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Integral points on algebraic subvarieties of period domains: from number fields to finitely generated fields

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Abstract. We show that for a variety which admits a quasi-finite period map, finiteness (resp. non-Zariski-density) of S-integral points implies finiteness (resp. non-Zariski-density) of points over all $\mathbb{Z}$-finitely generated integral domains of characteristic zero. Our proofs rely on foundational results in Hodge theory due to Deligne, Griffiths, and Schmid, and Bakker-Brunebarbe-Tsimerman. We give straightforward applications to arithmetic locally symmetric varieties, the moduli space of smooth hypersurfaces in projective space, and the moduli of smooth divisors in an abelian variety.

Key words. Period maps · Hyperbolicity · Function fields · Rational points · Shimura varieties · Hypersurfaces · Arakelov inequality · Hodge theory · Lang–Vojta conjecture

1. Introduction

The goal of this note is to give arithmetic applications of foundational results in Hodge theory. Our main abstract result (which is a combination of Theorems 1.6 and 3.7 below) reads as follows (see Definition 1.5 for the definition of admitting a quasi-finite period map).

Theorem 1.1. (Main Result, I) *Let $A \subset k = \overline{\mathbb{Q}}$ be a finitely generated subring and let $\mathcal{X}$ be a finite type A-scheme such that $\mathcal{X}_k$ is a quasi-projective variety over k which admits a quasi-finite complex-analytic period map. Then the following statements are equivalent.*

(1) For every finitely generated subring $A' \subset k$ containing A, the set $\mathcal{X}(A')$ is finite (resp. not Zariski-dense in $\mathcal{X}(k)$).
(2) For every finitely generated integral domain B containing A, the set $\mathcal{X}(B)$ is finite (resp. not Zariski-dense in $\mathcal{X}(\overline{Frac(B)})$) (where $\overline{Frac(B)}$ is a choice of algebraic closure of $Frac(B)$).

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In other words, for varieties admitting a quasi-finite period map, finiteness of $\mathscr{O}_{K,S}$-points (where K ranges over all number fields and S ranges over all finite collections of finite places of K) implies finiteness of A-points for all $\mathbb{Z}$-finitely generated integral domains A of characteristic zero, and a similar statement (which requires substantially deeper input) holds for non-Zariski-density of rational points. Both the finiteness and non-density results require input from Hodge theory. Arguably, the novel technical result in our proof of Theorem 1.1 is Theorem 3.7.

We proceed to give various applications of these results, by applying them to various moduli spaces/stacks. For example, our first result extends Lawrence–Sawin's finiteness result for smooth hypersurfaces in an abelian variety from number fields [36] to finitely generated fields of characteristic zero (see [30] for a further generalization).

Theorem 1.2. (Main Result, II, Lawrence–Sawin + ϵ) *Let K be a number field, let S be a finite set of finite places of K, and let $\mathcal{A}$ be an abelian scheme over $\mathcal{O}_{K,S}$. Let D be an ample divisor on $\mathcal{A}_K$. Then, for any $\mathcal{O}_{K,S}$-finitely generated normal integral domain R of characteristic zero, the set of R-smooth hypersurfaces $H \subset \mathcal{A}_R$ such that H represents $D_{\mathrm{Frac}(R)}$ on $\mathcal{A}_{\mathrm{Frac}(R)}$ is finite.*

We stress that the bulk of the work to prove Theorem 1.2 is contained in the paper of Lawrence–Sawin [36]. We only *combine* their work with ours to prove more general finiteness statements over finitely generated fields of characteristic zero (as opposed to only number fields).

Note that Theorem 1.2 shows that the Shafarevich conjecture for smooth hypersurfaces in an abelian variety over a number field (as proven by Lawrence–Sawin) persists over finitely generated fields of characteristic zero. Also, note that the proof of Theorem 1.2 is a straightforward consequence of our main abstract result (Theorem 1.1) and the following two facts due to Lawrence–Sawin:

(1) Lawrence–Sawin's main theorem [36, Theorem 1.1] which says that Theorem 1.2 holds whenever $\dim R = 1$;
(2) The moduli space of smooth hypersurfaces in an abelian variety admits a quasi-finite period map; see [36, Proposition 5.10];

Our next result is of similar nature, but is instead concerned with the Shafarevich conjecture for smooth hypersurfaces in projective space (as opposed to smooth hypersurfaces in a fixed abelian variety).

Theorem 1.3. (Main Result, III) *Let $d \geq 3$ be an integer and let $n \geq 2$. Assume that, for every number field K and every finite set of finite places S of K, the set of $\mathcal{O}_{K,S}$-isomorphism classes of smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}_{\mathcal{O}_{K,S}}$ is finite. Then, for every $\mathbb{Z}$-finitely generated normal integral domain A of characteristic zero, the set of A-isomorphism classes of smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}_A$ is finite.*

Note that Theorem 1.3 says that the Shafarevich conjecture for smooth hypersurfaces over number fields implies the analogous conjecture for smooth hypersurfaces over all finitely generated fields of characteristic zero; we refer the reader to [25,27,29] for related results on this conjecture.

We now briefly discuss our main abstract result (Theorem 1.1); we will discuss the finiteness and non-Zariski-density statements separately, as the proofs are somewhat different.

1.1. Finiteness results

For convenience, we rephrase the part of Theorem 1.1 about finiteness in terms of the notion of arithmetic hyperbolicity. Lang introduced the notion of arithmetic hyperbolicity over $\overline{\mathbb{Q}}$(sometimes also referred to as *Mordellicity*) to appropriately formalize the property of "having only finitely many rational points".

Definition 1.4. (Arithmetic hyperbolicity) Let k be an algebraically closed field of characteristic zero. A finite type separated scheme X over k is *arithmetically hyperbolic over* k if there is a $\mathbb{Z}$-finitely generated subring $A \subset k$, a finite type separated A-scheme $\mathcal{X}$ and an isomorphism of schemes $\mathcal{X}_k \cong X$ over k such that, for all $\mathbb{Z}$-finitely generated subrings $A' \subset k$ containing A, the set $\mathcal{X}(A')$ of A'-points on $\mathcal{X}$ is finite.

For example, by Faltings's finiteness theorem [13], a smooth quasi-projective connected curve X over k is arithmetically hyperbolic over k if and only if X is not isomorphic to $\mathbb{P}^1_k$, $\mathbb{A}^1_k$, $\mathbb{A}^1_k \setminus \{0\}$, nor a smooth proper connected genus one curve over k. Faltings also proved that a closed subvariety X of an abelian variety A over k is arithmetically hyperbolic over k if and only if X does not contain the translate of a positive-dimensional abelian subvariety of A; see [14].

A period domain (usually denoted by D) is a classifying space for polarized Hodge structures of some fixed type.

Definition 1.5. We say that a variety X over k *admits a quasi-finite complex-analytic period map (up to Galois conjugation)* if there exists a subfield $k_0 \subset k$, an embedding $k_0 \to \mathbb{C}$, a variety X_0 over k_0, an isomorphism of k-schemes $X_{0,k} \cong X$, a period domain D, a discrete arithmetic subgroup Γ of $\mathrm{Aut}(D)$, and a horizontal locally liftable holomorphic map $X^{\mathrm{an}}_{0,\mathbb{C}} \to \Gamma\backslash D$ with finite fibres.

We will follow [44] and recall some basics of the theory in Sect. 4.

The part of Theorem 1.1 about finiteness may be rephrased (and slightly generalized) as:

Theorem 1.6. (Main Result, IV) *Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero. Let X be a variety over k such that X admits a quasi-finite complex-analytic period map. If X is arithmetically hyperbolic over k, then X_L is arithmetically hyperbolic over L.*

We note that Lang-Vojta's conjecture on integral points of varieties (see [49, Conj. 4.3]) implies that a variety X over $\overline{\mathbb{Q}}$ which admits a quasi-finite complex-analytic period map is in fact arithmetically hyperbolic over $\overline{\mathbb{Q}}$, as all its subvarieties are of log-general type by a theorem of Kang Zuo [51].

Theorem 1.6 can be applied to curves of genus at least two, as such curves admit a quasi-finite period map up to a finite étale cover [39]. However, in this case, Faltings already remarked that the statement follows from Grauert-Manin's finiteness theorem (*formerly* the function field analogue of Mordell's conjecture); see [13, VI.4, p.215]. Similarly, if $g > 0$ is an integer and X is the moduli space of principally polarized abelian varieties of dimension g over $\overline{\mathbb{Q}}$ with level 3 structure, then Faltings showed that X_k is arithmetically hyperbolic over k by "re-doing" part of his proof that X is arithmetically hyperbolic over $\overline{\mathbb{Q}}$; the fact that X is arithmetically hyperbolic over $\overline{\mathbb{Q}}$ is precisely Shafarevich's arithmetic finiteness conjecture for principally polarized abelian schemes over $\mathbb{Z}$-finitely generated subrings of $\overline{\mathbb{Q}}$. Around the same time, in Szpiro's seminar [47], Martin-Deschamps gave a *different* proof of the arithmetic hyperbolicity of X_k by using a specialization argument on the moduli stack of principally polarized abelian schemes; see [38]. We stress that our proof of Theorem 1.6 is very close to Martin-Deschamps's line of reasoning. Indeed, Martin-Deschamps' proof crucially relies on Faltings's function field analogue of the Shafarevich conjecture for abelian varieties [12] and Grothendieck's theorem on monodromy representations of abelian schemes [18]. In our proof of Theorem 1.6, we replace these results of Faltings and Grothendieck by foundational results of Deligne, Griffiths, and Schmid in Hodge theory.

In fact, our proof of Theorem 1.6 relies on the following consequence of Deligne's finiteness theorem for monodromy representations [10] and the "Rigidity Theorem" in Hodge theory (see Theorem 4.1).

Theorem 1.7. (Deligne + Rigidity Theorem) *Let X be a variety over k which admits a quasi-finite period map (up to Galois conjugation). Then, for every variety Y over k, every y in $Y(k)$, and every x in $X(k)$, the set of morphisms $f : Y \to X$ with $f(y) = x$ is finite.*

Note that Theorem 1.7 is a finiteness statement about maps of pointed varieties to a period domain. It is crucial that we consider *pointed* maps here; see Remark 5.2 for a discussion of this.

Besides applying our results for varieties with a quasi-finite period to the moduli space of smooth hypersurfaces, we also give further applications to arithmetic locally symmetric varieties and Shimura varieties.

1.2. Non-density results

It is also natural to study the non-Zariski-density (as opposed to the finiteness) of integral points on certain moduli spaces. Our main novel result on non-density is Theorem 3.7.

We can apply Theorem 3.7 to give conditional results on the non-density of integral points on the Hilbert scheme $\mathrm{Hilb}_{d,n}$ of smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}$ over $\mathbb{Z}$.

By Vojta's extension of Lang's conjecture on non-density of integral points [49, Conj. 4.3], the finiteness of integral points on a variety is conjecturally closely related to its subvarieties being of log-general type. However, as the Hilbert scheme of smooth hypersurfaces has subvarieties which are not of log-general type, it is not reasonable to expect finiteness of integral points on this moduli space (and it is not hard to see that $\mathrm{Hilb}_{d,n}$ has infinitely many $\mathbb{Z}[1/d]$-points). Nonetheless, it follows from [26] that there is a finite étale cover $H' \to \mathrm{Hilb}_{d,n}$ such that H' dominates a positive-dimensional variety of log-general type. Therefore, the integral points of $\mathrm{Hilb}_{d,n}$ should not be dense (even though they can be infinite).

The expectation that the integral points on $\mathrm{Hilb}_{d,n}$ should not be dense was investigated by Lawrence–Venkatesh [37] for large enough d and n (see also [5]). The current state-of-the-art can be stated as follows (see [37, Proposition 10.2]). (Note that the following statement provides a non-density statement *only* for $\mathbb{Z}[1/S]$-points.)

Theorem 1.8. (Lawrence–Venkatesh) *There is an integer n_0 and a function $D_0(n)$ such that, for every $n \geq n_0$, every $d \geq D_0(n)$, and every positive integer $S \geq 1$, the set $\mathrm{Hilb}_{d,n}(\mathbb{Z}[1/S])$ is not dense in $\mathrm{Hilb}_{d,n,\overline{\mathbb{Q}}}$.*

Motivated by Lawrence–Venkatesh's recent breakthrough, we show that the non-density of integral points on the Hilbert scheme $\mathrm{Hilb}_{d,n}$ valued in a number field persists to non-density over finitely generated fields.

Theorem 1.9. (Main Result, V) *Let $d \geq 3$ be an integer and let $n \geq 2$ be an integer. Suppose that, for every number field K and every finite set of finite places S of K, the set $\mathrm{Hilb}_{d,n}(\mathcal{O}_{K,S})$ is not dense in $\mathrm{Hilb}_{d,n}$. Then, for every $\mathbb{Z}$-finitely generated regular integral domain of characteristic zero A, we have that $\mathrm{Hilb}_{d,n}(A)$ is not dense in $\mathrm{Hilb}_{d,n}$.*

The proof of Theorem 1.9 is more involved than the proof of Theorem 1.3. For example, due to the fact that the Hilbert scheme does not admit any period map with finite fibres, we are forced to argue on the stack $[\mathrm{PGL}_{n+2}\backslash \mathrm{Hilb}_{d,n}]$ and to relate the non-density of the integral points on the stack to that on the Hilbert scheme using finiteness results for PGL_{n+2}-torsors over number rings.

Conventions. We let k be an algebraically closed field of characteristic zero. A variety over k is a finite type separated scheme over k. If X is a variety over k and $A \subset k$ is a subring, then a model for X over A is a pair $(\mathcal{X}, \phi)$ with $\mathcal{X}$ a finite type separated scheme over A and $\phi : \mathcal{X}_k \to X$ an isomorphism of schemes over k. We will usually omit ϕ from our notation and simply refer to $\mathcal{X}$ as a model for X over A.

If K is a number field and S is a finite set of finite places of K, we let $\mathcal{O}_{K,S}$ be the ring of S-integers of K.

If $k \subset L$ is a field extension and X is a variety over k, we denote $X \times_k \operatorname{Spec} L$ by X_L.

2. Arithmetic hyperbolicity and geometric hyperbolicity

To prove Theorems 1.3 and 1.6 on the arithmetic hyperbolicity of certain varieties (as defined in Definition 1.4), we will use a geometric criterion for the persistence of arithmetic hyperbolicity of a variety along field extensions proven in [22]. To state this criterion, we introduce the notion of geometric hyperbolicity. We view this property as a "function field" analogue of arithmetic hyperbolicity.

Definition 2.1. A variety X over k is *geometrically hyperbolic over* k if, for every smooth integral curve C over k, every c in $C(k)$, and every x in $X(k)$, the set $\mathrm{Hom}_k((C,c),(X,x))$ of morphisms of k-schemes $f : C \to X$ with $f(c) = x$ is finite.

More generally, a finite type separated Deligne–Mumford algebraic stack X over k is *geometrically hyperbolic* if, for every smooth integral curve C over k, every c in $C(k)$, and every x in (the groupoid) $X(k)$, the set $\mathrm{Hom}_k((C,c),(X,x))$ of isomorphism classes of morphisms $f : C \to X$ with $f(c) = x$ is finite. (Here $f(c) = x$ means that $f(c)$ and x are isomorphic in $X(k)$.)

Example 2.2. (Urata's theorem) A proper variety X over $\mathbb{C}$ which is Brody hyperbolic (i.e., has no entire curves) is geometrically hyperbolic. Indeed, as X^{an} is a compact complex-analytic space with no entire curves, it follows from Brody's theorem that X^{an} is Kobayashi hyperbolic (as defined in [35]). Therefore, as X^{an} is a compact Kobayashi hyperbolic complex-analytic space, we conclude that X is geometrically hyperbolic from Urata's theorem [35, Theorem 5.3.10] (or the original [48]). (Note that Urata's theorem has been extended to the logarithmic case in [28].)

Example 2.3. Let $\mathcal{M}$ be the locally finite type separated Deligne-Mumford algebraic stack of smooth proper canonically polarized varieties over $\mathbb{Q}$. That is, for a scheme S over $\mathbb{Q}$, the objects of $\mathcal{M}(S)$ are smooth proper morphisms $\mathcal{X} \to S$ whose geometric fibres are connected and have ample canonical bundle. (For example, for every $g \geq 2$, the stack of smooth proper genus g curves $\mathcal{M}_g$ is an open and closed substack of $\mathcal{M}$. In fact, $\mathcal{M}$ is the disjoint union of the stacks $\mathcal{M}_h$, where h runs over all polynomials in $\mathbb{Q}[t]$ and $\mathcal{M}_h$ is the substack of smooth proper canonically polarized varieties with Hilbert polynomial h.) Let X be a quasi-projective scheme over $\mathbb{C}$ such that there exists a quasi-finite morphism $X \to \mathcal{M}_{\mathbb{C}}$. (In other words, there is a smooth proper morphism $f : Y \to X$ whose geometric fibres are canonically polarized varieties such that, for every x in $X(\mathbb{C})$, the set of y in $Y(\mathbb{C})$ with $Y_x \cong Y_y$ is finite. In particular, the family $f : Y \to X$ of canonically polarized varieties has "maximal variation in moduli".) Then, it follows from Viehweg–Zuo's theorem (see [50]) that X is Brody hyperbolic. In particular, if X is projective, then Urata's theorem (Example 2.2) implies that X is geometrically hyperbolic over $\mathbb{C}$. (It seems reasonable to suspect that the assumption that X is projective is unnecessary; see [31] for recent progress.)

We will prove the geometric hyperbolicity of some (not necessarily proper) varieties over $\mathbb{C}$ by appealing to their complex-analytic properties. The following lemma will then be applied to deduce the geometric hyperbolicity of these varieties over every algebraically closed field of characteristic zero.

Lemma 2.4. *Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero. If k is uncountable and X is a finite type separated geometrically hyperbolic Deligne-Mumford algebraic stack over k, then X_L is geometrically hyperbolic over L.*

Proof. Assume that X_L is not geometrically hyperbolic over L. We show that X is not geometrically hyperbolic over k. To do so, let C be smooth affine connected curve over L, let $c \in C(L)$, and let $x \in X(L)$ be such that the set $\mathrm{Hom}_L((C, c), (X_L, x))$ of (isomorphism classes of) morphisms $f : C \to X_L$ with $f(c) = x$ is infinite. Let $f_1, f_2, \ldots$ be a sequence of pairwise distinct (non-isomorphic) elements in $\mathrm{Hom}_L((C, c), (X_L, x))$. Let S be an integral variety over k and let $(\mathcal{C}, P)$ be a model for (C, c) over S. That is, the morphism $\mathcal{C} \to S$ is a smooth affine geometrically connected morphism of relative dimension one and $P \in \mathcal{C}(S)$ is a section such that there is a (fixed) isomorphism $\mathcal{C}_L \cong C$ and $P_L = c$. We now recursively descend every $f_i : C \to X$ to some "étale neighbourhood" of S (using for instance [43, Appendix B]). Thus, let $A_1 \subset L$ be a finitely generated k-algebra with $S_1 = \mathrm{Spec}\, A_1$, let $S_1 \to S$ be an étale morphism and let $F_1 : \mathcal{C}_{S_1} \to X \times_k S_1$ be a morphism with $F_1(P) = \{x\} \times S_1$ such that the morphism $f_1 : C \to X_L$ coincides with $F_{1,L} : C \cong \mathcal{C}_L \to X_L$. Now, we construct integral affine varieties $S_2, S_3, \ldots$ over k recursively, as follows. Assume S_{i-1} has been constructed. Then, for every $i = 2, 3, \ldots$, we choose a finitely generated k-algebra $A_i \subset L$ with $S_i = \mathrm{Spec}\, A_i$, an étale morphism $S_i \to S_{i-1}$ and a morphism $F_i \in \mathrm{Hom}_{S_i}((\mathcal{C}_i, P_i), (X \times_k S_i, x \times \{S_i\}))$ with $\mathcal{C}_i = \mathcal{C}_{S_i}$, $P_i = P_{S_i}$, and $F_{i,L} = f_i$ such that, for every $1 \le j < i$, every s in $S_j(k)$ and every s' in $S_i(k)$ lying over s, the morphism $F_{i,s'}$ does not equal $F_{j,s}$. Let Z_i be the (non-empty and open) image of $S_i \to S$. Since k is uncountable and every Z_i is a non-empty open of S, there is an s in $S(k)$ contained in $\cap_{i=1}^{\infty} Z_i$. Now, for every $i = 1, 2, \ldots$, let s_i be a point of $S_i(k)$ lying over s in $S(k)$. Define $D := \mathcal{C}_s$ and note that $D \cong \mathcal{C}_{S_i, s_i}$. Moreover, the morphisms $F_{i,s_i} : D \cong \mathcal{C}_{S_i, s_i} \to X \times \{s_1\} \cong X$ are, by construction, pairwise distinct. Finally, as $F_{i,s_i}(P_{s_i}) = x$, we see that X is not geometrically hyperbolic over k, as required. □

The notion of geometric hyperbolicity is studied in more generality in [32] building on [23] (see also [4,21]) in the case of varieties. We indicate how to extend some results needed in this note to the case of stacks.

Lemma 2.5. *(From pointed curves to pointed varieties) Let k be an* uncountable *algebraically closed field of characteristic zero, and let X be a finite type separated Deligne-Mumford geometrically hyperbolic algebraic stack over k. Then, for every integral variety Y over k, every y in $Y(k)$, and every x in $X(k)$, the set* $\mathrm{Hom}_k((Y, y), (X, x))$ *of morphisms $f : Y \to X$ with $f(y) = x$ is finite.*

Proof. Suppose that $f_1, f_2, \ldots$ are pairwise distinct morphisms from Y to X which map y to x. Let $Y^{i,j} \subset Y$ be the closed subset of points P such that $f_i(P) = f_j(P)$. Let w be a point of $Y(k)$ such that, for every $i \neq j$, the point w does not lie in $Y^{i,j}$. (Such a point exists as k is uncountable and $Y^{i,j} \neq Y$ whenever $i \neq j$.) Let C be a smooth connected curve, and let $C \to Y$ be a morphism whose image contains w and y. (Such a curve exists as Y is integral.) Then the morphisms $f_1|_C, f_2|_C, \ldots$ are pairwise distinct morphisms from C to X and send y to x. This shows that X is not geometrically hyperbolic, as required. □

Lemma 2.6. *(Descending along coverings) Let $X \to Y$ be a finite étale morphism of finite type separated Deligne-Mumford algebraic stacks over k. Then X is* *geometrically hyperbolic over k if and only if Y is geometrically hyperbolic over k.*

Proof. This is proven when X and Y are projective schemes in [23, 5], and the arguments in *loc. cit.* easily adapt to prove the more general statement for stacks. □

The relation between geometric hyperbolicity and arithmetic hyperbolicity is provided by the following consequence of the results proven in [22, 4].

Proposition 2.7. *Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero, and let X be an arithmetically hyperbolic variety over k such that X_L is geometrically hyperbolic over L. Then, X_L is arithmetically hyperbolic over L.*

3. Weakly bounded varieties and persistence of non-density

To prove Theorem 1.3 on the arithmetic hyperbolicity of the moduli of smooth hypersurfaces, we will use the geometric hyperbolicity of the moduli stack of smooth hypersurfaces. However, to prove Theorem 1.9 (which is concerned with the non-density of integral points on a certain Hilbert scheme), we will require an additional property of the moduli space. Namely, we will need that it is "weakly bounded". Here we follow the terminology of Kovács-Lieblich; see [34]. To be precise, we use the notion of "weak boundedness" to give a criterion for extending results on non-density of integral points valued in number fields to non-density of integral points valued in finitely generated fields (see Theorem 3.7).

Definition 3.1. (Kovács-Lieblich) Let $\overline{X}$ be a projective scheme over k, let $\mathcal{L}$ be an ample line bundle on $\overline{X}$, and let $X \subset \overline{X}$ be a dense open subscheme. We say that X is *weakly bounded over k in $\overline{X}$ with respect to $\mathcal{L}$* if, for every integer $g \geq 0$, and every $d \geq 0$, there is a real number $\alpha(X, \overline{X}, \mathcal{L}, g, d)$ such that, for every smooth projective connected curve $\overline{C}$ over k of genus g and every dense open subscheme $C \subset \overline{C}$ with $\#(\overline{C} \setminus C) = d$ and every morphism $f : C \to X$, the following inequality

$$\deg_{\overline{C}} \overline{f}^* \mathcal{L} \leq \alpha(\mathcal{L}, g, d) \tag{3.1}$$

holds, where $\overline{f} : \overline{C} \to \overline{X}$ is the unique morphism restricting to $f : C \to X$.

For Y and X projective schemes over k, we let $\underline{\mathrm{Hom}}_k(Y, X)$ be the moduli scheme parametrizing morphisms $Y \to X$; recall that $\underline{\mathrm{Hom}}_k(Y, X)$ is a disjoint union of quasi-projective schemes over k (see [8, 2]). We will make use of the following basic proposition.

Proposition 3.2. *Let $\overline{X}$ be a projective variety over k, let $\mathcal{L}$ be an ample line bundle on $\overline{X}$, and let $X \subset \overline{X}$ be a dense open subscheme. Let $\overline{C}$ be a smooth projective curve and let $C \subset \overline{C}$ be a dense open subscheme. If X is weakly bounded over k in $\overline{X}$ with respect to $\mathcal{L}$, then* $\mathrm{Hom}_k(C, X)$ *is a quasi-compact constructible subset of* $\underline{\mathrm{Hom}}_k(\overline{C}, \overline{X})(k)$.

Proof. Let g be the genus of $\overline{C}$ and let $d := \#(\overline{C} \setminus C)$. Let $\alpha := \alpha(\mathcal{L}, g, d)$ be the real number in Definition 3.1, and note that $\mathrm{Hom}_k(C, X)$ is a subset of the scheme $\underline{\mathrm{Hom}}_k^{\leq\alpha}(\overline{C}, \overline{X})$ parametrizing morphisms $\overline{C} \to \overline{X}$ of degree at most α (with respect to $\mathcal{L}$).

Consider the natural morphisms of finite type schemes

$$\mathrm{ev}\colon C \times \mathrm{Hom}_k^{\leq\alpha}(\overline{C}, \overline{X}) \to \overline{X}, \quad \mathrm{pr} : C \times \mathrm{Hom}_k^{\leq\alpha}(\overline{C}, \overline{X}) \to \mathrm{Hom}_k^{\leq\alpha}(\overline{C}, \overline{X}) \tag{3.2}$$

Let Δ be the boundary of X in $\overline{X}$. Let $\Delta' := \mathrm{ev}^{-1}\Delta$ be the (closed) inverse image of Δ in the finite type k-scheme $C \times \mathrm{Hom}_k^{\leq\alpha}(\overline{C}, \overline{X})$. Let $Z := \mathrm{pr}(\Delta')$ be the image of Δ' in $\mathrm{Hom}_k^{\leq\alpha}(\overline{C}, \overline{X})$, and note that Z is constructible [46, Tag 054J]. As the complement of Z is a constructible subset of the finite type k-scheme $\underline{\mathrm{Hom}}_k^{\leq\alpha}(\overline{C}, \overline{X})$ whose k-points equal $\mathrm{Hom}_k(C, X)$, this shows that $\mathrm{Hom}_k(C, X)$ is a finite union of locally closed subschemes of $\underline{\mathrm{Hom}}_k^{\leq\alpha}(\overline{C}, \overline{X})$. □

Definition 3.3. A quasi-projective scheme X over k is *weakly bounded over k* if there exists a projective scheme $\overline{X}$ over k, an ample line bundle $\mathcal{L}$ on $\overline{X}$, and an open immersion $X \subset \overline{X}$ such that X is weakly bounded over k in $\overline{X}$ with respect to $\mathcal{L}$.

Remark 3.4. (The choice of an ample line bundle) Let $\overline{X}$ be a projective scheme over k, let $\mathcal{L}$ be an ample line bundle on $\overline{X}$, and let $X \subset \overline{X}$ be a dense open subscheme such that X is weakly bounded in $\overline{X}$ over k with respect to $\mathcal{L}$. Then, for every ample line bundle $\mathcal{L}'$ on $\overline{X}$, the quasi-projective scheme X is weakly bounded over k in $\overline{X}$ with respect to $\mathcal{L}'$. To prove this, choose an integer n such that $\mathcal{L}^{\otimes n} \otimes \mathcal{L}^{\vee}$ is effective. Let $C \subset \overline{C}$ and $f : C \to X$ be as in Definition 3.1. Since $\deg_{\overline{C}} \overline{f}^*(\mathcal{L}^{\otimes n} \otimes \mathcal{L}^{\prime,\vee}) \geq 0$, it follows that

$$\deg_{\overline{C}} \overline{f}^* \mathcal{L}' \leq n \deg_{\overline{C}} \overline{f}^* \mathcal{L}.$$

Thus, the lefthandside is bounded by a constant depending only on g, d, $\mathcal{L}$ and $\mathcal{L}'$. This implies that X is weakly bounded over k in $\overline{X}$ with respect to $\mathcal{L}'$.

Proposition 3.5. *Let X be a weakly bounded quasi-projective scheme over k, and let $C \subset \overline{C}$ be a dense open of a smooth projective connected curve $\overline{C}$ over k. Then, for every projective variety $\overline{X}$ over k and every open immersion $X \to \overline{X}$, the subset* $\mathrm{Hom}_k(C, X)$ *of* $\underline{\mathrm{Hom}}_k(\overline{C}, \overline{X})(k)$ *is quasi-compact and constructible.*

Proof. Let $\overline{X}'$ be such that X is weakly bounded in $\overline{X}'$ with respect to some ample line bundle. Then, $\mathrm{Hom}(C, X)$ is quasi-compact constructible in $\underline{\mathrm{Hom}}_k(\overline{C}, \overline{X}')(k)$ by Proposition 3.2. Now, to prove the proposition, we choose a projective variety Z, an open immersion $X \subset Z$, a proper birational morphism $Z \to \overline{X}'$ which is an isomorphism over X, and a proper birational morphism $Z \to \overline{X}$ which is an isomorphism over X. Note that, the inverse image of the quasi-compact constructible subset $\mathrm{Hom}_k(C, X) \subset \mathrm{Hom}_k(\overline{C}, \overline{X}')(k)$ in $\underline{\mathrm{Hom}}_k(\overline{C}, Z)(k)$ along the finitely presented morphism of k-schemes $\underline{\mathrm{Hom}}_k(\overline{C}, Z) \to \underline{\mathrm{Hom}}_k(\overline{C}, \overline{X}')$ is again a quasi-compact constructible subset and equals $\mathrm{Hom}_k(C, X)$. Moreover, the image of the latter quasi-compact constructible subset in $\underline{\mathrm{Hom}}_k(\overline{C}, \overline{X})(k)$ along the finitely presented morphism of k-schemes is again equal to $\mathrm{Hom}_k(C, X)$ and quasi-compact constructible (by Chevalley's theorem). This concludes the proof.

Lemma 3.6. *Let $\mathbb{Z} \subset A$ be a finitely generated integral domain of characteristic zero and let $\mathcal{X}$ be a finite type scheme over A. Let $k := \overline{\mathrm{Frac}(A)}$ be an algebraic closure of* $\mathrm{Frac}(A)$*. Let $k \subset L$ be an extension of algebraically closed fields with L of transcendence degree one over k. Assume that $X := \mathcal{X}_k$ is quasi-projective over k. Assume the following two properties hold.*

1. *The variety X_L is weakly bounded and geometrically hyperbolic over L.*
2. *For every finitely generated subalgebra $A' \subset k$ containing A, the set*

$$\mathcal{X}(A') = \mathrm{Hom}_A(\mathrm{Spec}\, A', \mathcal{X})$$

is not dense in X.

Then, for any finitely generated subring $B \subset L$ containing A, the set $\mathcal{X}(B)$ is not dense in $X(L)$.

Proof. To prove the statement, let $B \subset L$ be a finitely generated subring containing A and define $K := \mathrm{Frac}(B)$. We now show that $\mathcal{X}(B)$ is not dense in $X(L)$.

Note that if K has transcendence degree zero over $\mathrm{Frac}(A)$, then it follows from (2) that $\mathcal{X}(B)$ is not dense in $X(L)$. Therefore, to prove the lemma, we may and do assume that $K = \mathrm{Frac}(B)$ has transcendence degree one over $\mathrm{Frac}(A)$. Moreover, replacing A by a finitely generated sub-A-algebra of k, we may and do assume that the scheme $\mathcal{C} := \mathrm{Spec}\, B$ over $\mathrm{Spec}\, A$ has a section $\sigma : \mathrm{Spec}\, A \to \mathcal{C}$ and that $\mathcal{C} \to \mathrm{Spec}\, A$ is a smooth morphism.

Define $C := \mathcal{C}_k = \mathcal{C} \times_A k$, and note that C is a smooth affine curve over k. Furthermore, since $\mathcal{C}(A) \neq \emptyset$ and $\mathcal{C}$ is an integral scheme, it follows that C is connected. We let σ_k be the k-rational point of C induced by the section $\sigma : \mathrm{Spec}\, A \to \mathcal{C}$; we will also view this as an L-point of C_L. Let $\overline{C}$ be the smooth projective connected model of C over k. Now, let Δ be the closure of the subset $\mathrm{Im}[\mathcal{X}(A) \to X(k)]$ in X with the reduced closed subscheme structure, and note that $\Delta \subsetneq X$ is a proper closed subscheme by our second assumption (2).

Let $\overline{X}$ be a projective variety over k with $X \subset \overline{X}$ an open immersion. Since X_L is weakly bounded over L, it follows that $\mathrm{Hom}_L(C_L, X_L)$ is a quasi-compact constructible subset of $\underline{\mathrm{Hom}}_L(\overline{C}_L, \overline{X}_L)(L)$; see Proposition 3.5. We define $Z \subset \mathrm{Hom}_L(C_L, X_L)(L)$ to be the closure of

$$\mathrm{Im}[\mathcal{X}(\mathcal{C}) \to \mathrm{Hom}_L(C_L, X_L)(L)]$$

in $\mathrm{Hom}_L(C_L, X_L)(L)$. Since $Z \subset \mathrm{Hom}_L(C_L, X_L)(L)$ is closed, it follows that Z is a quasi-compact constructible subset of $\underline{\mathrm{Hom}}_L(\overline{C}_L, \overline{X}_L)(L)$.

Note that the evaluation map $Z \to X(L)$ which sends f to $f(\sigma_k)$ has finite fibres by the geometric hyperbolicity of X_L over L. Moreover, since the dense subset $\mathrm{Im}[\mathcal{X}(\mathcal{C}) \to \mathrm{Hom}_L(C_L, X_L)]$ of Z lands in Δ, we see that the evaluation map $Z \to X(L)$ factors through $\Delta(L)$. In particular, it follows that

$$\dim Z \leq \dim \Delta.$$

Therefore, since $\dim \Delta < \dim X$, we see that

$$\dim Z < \dim X.$$

Consider the Cartesian diagram of morphisms of schemes

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_L(\overline{C}_L, \overline{X}_L) & \longrightarrow & \overline{X} \times_k \operatorname{Spec} L \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}_k(\overline{C}, \overline{X}) \times_k \overline{C} & \longrightarrow & \overline{X} \times \overline{C}, \end{array}$$

where the bottom horizontal arrow is given by the universal evaluation morphism $(f, c) \mapsto (f(c), c)$, and the right vertical morphism is induced by the geometric generic point $\operatorname{Spec} L \to \overline{C}$ of $\overline{C}$. Since Z is a quasi-compact constructible subset of the L-points $\underline{\mathrm{Hom}}_L(\overline{C}_L, \overline{X}_L)(L)$ of the scheme $\underline{\mathrm{Hom}}_L(\overline{C}_L, \overline{X}_L)$, its image Z' along the morphism $\underline{\mathrm{Hom}}_L(\overline{C}_L, \overline{X}_L) \to \overline{X}_L$ is a quasi-compact constructible subset of $\overline{X}(L)$. Note that $Z' \subset X(L)$ and that $\dim Z' \leq \dim Z < \dim X$. Since $\mathrm{Im}[\mathcal{X}(\mathcal{C}) \to X(L)]$ is contained in Z' and $\dim Z' < \dim X$, we conclude that $\mathcal{X}(\mathcal{C})$ is not dense in $X(L)$, as the dimension of the closure of Z' is at most that of Z', by constructibility. □

The following result provides a general criterion for proving the persistence of non-density of integral points on an algebraic variety. In fact, in Theorem 5.4, we verify that a variety with a quasi-finite period map verifies the first property necessary to apply this result.

Theorem 3.7. *Let A be a finitely generated integral domain of characteristic zero and let $\mathcal{X}$ be a finite type scheme over A. Let $k := \overline{\mathrm{Frac}(A)}$ be an algebraic closure of* $\mathrm{Frac}(A)$*, and let $k \subset L$ be an extension of algebraically closed fields. Let $X := \mathcal{X} \times_A k$. Assume the following two properties hold.*

1. *The variety X_L is weakly bounded and geometrically hyperbolic over L.*
2. *For every finitely generated subring $A' \subset k$ containing A, the set*

$$\mathcal{X}(A') = \mathrm{Hom}_A(\operatorname{Spec} A', \mathcal{X})$$

   is not dense in X.

Then, for any finitely generated subring $B \subset L$ containing A, the set $\mathcal{X}(B)$ is not dense in X_L.

Proof. Let K be the algebraic closure of $\mathrm{Frac}(B)$ in L, and note that K has finite transcendence degree over k. We proceed by induction on the transcendence degree d of K over k. If $d = 0$, then the required non-density statement holds by (2). Now, assume $d > 0$ and let $K_0 \subset K$ be an algebraically closed subfield of transcendence degree $d - 1$ over k. Define $Y := X_{K_0}$. Now, as X_L is weakly bounded and geometrically hyperbolic over L, we have that Y_K is weakly bounded and geometrically hyperbolic over K. Moreover, write $\mathcal{Y} = \mathcal{X}$ (for the sake of clarity) and note that, by the induction hypothesis, for every finitely generated subring $A' \subset K_0$ containing A, the set $\mathcal{Y}(A')$ is not dense in Y. Therefore, as K has transcendence degree one over K_0, we conclude that $X_K = Y_K$ satisfies the required non-density statement (Lemma 3.6). This concludes the proof. □

4. Hodge theory

We set notation by recalling the definition of a period domain, following [44, Sect. 3]. Let H be a finitely-generated free $\mathbb{Z}$-module, k an integer, and $\{h^{p,k-p}\}$ a collection of non-negative integers with $h^{p,k-p} = h^{k-p,p}$ for all p, such that

$$\sum_p h^{p,k-p} = \mathrm{rk}_{\mathbb{Z}} H.$$

Let $\hat{\mathscr{F}}$ be the flag variety parametrizing decreasing, exhaustive, separated filtrations of $H_{\mathbb{C}}$, $(F^\bullet)$, with $\dim F^p = \sum_{i \geq p} h^{i,k-i}$.

Let $\mathscr{F} \subset \hat{\mathscr{F}}$ be the analytic open subset of $\hat{\mathscr{F}}$ parametrizing those filtrations corresponding to $\mathbb{Z}$-Hodge structures of weight k, i.e. those filtrations with

$$H_{\mathbb{C}} = F^p + \overline{F^{k-p+1}}$$

for all p.

Now suppose q is a non-degenerate bilinear form on $H_{\mathbb{Q}}$, symmetric if k is even and skew-symmetric if k is odd. Let $D \subset \mathscr{F}$ be the locally closed analytic subset of $\mathscr{F}$ consisting of filtrations corresponding to polarized Hodge structures (relative to the polarization q), i.e. the set of filtrations $(F^\bullet)$ in $\mathscr{F}$ with

$$q_{\mathbb{C}}(F^p, F^{k-p+1}) = 0 \text{ for all } p$$

and

$$q_{\mathbb{C}}(Cv, \bar{v}) > 0$$

for all nonzero $v \in H_{\mathbb{C}}$, where C is the linear operator defined by $C(v) = i^{p-q}v$ for

$$v \in H^{p,q} := F^p \cap \overline{F^q}.$$

Let $G = O(q)$ be the orthogonal group of q; it is a $\mathbb{Q}$-algebraic group. We abuse notation to denote $G_{\mathbb{Z}} = G(\mathbb{Q}) \cap GL(H)$. Let $\Gamma \subset G_{\mathbb{Z}}$ be a finite index subgroup. Let h be a point of the complex-analytic space $\Gamma\backslash D$.

Let X be an integral variety over k. A complex-analytic map to $f : X \to \Gamma\backslash D$ is *locally liftable* if it locally factors through the quotient map $D \to \Gamma\backslash D$. If X is smooth and f is locally liftable, we say f is *horizontal* if for each $x \in X$ and each tangent vector v in $T_x X$, we have that for a local lifting $\tilde{f}$ of f, $\tilde{f}_*(v) \in T_{\tilde{f}(x)} D$ sends $F^p_{\tilde{f}(x)}$ into $F^{p-1}_{\tilde{f}(x)}$, for each p. Here we regard the tangent space to D at a point d as an element of

$$\mathrm{End}(H_{\mathbb{C}})/\mathrm{Lie}(\mathrm{Stab}_{G_{\mathbb{C}}}(F^\bullet_d)).$$

Horizontality is equivalent to the statement that the associated variation of Hodge structure satisfies Griffiths transversality. See [44, Sect. 3] for details.

If X is smooth, then we say that a holomorphic map $X^{\mathrm{an}} \to \Gamma\backslash D$ is *a period map* if it is locally liftable and horizontal (i.e., satisfies Griffiths transversality). More generally, a holomorphic map $X^{\mathrm{an}} \to \Gamma\backslash D$ is a period map if there is a desingularization $\tilde{X} \to X$ such that the composed morphism $\tilde{X}^{\mathrm{an}} \to X^{\mathrm{an}} \to \Gamma\backslash D$ is a period map.

Since Γ does not act freely on D, the fundamental group of $\Gamma\backslash D$ might be trivial (e.g., $D = \mathbb{H}$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. To study period maps, it is more natural to consider the orbifold fundamental group of $\Gamma\backslash D$, i.e., the fundamental group of the stacky quotient $[\Gamma\backslash D]$ in the category of complex-analytic stacks. Note that the orbifold fundamental group is the image of Γ in the group of biholomorphic automorphisms $\mathrm{Bihol}(D)$ of D. In particular, since the kernel of $\Gamma \to \mathrm{Bihol}(D)$ is finite (Selberg's Lemma REFERENCE), the orbifold fundamental group of $\Gamma\backslash D$ is the quotient of Γ by the finite subgroup of elements acting trivially on D.

Given a period map $p : X^{\mathrm{an}} \to \Gamma\backslash D$ and a point x in X^{an}, we refer to the induced homomorphism of orbifold fundamental groups $p_* : \pi_1(X^{\mathrm{an}}, x) \to \pi_1^{orb}(\Gamma\backslash D, p(x))$ as the monodromy representation of p. Two period maps with the same monodromy representation are equal if they also agree at a point; this is the content of the Rigidity Theorem.

Proposition 4.1. *(Rigidity Theorem) Let D, Γ, h, and X be as above. Let $f : X^{\mathrm{an}} \to \Gamma\backslash D$ be a period map and let $g : X^{\mathrm{an}} \to \Gamma\backslash D$ be a period map. Let $x \in X^{\mathrm{an}}$ such that $f(x) = g(x) = h$. Assume that the monodromy representation $f_* : \pi_1(X, x) \to \pi_1^{orb}(\Gamma\backslash D, h)$ of f is equal to the monodromy representation $g_* : \pi_1(X, x) \to \pi_1(\Gamma\backslash D, h)$ of g. Then, we have that $f = g$.*

Proof. This is the so-called rigidity theorem of Deligne-Griffiths-Schmid [44, 7.24]. (This was proven in [9, Sect. 4] in the case that the variation of Hodge structures comes from geometry.) □

The following result is based on "Arakelov's inequality". Historically, this started with Arakelov-Parshin [1] for families of curves of genus at least two, and was then subsequently generalized to more general variations of Hodge structures by Deligne, Faltings, Peters, and Jost-Zuo; see [12,33,41]. However, to obtain the desired statement for varieties with a quasi-finite period map, we need *in addition* to the aforementioned results the recently established algebraization theorem of Bakker-Brunebarbe-Tsimerman [2].

Theorem 4.2. (Consequence of Arakelov's inequality) *Let D, Γ, and X be as above. Assume X is quasi-projective over $\mathbb{C}$ and let $p : X^{\mathrm{an}} \to \Gamma\backslash D$ be a period map with finite fibres. Then X is weakly bounded over $\mathbb{C}$ (see Definition 3.3).*

Proof. If X is smooth, then Theorem 4.2 can be deduced directly from recent results of Deng [11]. Indeed, in *loc. cit.* Deng shows that X is in fact "algebraically hyperbolic". If X is singular, we argue as follows.

Let $g : C \to X$ be a map as in Definition 3.3. The map $f \circ g$ classifies a variation of Hodge structure on C; let $\mathscr{V}^{p,k-p}$ be the graded pieces of the associated Hodge bundle, and let $\mathscr{F}^p$ be the associated filtration. Consider the Griffiths bundle

$$\mathscr{L} = \bigotimes_i \det(\mathscr{F}^i).$$

By [2, Theorem 6.2], the line bundle $\mathscr{L}$ is ample. Thus, to conclude the proof, it suffices to show that X is weakly bounded (in some compactification) with respect to $\mathscr{L}$.

To show that the desired bound exists, we first appeal to a result of Peters. In fact, by [42, Theorem 3.1], we have that, for each p, the integer $\deg \mathscr{V}^{p,k-p}$ is bounded in terms of only the genus of $\overline{C}$ and $\#(\overline{C} \setminus C)$ (as well as invariants of the variation of Hodge structure on X, which is fixed).

Now, it follows from the aforementioned result of Peters that the degree of C with respect to the auxiliary variation of Hodge structures $\otimes_{p\in\mathbb{Z}}\Lambda^{r_p}\mathscr{V}$ with $r_p = \operatorname{rank} \mathscr{F}^p$ is bounded by a constant depending only on $\overline{C}$, $\#(\overline{C} \setminus C)$, n, and invariants of $\mathscr{V}$. In particular, since the Griffiths line bundle $\mathcal{L}$ is the lowest piece of the Hodge filtration of $\otimes_{p\in\mathbb{Z}}\Lambda^{r_p}\mathscr{V}$, the same holds for the degree of $C \to X$ with respect to $\mathcal{L}$. We conclude that X is weakly bounded, as required. □

Remark 4.3. In the proof of Theorem 4.2 we appeal to the recent work of Bakker-Brunebarbe-Tsimerman [2] to guarantee the ampleness of $\mathscr{L}$. If X is smooth, we expect that this fact can also be deduced from the methods of Sommese's classical paper [45].

5. Proof of Theorems 1.6 and 1.7

The geometric finiteness property we require in this paper to prove the persistence of arithmetic hyperbolicity for varieties with a quasi-finite period map is provided by the following finiteness theorem; see Definition 2.1 for the definition of geometric hyperbolicity.

We stress that the following theorem is proven by combining Deligne's finiteness result for monodromy representations [10] with Deligne-Grifitths-Schmid's 'Rigidity Theorem" (Proposition 4.1). Deligne's finiteness theorem for monodromy representations used below generalizes the result of Faltings for weight one monodromy representations [12], but itself does not immediately imply the desired finiteness result we require.

Theorem 5.1. [Deligne + Rigidity Theorem] *Let X be a variety over k which admits (up to Galois conjugation) a quasi-finite period map. Then X is geometrically hyperbolic over k.*

Proof. We may and do assume that $k = \mathbb{C}$ (by Lemma 2.4) and that X admits a period map $X^{\mathrm{an}} \to \Gamma\backslash D$ with finite fibres. We wish to show that for each c in C and x in X, the set $\operatorname{Hom}((C, c), (X, x))$ of maps $\phi : C \to X$ with $\phi(c) = x$ is finite. The map

$$\operatorname{Hom}((C, c), (X, x)) \to \operatorname{Hom}((C, c), (\Gamma\backslash D, f(x)))$$

has finite fibers, because f is quasi-finite. Thus it is enough to show that there are finitely many locally liftable, horizontal analytic maps

$$f' : (C, c) \to (\Gamma\backslash D, f(x)).$$

We denote the set of such maps by

$$\operatorname{Hom}_{per}((C, c), \Gamma\backslash D, f(x)).$$

By Proposition 4.1 above, the map

$$\mathrm{Hom}_{per}((C,c),\Gamma\backslash D, f(x)) \to \mathrm{Hom}(\pi_1(C,c),\pi_1(\Gamma\backslash D, f(x)))$$

has finite fibers, so it suffices to show that it has finite image. But this is precisely Deligne's finiteness theorem [10]. □

Remark 5.2. Let $g \geq 8$ be an integer, and let X be the fine moduli space of principally polarized g-dimensional abelian varieties with full level 3 structure over $\mathbb{C}$. In [12] Faltings showed that there is a smooth curve C such that the set of non-constant morphisms $f : C \to X$ is infinite. Therefore, as X admits a quasi-finite period map, this shows that one can not expect a strengthening of Theorem 5.1 for maps $C \to X$. That is, in Theorem 5.1 one needs to consider maps of *pointed* varieties to obtain finiteness.

Remark 5.3. Suppose that X is a proper scheme over $\mathbb{C}$ and that X admits a quasi-finite period map. In this case (as X is proper), there is a different proof of Theorem 5.1. Indeed, if X admits a quasi-finite period map, then X has no entire curves [19, Corollary 9.4]. Therefore, as X is also proper, it follows from Urata's theorem (Example 2.2) that X is geometrically hyperbolic.

Theorem 5.4. (Deligne + Rigidity Theorem + Arakelov inequality) *If X is a variety over k which admits a quasi-finite complex-analytic period map (Definition 1.5), then X is weakly bounded over k and, for every variety Y over k, every $y \in Y(k)$, and every $x \in X(k)$, the set of morphisms $f : Y \to X$ with $f(y) = x$ is finite.*

Proof. We first prove that X is geometrically hyperbolic over k.

If $k = \mathbb{C}$, then Theorem 5.1 says that X is geometrically hyperbolic over $\mathbb{C}$. By a standard specialization argument and Lemma 2.4, we have that X_L is geometrically hyperbolic over any algebraically closed field extension L of k (without any additional assumption on k).

Choosing L to be an uncountable algebraically closed field extension of k, it follows from Lemma 2.5 that, for every variety Y over L, every $y \in Y(L)$, and every $x \in X(L)$, the set of morphisms $f : Y \to X_L$ with $f(y) = x$ is finite. (As this holds over L, it certainly also holds over k, as required.)

To conclude the proof, it suffices to note that X is weakly bounded over k by Theorem 4.2. □

Proof (Proof of Theorem 1.7). This is part of the statement of Theorem 5.4. □

Proof (Proof of Theorem 1.6). Combine Proposition 2.7 and Theorem 5.4. □

Proof (Proof of Theorem 1.2). Replacing K by a finite field extension if necessary, we may choose an ample divisor $\mathcal{D}$ in $\mathcal{A}$ such that $\mathcal{D}_K = D$. Let $\mathcal{M}$ be the (fine) moduli space of smooth hypersurfaces in $\mathcal{A}$ over $\mathcal{O}_{K,S}$ representing $\mathcal{D}$, and note that $\mathcal{M}$ is a quasi-projective scheme over $\mathcal{O}_{K,S}$. Lawrence–Sawin showed that $\mathcal{M}_{\overline{\mathbb{Q}}}$ is arithmetically hyperbolic over $\overline{\mathbb{Q}}$ and that $\mathcal{M}_{\mathbb{C}}$ admits a quasi-finite period map (see [36, Theorem 1.1] and [36, Proposition 5.10], respectively). Therefore, by Theorem 1.6, the variety $\mathcal{M}_k$ is arithmetically hyperbolic over any algebraically closed field k of characteristic zero. This implies that $\mathcal{M}(R)$ is finite, as required. □

6. Arithmetic locally symmetric varieties

A (smooth connected) variety X over $\mathbb{C}$ is an *arithmetic locally symmetric* if there exists a bounded symmetric domain D, a torsionfree arithmetic subgroup of $\mathrm{Aut}(D)$ and an isomorphism of complex analytic spaces $X^{\mathrm{an}} \cong \Gamma\backslash D$ (see [6, Definition 4.3] for a precise definition). By Baily-Borel's theorem, an arithmetic locally symmetric variety is quasi-projective over $\mathbb{C}$.

If X is a variety over k, then X is an *arithmetic locally symmetric variety over* k if there exists a subfield $k_0 \subset k$, an embedding $k_0 \to \mathbb{C}$, a variety X_0 over k_0, and an isomorphism of k-schemes $X_{0,k} \cong X$ such that $X_{0,\mathbb{C}}$ is an arithmetic locally symmetric variety over $\mathbb{C}$ (as defined above).

Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero, and let X be an arithmetic locally symmetric variety over k. Then, as X admits a quasi-finite period map (see [40, p. 40-43]), by combining Theorems 1.6, 3.7, and 5.4, we obtain the following result.

Theorem 6.1. *The following statements hold.*

1. *The arithmetic locally symmetric variety X_L is geometrically hyperbolic over L and weakly bounded over L.*
2. *If X is arithmetically hyperbolic over k, then X_L is arithmetically hyperbolic over L.*
3. *Let $A \subset k$ be a finitely generated $\mathbb{Z}$-algebra, and let $\mathcal{X}$ be a model for X over A. Assume that, for every finitely generated subring $A' \subset k$ containing A, the set $\mathcal{X}(A')$ is not dense in X. Then, for any finitely generated subring $B \subset L$ containing A, the set $\mathcal{X}(B)$ is not dense in X_L.*

Remark 6.2. By the work of Margulis, most examples of arithmetic locally symmetric quasi-projective varieties are Shimura varieties (associated to a *torsionfree* congruence subgroup of $\mathrm{Aut}(D)$).

7. The moduli of smooth hypersurfaces

In this section we prove Theorem 1.3. Throughout this section, let $d \geq 3$ be an integer, let $n \geq 2$ be an integer, and let $\mathcal{C}_{d;n} = [\mathrm{PGL}_{n+2}\backslash \mathrm{Hilb}_{d,n}]$ be the stack of smooth hypersurfaces in $\mathbb{P}^{n+1}$ of degree d. The stack $\mathcal{C}_{d;n}$ is a finite type separated Deligne-Mumford algebraic stack over $\mathbb{Z}$ with affine coarse space; see [3]. Moreover, by [26], the stack $\mathcal{C}_{d;n,\mathbb{Q}}$ is uniformisable, i.e., there is a smooth affine scheme $U := U_{d,n}$ over $\mathbb{Q}$ and a finite étale morphism $U \to \mathcal{C}_{d;n,\mathbb{Q}}$. Now, if $(d,n) \neq (3,2)$ the natural period map on the smooth affine scheme $U^{\mathrm{an}}_{\mathbb{C}}$ is injective on tangent spaces by a theorem of Griffiths [15] (see also [17]), as smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}$ satisfy the infinitesimal Torelli property (as $(d,n) \neq (3,2)$). This implies that the associated period map on $U^{\mathrm{an}}_{\mathbb{C}}$ has finite fibres (see [25, Thm. 2.8]).

The next result says that, for k an algebraically closed field of characteristic zero, the stack $\mathcal{C}_{d;n,k}$ is geometrically hyperbolic over k. That is, the moduli space of pointed maps from any given pointed variety into the stack is finite, i.e., such maps to $\mathcal{C}_{d;n,k}$ are *rigid* and form a *bounded* moduli space.

Theorem 7.1. *Let k be an algebraically closed field of characteristic zero, and let X be a smooth hypersurface of degree d in $\mathbb{P}_k^{n+1}$ over k. Let Y be an integral variety over k and let y in $Y(k)$. Then, the set of Y-isomorphism classes of smooth hypersurfaces $\mathcal{X}$ of degree d in $\mathbb{P}_Y^{n+1}$ such that $\mathcal{X}_y$ is isomorphic to X over k is finite.*

Proof. Replacing k by a field extension if necessary, we may and do assume that k is uncountable. Let U be a smooth affine scheme over k such that there is a finite étale morphism $U \to \mathcal{C}_{d;n,k}$ of stacks (see [26]).

Assume that $(d, n) \neq (3, 2)$. Then U admits a quasi-finite complex-analytic period map (up to Galois conjugation) by Griffiths's theorem (see for instance [15]). Thus, it follows from Theorem 5.4 that U is geometrically hyperbolic over k. Now, by a standard descent argument (Lemma 2.6), we deduce that the finite type separated Deligne-Mumford algebraic stack $\mathcal{C}_{d;n,k}$ is geometrically hyperbolic over k. In particular, as k is uncountable, it follows from Lemma 2.5 that, for every integral normal variety Y over k, every point y in $Y(k)$, and every x in $\mathcal{C}_{d;n}(k)$, the set of morphisms $f : Y \to \mathcal{C}_{d;n,k}$ with $f(y)$ isomorphic to x is finite.

Now, suppose that $d = 3$ and $n = 2$. We use the cyclic covering trick to deduce the desired finiteness statement from the case of cubic threefolds. Namely, by the cyclic covering trick, there is a μ_3-gerbe $\mathcal{H} \to \mathcal{C}_{3;2,k}$ (see [29, Example 3.25]) and a quasi-finite (even unramified) morphism $\mathcal{H} \to \mathcal{C}_{3;3,k}$ (see [29, Proposition 4.2]). Moreover, the μ_3-gerbe $\mathcal{H} \to \mathcal{C}_{3;2,k}$ has a section [29, Proposition 3.26.(3)], so that there exists a quasi-finite morphism $\mathcal{C}_{3;2,k} \to \mathcal{C}_{3;3,k}$. (On the level of k-points, this morphism associates to the isomorphism class of a cubic surface defined by $F(x_0, x_1, x_2, x_3) = 0$ the isomorphism class of the cubic threefold defined by $x_4^4 = F(x_0, x_1, x_2, x_3)$.) Since $\mathcal{C}_{3;3,k}$ is geometrically hyperbolic and $\mathcal{C}_{3;2,k} \to \mathcal{C}_{3;3,k}$ is quasi-finite, it follows that $\mathcal{C}_{3;2,k}$ is geometrically hyperbolic, as required. □

We now use the geometric hyperbolicity of the stack to show that its arithmetic hyperbolicity persists over field extensions. Concerning arithmetically hyperbolic *stacks*, we follow the conventions of [24, 4].

Theorem 7.2. *Let L be an algebraically closed field of characteristic zero. The stack $\mathcal{C}_{d;n,\overline{\mathbb{Q}}}$ is arithmetically hyperbolic over $\overline{\mathbb{Q}}$ if and only if $\mathcal{C}_{d;n,L}$ is arithmetically hyperbolic over L.*

Proof. Let $U := U_{d,n} \to \mathcal{C}_{d;n,\overline{\mathbb{Q}}}$ be a finite étale surjective morphism with U a smooth affine scheme over $\overline{\mathbb{Q}}$ (see [26]). Since $\mathcal{C}_{d;n,\overline{\mathbb{Q}}}$ is arithmetically hyperbolic over $\overline{\mathbb{Q}}$ (by assumption) and $U \to \mathcal{C}_{d;n,\overline{\mathbb{Q}}}$ is quasi-finite, it follows from [24, Proposition 4.16] that U is arithmetically hyperbolic over $\overline{\mathbb{Q}}$. Since U admits a quasi-finite period map (as explained above), it follows from Theorem 5.4 that U_L is geometrically hyperbolic over L. Thus, as U is arithmetically hyperbolic over $k := \overline{\mathbb{Q}}$ and U_L is geometrically hyperbolic over L, we conclude that U_L is arithmetically hyperbolic over L (Proposition 2.7). Now, as $U_L \to \mathcal{C}_{d;n,L}$ is finite étale, it follows from stacky Chevalley-Weil [24, Theorem 5.1] that $\mathcal{C}_{d;n,L}$ is arithmetically hyperbolic over L. □

Lemma 7.3. *Let K be a number field and let S be a finite set of finite places of K. Then, for every finite type affine group scheme G over $\mathcal{O}_{K,S}$, the set of $\mathcal{O}_{K,S}$-isomorphism classes of G-torsors over $\mathcal{O}_{K,S}$ is finite.*

Proof. This is [16, Proposition 5.1]. □

We now show that the Shafarevich conjecture for hypersurfaces over number fields [25, Conjecture 1.4] *implies* a finiteness result for hypersurfaces over finitely generated fields of characteristic zero.

Proof (Proof of Theorem 1.3). The assumption is that, for every number field K and every finite set S of finite places of K, the set of $\mathcal{O}_{K,S}$-isomorphism classes of smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}_{\mathcal{O}_{K,S}}$ is finite. Now, an object of the groupoid $\mathcal{C}_{d;n}(\mathcal{O}_{K,S})$ is given by the data of a Brauer-Severi scheme P over $\mathcal{O}_{K,S}$ and a smooth hypersurface of degree d in P (see [29, Lemma 4.1] for a detailed explanation of this).

Our assumption says that, for every number field K and every finite set of finite places S of K, the set of $\mathcal{O}_{K,S}$-isomorphism classes of smooth hypersurfaces in $\mathbb{P}^{n+1}_{\mathcal{O}_{K,S}}$ is finite. The proof of (4) $\implies$ (1) in [29, Proposition 4.5] shows that $\mathcal{C}_{d;n,\overline{\mathbb{Q}}}$ is arithmetically hyperbolic over $\overline{\mathbb{Q}}$.

Now, let A be a $\mathbb{Z}$-finitely generated normal integral domain of characteristic zero with fraction field K and let $k := \overline{K}$ be an algebraic closure of K. Since $\mathcal{C}_{d;n,\overline{\mathbb{Q}}}$ is arithmetically hyperbolic over $\overline{\mathbb{Q}}$, it follows that $\mathcal{C}_{d;n,k}$ is arithmetically hyperbolic over k by Theorem 7.2. Therefore, since A is normal and $\mathcal{C}_{d;n}$ is a finite type separated Deligne-Mumford algebraic stack over $\mathbb{Z}$, we conclude that the set of isomorphism classes of objects of $\mathcal{C}_{d;n}(A)$ is finite from the twisting lemma [24, 4.3]; here we use that A is integrally closed in its fraction field. In particular, the set of A-isomorphism classes of smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}_A$ is finite. This concludes the proof. □

8. Non-density of integral points on the Hilbert scheme

In this section we prove Theorem 1.9 (which is a statement about integral points on the Hilbert scheme) by using that the *stack* of smooth hypersurfaces is weakly bounded and geometrically hyperbolic.

Recall that $\mathrm{Hilb}_{d,n}$ denotes the Hilbert scheme of smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}$ over $\mathbb{Z}$, and that this is a smooth affine scheme over $\mathbb{Z}$. As before, we write $\mathcal{C}_{d;n} = [\mathrm{PGL}_{n+2}\backslash\mathrm{Hilb}_{d,n}]$ for the stack of smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}$. In our proof of Theorem 1.9 we will use Lemma 7.3 to relate the non-density of integral points on the Hilbert scheme to the non-density of integral points on the stack.

Lemma 8.1. *Assume that, for every number field K and finite set of finite places S of K, the set $\mathrm{Hilb}_{d,n}(\mathcal{O}_{K,S})$ is not dense in $\mathrm{Hilb}_{d,n}$. Then, for every number field K and finite set of finite places S of K, the set of isomorphism classes of objects of $\mathcal{C}_{d;n}(\mathcal{O}_{K,S})$ is not dense in $\mathcal{C}_{d;n}$.*

Proof. Let K be a number field and let S be a finite set of finite places of K. Suppose that $\mathcal{C}_{d;n}(\mathcal{O}_{K,S})$ is dense in $\mathcal{C}_{d;n}$. Note that the set of $\mathcal{O}_{K,S}$-isomorphism classes of PGL_{n+2}-torsors over $\mathcal{O}_{K,S}$ is finite. Let $r \geq 1$ be an integer, and let $T_1, \ldots, T_r$ be representatives for all the PGL_{n+2}-torsors over $\mathcal{O}_{K,S}$, up to $\mathcal{O}_{K,S}$-isomorphism. Let L be a number field over K and let T be a finite set of finite places containing all the places of L lying over S such that, for every $i = 1, \ldots, r$, the PGL_{n+2}-torsor T_i is trivial over $\mathcal{O}_{L,T}$. For every x in $\mathcal{C}_{d;n}(\mathcal{O}_{K,S})$, the fibre of the torsor $\mathrm{Hilb}_{d,n} \to \mathcal{C}_{d;n}$ over x has a dense set of $\mathcal{O}_{L,T}$-points. This implies that the set of $\mathcal{O}_{L,T}$-points $\mathrm{Hilb}_{d,n}(\mathcal{O}_{L,T})$ of $\mathrm{Hilb}_{d,n}$ is dense. □

Lemma 8.2. *(Non-density à la Chevalley–Weil) Let $X \to Y$ be a finite étale surjective morphism of varieties over k. Assume that, for every $\mathbb{Z}$-finitely generated subring $A \subset k$ and every (finite type separated) model $\mathcal{X}$ for X over A, the set $\mathcal{X}(A)$ is not dense. Then, for every $\mathbb{Z}$-finitely generated subring $A \subset k$ and every model $\mathcal{Y}$ for Y over A, the set $\mathcal{Y}(A)$ is not dense in Y.*

Proof. We follow the pf of the Chevalley–Weil theorem (see for example [7, Theorem 3.8]).

We argue by contrapositive and assume that there is a $\mathbb{Z}$-finitely generated subring $A \subset k$ and a model $\mathcal{Y}$ for Y over A such that the set $\mathcal{Y}(A)$ is dense in Y. Replacing A by a larger $\mathbb{Z}$-finitely generated subring of k, we may assume that there is a model $\mathcal{X}$ for X over A and a finite étale surjective morphism $\mathcal{X} \to \mathcal{Y}$ extending the morphism $X \to Y$. Let $\mathrm{Spec}\, A \to \mathcal{Y}$ be an A-point. Note that its pull-back along $\mathcal{X} \to \mathcal{Y}$ is a B-point, where $B \to \mathrm{Spec}\, A$ is a finite étale surjective morphism of degree at most the degree of $X \to Y$.

Now, recall the following well-known extension of Hermite's finiteness theorem: if $D \geq 1$ is an integer and A is a $\mathbb{Z}$-finitely generated normal integral domain of characteristic zero, then the set of A-isomorphism classes of finite étale morphisms $T \to \mathrm{Spec}\, A$ of degree at most D is finite [20, Theorem, p.1].

In particular, it follows that there is a $\mathbb{Z}$-finitely generated extension (even finite étale) of A contained in k, say $A \subset B \subset k$, such that every A-point of $\mathcal{Y}$ lifts to a B-point of $\mathcal{X}$. This implies that $\mathcal{X}(B)$ is dense. Since $\mathcal{Y}(A)$ is dense, it follows that $\mathcal{X}(B)$ is dense, as required. □

Proof (Proof of Theorem 1.9). By [26], there is an integer $D \geq 3$, a smooth affine scheme U over $\mathbb{Z}[1/D]$ and a finite étale Galois morphism $U \to \mathcal{C}_{d;n,\mathbb{Z}[1/D]}$ of stacks over $\mathbb{Z}[1/D]$. The assumption on the non-density of integral points on the Hilbert scheme $\mathrm{Hilb}_{d,n}$ implies by Lemma 8.1 that, for every number field K and every finite set of finite places S of K, we have that $\mathcal{C}_{d;n}(\mathcal{O}_{K,S})$ is not dense in $\mathcal{C}_{d;n}$. Then, as $U \to \mathcal{C}_{d;n,\mathbb{Z}[1/D]}$ is surjective, we conclude that, for every number field K and every finite set of finite places S of K, the set $U(\mathcal{O}_{K,S})$ is not dense in U. Note that $U_{\mathbb{C}}$ is a smooth affine scheme over $\mathbb{C}$ which admits a quasi-finite period map (Sect. 7). Therefore, it follows from Theorem 3.7 and Theorem 5.4 that the non-density of integral points on U persists over finitely generated fields, i.e., for every $\mathbb{Z}$-finitely generated integral domain A of characteristic zero, the set $U(A)$ is not dense in U. Let $H' := \mathrm{Hilb}_{d,n,\mathbb{Z}[1/D]} \times_{\mathcal{C}} U$, where $\mathcal{C} := \mathcal{C}_{d;n,\mathbb{Z}[1/D]}$. Note that $H' \to H$ is a finite étale morphism of schemes over $\mathbb{Z}[1/D]$. Since $H' \to U$

is surjective, it follows that, for every $\mathbb{Z}$-finitely generated integral domain A of characteristic zero, the set $H'(A)$ is not dense in H'. Therefore, by Lemma 8.2, as $H' \to \mathrm{Hilb}_{d,n,\mathbb{Z}[1/D]}$ is finite étale, we conclude that, for every $\mathbb{Z}$-finitely generated integral domain A of characteristic zero, the set $\mathrm{Hilb}_{d,n}(A)$ is not dense. □

Remark 8.3. One may wonder whether arguing on the stack (or on U) is necessary and whether one could simply argue only on the Hilbert scheme in the proof of Theorem 1.9. The problem is that the Hilbert scheme $\mathrm{Hilb}_{d,n,\mathbb{C}}$ (over the complex numbers) does **not** admit a quasi-finite period map, is not geometrically hyperbolic, and is not weakly bounded. Thus, to use our results on varieties with a quasi-finite period map, one is forced (in our proof of Theorem 1.9) to argue on the moduli stack of smooth hypersurfaces (or its finite étale atlas U).

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest

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