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Polar germs, Jacobian ideal and analytic classification of irreducible plane curve singularities

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Abstract. We examine the relationship between the analytic type of an irreducible plane curve singularity with a single characteristic exponent and the germs of curve defined by the elements of its Jacobian ideal, in particular its polar germs.

1. Introduction

Let $\xi : f = 0$ be a reduced non-empty germ of analytic plane curve, defined in a neighbourhood of the origin of $\mathbb{C}^2$ by a convergent series $f \in \mathbb{C}\{x, y\}$ with no multiple factors. Here the ideal $\mathbf{J}(\xi) = (f_x, f_y, f)$, f_x and f_y the derivatives of f, will be called the *Jacobian ideal of* ξ; it does not depend on the series f defining ξ. The system of germs of curve defined by $\mathbf{J}(\xi)$, $\mathcal{J}(\xi) = \{\zeta : g = 0, g \in \mathbf{J}(\xi) - \{0\}\}$, will be called the *Jacobian system of* ξ. The polar germs of ξ –often called just *polars* in the sequel– are the germs of curve defined by equations $h_1 f_x + h_2 f_y + hf = 0$ with $h_1, h_2, h \in \mathbb{C}\{x, y\}$ and $h_i(0, 0) \neq 0$ for at least one $i = 1, 2$. Generic germs (in the sense of [6], 2.7) of $\mathcal{J}(\xi)$ are thus polar germs. In the sequel we will call *generalized polars* the elements of $\mathcal{J}(\xi)$. The weighted cluster of base points of the Jacobian ideal, $BP(\mathbf{J}(\xi))$, consists of the infinitely near points and multiplicities shared by generic polars ([6], 7.2.13, 7.2.15); we will often refer to it as the *(weighted cluster of) base points of the polars of* ξ. Generic polars have no singular points outside $BP(\mathbf{J}(\xi))$; therefore they all have the same topological type, which is determined by $BP(\mathbf{J}(\xi))$; it is called the *topological type of generic polars*. If a local analytic automorphism φ maps ξ to ξ', then it maps $BP(\mathbf{J}(\xi))$ onto $BP(\mathbf{J}(\xi'))$ preserving multiplicities; in particular, the topological type of generic polars is an analytic invariant of the germ.

Since the nineteenth century, polar curves and polar germs are known to enclose deep information: in a global context, they determine the singular points of a reduced projective plane curve and may be used to control the resolution of its singularities ([10], IV.II.14, for instance); in the local case, the base points of the polars of a reduced germ of curve ξ are known to determine the topological type of ξ ([6]

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8.6.4, [2] 2.4); in a higher-dimensional context, polar varieties provide the so called *polar invariants* ([11,13,18]).

Interest in polar germs was increased after an example due to Pham ([17]), showing that the topological type of generic polars of a germ of curve ξ depends on the analytic type of ξ, and not only on its topological type. As a consequence, rather evident characters of the polar germs of ξ may be used to unveil much more hidden analytic characters of ξ: for instance, topologically equivalent germs of curve may be shown not to be analytically equivalent by showing that their generic polars are not topologically equivalent. For irreducible germs general enough, the base points of the polar germs provide a number of continuous analytic invariants ([5]). Regarding the whole Jacobian ideal, Mather and Yau proved in [14] that the quotient algebra $\mathbb{C}\{x, y\}/\mathbf{J}(\xi)$ determines the analytic type of ξ. More recently, a deep result first claimed by Hefez and Hernándes [12], characterizes the analytically relevant coefficients of the Puiseux series of an irreducible germ ξ in terms of the intersection multiplicities of ξ and the germs in its Jacobian system $\mathcal{J}(\xi)$.

The main purpose of this paper is to go deeper into the relationship between polars –and generalized polars– and analytic classification of irreducible germs of plane curve with single characteristic exponent m/n, on the basis of the analytic information recently given in [7] and examining in particular how much of this information is retained by the polars.

Let us first recall some results from [7]. For fixed coprime integers $m, n, m > n > 1$, are considered there the germs of curve γ with equations

$$f = y^n - x^m + \sum_{ni+mj>nm} A_{i,j} x^i y^j = 0, \quad A_{i,j} \in \mathbb{C},$$

which represent all analytic classes of irreducible germs with single characteristic exponent m/n. As usual, the point $(i, j) \in \mathbb{R}^2$ is associated to the coefficient $A_{i,j}$. Then –with the exceptions noted below –are determined there the coefficients $A_{i,j}$ whose variation changes the analytic type of γ *(relevant coefficients)* and, among them, those whose value is –but for a finite conjugation– determined by the analytic type of γ (*continuous invariants*). The excepted coefficients are those corresponding to the integral points in the interior of a certain triangle T: each of them may be either non-relevant or a continuous invariant, depending on the values of preceding –by the ordering induced by the value of $ni + mj$– coefficients. The coefficients corresponding to the points in T are called *conditional invariants*. For the easiest representative example, see Fig. 1.

Now, for the main results obtained here, those relative to polars are essentially negative, showing that the topological type of generic polars and the base points of the polars retain only partial information on the analytic type of the germ. More precisely, for germs γ with single characteristic exponent m/n as above, we obtain:

- The Zariski invariant $\sigma(\gamma)$ –the easiest analytic, non-topological, invariant of γ, see Sect. 3– is not determined by the topological type of generic polars, see 3.1.
- The topological type of generic polars is the same for all germs γ with Zariski invariant $\sigma(\gamma) > m - n$, see 3.5. The inequality $\sigma(\gamma) > m - n$ is a moderate

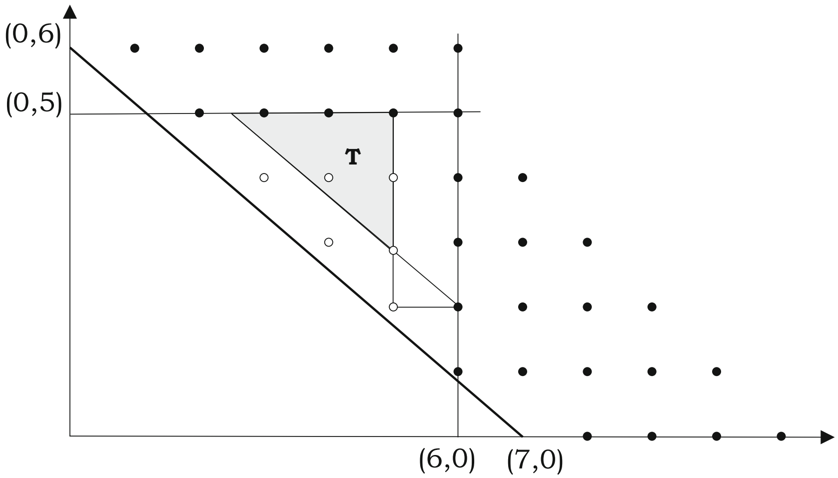

Fig. 1. The case $m/n = 7/6$, see [7], 12.14. The coefficient $A_{6,1}$ is not relevant and therefore may be turned into zero. This done, $A_{5,2}$ is relevant; if $A_{5,2} \neq 0$, then $A_{4,3}$ and $A_{3,4}$ are continuous invariants and $A_{4,4}$ is a conditional invariant: is a continuous invariant if $63A_{4,3}^2 - 56A_{3,4} - 20 = 0$, and is non-relevant otherwise. The other coefficients are all non-relevant. Standing and non-standing points (see Sect. 2) are represented by white and black dots, respectively. It is easy to check that in this case the topological type of generic polars does not depend on the values of the continuous invariants

restriction allowing the appearance of continuous and conditional invariants; in some cases –for instance for $m/n = 7/6$, see Fig. 1– it is satisfied by all germs.
- For any γ, the cluster of base points of its polars is unaffected by the arbitrary variation of any coefficient $A_{i,j}$ with $ni + mj > nm + m$, see 4.3. Some of these coefficients may be continuous or conditional invariants, as is the case of $A_{4,4}$ in the example of Fig. 1.

Generalized polars enclose much more information, as shown in the already cited [14] and [12]. Regarding them, here we obtain:

- A proof of a single characteristic exponent version of Hefez and Fernandes' characterization: a coefficient $A_{i,j}$, $ni + mj > nm + \sigma(\gamma)$, is relevant if and only if there is no generalized polar ζ with $[\zeta \cdot \gamma] = ni + mj$ (7.5). It results a rather explicit determination of analytic automorphisms causing the variation of each non-relevant coefficient and leaving the preceding coefficients invariant (7.7).
- The existence of a continuous broken line –the *staircase line* of γ– that separates the points in T corresponding to conditional invariants into those corresponding to continuous invariants and those corresponding to irrelevant coefficients, see Fig. 3, 7.10 and 7.12. The staircase line depends on the values of continuous invariants of γ and is directly related to a standard basis of $\mathbf{J}(\gamma)$ (8.4); in particular, it easily gives the Tjurina number $\tau(\gamma) = \dim \mathbb{C}\{x, y\}/\mathbf{J}(\gamma)$ (9.5).

The content is organized as follows: Sects. 3 and 4 are devoted to polars; besides the results quoted above, they contain a close examination of Newton polygons of polars and the way they are shaped by the values of certain continuous invariants. Sections 5 and 6 are preparatory; in the second one there is a new way to obtain coordinates on infinitesimal neighbourhoods which may be of independent interest. Section 7 deals with Hefez and Fernandes' characterization and the staircase line. Section 8 completes the information about the intersection multiplicities $[\zeta \cdot \gamma]$, ζ a generalized polar; this provides a characterization of the Zariski invariant (8.2). In Sect. 9, a couple of easy tools are adapted from computational algebra and applied to compute the Zariski invariant (9.3) and the staircase line (9.9). They are used in three examples: in 9.4 a non-evident Zariski invariant is computed, 9.10 sheds light on an already known case of jumping Tjurina number and 9.11 describes the different possibilities of relevance of coefficients for germs with characteristic exponent $17/6$ and minimal Zariski invariant.

2. Preliminaries

We place ourselves under the general conventions of [6] and, more especifically, under those of [7]. In particular, germs of complex analytic plane curves will often be called just *germs*. Positive coprime integers m, n, $m > n > 1$ will be fixed throughout and we will mainly consider irreducible germs γ, defined in a neighbourhood of the origin O of $\mathbb{C}^2$ by a convergent power series of the form

$$f = y^n - x^m + \sum_{ni+mj>nm} A_{i,j} x^i y^j, \quad A_{i,j} \in \mathbb{C}; \tag{1}$$

as it is well known ([7], Section 2, for instance), they are irreducible, have single characteristic exponent m/n, and a Puiseux series

$$S = x^{m/n} + \sum_{r>0} c_r x^{(m+r)/n}, \quad c_r \in \mathbb{C}. \tag{2}$$

Furthermore, any irreducible germ with single characteristic exponent m/n is analytically equivalent to a germ γ as before. Therefore, our analytically invariant conclusions will hold for arbitrary irreducible germs with a single characteristic exponent.

The multiplicity of a point O on a curve or germ of curve ξ will be denoted $e_O(\xi)$. The Newton polygon of a germ $\xi : g = 0$, $g = \sum_{i,j\geq 0} B_{i,j} x^i y^j$ will be taken to be the union of the compact sides and vertices of the border of the convex envelope of $\{(i, j) \in \mathbb{R}^2 | B_{i,j} \neq 0\} + (\mathbb{R}^+)^2$. Newton polygons will be drawn on a plane $\mathcal{N} = \mathbb{R}^2$ –the *Newton plane*– in which we conventionally assume that the first axis is horizontal, oriented from left to right, while the second axis is vertical, oriented from bottom to top. We will take the Newton polygons and their sides –all with negative slope– oriented from top left to bottom right. $\mathbb{N}^2$ will denote the set of points of $\mathcal{N}$ with non-negative integral coordinates. Non-zero vectors v on $\mathcal{N}$ with both components non-negative will be called positive vectors, indicated $v > 0$.

We will take $ni + mj$ as the twisted degree of any non-zero monomial $ax^i y^j$. The twisted degree td (resp. twisted order to) of a non-zero polynomial (resp. series) g will be the highest (resp. lowest) twisted degree of the non-zero monomials of g, and we will take $\mathrm{to}(0) = \infty$. The twisted initial form of $g \in \mathbb{C}\{x, y\} - \{0\}$ is the sum of all its monomials of minimal twisted degree, usually denoted $\bar{g}$ in the sequel. The integer $ni + mj$ will be also taken as being the *twisted degree* of the point (i, j), which represents $ax^i y^j$ on the Newton plane.

As in [7], a point $(i, j) \in \mathbb{N}^2$ with $ni + mj > nm$ will be called *standing* if $i < m - 1$ and $j < n - 1$, and *non-standing* otherwise. The monomials corresponding to standing points and their coefficients will be called *standing monomials* and *standing coefficients*, respectively.

Remark 2.1. The reader may easily check that if p is a standing point, then no point $p' \in \mathbb{N}^2$, $p' \neq p$, has $\mathrm{td}(p) = \mathrm{td}(p')$. In particular, if $g \in \mathbb{C}\{x, y\}$ has $\mathrm{to}(g) = \mathrm{td}(p)$, p a standing point, then the twisted initial form of g has a single monomial, and such a monomial corresponds to p. It easily turns out that the set of twisted degrees of the standing points and the set of twisted degrees of the non-standing points are disjoint, and also that the latter is

$$(m(n-1) + \langle n, m\rangle) \cup (n(m-1) + \langle n, m\rangle),$$

where $\langle n, m\rangle$ is the semigroup generated by n, m.

A coefficient $A_{h,k}$ of the equation (1) of a germ γ, say with $nh + mk = d$, is said to be *(analytically) irrelevant* if and only if for any $\alpha \in \mathbb{C}$ there is a germ γ_α, analytically equivalent to γ and defined by an equation

$$y^n - x^m + \sum_{d \geq ni+mj > nm} A_{i,j} x^i y^j + \alpha x^h y^k + \cdots = 0,$$

the dots meaning terms of higher twisted degree. Irrelevant coefficients may be turned into zero by the action of a suitable local automorphism, with no modification to the other coefficients of equal or smaller twisted degree. This is called *eliminating* the coefficient, see [7], Section 5 for details. All non-standing coefficients are irrelevant ([7], 6.1). *Relevant* coefficients are of course those which are not irrelevant.

The reader is referred to [7], Sect. 1, for a description of which coefficients in (1) are relevant. In particular, the coefficients whose variation causes a variation of the analytic type with at most finitely many occurrences of each analytic type, are called *continuous invariants*. For a precise definition, a coefficient $A_{h,k}$, again with $nh + mk = d > nm$, in (1) is an continuous invariant of the germ $\gamma : f = 0$ if and only if there is a non-constant polynomial $\phi \in \mathbb{C}[X]$ such that no germs

$$\gamma_\ell : y^n - x^m + \sum_{nm < ni+mj \leq d} A_{i,j} x^i y^j + \rho_\ell x^h y^k + \cdots = 0,$$

$\ell = 1, 2$, are analytically equivalent if $\phi(\rho_1) \neq \phi(\rho_2)$, see [7], Sect. 9. Obviously, continuous invariants are relevant. Coefficients which may or may not be continuous invariants, depending on the values of lower twisted degree coefficients, are called *conditional invariants*.

For the convenience of the reader, we sketch below some elementary facts relating Newton polygons and infinitely near points that will be used in the sequel. The origin O of $\mathbb{C}^2$, the points infinitely near to O on the x-axis and the satellite points of the latter will be called *initial points*. If ζ is an irreducible germ with origin at O and Puiseux series $S = cx^{s/r} + \ldots$, $c \neq 0$, $r = e_O(\zeta)$, we will call $cx^{s/r}$ the *initial term*, and s/r the *initial exponent*, of ζ. The next lemma contains well known facts, see for instance [6], 5.3.1.

Lemma 2.2. *The hypothesis and notations being as above, the initial points ζ is going through, and their multiplicities on ζ, depend only on the pair (s,r) and not on ζ itself: the quotients appearing in the Euclidean algorithm for* $\gcd(s,r)$ *determine the points themselves, while the remainders are the multiplicities of the points. Also, irreducible germs with different initial exponents share no points other than initial points.*

Assume that $\xi : \sum_{i,j\geq 0} \alpha_{i,j} x^i y^j = 0$ is a germ of curve with origin at O and not containing any of the two germs of the coordinate axes. Let us recall (see for instance [6], 2.2) that each branch of ξ comes associated with a side of the Newton polygon $\mathbf{N}(\xi)$ of ξ. If Ξ is a side of $\mathbf{N}(\xi)$ which has width s, height r and lowest end i_0, j_0, then the branches associated with Ξ are ζ_k, $k = 1, \ldots, h$, $h > 0$, where each ζ_k has initial term $c_k x^{s_k/r_k}$, $r_k = e_O(\zeta_k)$, $s_k/r_k = s/r$ and c_k is a solution of $\sum_{(i,j)\in\Xi} \alpha_{i,j} Z^{j-j_0} = 0$ –the *equation associated to* Ξ. Also, if the branches are repeated according to their multiplicities as irreducible components of ξ, $\sum_k s_k = s$ and $\sum_k r_k = r$.

Lemma 2.3. *Let ξ be a germ not containing the germ of the y-axis. Then the initial points ξ goes through, as well as their multiplicities on ξ, are determined by* $\mathbf{N}(\xi)$.

Proof. If the germ of the x-axis is a branch of ξ, then it is a smooth branch and its multiplicity as an irreducible component of ξ is the second coordinate of the lowest vertex of $\mathbf{N}(\xi)$. All the other branches of ξ correspond to sides of $\mathbf{N}(\xi)$; let Ξ be one of these sides. The notations being as above, take $\xi_\Xi = \sum_k \zeta_k$, where the ζ_k are the branches of ξ associated with Ξ, repeated according to multiplicities. Being $s/r = s_k/r_k$ for all k, the quotients in the Euclidean algorithms for $\gcd(s,r)$ and each $\gcd(s_k, r_k)$ are the same; hence (2.2) ξ_Ξ and any of the ζ_k have the same initial points, and these are determined by (s,r). Further, since $s = \sum_k s_k$ and $r = \sum_k r_k$, the remainders in the Euclidean algorithm for $\gcd(s,r)$ equal the sums of the corresponding remainders in the algorithms for the $\gcd(s_k, r_k)$; using 2.2 again, the former are the multiplicities of the initial points on ξ_Ξ. By adding up for the different sides of $\mathbf{N}(\xi)$ and taking into account the germ of the x-axis if it is a branch of ξ, the claim follows. □

3. Polars

The results in this section and the next one show that, for irreducible germs of curve with a single characteristic exponent, both the topological type of generic polars and the base points of the polars give only partial information on the analytic type of the germ.

If $\xi : g = 0$ is a reduced germ of plane curve, we will consider the pencil of polar germs

$$\mathcal{P}_g = \{\xi : \lambda g_x + \mu g_y = 0 \mid (\lambda, \mu) \in \mathbb{C}^2 - (0,0)\},$$

generated by the polars $\xi_x : g_x = 0$ and $\xi_y : g_y = 0$. The pencil $\mathcal{P}_g$ depends on the choices of the coordinates x, y and the series g. Nevertheless, by [6], 8.5.7, $\mathcal{P}_g$ and $\mathbf{J}(\xi)$ have the same cluster of base points and so, in particular (by [6], 7.2.1), all but finitely many polars in $\mathcal{P}_g$ have the topological type of generic polars. In the sequel, the base points of the polars of ξ and the topological type of generic polars of ξ will be obtained from a pencil $\mathcal{P}_g$. To this end we will often use the Newton polygon shared by all but finitely many members of $\mathcal{P}_g$; the series g being fixed, it will be noted $\mathbf{PN}(\xi)$.

Let γ be a germ defined by an equation as (1); as said, it is irreducible and has single characteristic exponent m/n. The easiest analytic –non-topological– invariant of γ is the *Zariski invariant* $\sigma = \sigma(\gamma)$; it may be defined as

$$\sigma = \sigma(\gamma) = \min\{ni + mj \mid (i, j) \text{ standing and } A_{i,j} \neq 0\} - nm$$

provided all $A_{i',j'}$ with either $i' \geq m-1$ or $j' \geq n-1$ (non-standing coefficients) and $ni' + mj' < nm + \sigma$ have been previously turned into zero by the action of a suitable analytic automorphism ([7], 7.14). We take $\min \emptyset = \infty$; the germs with $\sigma = \infty$ (*quasihomogeneous* germs) are all analytically equivalent to $y^n - x^m = 0$, and therefore have little interest regarding the analytic classification. If $\sigma \neq \infty$, then the only point $\omega = (\omega_1, \omega_2) \in \mathbb{N}^2$ with twisted degree $nm + \sigma$ is called the *Zariski point* of γ. When finite, the Zariski invariant obviously carries the same information as the Zariski point. The next example shows that the Zariski invariant is not determined by the topological type of generic polars:

Example 3.1. The germs of curve

$$\gamma_\alpha : y^7 - x^{17} + \alpha x^5 y^5 + x^{15} y = 0, \quad \alpha \in \mathbb{C},$$

have $\sigma(\gamma_\alpha) = 1$ if $\alpha \neq 0$ and $\sigma(\gamma_0) = 3$. The topological type of generic polars of γ_α is the same for all values of α but those satisfying $\alpha^3 = -1323/500$. Indeed, the values of the Zariski invariant are clear. The Newton polygons $\mathbf{PN}(\gamma_\alpha)$ are the same for all α; in addition, they have a single side and the equation associated to it has simple roots provided $\alpha^3 \neq -1323/500$. Excluding these values of α, this is enough to determine the topological type and the claim follows.

The variation of the topological type of generic polars of the same germs is also worth some attention:

Example 3.2. Take the same germs γ_α as in Example 3.1 above, this time with $\alpha \neq 0$ in order to have the same Zariski point $(5, 5)$ for all. As seen in 3.1, the topological type of generic polars is constant for $\alpha^3 \neq -1323/500$. However, it changes for $\alpha^3 = -1323/500$, because then the equation associated to the only side of $\mathbf{PN}(\gamma_\alpha)$ acquires a double root. Therefore, also the analytic type of γ_α changes for $\alpha^3 = -1323/500$. This change is, however, just part of a continuous variation of the analytic type of γ_α with α^3 which otherwise does not affect the topological type of generic polars.

Indeed, by [7], 11.1, the coefficient corresponding to (15, 1) is a continuous invariant of γ_α. For a correct reading of it, the equation needs to be normalized by turning the coefficient of the term corresponding to the Zariski point into 1. For each fixed $\alpha \neq 0$, this is achieved by replacing γ_α with its inverse image by the local automorphism $x^* = \alpha^7 x$, $y^* = \alpha^{17} y$, which is

$$\gamma_\alpha^* : y^7 - x^{17} + x^5 y^5 + \frac{1}{\alpha^3} x^{15} y = 0$$

and gives $1/\alpha^3$ as the value of the invariant.

Back to considering an arbitrary germ γ with equation (1), we will pay some attention to the Newton polygons of polar germs of γ, and in particular to $\mathbf{PN}(\gamma)$. As it is well known, the Newton polygon of a germ of plane curve depends on the relative position of the germ and the coordinate axes. Therefore $\mathbf{PN}(\gamma)$ cannot be expected to be an analytic invariant of γ, and in fact it is not, see Example 3.10 below.

For generic germs γ (that is, for all γ with equation (1) but those whose coefficients $A_{i,j}$ satisfy certain finitely many polynomial equations), the Newton polygon $\mathbf{PN}(\gamma)$ was determined in [4]. Nevertheless, examining the relationship between the polars and the analytic type of γ requires considering all germs γ, as we will do next.

First of all, it is worth noting that if $\omega = (\omega_1, \omega_2)$ is the Zariski point, y-derivation of the corresponding monomial gives rise to the point $\omega = (\omega_1, \omega_2 - 1)$, which, in spite of the minimality of $\mathrm{td}(\omega)$, may not belong to $\mathbf{PN}(\gamma)$. Here is an example:

Example 3.3. Take $\gamma : y^{13} - x^{43} + x^7 y^{11} + x^{37} y^2 = 0$. Its Zariski point is (7, 11) and the point (7, 10) does not belong to any of the sides of $\mathbf{PN}(\gamma)$, whose supporting lines are defined by the equations $11i + 37j = 444$ and $i + 5j = 42$.

If $g \in \mathbb{C}\{x, y\}$, the set of points of $\mathcal{N}$ corresponding to the non-zero coefficients of g –the *Newton diagram* of g– will be denoted $\Delta(g)$ in the sequel. First of all note that the derivation $\partial/\partial y$ (resp. $\partial/\partial x$) acts on the points of $\Delta(f)$ by deleting the points on the first (resp. second) axis and shifting the other points one unit downwards (resp. leftwards). Therefore, no matter what the values of the coefficients $A_{i,j}$ are, $\Delta(f_y)$ always has $(0, n-1)$ as its lowest point on the second axis, while $\Delta(f_x)$ contains $(m-1, 0)$ and, being $m/n > 1$, no point $(0, r)$, $r \leq n-1$. It follows:

Lemma 3.4. $\mathbf{PN}(\gamma)$ *has* $(0, n-1)$ *as first vertex and no vertex strictly above the line segment* Γ *with ends the points* $(0, n-1)$ *and* $(m-1, 0)$.

As a consequence, the topological type of generic polars of γ ceases giving analytic information on γ if the Zariski invariant of γ is $m - n$ or higher:

Proposition 3.5. *Generic polars of irreducible germs of curve with single characteristic exponent m/n and Zariski invariant $\sigma \geq m-n$ have all the same topological type: if $d = \gcd(n-1, m-1)$, $n-1 = rd$ and $m-1 = sd$, then they are composed of branches ξ_k, $k = 1, \dots, h$, each ξ_k with a Puiseux series $S_k = \alpha_k x^{s/r} + \cdots$, $\alpha_k \neq 0$ and $\alpha_k^r \neq \alpha_\ell^r$ for $k \neq \ell$.*

Proof. By [7] 6.3, up to replacing γ with an analytically equivalent germ, we may assume that γ is given by an equation as (1) in which all non-standing coefficients $A_{i,j}$ (i.e., those with either $i \geq m-1$ or $j \geq n-1$) are zero. After this, since in particular $A_{m-1,1} = 0$, both ends of Γ belong to the Newton diagram of $\lambda(f_x) + \mu(f_y) = 0$ for $\lambda, \mu \neq 0$. Furthermore, by [7], 7.14, if $A_{i,j} \neq 0$, then $ni + mj \geq nm + \sigma \geq nm + m - n$. The monomials produced by deriving a non-zero monomial $A_{i,j}x^i y^j$ correspond to the points $(i-1, j)$ and $(i, j-1)$, and have twisted degrees

$$ni + mj - n \geq nm + \sigma - n \geq nm + m - 2n > nm - n \quad \text{and}$$
$$ni + mj - m \geq nm + \sigma - m \geq nm - n.$$

Since the maximum of the twisted degrees of the points on Γ is $nm - n$, reached at the end $(m-1, 0)$ only, and neither of the above points is $(m-1, 0)$ (because $A_{m-1,1} = 0$), they both lie strictly above Γ. This proves that Γ is the only side of $\mathbf{PN}(\gamma)$ and also that no point on Γ other than its ends corresponds to a non-zero coefficient of the equation of a polar in $\mathcal{P}_f$. Then the equations associated to Γ have no multiple roots and the claim follows by just computing the initial terms of the Puiseux series of the polars $\lambda(f_x) + \mu(f_y) = 0$ with $\lambda, \mu \neq 0$. □

For an arbitrary γ, $\mathbf{PN}(\gamma)$ is not far away from Γ, see Fig. 2:

Lemma 3.6. *For any germ γ with equation (1), the first vertex of $\mathbf{PN}(\gamma)$ is $(0, n-1)$, and the last one lies on the first axis. $\mathbf{PN}(\gamma)$ is contained in the (closed) triangle Λ limited by the line $ni + mj = nm - m$, the segment Γ and the first axis.*

Proof. Obviously $\mathbf{PN}(\gamma)$ is contained in the half-plane $j \geq 0$. It has been seen in 3.4 that $\mathbf{PN}(\gamma)$ has $(0, n-1)$ as first vertex and no vertex strictly above Γ. Since all non-zero monomials in f have bidegree (i, j) with $ni + mj \geq nm$, no point in the Newton diagram of f_y lies strictly below the line $ni + mj = nm - m$ and no point in the Newton diagram of f_x lies strictly below the line $ni + mj = nm - n$. This proves that $\mathbf{PN}(\gamma) \subset \Lambda$. Finally, for $\lambda A_{m-1,1} - m\mu \neq 0$, the monomial of bidegree $(m-1, 0)$ in the equation of the polar is not zero, and so, necessarily, the last vertex of $\mathbf{PN}(\gamma)$ lies on the first axis. □

Lemma 3.7. *The slope $-r/s$ of any side of $\mathbf{PN}(\gamma)$ satisfies $s/r > m/n$.*

Proof. the integers n, m being coprime, no point in $\mathbb{N}^2$ other than $(0, n-1)$ lies on the line $ni + mj = nm - m$. Hence, by 3.6, the first side of $\mathbf{PN}(\gamma)$ has slope strictly higher than $-n/m$, and the same holds for the other sides due to the convexity of the polygon. □

The equations of the polars in $\mathcal{P}_f$ other than $f_x = 0$ may be written in the form $\theta f_x + f_y$, $\theta \in \mathbb{C}$; then, some of their coefficients remain constant:

Lemma 3.8. *The coefficients of $h_\theta = \theta f_x + f_y$ corresponding to the points in Λ other than $(m-1, 0)$, are all independent of θ.*

Proof. As noticed in the proof of 3.6, the Newton diagram of f_x lies in the half-plane $ni + mj \geq nm - n$, whose only common point with Λ is $(m-1, 0)$. □

Let $\hat{\Lambda}$ be the image of Λ by the translation of vector $(0, 1)$. By 3.8, the coefficients of h_θ corresponding to the points $(i, j-1) \in \Lambda$, $(i, j-1) \neq (m-1, 0)$, are $jA_{i,j}$, $(i, j) \in \hat{\Lambda}$, $(i, j) \neq (m-1, 1)$. Regarding the $A_{i,j}$ involved, we have:

Lemma 3.9. *Neither of the coefficients $A_{i,j}$, $(i, j) \in \hat{\Lambda}$, $(i, j) \neq (m-1, 1)$, is a conditional invariant.*

Proof. The Zariski point $\omega = (\omega_1, \omega_2)$ being a standing point, it holds $n - 2 \geq \omega_2 \geq 1$, and hence, using that $n\omega_1 + m\omega_2 > nm$, also $\omega_1 > 2m/n$. Therefore, according to their definition (see [7], Section 12), all points (i, j) corresponding to conditional invariants satisfy either

$$ni + mj > n\omega_1 + m(n-1) > nm + m$$

or

$$ni + mj > n(m-1) + m\omega_2 \geq nm + m - n.$$

Since the maximum of the twisted degrees of the points in $\hat{\Lambda}$ is $nm + m - n$, the claim follows. □

There may be non-standing points in the interior of $\hat{\Lambda}$; the elimination of their corresponding –necessarily irrelevant– coefficients by local automorphisms may change $\mathbf{PN}(\gamma)$, but of course not the topological type of generic polars. Here is an example:

Example 3.10. Take $\gamma : f = y^4 - x^{21} + 4x^6y^3 = 0$. The point $(6, 3)$ is non-standing and belongs to both $\Delta(f)$ and $\hat{\Lambda}$. The Newton polygon $\mathbf{PN}(\gamma)$ has vertices $(3, 0)$, $(6, 2)$ and $(20, 0)$, the intermediate one being originated by $(6, 3)$ by derivation. The inverse image of γ by the local automorphism $x^* = x$, $y^* = y - x^6$ is $\gamma^* : y^4 - x^{21} - 6x^{12}y^2 + 8x^{18}y - 3x^{24} = 0$, which has not the irrelevant monomial corresponding to $(6, 3)$. The Newton polygon $\mathbf{PN}(\gamma^*)$ has a single side, with vertices $(3, 0)$ and $(18, 0)$. Needless to say, see Sect. 1, generic polars of γ and γ^* have the same topological type, which may be directly checked by the reader in this case. The difference between $\mathbf{PN}(\gamma)$ and $\mathbf{PN}(\gamma^*)$ is due to the different positions of the polars with respect to the coordinate axes.

To avoid the effect of non-standing coefficients, assume that those with twisted degree less than $2nm - n - m + 1$ (or all) have been turned into zero by replacing γ with an analytically equivalent germ, still named γ ([7], 6.2 and 6.3). Then, by [7] 7.14, all critical coefficients are zero, but the one corresponding to the Zariski point $\omega = (\omega_1, \omega_2)$. By 3.9 and the previous elimination of non-standing coefficients, the points in $\hat{\Lambda}$ corresponding to a non zero $A_{i,j}$ are ω, in case it belongs

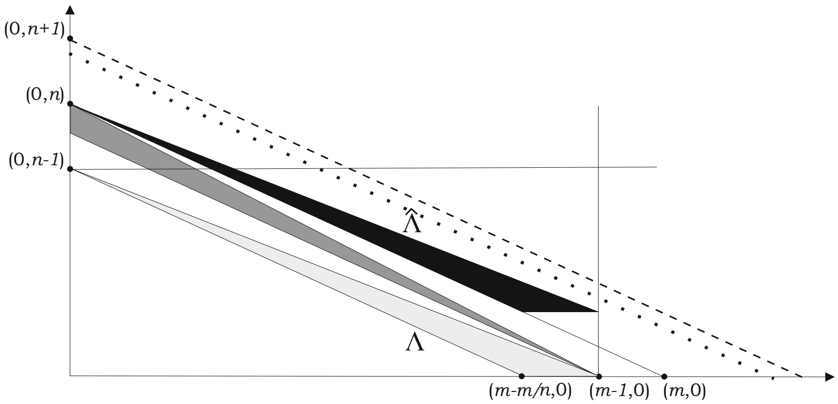

Fig. 2. The topological type of generic polars is the same for all germs γ with Zariski point above the dotted line $ni + mj = nm + m - n$ (3.5). The Newton polygon $\mathbf{PN}(\gamma)$ of generic polars of any germ γ lies inside the light grey triangle Λ (3.6). $\mathbf{PN}(\gamma)$ is determined by the coefficients corresponding to points inside the black triangle $\hat{\Lambda}$ (3.11). The free variation of the coefficients of the equation of γ corresponding to points above the dashed line $ni + mj = nm + m$ does not modify the base points of the polars (4.3). The unnamed medium grey triangle is the one described in 4.1

to $\hat{\Lambda}$, points corresponding to continuous invariants, and points corresponding to irrelevant monomials $A_{i,j}x^i y^j$, $(i, j) = \omega + v\ v > 0$, see [7] 8.1. By the positivity of v, no one of the latter gives rise, after y-derivation, to a point $(i, j-1)$ belonging to $\mathbf{PN}(\gamma)$. The Zariski point and the points in $\hat{\Lambda}$ corresponding to continuous invariants suffice thus to determine $\mathbf{PN}(\gamma)$. More precisely:

Proposition 3.11. *Assume that all the non-standing coefficients $A_{i,j}$ of the equation of γ up to the twisted degree $2nm - n - m + 1$ are zero. Let $\hat{L}$ be the subset of the Newton plane $\mathcal{N}$ composed of* $(0, n)$, $(m-1, 1)$, *the Zariski point and all points in $\hat{\Lambda}$ corresponding to non-zero continuous invariants. Take L to be the image of $\hat{L}$ by the translation of vector* $(0, -1)$. *Then the border of the convex envelope of $L + (\mathbb{R}^+)^2$ is composed of two half-lines, one on each coordinate axis, and the Newton polygon* $\mathbf{PN}(\gamma)$ *joining their ends.*

The reader may note in particular how, under the hypothesis of 3.11 and for fixed Zariski invariant, the different shapes of $\mathbf{PN}(\gamma)$ result from the annhilation of certain continuous invariants of γ.

4. Base points of polars

We will write $BP(\mathbf{J}(\gamma)) = \mathbf{PBP}$ if no confusion may arise. Fix $\xi : h = \lambda f_x + \mu f_y = 0$ to be a polar with Newton polygon $\mathbf{PN}(\gamma)$ and going sharply through $\mathbf{PBP}$. We take $h' = f_x$ if $A_{1,n} \neq 0$ and $h' = f_x + f$ otherwise, in order to have $(0, n) \in \Delta(h')$ in all cases.

Lemma 4.1. *The Newton polygon* $\mathbf{PN}(h')$ *has* $(0, n)$ *and* $(m-1, 0)$ *as its first and last vertices; it is contained in the triangle with sides* $ni + mj = nm - n$, $ni + (m-1)j = nm - n$ *and the second axis; the slope* $-r/s$ *of any of the sides of* $\mathbf{PN}(h')$ *satisfies* $s/r < m/n$.

Proof. The first claim is clear after the definition of h'. The second one follows from it, as we already know (proof of 3.6) that the Newton diagram of f_x is contained in the half-plane $ni + mj \geq nm - n$. The third claim follows then from an easy argument similar to the one used in the proof of 3.7. □

Lemma 4.2. *Any monomial* $x^i y^j$, *with* $ni + (m-1)j \geq nm - n$ *defines a germ of curve going through* **PBP**.

Proof. The polar $\xi' : h' = 0$ obviously goes through **PBP**. By 3.7 and 4.1, no branch of ξ' has the same initial exponent as a branch of ξ. As a consequence the points shared by ξ and ξ' are all initial points (by 2.2). Then, since all points in **PBP** belong to ξ, all the points in **PBP** the polar ξ' is effectively going through, are initial points. For any non-initial point $q \in$ **PBP**, ξ' does not effectively go through q and so, the multiplicity at q of the virtual transform with origin at q remains the same if ξ' is replaced with any germ ξ'' having the same effective multiplicities as ξ' at the initial points. As a consequence any such ξ'' also goes through **PBP** (see [6], 4.1, 4.2). By 2.3, this occurs if $\mathbf{N}(\xi'') = \mathbf{N}(\xi')$.

By the hypothesis and 4.1, the point $p = (i, j)$ does not lie below $\mathbf{N}(h')$; then, after taking $h'' = h' + \alpha x^i y^j$, $\alpha \neq 0$, it holds $\mathbf{N}(h'') = \mathbf{N}(h')$ for all values of α but at most one, in particular for a certain non-zero α. By the above, the germ $\xi'' : h'' = 0$ goes through **PBP**, after which so does the germ defined by $x^i y^j = \alpha^{-1}(h' - h)$. □

Theorem 4.3. *If* γ *is an irreducible germ of curve with single characteristic exponent* m/n, $m > n > 1$, m, n *coprime, and equation*

$$y^n - x^m + \sum_{ni+mj>nm} A_{i,j} x^i y^j = 0,$$

then the weighted cluster of base points of the polars of γ *and the topological type of its generic polars remain the same if any of the coefficients* $A_{i,j}$, $ni + mj \geq nm + m$ *is allowed to take arbitrary values.*

Some of the above coefficients $A_{i,j}$, $ni + mj \geq nm + m$ may be continuous invariants; then their values cannot be read from $\mathbf{PBP}(\gamma)$. This is the case for $m/n = 13/6$, see [7], Example 12.18.

Proof of 4.3:. Of course, only the part of the claim regarding **PBP** needs to be proved; the other has been included for the sake of completeness. We will prove that the germ obtained from γ by an arbitrary modification of $A_{i,j}$ still has two polars going sharply through **PBP** and sharing no points outside of it; from this, the claim directly follows.

Fix i, j satisfying the inequality of the claim and for any $\alpha \in \mathbb{C}$ let γ_α be the germ $\gamma_\alpha : f + \alpha x^i y^j = 0$. For each polar of γ, $\xi : g = \lambda f_x + \mu f_y = 0$, going sharply through **PBP** and with Newton polygon $\mathbf{PN}(\gamma)$, we will prove next that the polar of γ_α

$$\xi_\alpha : g_\alpha = \lambda f_x + \mu f_y + \alpha(i\lambda x^{i-1}y^j + j\mu x^i y^{j-1}) = 0$$

is also going sharply through **PBP**.

Note first that all branches of ξ are tangent to the x axis, because their initial exponents are higher than m/n (by 3.7). Also, all the infinitely near points shared by ξ and the x-axis belong to **PBP**, because these points depend only on the Newton polygon the germ (by 2.3), and therefore are also shared by any other polar in $\mathcal{P}_f$ going sharply through **PBP** and with Newton polygon $\mathbf{PN}(\gamma)$.

The coefficients λ, μ still being fixed, the polars ξ_α, together with the germ $\xi_\infty : i\lambda x^{i-1}y^j + j\mu x^i y^{j-1} = 0$, describe a pencil of germs of curve $\mathcal{B}$, generated by $\xi_0 = \xi$ and ξ_∞. The germ ξ obviously goes through **PBP**, and, after direct computation, so does ξ_∞ due to the hypothesis $ni + mj \geq nm + m$ and 4.2. All germs in $\mathcal{B}$ are thus going through **PBP**, and in particular so do the polars ξ_α.

Select a branch ζ of ξ. By 3.7, it has a Puiseux parameterization $x = t^r$, $y = ct^s + \cdots$, with $s/r > m/n$. Then the only non-zero monomial of minimal (r, s)-twisted degree in f_x is $-mx^{m-1}$ and hence, after substitution, the intersection multiplicity of ζ and the polar $\hat{\xi} : f_x = 0$ is $[\hat{\xi} \cdot \zeta] = r(m-1)$. Since ζ is contained in ξ, any polar $\tilde{\xi} \in \mathcal{P}_f$, $\tilde{\xi} \neq \xi$, has $[\tilde{\xi} \cdot \zeta] = r(m-1)$. In particular we chose such a $\tilde{\xi}$ to be going sharply through **PBP**. Being $\tilde{\xi} \neq \xi$, ζ and $\tilde{\xi}$ share no point outside **PBP**, otherwise such a point would be a base point. Using then the Noether formula ([6], 3.3.1), it results

$$r(m-1) = [\tilde{\xi} \cdot \zeta] = \sum_{p \in \mathbf{PBP}} \nu_p e_p(\zeta),$$

where ν_p is the virtual multiplicity of p in **PBP** and $e_p(\zeta)$ its (effective) multiplicity on ζ.

By using again the above Puiseux parameterization and the hypothesis, we obtain:

$$[\xi_\infty \cdot \zeta] = r\left(i + \frac{s}{r}(j-1)\right) > r\left(i + \frac{m}{n}(j-1)\right) \geq r(m-1) = \sum_{p \in \mathbf{PBP}} \nu_p e_p(\zeta). \tag{3}$$

Let q be the first point on ζ not in **PBP**. Since ξ goes sharply through **PBP**, q and all points following it on ζ are free and have multiplicity one on ζ. The point q clearly does not belong to $\xi_\infty : x^{i-1}(i\lambda y + j\mu x)y^{j-1} = 0$ because, as noted at the beginning of the proof, ζ is tangent to the x axis and all points shared by ξ and the x-axis are base points. Nevertheles, iterated use of the virtual Noether formula ([6], 4.1.2) and the inequality (3) show that the virtual transform with origin at q, $\hat{\xi}_\infty$, of ξ_∞ (relative to the virtual multiplicities in **PBP**) is non empty, and therefore equal to kE, where E is the germ of the exceptional divisor at q (a smooth germ because q is free) and k a positive integer.

The virtual transforms with origin at q of the germs in $\mathcal{B}$ describe a pencil $\hat{\mathcal{B}}$ generated by $\hat{\xi}_\infty$ and the virtual transform $\hat{\xi}$ of ξ. The latter is in fact the strict transform of ξ –because ξ goes sharply through **PBP**– and therefore is a smooth germ transverse to E. Using suitable equations z_1 of $\hat{\xi}$ and z_2 of E as local coordinates at q, the virtual transform $\hat{\xi}_\alpha$ of ξ_α has equation $z_1 + \alpha z_2^k = 0$, k a positive integer, and therefore is a smooth germ transverse to E. In particular $\hat{\xi}_\alpha$ has not E as a component and therefore is also the strict transform of ξ_α, which has then q as a non-singular point. Call ζ_α the only branch of ξ_α containing q: ζ and ζ_α, as any two irreducible germs sharing a non-singular point q, have the same points preceding q and the same multiplicities at them.

We have thus associated to each branch ζ of ξ, a branch ζ_α of ξ_α in such a way that both ζ and ζ_α have the same effective multiplicities at the points in **PBP** and the same first point outside **PBP**; such a point is non-singular for both branches and does not belong to other branches of ξ. Since, clearly from its definition, ξ_α has the same multiplicity $n-1$ as ξ, ξ_α has no branch other than those associated to the branches of ξ. In particular ξ_α has no singular points outside **PBP**. It suffices to add up the multiplicities of their branches at each point of **PBP** to see that ξ_α and ξ have the same multiplicities at the points of **PBP** and therefore that ξ_α goes sharply through **PBP**.

Now, to close, take λ', μ', $\lambda'/\mu' \neq \lambda/\mu$, such that still the polar of γ, $\xi' : g = \lambda' f_x + \mu' f_y = 0$ goes sharply through **PBP** and has Newton polygon $\mathbf{PN}(\gamma)$. Then, the above arguments applying also to ξ', the corresponding polar of γ_α, ξ'_α, goes sharply through **PBP** too.

As it is well known ([6], 6.4, for instance), the intersection number of two different polars of a reduced germ of curve is the Milnor number of the germ, which in our case, for both γ and γ_α, equals $nm - n - m + 1$. Therefore, if still ν_p denotes the virtual multiplicity of p in **PBP**,

$$[\xi_\alpha \cdot \xi'_\alpha] = [\xi \cdot \xi'] = \sum_{p \in \mathbf{PBP}} \nu_p^2,$$

the last equality using the Noether's formula ([6], 3.3.1) and the fact that both ξ and ξ' go sharply through **PBP** and share no points outside it. Since also ξ_α and ξ'_α go sharply through **PBP**, the equality displayed above shows that they share no point outside **PBP**. Therefore the weighted cluster of base points of the polars of γ_α is **PBP**, as claimed. □

5. Intersecting with generalized polars

Still taking $\gamma : f = 0$, with

$$f = y^n - x^m + \sum_{ni+mj>nm} A_{i,j} x^i y^j,$$

in this section we will consider the finite intersection multiplicities of γ and its generalized polars, and prove that these intersection multiplicities may be replaced with twisted orders of equations of generalized polars, and conversely.

We take

$$\Theta = \Theta(\gamma) = \{[\zeta \cdot \gamma] \mid \zeta \in \mathcal{J}(\gamma) \text{ and } \zeta \not\supset \gamma\}.$$

Remark 5.1. If we denote by Υ the semigroup of γ, namely

$$\Upsilon = \{[\zeta \cdot \gamma] \mid \zeta \text{ a germ of curve and } \zeta \not\supset \gamma\},$$

which in our case is the semigroup generated by n, m ([6], 5.8.2, for instance), we obviously have $\Theta \subset \Upsilon$ and $\Theta + \Upsilon \subset \Theta$.

In particular, Θ is a semigroup. The positive integers that do not belong to Θ will be called the *gaps* of $\mathbf{J}(\gamma)$. The Jacobian ideal being intrinsically related to the germ, analytically equivalent germs γ, γ' have $\Theta(\gamma) = \Theta(\gamma')$.

Remark 5.2. Both $nm-m$ and $nm-n$ belong to Θ, because they are the intersection multiplicities of γ and the polars $f_y = 0$ and $f_x = 0$ (by either the Plücker formula ([6], 6.3.2) or a direct substitution). If γ has finite Zariski invariant σ, then, $nm + \sigma \in \Theta$. Indeed, assuming, as allowed, all non-standing coefficients of f to be zero,

$$f = y^n - x^m + A_{\omega_1,\omega_2} x^{\omega_1} y^{\omega_2} + \dots, \quad A_{\omega_1,\omega_2} \neq 0,$$

where (ω_1, ω_2) is the Zariski point of γ. Then

$$nm + \sigma = n\omega_1 + m\omega_2 = [\zeta \cdot \gamma]$$

for $\zeta : nxf_x + myf_y - nmf = 0$.

Using the equality of differentials on γ,

$$Adx + Bdy = \frac{Af_y - Bf_x}{f_y} dx, \quad A, B \in \mathbb{C}\{x, y\},$$

it is direct to check that a suitable shifting of Θ is the set of the orders of the non-zero differentials on γ, which is the way in which Θ appears in [12].

In our case, Θ may be handled as the set of the twisted orders of the non-zero elements of the Jacobian ideal, which will be quite useful in the sequel:

Lemma 5.3. *A positive integer r belongs to Θ if and only if there is $h \in \mathbf{J}(\gamma)$ which has twisted order r.*

Proof. For an arbitrary germ of curve $\zeta : g = 0$, assume $\mathrm{to}(g) = d$ and $g = g_d + \dots$ where g_d is the twisted initial form of g and the dots represent terms of higher twisted degree. Substituting the Puiseux parameterization of γ, $x = t^n$, $y = t^m + \dots$, it is

$$[\zeta \cdot \gamma] = o_t(g(t^n, t^m + \dots)) = o_t(t^d(g_d(1, 1) + \dots)),$$

the dots meaning now terms of higher degree in t. Therefore we have $[\zeta \cdot \gamma] \geq \mathrm{to}(g)$ and the equality holds if and only if $g_d(1, 1) \neq 0$.

Now assume that $r \in \Theta$. Then there is a germ $\zeta : g = 0$, $g \in \mathbf{J}(\gamma)$, for which $r = [\zeta \cdot \gamma]$. By the above, $d = \mathrm{to}(g) \leq r$. If the equality holds, r is indeed the twisted order of a non-zero element of $\mathbf{J}(\gamma)$. Otherwise the twisted initial form g_d of g satisfies $g(1, 1) = 0$. An easy computation shows that then g_d has a factor $y^n - x^m$, say $g_d = (y^n - x^m)Q$. Substracting fQ cancells the twisted initial form of g, hence $g' = g - fQ \in \mathbf{J}(\gamma)$ has twisted order at least $d+1$; the germ it defines, $\zeta' : g' = 0$, still has $[\zeta' \cdot \gamma] = r$ and it is enough to use decreasing induction on $r - d$.

Assume $d = \mathrm{to}(g)$, $g \in \mathbf{J}(\gamma) - \{0\}$. Take $\zeta : g = 0$. Again it is $d \leq [\zeta \cdot \gamma]$ and, if the equality holds, $d \in \Theta$. Otherwise, the twisted initial form g_d of g satisfies $g(1,1) = 0$, and therefore, as above, has factor $y^n - x^m$. Assume $g_d = (y^n - x^m)^k H$, where $k > 0$, all monomials of H have twisted degree $d - knm$ and $H(1,1) \neq 0$. The twisted initial form of yf_y is ny^n, after which it is clear that

$$\left(g_d + (ny^n)^k H\right)(1,1) \neq 0. \tag{4}$$

In particular $g_d + (ny^n)^k H \neq 0$ and so, it has twisted degree d and is the twisted initial form of $g' = g + (yf_y)^k H$; then, obviously, $g' \in J(\gamma)$ and has twisted order d. If ζ' is the germ $\zeta' : g' = 0$, then, due to the inequality (4), $[\zeta' \cdot \gamma] = d$, as wanted. □

Remark 5.4. Since $\mathbf{J}(\gamma)$ is an (x, y)-primary ideal, any $g \in \mathbb{C}\{x, y\}$ with $\mathrm{td}(g)$ high enough belongs to $\mathbf{J}(\gamma)$: it follows thus from 5.3 that $\mathbf{J}(\gamma)$ has finitely many gaps.

Remark 5.5. Lemma 5.3 is not true for irreducible germs with two or more characteristic exponents: if the characteristic exponents are $6/4$, $9/4$, then $n = 4$, $m = 6$ and, by the Plücker formula ([6], 6.3.2), the intersection multiplicity with a generic polar is 21.

6. Coordinates and translations in first neighbourhoods

Let s be a positive integer. We will call q_s the point on γ in the s-th neighbourhood of the last satellite point, and E_s the first neighbourhood of q_s. E_s is a projective line and, q_s being free, contains a single satellite point $\bar{q}$; $E_s - \bar{q}$, which is the set of free points in E_s, may thus be taken as an affine line with improper (or ideal) point $\bar{q}$. In [7], Section 4, it is shown that coefficients of twisted degree $nm + s$ of the equations of irreducible germs through q_s may be taken as affine coordinates of their points in E_s. In this section we will show a different way of getting affine coordinates on E_s, and use them to control the action of local automorphisms of $\mathbb{C}^2$ on E_s.

Let $\eta : h = 0$, $h = y^n - x^m + \ldots$, the dots meaning terms of higher twisted degree, be an irreducible germ with single characteristic exponent m/n going through q_s. Let $x = t^n$, $y = t^m u(t)$, $u(0) = 1$, be a Puiseux parameterization of a second germ η', also irreducible, with single characteristic exponent m/n and going through q_s. The germs η and η' sharing the point q_s, by the Noether formula,

$$o_t(h(t^n, t^m u(t)) = [\eta \cdot \eta'] \geq nm + s.$$

and we have:

Proposition 6.1. *There is an affine coordinate on E_s such that, for any η' as above, if*

$$h(t^n, t^m u(t)) = \alpha t^{nm+s} + \dots,$$

then the point on η' in E_s has coordinate α.

Proof. Pick non-negative integers i, j with $ni + mj = nm + s$ and take $h_a = h + ax^i y^j$. By [7], 4.3, there is an affine coordinate on E_s such that for any $a \in \mathbb{C}$, a is the affine coordinate of the point on $\eta_a : h_a = 0$ in E_s. By substituting the Puiseux parameterization of η' in h_a,

$$h_a(t^n, t^m u(t)) = h(t^n, t^m u(t)) + at^{nm+s} u^j = (\alpha + a)t^{nm+s} + \dots.$$

Again by the Noether formula, η' and η_a have the same point in E_s if and only if $[\eta' \cdot \eta_a] > nm + s$, which, by the above equality, is equivalent to $\alpha = -a$, hence the claim. □

The coordinate of 6.1 depends on the choice of the series h defining η. It will be called the *coordinate associated to* h, or just the *h-coordinate*. The series h will always be taken of the form $h = y^n - x^m + \dots$. Obviously the h-coordinate of the point on η is $\alpha = 0$. A change between affine coordinates $z, \hat{z}$, of the form $\hat{z} = z + b$, $b \in \mathbb{C}$, will be called *unimodular*. We have:

Lemma 6.2. *The change between the affine coordinates $z, \hat{z}$ on E_s associated respectively to the equations $h = y^n - x^m + \dots$ and $\hat{h} = y^n - x^m + \dots$, of irreducible germs η and $\hat{\eta}$, each with single characteristic exponent m/n and both going through q_s, is unimodular.*

Proof. By [7], 4.5, there is an invertible $v \in \mathbb{C}\{x, y\}$ such that h and $v\hat{h}$ have the same partial sum of twisted degree $nm + s - 1$; it holds thus $v\hat{h} = h + B$, where $\mathrm{to}(B) \geq nm + s$. Since h and $\hat{h}$ have the same twisted initial form, of degree nm, necessarily $v(0) = 1$.

Let $x = t^n$, $y = t^m u(t)$, $u(0) = 1$, be a Puiseux parameterization of an irreducible germ η', with single characteristic exponent m/n and going through q_s. By substituting, on one hand

$$(v\hat{h})(t^n, t^m u) = (1 + \dots)(\hat{\alpha} t^{nm+s} + \dots) = \hat{\alpha} t^{nm+s} + \dots,$$

where $\hat{\alpha}$ is the $\hat{h}$-coordinate of the point on η' in E_s. On the other hand,

$$(h + B)(t^n, t^m u) = h(t^n, t^m u) + B(t^n, t^m u) = (\alpha + \widetilde{B}(1, 1))t^{nm+s} + \dots,$$

where α is the h-coordinate of the point on η' in E_s and $\widetilde{B}$ is the quasihomogeneous form of twisted degree $nm + s$ of B. This yields $\hat{\alpha} = \alpha + \widetilde{B}(1, 1)$, which proves the claim. □

If, in an affine line with a fixed affine coordinate z, a translation is given by the equation $z^* = z + b$, we will call the complex number b the *modulus* of the translation. As it is clear, the modulus may change by changing the affine coordinate, but it remains the same if the change of coordinate is unimodular.

The next proposition is our main goal in this section. Recall from [7], 9.2 that the principal automorphisms are the local automorphisms of $\mathbb{C}^2$ whose equations

$$x^* = \sum_{(i,j)\in\mathbb{N}^2-\{(0,0)\}} a_{i,j}x^i y^j, \quad y^* = \sum_{(i,j)\in\mathbb{N}^2-\{(0,0)\}} b_{i,j}x^i y^j,$$

satisfy $b_{i,0} = 0$ for $i < m/n$ and $a_{1,0} = b_{0,1} = 1$.

Proposition 6.3. *Assume that φ is a principal automorphism leaving invariant the point q_s. Take f to be a series defining γ of the form $f = y^n - x^m + \dots$ and let $x = t^n$, $y = t^m u$, $u(0) = 1$, be a Puiseux parameterization of γ. If on the first neighbourhood E_s of q_s we take the f-coordinate, then*

$$(\varphi^*(f))(t^n, t^m u(t)) = \beta(\varphi)t^{mn+s} + \dots, \quad \beta(\varphi) \in \mathbb{C},$$

and $\beta(\varphi)$ is the modulus of the translation (see [7], 12.2) induced by φ on E_s.

Proof. Since φ leaves invariant q_s, again by the Noether formula, $[\gamma \cdot \varphi^*(\gamma)] \geq nm + s$ and an equality as the one in the claim does hold. The automorphism φ being principal, the series $\varphi^*(f)$ still has the form $\varphi^*(f) = y^n - x^m + \dots$ and we may consider the coordinate associated to it. Using this coordinate, by 6.1, $\beta(\varphi)$ is the coordinate of the point on γ in E_s. This point is the image by φ of the point on $\varphi^*(\gamma)$, which has coordinate 0; thus, still using the coordinate associated to $\varphi^*(f)$, the modulus of the translation is $\beta(\varphi)$. The same holds using the coordinate relative to f due to 6.2. □

In particular:

Corollary 6.4. *The mapping $\varphi \mapsto \beta(\varphi)$ is a group homomorphism between the group of principal automorphisms at O leaving q_s invariant and the additive group of $\mathbb{C}$.*

7. Analytic relevance and gaps of $\mathbf{J}(\gamma)$

This section is devoted to give a proof of the theorem below, which sets a very interesting link between analytically relevant coefficients of a germ and gaps of its Jacobian ideal. It was first proved by Hefez and Hernandez in 2011. For the original version, which applies to germs with many characteristic exponents, the reader is referred to [12]. As an application, we will also prove the existence of a continuous broken line –the *staircase line* of the germ– that separates the points representing conditional invariants into points representing continuous invariants and points representing irrelevant coefficients.

Theorem 7.1. *Let γ be an irreducible analytic germ of plane curve with a single characteristic exponent m/n, finite Zariski invariant σ and equation*

$$f = y^n - x^m + \sum_{ni+mj>nm} A_{i,j} x^i y^j.$$

Then, for any integer $s > \sigma$, a coefficient $A_{i,j}$, with twisted degree $ni+mj = nm+s$, is analytically relevant if and only if there is no generalized polar $\zeta : g = 0$, $g \in \mathbf{J}(\gamma) - \{0\}$, with $[\zeta \cdot \gamma] = nm + s$.

Remark 7.2. The theorem does not cover the coefficients $A_{i,j}$ with twisted degree $nm+s$, $s \leq \sigma$, but the relevance of these is clear after [7], Section 7: if the inequality is strict, they are either non-standing coefficients –and therefore analytically irrelevant– or critical –hence relevant– coefficients taking their critical values; the only coefficient $A_{i,j}$ with twisted degree $nm + \sigma$ is critical and therefore relevant; it is the critical coefficient of highest twisted degree ([7], 9.3). Therefore, when the theorem applies, the coefficients corresponding to gaps are all continuous invariants (see [7], Sect. 1).

Regarding the coefficients $A_{i,j}$, with twisted degree $nm + s$, $s > \sigma$, using 5.1 and 5.2, it is direct to reprove from 7.1 that the non-standing ones are irrelevant, and also that so are those with $(i, j) = (\omega_1, \omega_2) + v$, where (ω_1, ω_2) is the Zariski point and $v > 0$ (Zariski's elimination criteria, see [7], 6.1 and 8.1).

It is worth noting that, in general, just because of 7.1, setting a certain positive integer r to be, or to be not, a gap of $\mathbf{J}(\gamma)$ may impose constraints on coefficients of lesser twisted degree that are continuous invariants. For instance, back to Zariski's example $m/n = 7/6$ ([19] V.5), as presented in [7] 12.15, all classes of germs with minimal Zariski invariant are represented by germs

$$y^6 - x^7 + x^5y^2 + A_{4,3}x^4y^3 + A_{3,4}x^3y^4 + A_{4,4}x^4y^4 = 0, \quad A_{4,3}, A_{3,4}, A_{4,4} \in \mathbb{C},$$

and $A_{4,3}$ and $A_{3,4}$ are continuous invariants. By the theorem, setting 52 as a gap is equivalent to the relevance of the coefficient $A_{4,4}$, which in turn, by [7] 12.15, is equivalent to the equality $63A_{4,3}^2 - 56A_{3,4} = 20$. Therefore, setting the semigroup Θ as a first invariant in order to analytically classify irreducible germs, makes clear which coefficients are continuous invariants, but leaves obscure which is the range of variation of each of them.

Proof of 7.1:. For the *only if* part we will assume that there is a generalized polar ζ as in the claim and construct from it a local flow whose members act transitively on the first neighbourhood of the point in the s-th neighbourhood of the last satellite, thus proving the irrelevance of the coefficient. For the converse, assuming the coefficient irrelevant, we will use results from [7] to construct a particular family of local automorphisms, and from it a generalized polar as wanted.

We will make frequent use of the fact, already seen in the proof of 5.3, that any germ $\delta : h = 0$ has $[\delta \cdot \gamma] \geq \mathrm{to}(h)$, the inequality being strict if and only if the twisted initial form of h has a factor $y^n - x^m$.

The claim being invariant by local automorphisms at O, in the sequel we assume that the series f defining γ has the form

$$y^n - x^m + x^{\omega_1} y^{\omega_2} + \sum_{ni+mj>nm+\sigma} A_{i,j} x^i y^j, \tag{5}$$

where (ω_1, ω_2) is the Zariski point and therefore $nm + \sigma = n\omega_1 + m\omega_2$ ([7], 7.15). Fix $s > \sigma$, and assume that there is a generalized polar $\zeta : g = 0$ that has $[\zeta \cdot \gamma] = nm + s$. Since subtracting from g a multiple of f does not change $[\zeta \cdot \gamma]$, we assume

$$g = Cf_x + Df_y, \quad C = \sum_{i,j\geq 0} c_{i,j} x^i y^j, \quad D = \sum_{i,j\geq 0} d_{i,j} x^i y^j.$$

We will need:

Lemma 7.3. *Hypothesis and notations being as above, one has $c_{0,0} = c_{1,0} = 0$, $d_{0,1} = 0$ and $d_{i,0} = 0$ for $i < m/n$.*

Proof of 7.3:. Along the proof we assume chosen a Puiseux parameterization of γ, say $x = t^n$, $y = t^m h(t)$, $h(0) = 1$, and for any $A \in \mathbb{C}\{x, y\}$ we write $o_t A = o_t A(t^n, t^m h(t))$. If $A \neq 0$ and δ is the germ $\delta : A = 0$, then $o_t(A) = [\delta \cdot \gamma]$. Note that, for any A, $o_t(A) \geq$ to A, and that, by either the Plücker formula ([6], 6.3.2) or a direct checking, $o_t f_x = nm - n$ and $o_t f_y = nm - m$.

Assume first that $c_{0,0} \neq 0$ and there is $i < m/n$ for which $d_{i,0} \neq 0$. Take such i to be the minimal one. Then, since $o_t y = m > in$, it holds $o_t D = in$.

If $in < m - n$, by the above,

$$o_t(Df_y) = in + nm - m < nm - n = o_t(f_x) = o_t(Cf_x)$$

and so

$$[\zeta \cdot \gamma] = o_t(Cf_x + Df_y) = o_t(Df_y) < nm - m < nm$$

against the hypothesis.

If $m - n < in < m$, then

$$o_t(Df_y) = in + nm - m > nm - n = o_t(f_x) = o_t(Cf_x)$$

and so, this time,

$$[\zeta \cdot \gamma] = o_t(Cf_x + Df_y) = o_t(Cf_y) = nm - m < nm$$

against the hypothesis.

If still $c_{0,0} \neq 0$ and there is no i, $ni < m$, for which $d_{i,0} \neq 0$, then, clearly, $o_t D \geq \text{to}(D) \geq m$. After this,

$$o_t(Df_y) \geq m + nm - m > nm - n = o_t(Cf_x),$$

leading to the same contradiction as above. Thus, $c_{0,0} = 0$.

Now, since $c_{0,0} = 0$, one has $o_t C \geq n$ and so

$$o_t(C f_x) \geq n + nm - n = nm.$$

If there is i, $in < m$, for which $d_{i,0} \neq 0$, then, assuming again that it is the minimal one, still $o_t D = in$ and we have

$$o_t(D f_y) = in + nm - m < nm,$$

which gives

$$[\zeta \cdot \gamma] = o_t(C f_x + D f_y) = o_t(D f_x) < nm,$$

once again against the hypothesis. Thus $d_{i,0} = 0$ for all $i < m/n$.

From what we have proved till now, $C f_x + D f_y = -mc_{1,0} x^m + nd_{0,1} y^n + \dots$, the dots meaning terms of higher twisted degree. If $mc_{1,0} \neq nd_{0,1}$, then direct substitution yields $[\zeta \cdot \gamma] = nm$, and so again a contradiction.

Assume to have thus $mc_{1,0} = nd_{0,1} \neq 0$. Then, up to dividing $g = C f_x + D f_y$ by $mc_{1,0}$, we may assume that $c_{1,0} = 1/m$ and $d_{0,1} = 1/n$, after which

$$g = y^n - x^m + \left(C - \frac{1}{m} x \right) m x^{m-1} + (D - \frac{1}{n} y) n y^{n-1} + \frac{nm + \sigma}{nm} x^{\omega_1} y^{\omega_2} + \dots,$$

where the dots indicate terms of twisted degree higher than $n\omega_1 + m\omega_2 = nm + \sigma$. As it is clear, g has twisted initial form $y^n - x^m$ and therefore ζ is an irreducible germ with single characteristic exponent m/n that goes through the first point on γ after the last satellite.

If ζ does not go through the point q_σ, on γ and in the σ-th neighbourhood of the last satellite, then, by the Noether formula, $[\zeta \cdot \gamma] < nm + \sigma$ against the hypothesis. Otherwise, by [7], 4.5, there is an invertible series, necessarily of the form $1 + u$, $u(0, 0) = 0$, such that $(1 + u)g$ and f have the same partial sum of twisted degree $nm + \sigma - 1$. Note that the monomials each monomial $\alpha x^i y^j$ of g gives rise to by multiplication by $1 + u$ are $\alpha x^i y^j$ itself, plus monomials $\alpha' x^k y^r$ with $(k, r) = (i, j) + v$, v a positive vector. Then, all monomials given rise to by monomials of

$$y^n - x^m + \left(C - \frac{1}{m} x \right) m x^{m-1} + \left(D - \frac{1}{n} y \right) n y^{n-1}$$

correspond to non-standing points. Since (ω_1, ω_2) is a standing point, $(1+u)g$ has the same monomial of bidegree (ω_1, ω_2) as g, and therefore the form

$$(1 + u)g = y^n - x^m + \frac{nm + \sigma}{nm} x^{\omega_1} y^{\omega_2} + \dots,$$

the dots still indicating terms of twisted degree higher than $n\omega_1 + m\omega_2 = nm + \sigma$. Then,

$$(1 + u)g - f = \frac{\sigma}{nm} x^{\omega_1} y^{\omega_2} + \dots$$

and considering the germ $\zeta' : (1 + u)g - f = 0$,

$$[\zeta \cdot \gamma] = [\zeta' \cdot \gamma] = o_t((1 + u)g - f) = nm + \sigma,$$

the last contradiction needed in order to prove the claim. $\square$

Lemma 7.4. *Still assume that $\zeta : g = Cf_x + Df_y = 0$ has $[\zeta \cdot \gamma] = nm + s$, $s > \sigma$. Consider the vector field $(C\partial_x + D\partial_y)$, defined in a neighbourhood of the origin, and its associated local flow*

$$\varphi_u : x^* = x^*(x, y, u), \quad y^* = y^*(x, y, u).$$

Then each local automorphism φ_u is principal.

Proof of 7.4:. Write

$$x^*(x, y, u) = \sum_{i,j \geq 0} a_{i,j}(u) x^i y^j, \quad y^*(x, y, u) = \sum_{i,j \geq 0} b_{i,j}(u) x^i y^j.$$

By the hypothesis,

$$\frac{dx^*(x, y, u)}{du} = C(x, y) \quad and \quad \frac{dy^*(x, y, u)}{du} = D(x, y),$$

for x, y, u small enough. Since, by 7.3, it is $c_{0,0} = c_{1,0} = d_{0,1} = d_{i,0} = 0$ for $i < m/n$, we get

$$da_{0,0}/du = 0, \quad da_{1,0}/du = 0, \quad db_{0,1}/du = 0$$

and

$$db_{i,0}/du = 0 \quad \text{for} \quad 0 \leq i < m/n,$$

locally at $u = 0$. Therefore $a_{0,0}, a_{1,0}, b_{0,1}$ and the $b_{i,0}$, $0 \leq i < m/n$, are all constant for u small enough. Using that $\varphi_0 = Id$ yields

$$a_{1,0} = b_{0,1} = 1, \quad a_{0,0} = 0 \quad \text{and} \quad b_{i,0} = 0 \text{ for } 0 \leq i < m/n,$$

as wanted. □

Proof of 7.1, continued:. Still consider the local flow

$$\varphi_u : x^* = x^*(x, y, u), \quad y^* = y^*(x, y, u).$$

associated to $C\partial_x + D\partial_y$. Assume that

$$x = x(t) \quad , \quad y = y(t)$$

is a Puiseux parameterization of γ and substitute it in the series $\varphi_u^*(f) = f \circ \varphi_u$ defining $\varphi_u^*(\gamma)$ to get

$$\varphi_u^*(f)(x(t), y(t)) = \theta(u) t^{r'} + \dots \tag{6}$$

where the power series $\theta(u)$ is non-zero and the dots indicate terms of higher degree in t. Then $r' = [\varphi_u^*(\gamma) \cdot \gamma]$ for $u \neq 0$ and close enough to 0. Since, by 7.4, φ_u is principal, it leaves invariant all points on γ up to the first free point after the satellite points and therefore $r' > nm$. The difference $r = r' - nm$ is thus a positive integer. For all u close enough to 0, it holds $[\varphi_u^*(\gamma) \cdot \gamma] \geq nm + r$ and therefore φ_u leaves invariant all points on γ up to the r-th free point p_r after the satellite points. We need a further lemma:

Lemma 7.5. $\theta(u) = cu$, $c \in \mathbb{C} - \{0\}$.

Proof of 7.5:. By 6.3, for u small enough, $\theta(u)$ is the modulus of the translation induced by φ_u^* in the first neighbourhood of p_r (relative to an affine coordinate independent of u). After this, by 6.4, it holds

$$\theta(u_1 + u_2) = \theta(u_1) + \theta(u_2)$$

for u_1 and u_2 small enough. Then, using just the definition of derivative, $\theta(u) = [d\theta/du]_{|u=0}u$, hence the claim. □

End of the proof of 7.1. After taking derivatives with respect to u at $u = 0$ in (6),

$$\begin{aligned} ct^{nm+r} + \ldots &= \left[\frac{\partial}{\partial u} f(x^*(x(t), y(t), u), y^*(x(t), y(t), u))\right]_{|u=0} \\ &= \left[\frac{\partial}{\partial u} f(x^*(x, y, u), y^*(x, y, u))\right]_{|x=x(t), y=y(t), u=0} \\ &= \left[\frac{\partial f}{\partial x}\frac{\partial x^*}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y^*}{\partial u}\right]_{|x=x(t), y=y(t), u=0} \\ &= \left[C\frac{\partial f}{\partial x} + D\frac{\partial f}{\partial y}\right]_{|x=x(t), y=y(t)}. \end{aligned}$$

This yields

$$nm + r = [\zeta \cdot \gamma] = nm + s$$

and thus $r = s$. After this, 6.3 and the equality (6) show that, for u small enough, $u \neq 0$, φ_u^* induces a non-identical translation in the first neighbourhood of q_s, and therefore ([7], 5.5) that the coefficients of twisted degree $nm + s$ are irrelevant, as claimed.

For the converse, assume that, for a certain integer $s > \sigma$, the coefficients of twisted degree $nm + s$ are irrelevant. Using the notations of [7]. Sect. 12, denote by $[\varphi]_s$ the twisted s-jet of a principal automorphism φ, by $\mathcal{B}_s$ the group of twisted s-jets of the principal automorphisms and by $\overline{\mathcal{W}}_s$ the subgroup of the twisted s-jets leaving fixed q_s. Consider the group homomorphism δ'_s mapping each twisted s-jet to the modulus of the translation induced by it on the first neighbourhood of q_s ([7], 12.2), namely

$$\begin{aligned} \delta'_s : \overline{\mathcal{W}}_s &\longmapsto \mathbb{C} \\ [\varphi]_s &\mapsto \overline{\Psi}_s(a_v, b_w), \end{aligned}$$

where $\overline{\Psi}_s(X_v, Y_w)$ is a polynomial in variables X_v, Y_w, for v, w admissible of twisted degree at most s. In particular, the zeros of $\overline{\Psi}_s(X_v, Y_w)$ in $\overline{\mathcal{W}}_s$ are the twisted s-jets of the principal automorphims leaving fixed q_{s+1}.

The coefficients of twisted degree s being irrelevant, δ'_s is not constant (i. e., not zero) on $\overline{\mathcal{W}}_s$ ([7], 12.8). Being a group homomorphism, δ'_s, seen as a function on the smooth variety $\overline{\mathcal{W}}_s$, has no critical points, in particular its differential at $[Id]_s$

is not zero. We may thus choose an analytic family of twisted s-jets $\hat{\varphi}_u \in \overline{\mathcal{W}_s}$, defined for u small enough and with $\hat{\varphi}_0 = [Id]_s$, which is transverse at $[Id]_s$ to the variety of zeros of $\overline{\Psi_s}$ in $\overline{\mathcal{W}}_s$. The coefficients up to twisted degree s of the local automorphisms being affine coordinates of the corresponding twisted jet in $\mathcal{B}_s$, assume that the above family is given by analytic functions

$$a_v = a_v(u) \quad , \quad b_w = b_w(u), \qquad a_v(0) = b_w(0) = 0,$$

for v, w admissible and with twisted degree at most s. By the transversality above,

$$\overline{\Psi}(a_v(u), b_w(u)) \neq 0, \tag{7}$$

for $u \neq 0$ small enough, and

$$\left[\frac{d}{du}\overline{\Psi}(a_v(u), b_w(u))\right]_{u=0} \neq 0. \tag{8}$$

For u small enough, let φ_u be the representative of $\hat{\varphi}_u$ with all its coefficients a_v, b_w equal to zero for $\operatorname{td}(v) > nm + s$, $\operatorname{td}(w) > nm + s$. Assume it to be given by equalities

$$x^* = x^*(a_v(u), b_w(u), x, y) \quad , \quad y^* = y^*(a_v(u), b_w(u), x, y).$$

For $u \neq 0$, the germ $\varphi_u^*(\gamma)$ goes through q_s and, by (7), misses q_{s+1}. Therefore $[\varphi_u^*(\gamma) \cdot \gamma] = nm + s$, and so, if still $x = x(t)$, $y = y(t)$ is a Puiseux parameterization of γ,

$$\varphi_u^*(f)(x(t), y(t)) = \beta(u)t^{nm+s} + \ldots,$$

where $\beta(u) \neq 0$ and the dots indicate terms of higher degree in t.

Now, by 6.3, both $\beta(u)$ and $\overline{\Psi}(a_v(u), b_w(u))$ are moduli of the action of φ_u on the first neighbourhood of q_s relative to affine coordinates independent of u. Therefore, there is $c \in \mathbb{C} - \{0\}$ such that $\beta(u) = c\overline{\Psi}(a_v(u), b_w(u))$ for any u small enough. The above may thus be equivalently written

$$f(x^*((x(t), y(t), u), y^*((x(t), y(t), u)) = c\overline{\Psi}(a_v(u), b_w(u))t^{nm+s} + \ldots.$$

Taking derivatives with respect to u at $u = 0$ yields

$$\left[\frac{\partial f}{\partial x}\left[\frac{\partial x^*}{\partial u}\right]_{|u=0} + \frac{\partial f}{\partial y}\left[\frac{\partial y^*}{\partial u}\right]_{|u=0}\right]_{|x=x(t), y=y(t)}$$
$$= c\left[\frac{d}{du}\overline{\Psi}(a_v(u), b_w(u))\right]_{u=0} t^{nm+s} + \ldots.$$

This equality, together with (8), shows that the generalized polar

$$\zeta : \left[\frac{\partial x^*}{\partial u}\right]_{|u=0}\frac{\partial f}{\partial x} + \left[\frac{\partial y^*}{\partial u}\right]_{|u=0}\frac{\partial f}{\partial y} = 0$$

has $[\zeta \cdot \gamma] = nm + s$, as wanted. □

Example 7.6. From Zariski's 7/6 example ([7], 12.14), we know that the germs

$$\gamma : f = y^6 - x^7 + x^5y^2 + A_{4,3}x^4y^3 + A_{3,4}x^3y^4 + A_{4,4}x^4y^4 = 0,$$

$A_{4,3}, A_{3,4}, A_{4,4} \in \mathbb{C}$, represent all analytic classes of irreducible germs with single characteristic exponent 7/6 and minimal Zariski invariant, and also that their coefficients $A_{4,3}$ and $A_{3,4}$ are continuous invariants. Consider the generalized polar ζ_0 defined by

$$\begin{aligned}&(84x^2 - 126A_{4,3}xy + 4y^2)f_x + 7y(14x - 21A_{4,3}y)f_y\\ &\quad = (20 - 63A_{4,3}^2 + 56A_{3,4})x^4y^4 + (16A_{4,3} - 84A_{4,3}A_{3,4})x^3y^5 + 12A_{3,4}x^2y^6\\ &\qquad -210A_{4,3}A_{4,4}x^4y^5 + 140A_{4,4}x^5y^4 + 16A_{4,4}x^3y^6.\end{aligned}$$

If $20 - 63A_{4,3}^2 + 56A_{3,4} \neq 0$, using the Puiseux parameterization of γ

$$x = t^6 \quad , \quad y = t^7 + \dots$$

it easily turns out that $[\zeta_0 \cdot \gamma] = 52$ and therefore, by 7.1, the coefficient $A_{4,4}$ is irrelevant, as already seen in [7], 12.14.

The proof of 7.1 provides local automorphisms having a non-trivial action on the first neighbourhood of q_s:

Corollary 7.7. (of the proof of 7.1) *If the generalized polar $\zeta : Cf_x + Df_y = 0$ has $[\zeta \cdot \gamma] = nm + s$, $s > 0$, then the non-identical local automorphisms in the flow associated to the vector field $C\partial_x + D\partial_y$ leave invariant the point on γ in the s-th neighbourhood of the last satellite and induce non-trivial translations in the first neighbourhood of it.*

Given a point $p = (i, j)$ on the Newton plane $\mathcal{N}$, the set

$$\mathcal{Q}_p = \{p' = (i', j') \in \mathcal{N} | i' \geq i, j' \geq j\}$$

will be called the *quadrant* with *vertex* p.

Remark 7.8. Assume that for a certain $p = (i, j) \in \mathcal{N}$ there is $\zeta : g = 0$, $g \in \mathbf{J}(\gamma)$ with $[\zeta \cdot \gamma] = \mathrm{td}(p)$. Then, for any $p' = (i + k, j + r) \in \mathcal{Q}_p$, $x^ky^rg \in \mathbf{J}(\gamma)$ defines a germ ζ' with $[\zeta' \cdot \gamma] = \mathrm{td}(p')$. In other words, if $\mathrm{td}(p)$ is not a gap of $\mathbf{J}(\gamma)$, neither is $\mathrm{td}(p')$ for any $p' \in \mathcal{Q}_p$. Using 7.1, this directly gives new information regarding relevance of coefficients:

Corollary 7.9. *Let γ be as above, defined by the equation $f = y^n - x^m + \sum_{ni+mj>nm} A_{i,j}x^iy^j = 0$. If the coefficient $A_{i,j}$ of γ, corresponding to $p = (i, j)$, $ni + mj > nm + \sigma$, is irrelevant, then so are all coefficients corresponding to points in $\mathcal{Q}_p$.*

Still assume that γ has finite Zariski invariant σ and Zariski point $\omega = (\omega_1, \omega_2)$. As in [7], Section 12, take $\kappa = \min\{\mathrm{td}(m-1, \omega_2), \mathrm{td}(\omega_1, n-1)\} - nm$ and, in the Newton plane $\mathcal{N}$, let the triangle $\mathbf{T}$ be either

$$\mathbf{T} = \{(\alpha, \beta) \mid n\alpha + m\beta \geq \kappa + mn, \alpha \leq m-1, \beta \leq \omega_2\}$$

if $n\omega_1 + m(n-1) < n(m-1) + \omega_2$, or, otherwise,

$$\mathbf{T} = \{(\alpha, \beta) \mid n\alpha + m\beta \geq \kappa + mn, \alpha \leq \omega_1, \beta \leq n-1\}.$$

Then:

Corollary 7.10. *Still assume the Zariski invariant of γ to be finite. Take $p_1 = (m-1, 0)$, $p_2 = (0, n-1)$ and rename p_3 the Zariski point ω of γ. Then there are points $p_k \in \mathbb{N}^2$, $k = 4, \ldots, \ell$, $\ell \geq 3$, in the interior of $\mathbf{T}$, such that the set $\mathcal{U} = \bigcup_{k=1}^{\ell} \mathcal{Q}_{p_k}$ is composed of p_1, p_2, p_3 and all points corresponding to irrelevant coefficients.*

Proof. By [7], 6.1 and [7], 8.1, all coefficients corresponding to points in $\mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \mathcal{Q}_3$ are irrelevant. By [7], 7.9 and [7], 11.1, all the other coefficients are relevant but maybe those corresponding to interior points of $\mathcal{T}$. The claim follows then from 7.9 by taking the p_i, $i = 4, \ldots, \ell$ to be the integral points in the interior of $\mathcal{T}$ corresponding to irrelevant coefficients. □

Remark 7.11. Of course, from the quadrants $\mathcal{Q}_{p_k}$ of 7.10, those with p_k in another quadrant are redundant. In fact, one may choose p_4 to be the lowest twisted degree point in T corresponding to an irrelevant coefficient and, inductively, p_k to be the lowest twisted degree point in $T - \bigcup_{k'<k} \mathcal{Q}_{p_{k'}}$ corresponding to an irrelevant coefficient. Then still $\mathcal{U} = \bigcup_{k=1}^{\ell} \mathcal{Q}_{p_k}$ and the decomposition is irredundant.

The border of $\mathcal{U}$ in 7.10 is composed of two half-lines on the coordinate axes, with ends $(0, n-1)$ and $(m-1, 0)$, and a stairs-shaped line joining their ends: we will call this line the *staircase line* of γ, denoted $SL(\gamma)$. An irredundant decomposition $\mathcal{U} = \bigcup_{k=1}^{\ell} \mathcal{Q}_{p_k}$ being obviously unique, we will call the points p_k, $k = 1, \ldots, \ell$, the *corners* of $SL(\gamma)$.

Remark 7.12. After 7.10, the coefficients corresponding to either the Zariski point ω or a point strictly below $SL(\gamma)$ are relevant; those corresponding to a point other than ω and placed on or above $SL(\gamma)$ are irrelevant.

Since the coefficients corresponding to the points in the interior of $\mathbf{T}$ are the only ones whose relevance is not determined by m/n and σ (see the proof of 7.10 above), the interesting part of the staircase line is $SL(\gamma) \cap \mathbf{T}$, which is of course not determined by m/n and σ. For instance in case $m/n = 7/6$, $\sigma = 2$ (see [7], 12.14), $SL(\gamma)$ has corners $(0, 5)$, $(4, 4)$, $(5, 2)$, $(6, 0)$ if $20 - 63A_{4,3}^2 + 56A_{3,4} \neq 0$, while it has corners $(0, 5)$, $(5, 2)$, $(6, 0)$ otherwise. We will see more interesting examples in the forthcoming Section 9.

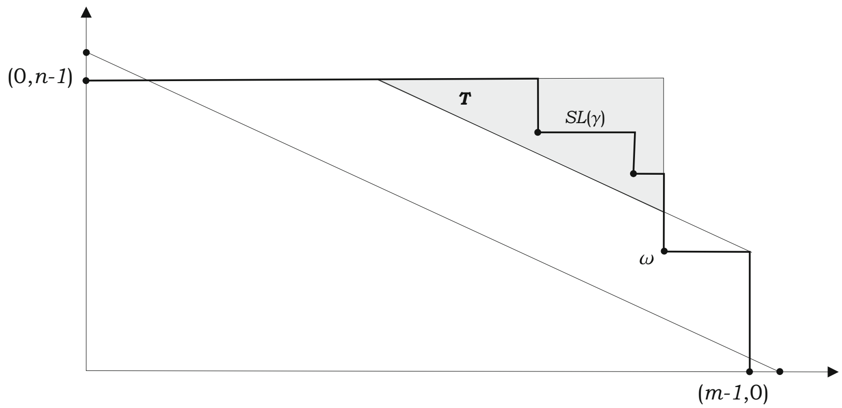

Fig. 3. The Zariski point ω, the triangle **T** –in grey– and the staircase line $SL(\gamma)$ of a germ γ

Remark 7.13. Not all possible candidates for the part of $SL(\gamma)$ inside T do actually occur, due to the constraints on the number of points corresponding to irrelevant coefficients of [7], 12.17. Indeed, as seen in [7], 12.18, in case $m/n = 13/6$ and $\sigma(\gamma) = 1$, at most one of the points (10, 2), (8, 3), (6, 4) corresponds to an irrelevant coefficient, and therefore in no case $SL(\gamma)$ can have two of them as corners. In particular no $SL(\gamma)$ can have all three as corners, as one could wrongly expect to be the case for "generic" germs γ.

Remark 7.14. Corollary 7.9 and [7], 12.17 combine to give non-obvious information. Still in the case of [7], 12.18, if $A_{8,3}$ is irrelevant, then so is $A_{9,3}$ and, as a consequence, $A_{7,4}$ is invariant. Similarly, if $A_{6,4}$ is irrelevant, so is $A_{7,4}$ and then $A_{9,3}$ is invariant.

The staircase line of a quasihomogeneous germ may be taken to have corners $(m-1, 0)$ and $(0, n-1)$; in such a way, by [7], 7.13, after dropping the mentions to the Zariski point, Remark 7.12 still applies.

8. Complements regarding the gaps of $\mathbf{J}(\gamma)$

As seen in 7.1, the gaps d of $\mathbf{J}(\gamma)$ with $d > nm + \sigma$ are the twisted degrees of the points corresponding to relevant coefficients. In this section we will determine the gaps d of $\mathbf{J}(\gamma)$ with $d \leq nm + \sigma$. This will provide a useful characterization of the Zariski invariant $\sigma(\gamma)$.

In this section and the next one we will make frequent use of the fact that, if finite, $\sigma = \sigma(\gamma)$ satisfies $\sigma + mn < 2nm - n - m + 1$, just because the Zariski point is a standing point.

Proposition 8.1. *If the germ γ has finite Zariski invariant σ, then an integer $d \leq nm + \sigma$ belongs to $\Theta(\gamma)$ if and only if either $d = nm + \sigma$ or d is the twisted degree of a non-standing point.*

Proof. As in former occasions, up to replacing γ with an analytically equivalent germ, we assume all non-standing coefficients of the equation of γ up to the twisted degree $nm + \sigma$ to be zero, and therefore an equation of γ to be

$$f = y^n - x^m + A_{\omega_1,\omega_2} x^{\omega_1} y^{\omega_2} + \cdots = 0, \quad A_{\omega_1,\omega_2} \neq 0,$$

where (ω_1, ω_2) is the Zariski point of γ and the dots stand for terms of higher twisted degree.

First, as already noticed, the polars $f_x = 0$ and $f_y = 0$ have intersection multiplicities $((m-1)n = \mathrm{td}(m-1,0))$ and $(n-1)m = \mathrm{td}(n-1,0))$ with γ; then, by 7.8, the twisted degree of any non-standing point belongs to Θ.

By substituting a Puiseux parameterization of γ it is direct to check that the generalized polar

$$\eta : nmf + nf_x - mf_y = 0$$

has $[\eta \cdot \gamma] = n\omega_1 + m\omega_2 = nm + \sigma$.

Assume now that $d \in \Theta(\gamma)$, $d < nm + \sigma$, is not the twisted degree of a non-standing point. By 5.3, there is $h = af + bf_x + cf_y$ which has $\mathrm{to}(h) = d$. The twisted initial forms of the three summands above are

$$\bar{a}(y^n - x^m), \quad -m\bar{b}x^{m-1}, \quad n\bar{c}y^{n-1}.$$

Since all their monomials correspond to non-standing points, neither of them has twisted degree d. Therefore, either two of them or all three cancel.

In the first case, assume for instance

$$\bar{a}(y^n - x^m) - m\bar{b}x^{m-1} = 0.$$

Then $\bar{a}$ is a multiple of x^{m-1} and so

$$\begin{aligned} \mathrm{to}(af + bf_x) &> \mathrm{td}(\bar{a}(y^n - x^m)) = \mathrm{td}(m\bar{b}x^{m-1}) \\ &\geq 2nm - n > 2nm - n - m + 1 > nm + \sigma > d, \end{aligned}$$

after which $d = \mathrm{to}(cf_y)$ would be the twisted order of a non-standing point, against the hypothesis. The other two possibilities in this case may be dealt with similarly.

In the second case,

$$(\bar{a}y + n\bar{c})y^{n-1} = (\bar{a}x + m\bar{b})x^{m-1}.$$

If both sides are not zero then $\bar{a}y + n\bar{c}$ is a non-zero multiple of x^{m-1}, after which, a computation as above gives again $d > nm + \sigma$. Otherwise,

$$\bar{b} = -\frac{\bar{a}x}{m} \quad \text{and} \quad \bar{c} = -\frac{\bar{a}y}{n}.$$

In particular, $\mathrm{td}(\bar{b}) \geq n$ and $\mathrm{td}(\bar{c}) \geq m$, after which it is clear that

$$\mathrm{to}(a(f - y^n + x^m) + b(f_x + mx^{m-1}) + c(f_y - ny^{n-1}) \geq n\omega_1 + m\omega_2 = nm + \sigma.$$

The monomials of twisted degree d of h are thus monomials of

$$a(y^n - x^m) - mbx^{m-1} + ncy^n$$

which have each the twisted degree of a non-standing point, against our hypothesis on d. □

Proposition 8.1 directly provides a characterization of the Zariski invariant in terms of $\mathbf{J}(\gamma)$:

Corollary 8.2. *If the germ γ has finite Zariski invariant σ, then*

$$nm + \sigma = \min\left[\Theta(\gamma) - (m(n-1) + \langle n, m\rangle) \cup (n(m-1) + \langle n, m\rangle)\right]$$

Remark 8.3. As said, quasihomogeneous germs have little interest, which is the reason why they have been seldom considered before. Anyway, for the sake of completeness, any quasihomogeneous germ τ being analytically equivalent to $y^n - x^m = 0$, it is direct to check using the latter that

$$\Theta(\tau) = (m(n-1) + \langle n, m\rangle) \cup (n(m-1) + \langle n, m\rangle),$$

which is the set of the twisted degrees of the non-standing points (2.1).

Corollary 8.4. (of 7.1) *Select monomials X_k, $k = 1, \dots, \ell$, corresponding to the corners p_k, $k = 1, \dots, \ell$, of $SL(\gamma)$. The ideal $(X_1, \dots, X_\ell) \subset \mathbb{C}\{x, y\}$ contains the twisted initial forms of all series $g \in \mathbf{J}(\gamma)$.*

Proof. If the twisted initial form $\bar{g}$ of g has two or more monomials, then each of these monomials corresponds to a non-standing point (by 2.1) and therefore is a multiple of either x^{m-1} or y^{n-1}. Otherwise $\bar{g}$ is a monomial, say corresponding to a point (i, j). Then, by 7.1 and 8.1, either (i, j) is one of the points $(m-1, 0)$, $(0, n-1)$, ω or $ni + mj > nm + \sigma$ and the coefficient $A_{i,j}$ of the equation of γ is irrelevant. In any case, by 7.10, $(i, j) \in \mathcal{Q}_k$ for some k and $\bar{g}$ is then a multiple of X_k. □

Using terms from commutative algebra, 8.4 asserts that any $g_1, \dots, g_\ell \in \mathbf{J}(\gamma)$ with $\bar{g}_k = X_k$, $k = 1, \dots, \ell$, make a standard basis (relative to the twisted grading) of $\mathbf{J}(\gamma)$.

9. Computations and examples

In this section we will present some computation procedures that allow to compute the Zariski invariant $\sigma(\gamma)$ and to examine, for fixed n, m and $\sigma(\gamma)$, the different possibilities of relevance of coefficients through Theorem 7.1. They will be used in three examples. Our procedures are simplified adaptations of procedures that are usual in computational algebra (see for instance [9]); nevertheless, since they are not direct applications, we will provide the –rather simple– arguments needed to support them.

As before, γ is the germ of curve defined by $f = y^n - x^m + \sum_{ni+mj>nm} A_{i,j} x^i y^j = 0$. Assume to have fixed $g_1, \dots, g_r \in \mathbb{C}\{x, y\}$, $r \geq 2$, each with a monomial as twisted initial form. Assume furthermore that the twisted initial forms of g_1 and g_2 are scalar multiples of powers of x and y, respectively. For $i = 1, \dots, r$, let p_i be the point corresponding to the twisted initial form of g_i, and $\mathcal{Q}$ the union of quadrants $\mathcal{Q} = \mathcal{Q}_{p_1} \cup \dots \cup \mathcal{Q}_{p_r}$. To avoid trivialities, we assume also $\mathcal{Q}_{p_i} \not\subset \mathcal{Q}_{p_j}$ for $i \neq j$. Note that there are finitely many points in $\mathbb{N}^2 - \mathcal{Q}$.

Given any $g \in \mathbb{C}\{x, y\}$, substract from g multiples of the g_i whose initial forms cancel the monomials of g corresponding to the points in $\mathcal{Q}$ of minimal twisted degree; then do the same with the difference and so on, until having cancelled all monomials of g corresponding to points in $\mathcal{Q}$ of twisted degree, say, ρ or less. This will give an expression

$$g = R + \sum_{k=0}^{r} h_k g_k + K, \tag{9}$$

where all the monomials of R correspond to points not in $\mathcal{Q}$, $\mathrm{to}(K) > \rho$ and, clearly, $\mathrm{to}(h_k g_k) \geq \mathrm{to}(g)$, $k = 1, \ldots, r$, provided $h_k g_k \neq 0$.

As soon as ρ is higher than the maximum of the twisted degrees of the points in $\mathbb{N}^2 - \mathcal{Q}$, all monomials of the multiples of the g_k used for the cancellations correspond to points in $\mathcal{Q}$ and therefore R becomes independent of ρ. This satisfied, we will say that the procedure leading to the equality (9) is a *division* of g by $g_1, \ldots, g_r$; the h_k will be called *quotients* of the division and R a *remainder* of dividing g by $g_1, \ldots, g_r$. Neither the quotients nor the remainder are uniquely determined by g and the g_k, as it is easy to check. The usual computational algebra division in the local case is different and quite more complicated, see for instance [8], 4.3.

We will make use of the following fact:

Lemma 9.1. *Any* $g \in \mathbb{C}\{x, y\}$ *with* $\mathrm{to}(g) \geq 2nm - n - m + 1$ *belongs to the ideal* (f_x, f_y)*, and so also to* $\mathbf{J}(\gamma)$*.*

Proof. Perform a division of g by f_x, f_y as (9). Note that the points of $\mathcal{Q}$ with integral coordinates are the non-standing points. The ideal (f_x, f_y) being (x, y)-primary, we take the positive integer ρ such that any series with twisted order higher than ρ belongs to (f_x, f_y), and so, in particular, so does the complementary term K. Since g has twisted order $2nm - n - m + 1$ or higher, the same holds for any of the multiples of f_x or f_y used to cancel monomials in the division procedure. On the other hand, any standing monomial has twisted degree strictly less than $2nm - n - m + 1$ and therefore no standing monomial does appear in g or along the division procedure, which forces the remainder R to be zero. Then the claim follows from equality (9). □

Remark 9.2. In our applications we will always take $g_1 = f_x$ and $g_2 = f_y$, in which case all the non-standing points belong to $\mathcal{Q}$. Also, unless otherwise said, we will take $\rho = 2nm - n - m + 1$: then all points p with $\mathrm{td}(p) \geq \rho$ are non-standing, and therefore belong to $\mathcal{Q}$; furthermore, $K \in \mathbf{J}(\gamma)$ by 9.1.

The next proposition provides an easy way of computing the Zariski invariant $\sigma(\gamma)$. Of course, it is useful only in case the equation has some non-zero non-standing monomial, as otherwise the Zariski invariant may be directly read from the equation, see Section 3.

Proposition 9.3. *Let* R *be any remainder of a division of the series* f *defining* γ *by* f_x *and* f_y*:*

(a) $R = 0$ *if and only if* γ *is quasihomogeneous.*
(b) If $R \neq 0$, *then* $\mathrm{to}(R) = nm + \sigma(\gamma)$.

Proof. The points of $\mathcal{Q}$ with integral coordinates are the non-standing points, and so all the monomials of R are standing monomials. In particular, if $R \neq 0$, $\mathrm{to}(R)$ is the twisted degree of a standing point. By the choices of 9.2 $R \in \mathbf{J}(\gamma)$. Then, if γ is quasihomogeneous, using 5.3 contradicts 8.3.

Assume now that γ is not quasihomogeneous. Then, by 5.3 and 8.2, there is $g \in \mathbf{J}(\gamma)$ with $\mathrm{to}(g) = nm + \sigma$. Assume

$$g = af + bf_x + cf_y, \quad a, b, c \in \mathbb{C}\{x, y\}.$$

Using the division of f by f_x, f_y gives an equality

$$g = aR + b' f_x + c' f_y + U$$

where $b', c', U \in \mathbb{C}\{x, y\}$ and $\mathrm{to}(U) \geq 2nm - n - m + 1 > \mathrm{to}(g)$. If the twisted initial forms of $b' f_x$ and $c' f_y$, namely $-m\bar{b}' x^{m-1}$ and $n\bar{c}' y^{n-1}$, cancel, then, arguing as in the proof of 8.1

$$\mathrm{to}(b' f_x + c' f_y) > \mathrm{to}(b' f_x) \geq 2nm - n - m > nm + \sigma = \mathrm{to}(g),$$

and so $\mathrm{to}(g) = \mathrm{to}(a) + \mathrm{to}(R)$.

Otherwise, the twisted initial form of $b' f_x + c' f_y$ is a sum of non-standing monomials. Therefore $\mathrm{to}(b' f_x + c' f_y) \neq \mathrm{to}(g)$ and, again, $\mathrm{to}(g) = \mathrm{to}(a) + \mathrm{to}(R)$.

Thus, in both cases, by the minimality in 8.2, $\mathrm{to}(a) = 0$ and the claim follows. □

Example 9.4. As in [7], 7.12, take γ defined by $f = y^5 - x^7 - 7x^6 y - 21x^5 y^2$. A division of f by its derivatives is

$$f = -6x^4 y^3 + \frac{1}{35}(5x - 2y) f_x + \frac{1}{5} y f_y.$$

Hence, the Zariski invariant is 6 and so the Zariski point is (4,3). Note the monomial $-21x^5 y^2$, which is standing, appears in the equation of γ, has twisted degree $39 < 41 = \mathrm{td}(4, 3)$ and does not correspond to the Zariski point. On the other hand, no non-zero monomial of f corresponds to the Zariski point.

The Tjurina number of γ is an analytic invariant, usually denoted $\tau(\gamma)$, which may be defined by the rule $\tau(\gamma) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/\mathbf{J}(\gamma)$. As one can expect in view of 8.4, $\tau(\gamma)$ can be directly read from $SL(\gamma)$:

Corollary 9.5. *If* $p_1, \ldots, p_\ell$ *are the corners of the staircase line of* γ, *then*

$$\tau(\gamma) = \sharp(\mathbb{N}^2 - \bigcup_{k=1}^{\ell} \mathcal{Q}_{p_k}).$$

Proof. Write $\mathcal{L} = \mathbb{N}^2 - \bigcup_{k=1}^{\ell} \mathcal{Q}_{p_k}$ and, for any $p = (i, j) \in \mathbb{N}^2$, $X_p = x^i y^j$. We will prove that the classes of the X_p, $p \in \mathcal{L}$, make a basis of $\mathbb{C}\{x, y\}/\mathbf{J}(\gamma)$. Indeed, in case of they being linearly dependent, it would be

$$g = \sum_{p \in \mathcal{L}} a_p X_p \in \mathbf{J}(\gamma)$$

for some $a_p \in \mathbb{C}$, not all zero; then, $g \neq 0$ and $\bar{g} = a_p X_p$ for some $p \in \mathcal{L}$ (due to 2.1), which contradicts 8.4. To prove that the classes of the X_p, $p \in \mathcal{L}$, generate the quotient, take $g_k \in \mathbf{J}(\gamma)$ with $\bar{g}_k$ a monomial corresponding to the corner p_k, $k = 1, \dots, \ell$. Then any $g \in \mathbb{C}\{x, y\}$ is congruent mod. $\mathbf{J}(\gamma)$ to a remainder of its division by $p_1, \dots, p_\ell$. □

The above, together with Remark 7.13, may explain the difficulties in computing the minimal Tjurina number for a given equisingularity class: recursive procedures were given in [3] and [15], and only recently a closed formula has been given in [1].

Assume given $g_1, g_2 \in \mathbb{C}\{x, y\} - \{0\}$, both with a monomial as twisted initial form. Define

$$[g_1 * g_2] = \frac{\bar{g}_2}{\gcd(\bar{g}_1, \bar{g}_2)} g_1 - \frac{\bar{g}_1}{\gcd(\bar{g}_1, \bar{g}_2)} g_2.$$

Remark 9.6. Since the twisted initial forms of the summands above cancel,

$$\operatorname{to}([g_1 * g_2]) > \operatorname{to}(g_1) + \operatorname{to}(g_2) - \operatorname{td}(\gcd(\bar{g}_1, \bar{g}_2)) = \operatorname{td}(\operatorname{lcm}(\bar{g}_1, \bar{g}_2)),$$

the latter being called the *cancelled twisted degree* of the pair g_1, g_2 in the sequel.

Lemma 9.7. *If $g_1, g_2 \in \mathbb{C}\{x, y\} - \{0\}$ and $g'_k = M_k g_k$, M_k a non-zero monomial, $k = 1, 2$, then both g'_1, g'_2 have monomials as twisted initial forms and*

$$[g'_1 * g'_2] = M[g_1 * g_2],$$

where M is a non-zero monomial.

Proof. Direct from the definition of $[\,*\,]$. □

Lemma 9.8. *Assume that the twisted initial forms of $g_k \in \mathbb{C}\{x, y\} - \{0\}$, $k = 1, \dots, r$, $r \geq 2$, are all monomials of the same twisted degree ρ. If $\operatorname{to}(\sum_{k=1}^{r} g_k) > \rho$, then there exist $a_{i,j} \in \mathbb{C}$ for which*

$$\sum_{k=1}^{r} g_k = \sum_{1 \leq i < j \leq r} a_{i,j} [g_i * g_j].$$

Proof. The hypothesis is that the initial forms of the g_k cancel: $\sum_{k=1}^{r} \bar{g}_k = 0$. Then the same happens if the summation is restricted to the g_k whose initial forms are scalar multiples of an arbitrarily fixed monomial M. Therefore, by splitting the given set $\{g_1, \dots, g_r\}$, it suffices to prove the claim with the supplementary hypothesis that all the $\bar{g}_k$ are scalar multiples of the same monomial, say $\bar{g}_k = b_k M$,

$k = 1, \ldots, r$. Then $\sum_{k=1}^{r} b_k = 0$ and $[g_i * g_j] = b_j g_i - b_i g_j$, after which it is enough to check that it holds

$$\sum_{k=1}^{r-1} \frac{1}{b_r}[g_k * g_r] = \sum_{k=1}^{r} g_r.$$

□

Proposition 9.9. *Let γ be an irreducible germ with single characteristic exponent m/n and finite Zariski invariant σ, defined by*

$$f = y^n - x^m + \sum_{ni+mj>nm} A_{i,j} x^i y^j, \quad A_{i,j} \in \mathbb{C}.$$

Take $g_1 = f_x$, $g_2 = f_y$ and g_3 a remainder of a division of f by f_x, f_y. Proceeding inductively, for $k \geq 3$, assume that the initial forms of $g_1, \ldots, g_k$ are monomials. For each pair r, s, $1 \leq r < s \leq k$, take $R_{r,s}$ to be a remainder of a division of $[g_r * g_s]$ by $g_1, \ldots, g_k$.

(a) *If $R_{r,s} = 0$ for $1 \leq r < s \leq k$, define $\ell = k$ and stop the procedure.*
(b) *Otherwise, take g_{k+1} to be one of the remainders $R_{r,s}$ with minimal twisted order; $g_{k+1} \neq 0$ and has a monomial as twisted initial form. Then, repeat from $g_1, \ldots, g_{k+1}$.*

After finitely many steps, possibility (a) occurs. Furthermore, if p_k is the point corresponding to the twisted initial form of g_k, $1 \leq k \leq \ell$, then $\{p_1, \ldots, p_\ell\}$ is the set of corners of the staircase line of γ.

Proof. Obviously, the initial forms of $g_1 = f_x$ and $g_2 = f_y$ are monomials. By 9.3, $g_3 \neq 0$ and its twisted initial form is a monomial corresponding to the Zariski point. Assume inductively that for a given k, $3 \leq k < \ell$, $g_1, \ldots, g_k$ are non-zero and have monomials as twisted initial forms. Then g_{k+1} is defined and non-zero because $k < \ell$. Since $\mathcal{Q}_{p_1} \cup \mathcal{Q}_{p_2}$ is the set of the non-standing points, the union of quadrants $\mathcal{U}_k = \mathcal{Q}_{p_1} \cup \cdots \cup \mathcal{Q}_{p_k}$, used in the division whose remainder is g_{k+1}, contains all the non-standing points. Then all non-zero monomials of the remainder g_{k+1} are standing monomials and no two have the same twisted degree, by 2.1. This in particular assures that the twisted initial form of g_{k+1} is a monomial.

As far as option (a) does not occur, the sets $\mathbb{N}^2 - \mathcal{U}_k$ are finite and make an strictly decreasing sequence. Having $\mathbb{N}^2 - \mathcal{U}_k = \emptyset$ obviously forces $R_{r,s} = 0$ for $1 \leq r < s \leq k$, hence the finiteness of the procedure.

By the definition of g_3 and 9.1, g_1, g_2, g_3 generate $\mathbf{J}(\gamma)$. Inductively, by its definition and again 9.1, $g_k \in \mathbf{J}(\gamma)$; hence also $g_1, \ldots, g_k$ generate $\mathbf{J}(\gamma)$ if $k > 3$.

Fix k, $3 \leq k \leq \ell$ and assume to have chosen $g_1, \ldots, g_k$ as in the claim. Take $\epsilon = \min\{\mathrm{to}(R_{r,s})\}_{1 \leq r < s \leq k}$, $\epsilon \leq \infty$. We will prove that no $g \in \mathbf{J}(\gamma)$ with initial form a monomial corresponding to a point in $\mathbb{N}^2 - \mathcal{U}_k$ has $\mathrm{to}(g) < \epsilon$. This proved, on one hand, for $3 \leq k < \ell$, p_{k+1} is the point of minimal twisted degree in $\mathbb{N}^2 - \mathcal{U}_k$ corresponding to the twisted initial form of an element of $\mathbf{J}(\gamma)$. On the other, no point in $\mathbb{N}^2 - U_\ell$ corresponds to the twisted initial form of an element of $\mathbf{J}(\gamma)$.

Then, after adding that $p_1 = (m-1, 0)$, $p_2 = (0, n-1)$ and p_3 is the Zariski point by 9.3, the proof will be complete.

Assume thus that the initial form of $g \in \mathbf{J}(\gamma)$ is a monomial whose corresponding point is $p \in \mathbb{N}^2 - \mathcal{U}_k$, and also that $\mathrm{to}(g) < \epsilon$. Note that then p is the only point in $\mathbb{N}^2$ with $\mathrm{td}(p) = \mathrm{to}(g)$, by 2.1. Since $g_1, \ldots, g_k$ generate $\mathbf{J}(\gamma)$, there is at least an expression

$$g = h_1 g_1 + \cdots + h_k g_k, \quad h_1, \ldots, h_k \in \mathbb{C}\{x, y\}. \tag{10}$$

Then take $\delta = \min\{\mathrm{to}(h_1 g_1), \ldots, \mathrm{to}(h_\ell g_\ell\}$ and note that $\delta \leq \epsilon$. Then, among all g as above and all their expressions as (10), choose a pair for which δ is maximal and still use for them the notations as in (10).

Up to reordering, assume that $\mathrm{to}(h_i g_i) = \delta$ for $i = 1, \ldots, s$ and $\mathrm{to}(h_i g_i) > \delta$ for $i = s+1, \ldots, k$. Split

$$g = g' + g'', \tag{11}$$

where

$$g' = \sum_{i=1}^{s} \bar{h}_i g_i$$

and

$$g'' = \sum_{i=1}^{s} (h_i - \bar{h}_i) g_i + \sum_{i=s+1}^{k} h_i g_i.$$

We will fix our attention on g', just retaining from g'' that all its summands have twisted order strictly higher than δ; in particular, $\mathrm{to}(g'') > \delta$. Just because of this, it cannot be $\mathrm{to}(g') = \delta$, as then $\mathrm{to}(g) = \delta$ would be the twisted order of a point in $\mathcal{U}_k$.

Decompose each $\bar{h}_i$, $1 \leq i \leq s$, into the sum of its monomials,

$$\bar{h}_i = \sum_{j=1}^{r_i} h_{i,j}$$

and rewrite

$$g' = \sum_{i=1}^{s} \sum_{j=1}^{r_i} h_{i,j} g_i.$$

Since, as noted above, $\mathrm{to}(g') > \delta$, by 9.8 there are $a_{i,j,i',j'} \in \mathbb{C}$ for which

$$g' = \sum_{i,i',j,j'} a_{i,j,i',j'} [h_{i,j} g_i * h_{i',j'} g_{i'}].$$

Using 9.7 this yields

$$g' = \sum_{i,i',j,j'} M_{i,j,i',j'} [g_i * g_{i'}]. \tag{12}$$

where each $M_{i,j,i',j'}$ is a monomial and

$$\operatorname{td}(M_{i,j,i',j'}) = \operatorname{to}([h_{i,j}g_i * h_{i',j'}g_{i'}]) - \operatorname{to}([g_i * g_{i'}]) > \delta - to([g_i * g_{i'}]). \quad (13)$$

Write the chosen division of $[g_i * g_{i'}]$ by $g_1, \ldots, g_k$ in the form

$$[g_i * g_{i'}] = R_{i,i'} + B_{i,i'} + K_{i,i'},$$

where $R_{i,i'}$ is the remainder, $B_{i,i'}$ a sum of multiples of $g_1, \ldots, g_k$, each with twisted order non-less than the twisted order of $[g_i * g_{i'}]$, and $K_{i,i'}$ a series with twisted order at least $2nm - n - m + 1$. By replacing in (12) and going back to (11) we get

$$g = \sum_{i,i',j,j'} M_{i,j,i',j'} R_{i,i'} + \sum_{i,i',j,j'} M_{i,j,i',j'} B_{i,i'} + \sum_{i,i',j,j'} M_{i,j,i',j'} K_{i,i'} + g''.$$

Our hypothesis regarding $\operatorname{to}(g)$ forces

$$\operatorname{to}(g) < \operatorname{to}(\sum_{i,i',j,j'} M_{i,j,i',j'} R_{i,i'})$$

and, clearly, it holds

$$\operatorname{to}(g) < \operatorname{to}(\sum_{i,i',j,j'} M_{i,j,i',j'} K_{i,i'}).$$

Therefore, if

$$\hat{g} = \sum_{i,i',j,j'} M_{i,j,i',j'} B_{i,i'} + g'',$$

then, necessarily, $\operatorname{to}(\hat{g}) = \operatorname{to}(g)$, which in turn forces g and $\hat{g}$ to have the same monomial as initial form (by 2.1). Obviously, $\hat{g} \in \mathbf{J}(\gamma)$. In addition, as noticed when defining it, any of the multiples of the g_i whose sum is $B_{i,i'}$ has twisted order $\operatorname{to}([g_i * g_{i'}])$ or higher. Using (13), any of the multiples of the g_i whose sum is $M_{i,j,i',j'} B_{i,i'}$ has twisted order strictly higher than δ. The same being true for g'', $\hat{g}$ contradicts our choice of g and its expression. $\square$

The series g_k of 9.9 may be computed using any computer algebra system. In practice, many of the $[g_i * g_k]$ involved may be discarded a priori due to a too high cancelled twisted degree. Coefficients that are continuous or conditional invariants may be taken as free parameters in order to discuss the different possibilities; in such a case the twisted initial forms of the g_i, $i > 3$, provide the conditions for the relevance of the conditional invariants. Next are two examples. The first one already appeared in [16], 3.7, as an example of jumping Tjurina number. The second one shows different possibilities with fixed characteristic exponent and Zariski invariant.

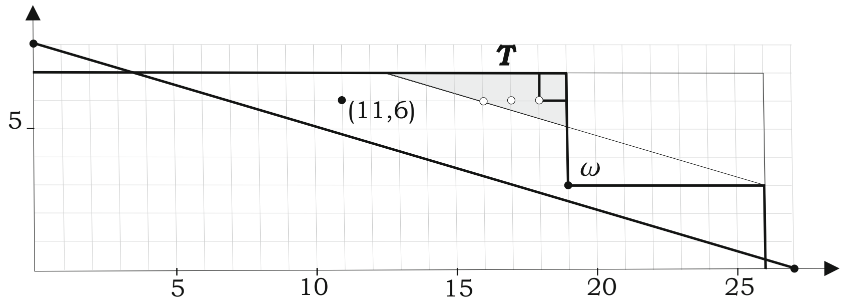

Fig. 4. Example 9.10

Example 9.10. Consider the family of germs

$$\gamma_A : y^8 - x^{27} + x^{19}y^3 + Ax^{11}y^6 = 0$$

For all of them, the Zariski point is $\omega = (19, 3)$, the coefficient A is a continuous invariant and there are three points in the interior of the triangle $\mathbf{T}$, namely $(16, 6)$, $(17, 6)$, $(18, 6)$. See Fig. 4. Computing as described in 9.9, it results

$$g_3 = -17x^{19}y^3 - 34Ax^{11}y^6 \quad \text{and} \quad g_4 = 17(19 + 54A)x^{18}y^6.$$

The first equality confirms the Zariski point, already evident from the equation. The second one shows that the coefficients corresponding to $(16, 6)$ and $(17, 6)$ are continuous invariants (both with value zero) for all values of the invariant A. The coefficient corresponding to $(18, 6)$ is irrelevant for $A \neq -19/54$, while it is also a continuous invariant for $A = -19/54$. Figure 4 shows the staircase lines for both cases; they in particular explain the jumping of the Tjurina number, from 153 to 154, for $A = -19/54$.

The reader may have noticed how the coefficient corresponding to $(18, 6)$ plays an important role in example 9.4, even if its value is zero in all cases. This shows the convenience of considering analytic relevance rather than just the possibility of turning a coefficient into zero by an analytic automorphism (elimination of coefficients), which may be confusing.

Example 9.11. The germs

$$\begin{aligned}\gamma : y^6 - x^{17} &+ x^6y^4 + A_{9,3}x^9y^3 + A_{12,2}x^{12}y^2 + A_{15,1}x^{15}y \\ &+A_{10,3}x^{10}y^3 + A_{13,2}x^{13}y^2 + A_{11,3}x^{11}y^3 + A_{14,2}x^{14}y^2 + A_{12,3}x^{12}y^3 \\ &+A_{15,2}x^{15}y^2 + A_{13,3}x^{13}y^3 + A_{14,3}x^{14}y^3 + A_{15,3}x^{15}y^3 = 0\end{aligned}$$

represent all the analytic types of germs with single characteristic exponent $17/6$ and (minimal) Zariski invariant 2. The coefficients

$$A_{9,3}, A_{12,2}, A_{15,1}, A_{10,3}, A_{13,2}, A_{11,3}, A_{14,2}$$

are continuous invariants by [7], 11.1, while

$$A_{12,3}, A_{15,2}, A_{13,3}, A_{14,3}, A_{15,3}$$

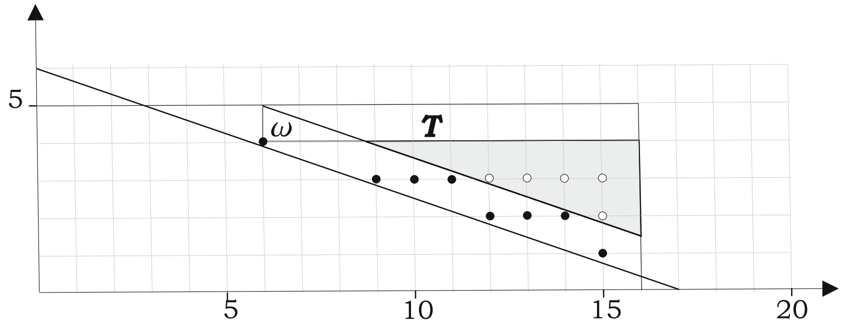

Fig. 5. Example 9.11: The characteristic exponent is $17/6$, ω is the Zariski point and the other black points correspond to continuous invariants. The points in the interior of the triangle **T**, in white, correspond to the conditional invariants, whose relevance is discussed in the text

are conditional invariants (see Fig. 5). After computing, it results $g_3 = -2x^6y^4 + \ldots$, as expected, and

$$g_4 = (8 - 24A_{12,2} + 27A_{9,3}^2)x^{12}y^3 + \ldots.$$

Assume first $8 - 24A_{12,2} + 27A_{9,3}^2 \neq 0$ (Case 1). Then $A_{12,3}$ is irrelevant, and hence so are $A_{13,3}$ $A_{14,3}$ and $A_{15,3}$. The remaining $A_{15,2}$ is invariant due to the constraints of [7] 12.7; further computation confirms this fact, as it gives $g_5 = 0$. Note that there is thus no staircase line with corners $(12, 3)$, $(15, 2)$.

Assume now otherwise (Case 2), and so fix $A_{12,2}$ to be

$$A_{12,2} = \frac{8 + 27A_{9,3}^2}{24}. \tag{14}$$

Then,

$$g_4 = (-30A_{15,1} + 18A_{9,3} + \frac{81}{2}A_{9,3}^3)x^{15}y^2 + \ldots$$

and so, as far as

$$-30A_{15,1} + 18A_{9,3} + \frac{81}{2}A_{9,3}^3 \neq 0$$

(Case 2.1), the coefficient $A_{15,2}$ is irrelevant, and therefore so is $A_{15,3}$. Further computation gives

$$\begin{aligned} g_5 = (&-1020A_{13,2} - 450A_{15,1}^2 + 2754A_{10,3}A_{9,3} + 540A_{15,1}A_{9,3} \\ &-162A_{9,3}^2 + 1215A_{15,1}A_{9,3}^2 - 729A_{9,3}^4 - \frac{6561}{8}A_{9,3}^6)x^{14}y^3 + \ldots \end{aligned} \tag{15}$$

showing that $A_{13,3}$ is necessarily invariant. This time the invariance is not due to the constraints of [7] 12.7; actually, $A_{18,3}$ cannot be irrelevant because its twisted degree is lesser than the cancelled twisted degrees appearing in the computation of g_5. The reader may note that, due to the invariance of $A_{18,3}$ and taking in account the restriction (14), the number of free continuous invariants is 8 in both Case 2.1 and the "generic" Case 1. Still in Case 2.1 and depending on the coefficient in (15), $A_{14,3}$ may, or may not, be irrelevant.

There remains another subcase of Case 2, namely when $-30A_{15,1} + 18A_{9,3} + \frac{81}{2}A_{9,3}^3 = 0$ (Case 2.2). Then $A_{15,2}$ is invariant and all possibilities for the remaining coefficients occur: either $A_{13,3}$, $A_{14,3}$, $A_{15,3}$ are irrelevant, or $A_{13,3}$ is invariant and $A_{14,3}$, $A_{15,3}$ are irrelevant, or $A_{13,3}$, $A_{14,3}$ are invariant and $A_{15,3}$ is irrelevant, or $A_{13,3}$, $A_{14,3}$, $A_{15,3}$ are all invariant. Details are left to the reader.

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References

[1] Alberich-Carramiñana, M., Almirón, P., Blanco, G., Melle-Hernández, A.: The minimal Tjurina number of irreducible germs of plane curve singularities. Indiana Univ. Math. J. **70**(4), 1211–1220 (2021)

[2] Alberich-Carramiñana, M., González, V.: Determining plane curve singularities from its polars. Adv. Math. **287**, 788–822 (2016)

[3] Briançon, J., Granger, M., Maisonobe, Ph.: Le nombre de modules du germe de courbe plane $x^a + y^b = 0$. Math. Ann. **279**, 535–551 (1988)

[4] Casas-Alvero, E.: On the singularities of polar curves. Manuscripta Math. **43**, 167–190 (1983)

[5] Casas-Alvero, E.: Base points of polar curves. Ann. Inst. Fourier **41**(1), 1–10 (1991)

[6] Casas-Alvero, E.: Singularities of plane curves, volume 276 of *London Math. Soc. Lecture Note Series*. Cambridge University Press, 2000

[7] Casas-Alvero, E.: On the analytic classification of plane curve singularities. *Asian J. of Math.*, **25**(5), 597–640 (2021)

[8] Cox, D.A., Little, J., O'Shea, D.: Using Algebraic Geometry, 2nd edn. Springer, Switzerland (2004)

[9] Cox, D.A., Little, J., O'Shea, D.: Ideals, Varieties and Algorithms, 4th edn. Springer, Switzerland (2015)

[10] Enriques, F., Chisini, O.: Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche. N. Zanichelli, Bologna, 1915-34

[11] Giles Flores, A., Teissier, B.: Local polar varieties in the geometric study of singularities. Annales de la Faculté des Sciences de Toulouse **27**(4), 679–775 (2018)

[12] Hefez, A., Hernandes, M.E.: The analytic classification of plane branches. Bull. Lond. Math. Soc. **43**, 289–298 (2011)

[13] Lê, D.T., Teissier, B.: Variétés polaires locales et classes de Chern des variétés singulières. Ann. Math. **114**, 457–491 (1981)

[14] Mather, J.N., Yau, S.T.: Classification of isolated hypersuperface singularities by their moduli algebras. Invent. Math. **69**, 243–251 (1982)

[15] Peraire, R.: Tjurina number of a generic irreducible curve singularity. J. Algebra **196**(1), 114–157 (1997)

[16] Peraire, R.: Moduli of plane curve singularities with a single characteristic exponent. Proc. Amer. Math. Soc. **126**(1), 25–34 (1998)

[17] Pham, F.: Deformations equisingulières des idéaux jacobiens des courbes planes. In: Proc. of Liverpool Symposium on Singularities II, volume 209 of Lect. Notes in Math., pages 218–233. Springer Verlag, Berlin, London, New York, 1971

[18] Teissier, B.: Variétés polaires I. Invariants polaires des singularités d'hypersurfaces. Invent. Math. **40**, 267–292 (1977)

[19] Zariski, O.: The moduli problem for plane branches. American Mathematical Society, Providence, Rhode Island (2006)