+q^2T)(1-q^2T)^{20}(1-q^3T)}$
(1, 1, 1, 2)(1, 1, 2)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1+a_qqT+q^3T^2\right)^5\left(1-a_qqT+q^3T^2\right)^4}{(1-T)(1-qT)^{17}(1+qT)^4(1+q^2T)^4(1-q^2T)^{17}(1-q^3T)}$
(1, 1, 1, 2)(1, 3)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1+a_qqT+q^3T^2\right)^5\left(1-a_qqT+q^3T^2\right)^4}{(1-T)(1-qT)^{14}(1+qT)(1+qT+q^2T^2)^3(1+q^2T+q^4T^2)^3(1+q^2T)(1-q^2T)^{14}(1-q^3T)}$
(1, 2, 2)(1, 1, 1, 1)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1+a_qqT+q^3T^2\right)^7\left(1-a_qqT+q^3T^2\right)^2}{(1-T)(1-qT)^{19}(1+qT)^2(1+q^2T)^2(1-q^2T)^{19}(1-q^3T)}$
(1, 2, 2)(1, 1, 2)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1+a_qqT+q^3T^2\right)^7\left(1-a_qqT+q^3T^2\right)^2}{(1-T)(1-qT)^{16}(1+qT)^5(1+q^2T)^5(1-q^2T)^{16}(1-q^3T)}$
(1, 2, 2)(1, 3)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1+a_qqT+q^3T^2\right)^7\left(1-a_qqT+q^3T^2\right)^2}{(1-T)(1-qT)^{13}(1+qT)^2(1+qT+q^2T^2)^3(1+q^2T+q^4T^2)^3(1+q^2T)^2(1-q^2T)^{13}(1-q^3T)}$

(1, 1, 3)(1, 1, 1, 1)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^7\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)}{(1-T)(1-qT)^{19}(1+qT+q^2T^2)(1+q^2T+q^4T^2)(1-q^2T)^{19}(1-q^3T)}$
(1, 1, 3)(1, 1, 2)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^7\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)}{(1-T)(1-qT)^{16}(1+qT)^3(1+qT+q^2T^2)(1+q^2T+q^4T^2)(1+q^2T)^3(1-q^2T)^{16}(1-q^3T)}$
(1, 1, 3)(1, 3)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^7\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)}{(1-T)(1-qT)^{13}(1+qT+q^2T^2)^4(1+q^2T+q^4T^2)^4(1-q^2T)^{13}(1-q^3T)}$
(1, 4)(1, 1, 1, 1)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^6\left(1+a_qqT+q^3T^2\right)\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)}{(1-T)(1-qT)^{18}(1+qT)(1+q^2T^2)(1+q^4T^2)(1+q^2T)(1-q^2T)^{18}(1-q^3T)}$
(1, 1, 3)(1, 4)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^6\left(1+a_qqT+q^3T^2\right)\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)}{(1-T)(1-qT)^{15}(1+qT)^4(1+q^2T^2)(1+q^4T^2)(1+q^2T)^4(1-q^2T)^{15}(1-q^3T)}$
(1, 4)(1, 3)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^6\left(1+a_qqT+q^3T^2\right)\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)}{(1-T)(1-qT)^{12}(1+qT)(1+q^2T^2)(1+qT+q^2T^2)^3(1+q^2T+q^4T^2)^3(1+q^4T^2)(1+q^2T)(1-q^2T)^{12}(1-q^3T)}$

$$G_{12.2}\colon \left\langle \begin{pmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{pmatrix}, \begin{pmatrix} -1&0&0\\ 0&1&0\\ 0&0&-1 \end{pmatrix} \right\rangle \simeq A_4$$

Hodge numbers:

$$h^{1,1}(\mathrm{Kum}_3(E, G_{12.2})) = 19 \quad \text{and} \quad h^{2,1}(\mathrm{Kum}_3(E, G_{12.2})) = 3$$

Zeta function:

Case:	$Z_q\,(\mathrm{Kum}_3(E, G_{12.2}))$
(1, 1, 1, 1)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^{19}(1-q^2T)^{19}(1-q^3T)}$
(1, 1, 2)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^{15}(1+qT)^4(1+q^2T)^4(1-q^2T)^{15}(1-q^3T)}$
(1, 3)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^9(1+qT+q^2T^2)^5(1+q^2T+q^4T^2)^5(1-q^2T)^9(1-q^3T)}$

$$G_{12.3}\colon \left\langle \begin{pmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{pmatrix}, \begin{pmatrix} 0&-1&1\\ 0&-1&0\\ 1&-1&0 \end{pmatrix} \right\rangle \simeq A_4$$

$$G_{12.4}\colon \left\langle \begin{pmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{pmatrix}, \begin{pmatrix} -1&-1&-1\\ 0&0&1\\ 0&1&0 \end{pmatrix} \right\rangle \simeq A_4$$

Hodge numbers:

$$h^{1,1}(\mathrm{Kum}_3(E, G_{12.3})) = h^{1,1}(\mathrm{Kum}_3(E, G_{12.4})) = 7$$
$$h^{2,1}(\mathrm{Kum}_3(E, G_{12.3})) = h^{2,1}(\mathrm{Kum}_3(E, G_{12.4})) = 3$$

Zeta function:

Case:	$Z_q\,(\mathrm{Kum}_3(E, G_{12.3})) = Z_q\,(\mathrm{Kum}_3(E, G_{12.4}))$
(1, 1, 1, 1)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^7(1-q^2T)^7(1-q^3T)}$

(1, 1, 2)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^6(1+qT)(1+q^2T)(1-q^2T)^6(1-q^3T)}$
(1, 3)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^5(1+qT+q^2T^2)(1+q^2T+q^4T^2)(1-q^2T)^5(1-q^3T)}$

5.3. Order 8

$$G_{8.1}: \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle \simeq D_4$$

Hodge numbers:

$$h^{1,1}(\mathrm{Kum}_3(E, G_{8.1})) = 36 \quad \text{and} \quad h^{2,1}(\mathrm{Kum}_3(E, G_{8.1})) = 6$$

Zeta function:

Case:	$Z_q\ (\mathrm{Kum}_3(E, G_{8.1}))$
(1, 1, 1, 1)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^6}{(1-T)(1-qT)^{36}(1-q^2T)^{36}(1-q^3T)}$
(1, 1, 2)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1+a_qqT+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^5}{(1-T)(1-qT)^{27}(1+qT)^9(1+q^2T)^9(1-q^2T)^{27}(1-q^3T)0}$
(1, 3)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^4\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)}{(1-T)(1-qT)^{18}(1+qT+q^2T^2)^9(1+q^2T+q^4T^2)^9(1-q^2T)^{18}(1-q^3T)}$
$G_{8.2}:$	$\left\langle \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle \simeq D_4$

Hodge numbers:

$$h^{1,1}(\mathrm{Kum}_3(E, G_{8.2})) = 15 \quad \text{and} \quad h^{2,1}(\mathrm{Kum}_3(E, G_{8.2})) = 3$$

Zeta function:

Case:	$Z_q\ (\mathrm{Kum}_3(E, G_{8.2}))$
(1, 1, 1, 1)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^{15}(1-q^2T)^{15}(1-q^3T)}$

(1, 1, 2)	$\dfrac{\left(1-(a_q^3-3a_q q)T+q^3T^2\right)\left(1-a_q qT+q^3T^2\right)^3}{(1-T)(1-qT)^{12}(1+qT)^3(1+q^2T)^3(1-q^2T)^{12}(1-q^3T)}$
(1, 3)	$\dfrac{\left(1-(a_q^3-3a_q q)T+q^3T^2\right)\left(1-a_q qT+q^3T^2\right)^3}{(1-T)(1-qT)^9(1+qT+q^2T^2)^3(1+q^2T+q^4T^2)^3(1-q^2T)^9(1-q^3T)}$

5.4. Order 6

$G_{6.1}$:	$\left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle \simeq S_3$
$G_{6.2}$:	$\left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle \simeq S_3$

Hodge numbers:

$$h^{1,1}(\mathrm{Kum}_3(E, G_{6.1})) = h^{1,1}(\mathrm{Kum}_3(E, G_{6.2})) = 15, \quad h^{2,1}(\mathrm{Kum}_3(E, G_{6.1})) = h^{2,1}(\mathrm{Kum}_3(E, G_{6.2})) = 15$$

Zeta function: We get 21 possibilities for the zeta function of $\mathrm{Kum}_3(E, G_{6.1})$ and $\mathrm{Kum}_3(E, G_{6.2})$:

$$\frac{\left(1-(a_q^3-3a_q q)T+q^3T^2\right)\left(1-a_q qT+q^3T^2\right)^2}{(1-T)(1-qT)^2(1-q^2T)^2(1-q^3T)} \cdot \left(\left(\frac{1-a_q qT+q^3T^2}{(1-qT)(1-q^2T)}\right)\otimes P_{(\eta)}\right)\cdot\left(\left(\frac{1-a_q qT+q^3T^2}{(1-qT)(1-q^2T)}\right)\otimes P_{(\lambda)}\right),$$

where

$$(\lambda) \in \left\{(1,1,1,1,1,1,1,1,1), (1,1,1,2,2,2), (1,2,2,2,2), (1,1,1,3,3), (1,4,4), (1,2,6), (1,8)\right\}$$
$$(\eta) \in \{(1,1,1,1), (1,1,2), (1,3)\}$$

Case:	$Z_q\left(\mathrm{Kum}_3(E, G_{6.1})\right) = Z_q\left(\mathrm{Kum}_3(E, G_{6.2})\right)$
$(1)^9$ $(1, 1, 1, 1)$	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^{15}}{(1-T)(1-qT)^{15}(1-q^2T)^{15}(1-q^3T)}$
$(1)^9$ $(1, 1, 2)$	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1+a_qqT+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^{14}}{(1-T)(1-qT)^{14}(1+qT)(1+q^2T)(1-q^2T)^{14}(1-q^3T)}$
$(1)^9$ $(1, 3)$	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^{13}\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)}{(1-T)(1-qT)^{13}(1+qT+q^2T^2)(1+q^2T+q^4T^2)(1-q^2T)^{13}(1-q^3T)}$
$(1, 1, 1, 2, 2, 2)$ $(1, 1, 1, 1)$	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1+a_qqT+q^3T^2\right)^{3}\left(1-a_qqT+q^3T^2\right)^{12}}{(1-T)(1-qT)^{12}(1+qT)^3(1+q^2T)^3(1-q^2T)^{12}(1-q^3T)}$
$(1, 1, 1, 2, 2, 2)$ $(1, 1, 2)$	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1+a_qqT+q^3T^2\right)^{4}\left(1-a_qqT+q^3T^2\right)^{11}}{(1-T)(1-qT)^{11}(1+qT)^4(1+q^2T)^4(1-q^2T)^{11}(1-q^3T)}$
$(1, 1, 1, 2, 2, 2)$ $(1, 3)$	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^{10}\left(1+a_qqT+q^3T^2\right)^{3}\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)}{(1-T)(1-qT)^{10}(1+qT)^3(1+qT+q^2T^2)(1+q^2T+q^4T^2)(1+q^2T)^3(1-q^2T)^{10}(1-q^3T)}$
$(1, 2, 2, 2, 2)$ $(1, 1, 1, 1)$	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1+a_qqT+q^3T^2\right)^{4}\left(1-a_qqT+q^3T^2\right)^{11}}{(1-T)(1-qT)^{11}(1+qT)^4(1+q^2T)^4(1-q^2T)^{11}(1-q^3T)}$
$(1, 2, 2, 2, 2)$ $(1, 1, 2)$	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1+a_qqT+q^3T^2\right)^{5}\left(1-a_qqT+q^3T^2\right)^{10}}{(1-T)(1-qT)^{10}(1+qT)^5(1+q^2T)^5(1-q^2T)^{10}(1-q^3T)}$

Case:	$Z_q\,(\mathrm{Kum}_3(E, G_{6.1})) = Z_q\,(\mathrm{Kum}_3(E, G_{6.2}))$
(1, 2, 2, 2, 2) (1, 3)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^9\left(1+a_qqT+q^3T^2\right)^4\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)}{(1-T)(1-qT)^9(1+qT)^4(1+qT+q^2T^2)(1+q^2T+q^4T^2)(1+q^2T)^4(1-q^2T)^9(1-q^3T)}$
(1, 1, 1, 3, 3) (1, 1, 1, 1)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^{11}\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)^2}{(1-T)(1-qT)^{11}(1+qT+q^2T^2)^2(1+q^2T+q^4T^2)^2(1-q^2T)^{11}(1-q^3T)}$
(1, 1, 1, 3, 3) (1, 1, 2)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^{10}\left(1+a_qqT+q^3T^2\right)\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)^2}{(1-T)(1-qT)^{10}(1+qT)(1+qT+q^2T^2)^2(1+q^2T+q^4T^2)^2(1+q^2T)(1-q^2T)^{10}(1-q^3T)}$
(1, 1, 1, 3, 3) (1, 3)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^9\left(1+a_qqT+q^3T^2\right)^4\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)}{(1-T)(1-qT)^9(1+qT+q^2T^2)^3(1+q^2T+q^4T^2)^3(1-q^2T)^9(1-q^3T)}$
(1, 4, 4) (1, 1, 1, 1)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^9\left(1+a_qqT+q^3T^2\right)^2\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)^2}{(1-T)(1-qT)^9(1+qT)^2(1+q^2T^2)^2(1+q^4T^2)^2(1+q^2T)^2(1-q^2T)^9(1-q^3T)}$
(1, 4, 4) (1, 1, 2)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^8\left(1+a_qqT+q^3T^2\right)^3\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)^2}{(1-T)(1-qT)^8(1+qT)^3(1+q^2T^2)^2(1+q^4T^2)^2(1+q^2T)^3(1-q^2T)^8(1-q^3T)}$
(1, 4, 4) (1, 3)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^7\left(1+a_qqT+q^3T^2\right)^2\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)^3}{(1-T)(1-qT)^7(1+qT)^2(1+q^2T^2)^2(1+qT+q^2T^2)(1+q^2T+q^4T^2)(1+q^4T^2)^2(1+q^2T)^2(1-q^2T)^7(1-q^3T)}$
(1, 2, 6) (1, 1, 1, 1)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^9\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)}{(1-T)(1-qT)^9(1+qT)^2(1+qT+q^2T^2)(1-qT+q^2T^2)(1-q^2T+q^4T^2)}\times$ $\times\dfrac{\left(1+a_qqT+q^3T^2\right)^2\left(1-a_qqT-(q^3-a_q^2q^2)T^2-a_qq^4T^3+q^6T^4\right)}{(1+q^2T+q^4T^2)(1+q^2T)^2(1-q^2T)^9(1-q^3T)}$

Case:	$Z_q\ (\mathrm{Kum}_3(E, G_{6.1})) = Z_q\ (\mathrm{Kum}_3(E, G_{6.2}))$
(1, 2, 6) (1, 1, 2)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^8\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)}{(1-T)(1-qT)^8(1+qT)^3(1+qT+q^2T^2)(1-qT+q^2T^2)(1-q^2T+q^4T^2)}\times$ $\times\frac{\left(1+a_qqT+q^3T^2\right)^3\left(1-a_qqT-(q^3-a_q^2q^2)T^2-a_qq^4T^3+q^6T^4\right)}{(1+q^2T+q^4T^2)(1+q^2T)^3(1-q^2T)^8(1-q^3T)}$
(1, 2, 6) (1, 3)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^7\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)^2}{(1-T)(1-qT)^7(1+qT)^2(1+qT+q^2T^2)^2(1-qT+q^2T^2)(1-q^2T+q^4T^2)}\times$ $\times\frac{\left(1+a_qqT+q^3T^2\right)^2\left(1-a_qqT-(q^3-a_q^2q^2)T^2-a_qq^4T^3+q^6T^4\right)}{(1+q^2T+q^4T^2)^2(1+q^2T)^2(1-q^2T)^7(1-q^3T)}$
(1, 8) (1, 1, 1, 1)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^8\left(1+a_qqT+q^3T^2\right)\left(1-2q^3T^2+a_q^2q^2T^2+q^6T^4\right)}{(1-T)(1-qT)^8(1+qT)(1+q^2T^2)(1+q^4T^2)(1+q^4T^4)}\times$ $\times\frac{\left(1+2q^6T^4-4a_q^2q^5T^4+a_q^4q^4T^4+q^{12}T^8\right)}{(1+q^8T^4)(1+q^4T^2)(1+q^2T)(1-q^2T)^8(1-q^3T)}$
(1, 8) (1, 1, 2)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^7\left(1+a_qqT+q^3T^2\right)^2\left(1-2q^3T^2+a_q^2q^2T^2+q^6T^4\right)}{(1-T)(1-qT)^7(1+qT)^2(1+q^2T^2)(1+q^4T^2)(1+q^4T^4)}\times$ $\times\frac{\left(1+2q^6T^4-4a_q^2q^5T^4+a_q^4q^4T^4+q^{12}T^8\right)}{(1+q^8T^4)(1+q^4T^2)(1+q^2T)^2(1-q^2T)^7(1-q^3T)}$
(1, 8) (1, 3)	$\frac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^6\left(1+a_qqT+q^3T^2\right)\left(1-2q^3T^2+a_q^2q^2T^2+q^6T^4\right)}{(1-T)(1-qT)^6(1+qT)(1+q^2T^2)(1+q^4T^2)(1+qT+q^2T^2)(1+q^4T^4)}\times$ $\times\frac{\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)\left(1+2q^6T^4-4a_q^2q^5T^4+a_q^4q^4T^4+q^{12}T^8\right)}{(1+q^8T^4)(1+q^2T+q^4T^2)(1+q^4T^2)(1+q^2T)(1-q^2T)^6(1-q^3T)}$

$$G_{6.3}: \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right\rangle \simeq S_3$$

Hodge numbers:

$$h^{1,1}(\mathrm{Kum}_3(E, G_{6.3})) = 7 \quad \text{and} \quad h^{2,1}(\mathrm{Kum}_3(E, G_{6.3})) = 7$$

Zeta function:

Case:	$Z_q\ (\mathrm{Kum}_3(E, G_{6.3}))$
(1, 1, 1, 1)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^7}{(1-T)(1-qT)^7(1-q^2T)^7(1-q^3T)}$
(1, 1, 2)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^{12}(1+qT)^3(1+q^2T)^3(1-q^2T)^{12}(1-q^3T)}$
(1, 3)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^5\left(1+a_qqT-(q^3-a_q^2q^2)T^2+a_qq^4T^3+q^6T^4\right)}{(1-T)(1-qT)^5(1+qT+q^2T^2)(1+q^2T+q^4T^2)(1-q^2T)^5(1-q^3T)}$

5.5. *Order 4*

$$G_{4.1}: \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

Hodge numbers:

$$h^{1,1}\ (\mathrm{Kum}_3(E, G_{4.1})) = 51 \quad \text{and} \quad h^{2,1}(\mathrm{Kum}_3(E, G_{4.1})) = 3$$

Zeta function:

Case:	$Z_q\left(\mathrm{Kum}_3(E, G_{4.1})\right)$
(1, 1, 1, 1)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^{51}(1-q^2T)^{51}(1-q^3T)}$
(1, 1, 2)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^{39}(1+qT)^{12}(1+q^2T)^{12}(1-q^2T)^{39}(1-q^3T)}$
(1, 3)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^{21}(1+qT+q^2T^2)^{15}(1+q^2T+q^4T^2)^{15}(1-q^2T)^{21}(1-q^3T)}$

$$G_{4.2}\colon \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

Hodge numbers:

$$h^{1,1}(\mathrm{Kum}_3(E, G_{4.2})) = 21 \quad \text{and} \quad h^{2,1}(\mathrm{Kum}_3(E, G_{4.2})) = 9$$

Zeta function:

Case:	$Z_q\left(\mathrm{Kum}_3(E, G_{4,2})\right)$
(1, 1, 1, 1)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^9}{(1-T)(1-qT)^{21}(1-q^2T)^{21}(1-q^3T)}$
(1, 1, 2)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^9}{(1-T)(1-qT)^{18}(1+qT)^{3}(1+q^2T)^{3}(1-q^2T)^{18}(1-q^3T)}$
(1, 3)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^9}{(1-T)(1-qT)^{15}(1+qT+q^2T^2)^{3}(1+q^2T+q^4T^2)^{3}(1-q^2T)^{15}(1-q^3T)}$

$$G_{4.3}\colon \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \right\rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

$$G_{4.4}\colon \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

Hodge numbers:

$$h^{1,1}(\mathrm{Kum}_3(E, G_{4.3})) = h^{1,1}(\mathrm{Kum}_3(E, G_{4.4})) = 15$$
$$h^{2,1}(\mathrm{Kum}_3(E, G_{4.3})) = h^{2,1}(\mathrm{Kum}_3(E, G_{4.4})) = 3$$

Zeta function:

Case:	$Z_q\left(\mathrm{Kum}_3(E, G_{4,3})\right) = Z_q\left(\mathrm{Kum}_3(E, G_{4,4})\right)$
(1, 1, 1, 1)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^{15}(1-q^2T)^{15}(1-q^3T)}$
(1, 1, 2)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^{12}(1+qT)^3(1+q^2T)^3(1-q^2T)^{12}(1-q^3T)}$
(1, 3)	$\dfrac{\left(1-(a_q^3-3a_qq)T+q^3T^2\right)\left(1-a_qqT+q^3T^2\right)^3}{(1-T)(1-qT)^9(1+qT+q^2T^2)^3(1+q^2T+q^4T^2)^3(1-q^2T)^9(1-q^3T)}$

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Declarations

Data sharing Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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