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The perfect cone compactification of quotients of type IV domains

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Abstract. The perfect cone compactification is a toroidal compactification which can be defined for locally symmetric varieties. Let $\overline{D_L/\tilde{O}^+(L)}^p$ be the perfect cone compactification of the quotient of the type IV domain D_L associated to an even lattice L . In our main theorem we show that the pair $(\overline{D_L/\tilde{O}^+(L)}^p, \Delta/2)$ has klt singularities, where Δ is the closure of the branch divisor of $D_L/\tilde{O}^+(L)$. In particular this applies to the perfect cone compactification of the moduli space of $2d$ -polarised $K3$ surfaces with ADE singularities when d is square-free.

1. Introduction

In [1, 13] Shepherd-Barron and Armstrong found a canonical model in the sense of the minimal model programme for the moduli space of principally polarised abelian varieties of dimension g for $g \geq 12$. Namely they showed that the perfect cone (or 1st Voronoi) compactification $\overline{A_g}^p$ has canonical singularities and that its canonical bundle is ample. It is natural to ask whether a canonical model can be found as well for quotients of type IV domains by the action of an arithmetic group. The analogue of the perfect cone compactification seems to be a good candidate. Let L be an even lattice of signature $(2, n)$ and D_L the associated hermitian symmetric domain of type IV. We consider the quotient $\mathcal{F}_L(\tilde{O}^+(L)) := D_L/\tilde{O}^+(L)$ and we study the singularities of its perfect cone compactification $\overline{\mathcal{F}_L(\tilde{O}^+(L))}^p$. Under certain assumptions on the reflections in $\tilde{O}^+(L)$ that fix a primitive isotropic vector in L we are able to prove that the pair $(\overline{\mathcal{F}_L(\tilde{O}^+(L))}^p, \frac{1}{2}\Delta)$ has klt singularities, where Δ is the closure of the branch divisor of $D_L \rightarrow \mathcal{F}_L(\tilde{O}^+(L))$. We recall that klt singularities form an important class in the study of higher dimensional algebraic varieties, in particular if (X, D) is a klt pair, then X has rational singularities. In Sect. 2.3 we apply our result to the perfect cone compactification of the moduli space of $2d$ -polarised $K3$ surfaces with ADE singularities when d is square-free.

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2. Compactifications

Finding compactifications of locally symmetric varieties is a classical problem in geometry which has been extensively studied. We start by recalling the construction of the Baily-Borel and of the toroidal compactifications with the focus on the setting of our main theorem. Our main references are [2,5].

2.1. Baily–Borel and toroidal compactifications

The Baily-Borel compactification and toroidal compactifications are all defined in general for quotients of Hermitian symmetric spaces by the action of an arithmetic subgroup of their automorphism group. We treat quotients of type IV domains. Let L be an even lattice of signature $(2, n)$ with $n \geq 1$ and let

$$D_L = \{[x] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (x, x) = 0 \ (x, \bar{x}) > 0\}^+$$

(where “+” denotes a connected component) be the associated Hermitian symmetric domain of type IV. Let $O^+(L)$ denote the subgroup of $O(L)$ preserving D_L and let $D_L^\bullet$ denote the affine cone over D_L . We consider modular forms:

Definition 2.1. Let $n \geq 3$, $k \in \mathbb{Z}$, $\Gamma < O^+(L)$ be a subgroup of finite index and $\chi : \Gamma \rightarrow \mathbb{C}^*$ be a character. A holomorphic function $F : D_L^\bullet \rightarrow \mathbb{C}$ is called a modular form of weight k and character χ for the group Γ if:

$$\begin{aligned} F(tZ) &= t^{-k} F(Z) \quad \forall t \in \mathbb{C}^* \\ F(gZ) &= \chi(g) F(Z) \quad \forall g \in \Gamma. \end{aligned}$$

In the case $n < 3$ an analogous definition would not imply the holomorphicity of F at the boundary, for more details we refer to Definition 6.4 and its discussion in [5]. Let $M_k(\Gamma, 1)$ denote the space of weight k modular form with trivial character. The Baily-Borel compactification of the quotient $\mathcal{F}_L(\Gamma) := D_L/\Gamma$ can be defined as:

$$\overline{D_L/\Gamma}^{BB} := \text{Proj}(\oplus_k M_k(\Gamma, 1)).$$

The Baily-Borel compactification is set-theoretically well understood, indeed the following holds:

Theorem 2.2. ([5] Theorem 5.5) *The Baily-Borel compactification $\overline{\mathcal{F}_L(\Gamma)}^{BB}$ decomposes as:*

$$\overline{\mathcal{F}_L(\Gamma)}^{BB} = \mathcal{F}_L(\Gamma) \sqcup \bigsqcup_{\pi} X_{\pi} \sqcup \bigsqcup_l Q_l$$

where l and π run through representatives of the finitely many Γ -orbits of isotropic lines and isotropic planes in $L \otimes \mathbb{Q}$ respectively. Each X_{π} is a modular curve, each Q_l is a point, and Q_l is contained in the closure of X_{π} if and only if the representatives may be chosen so that $l \subset \pi$.

The curves X_π and the points Q_l are called cusps. We introduce now toroidal compactifications. In comparison with the Baily-Borel compactification they have the advantage that the boundary consists of a divisor; on the other hand they are not unique: they depend on the choice of certain fans.

Let F be a cusp and $f \subseteq L \otimes \mathbb{Q}$ an isotropic space in the Γ -orbit corresponding to F . To every such f we associate the stabiliser group $N(f) \subseteq O^+(L_{\mathbb{R}})$ of the isotropic space f and the centre of the unipotent radical $U(f) \subseteq N(f)$. The group $U(f)$ has the structure of a vector space and $U(f) \cap \Gamma$ is a lattice in it. For every such f we will consider in the next section an open convex cone $C(f) \subseteq U(f)$ (see Theorem III.4.1 in [2] for the definition of $C(f)$). Let $\overline{C(f)}^\Gamma$ be the union of $C(f)$ and the rational rays lying in its closure. For every representative f of a cusp let $\Sigma(f)$ be a fan whose support is $\overline{C(f)}^\Gamma \subseteq U(f)$ and which is invariant under the action of the group $N(f)_\Gamma := N(f) \cap \Gamma \subseteq O^+(L)$; we denote by $X_{\Sigma(f)}$ the associated toric variety. The domain D_L can be embedded in an ambient space $D(f)$ and via this embedding we consider the compactification $(D_L/(U(f) \cap \Gamma))_{\Sigma(f)}$ of the quotient $D_L/(U(f) \cap \Gamma)$ inside an $X_{\Sigma(f)}$ -fibre bundle over F (a toric variety in the case F is a 0-dimensional cusp Q_l). If such toric varieties are given by a Γ -admissible collection of polyhedra Σ in the sense of Definition III.5.1 of [2], their quotients by the induced action of Γ glue to give a compactification of $\mathcal{F}_L(\Gamma)$. In greater detail, let $\widetilde{D}_L^\Gamma = \sqcup_f (D_L/(U(f) \cap \Gamma))_{\Sigma(f)}$ denote the union of the partial compactifications defined by the admissible collection of polyhedra Σ , where the union is taken over the set of all representatives f of cusps. Then the toroidal compactification $\overline{\mathcal{F}_L(\Gamma)}^\Sigma$ is the quotient of $\widetilde{D}_L^\Gamma$ by a certain equivalence relation Λ induced by Γ . The definition and analysis of the relation Λ is the subject of section III.6 of [2]. Thanks to Theorem III.5.2 of [2] the quotient $\overline{\mathcal{F}_L(\Gamma)}^\Sigma$ has the structure of a complex algebraic space. The compactification $\overline{\mathcal{F}_L(\Gamma)}^\Sigma$ comes with a surjection onto the Baily Borel compactification and analytic neighbourhoods of points lying over 0-cusps can be described as follows. Let $l \subseteq L$ be an isotropic line and Q_l be the associated cusp. The group $N(l)$ acts on the toric variety $X_{\Sigma(l)}$ preserving the partial compactification of $D_L/(U(l) \cap \Gamma)$. The quotient of the latter is isomorphic to an analytic open subset of the toroidal compactification $\overline{\mathcal{F}_L(\Gamma)}^\Sigma$ containing the points lying over the 0-cusp Q_l .

When Γ does not act freely it is meaningful to consider a neat normal subgroup of finite index: indeed given such a subgroup $\Gamma' \trianglelefteq \Gamma$, a Γ -admissible collection of polyhedra Σ is also Γ' -admissible. The finite group Γ/Γ' acts on the associated toroidal compactification $\overline{\mathcal{F}_L(\Gamma')}^\Sigma$ and has quotient $\overline{\mathcal{F}_L(\Gamma)}^\Sigma$.

Our main result concerns the quotient of D_L by the action of a particular group that we now introduce. Recall that a non-degenerate lattice Λ naturally injects in its dual $\Lambda^\vee := \text{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ and that the quotient $A_\Lambda := \Lambda^\vee/\Lambda$ is a finite group. The group A_Λ is called the discriminant group. The pairing of Λ induces a pairing on the dual with values in $\mathbb{Q}$ which in turn gives a $\mathbb{Q}/\mathbb{Z}$ -valued form on A_Λ . If Λ is even the form on $\Lambda^\vee$ yields:

$$q : A_\Lambda \rightarrow \mathbb{Q}/2\mathbb{Z}$$

We denote by $O(A_\Lambda)$ the group of automorphisms of A_Λ preserving this form. Our main result concerns the group

$$\tilde{O}^+(L) := \ker \left(O^+(L) \rightarrow O(L^\vee/L) \right).$$

We recollect some results contained in [9], Appendix A:

Proposition 2.3. *Given an isotropic line $l \subset L$, there is an isomorphism of lattices:*

$$U(l) \cap \tilde{O}^+(L) \simeq l^\perp/l.$$

The action of $N(l)$ on the variety $X_{\Sigma(l)}$ factors via the quotient $\overline{N(l)} := \frac{N(l) \cap \Gamma}{U(l) \cap \Gamma}$, which is a subgroup of the orthogonal group $O(l^\perp/l)$.

2.2. The perfect cone compactification

In this paragraph we introduce the toroidal compactification that is the object of our main result, the perfect cone compactification.

A cone in a real finite-dimensional vector space V is a set closed under multiplication by a positive scalar. Given an open cone $C \subseteq V \setminus \{0\}$ whose closure does not contain any line, we denote by $\overline{C}^* \subseteq V^*$ the linear forms that are non-negative on C ; we define the dual cone C^* to be the interior of $\overline{C}^* \setminus \{0\}$. We say that a cone is self-adjoint if there is a positive definite form on V whose induced isomorphism between V and V^* transforms C in C^* . A self-adjoint cone is necessarily convex. We say that C is homogeneous if the group of automorphisms of V preserving the cone C acts transitively on C .

We assume that $V \simeq L \otimes \mathbb{R}$ for a finitely generated free abelian group L , in particular V has a rational structure. Let G be an arithmetic subgroup of the automorphism group $\text{Aut}(V, C)$ of V fixing the cone C . Recall that a rational polyhedral cone $\sigma \subseteq V$ is a convex set of the form $\sigma = \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2 + \dots + \mathbb{R}_{\geq 0}v_n$ for vectors $v_1, \dots, v_n \in L \otimes \mathbb{Q}$.

Defining a toroidal compactification reduces to the problem of finding fans for certain homogeneous cones C with an action of G which are G -admissible polyhedral decompositions in the sense of the following

Definition 2.4. Given a self-adjoint homogeneous cone C in a real vector space V with integral structure $L \subseteq V$ and $G \subseteq \text{Aut}(V, C)$ an arithmetic subgroup, a G -admissible polyhedral decomposition for the cone C is a family $S = \{\sigma_\alpha\}$ of rational polyhedral cones $\sigma_\alpha \subseteq \overline{C}$ such that:

- The family S is closed under taking faces, i.e. for every $\sigma_\alpha \in S$, every face of σ_α is a cone in S ;
- For every pair of cones $\sigma_\alpha, \sigma_\beta \in S$ the intersection is a face of both;
- The group G acts on S , i.e. for every element $g \in G$ and for every cone $\sigma_\alpha \in S$, $g\sigma_\alpha$ is a cone in S ;
- The family S is finite modulo the action of the group G ;
- $C = \cup_\alpha (\sigma_\alpha \cap C)$.

Remark 2.5. The reader may wonder what the relation between the support of S and the rational closure of the cone C is. When they coincide, the toroidal space defined by S is compact.

On the one hand, if C is a cone and S a G -admissible polyhedral decomposition as in Definition 2.4, then their union is contained in the rational closure $\overline{C'}$, because all the cones in S are rationally polyhedral.

On the other hand, given the quotient $\mathcal{F}_L(\Gamma) := D_L/\Gamma$ and a Γ -admissible collection of polyhedra Σ , consisting of a $\Gamma(f)$ -admissible polyhedral decomposition of $C(f)$, one for each isotropic space f , then the associated toroidal space $\overline{\mathcal{F}_L(\Gamma)}^\Sigma$ is compact if and only if the equality

$$\bigcup_{\Sigma(f)} \sigma = \overline{C(f)}'$$

holds for every f .

Finally we recall Corollary II.5.23 of [2]

Corollary 2.6. *Let C be a self-adjoint homogeneous cone in a real vector space V with integral structure $L \subseteq V$ and let $G \subseteq \text{Aut}(V, C)$ be an arithmetic subgroup. Taking cones over the faces of the closed convex hull of $\overline{C} \cap L \setminus \{0\}$ yields a G -admissible polyhedral decomposition of C .*

Definition 2.7. Given a self-adjoint homogeneous cone C in a real vector space V with integral structure $L \subseteq V$, we refer to the decomposition given above as the perfect cone decomposition $\Sigma_p(C)$.

We define in the following the perfect cone compactification of $\mathcal{F}_L(\Gamma)$. Thanks to Lemma 2.25 of [4] for every 1 dimensional cusp X_π the vector space $U(\pi)$ has dimension 1, therefore the definition of a toroidal compactification does not involve choices for the fans in $C(\pi)$.

The definition of a toroidal compactification for $\mathcal{F}_L(\Gamma)$ then boils down to choosing an $N(l)_\Gamma$ -admissible decomposition of $C(l)$ for every Γ -orbit of isotropic vectors $l \in L$. We come to the following

Definition 2.8. The perfect cone compactification $\overline{\mathcal{F}_L(\Gamma)}^p$ is the toroidal compactification corresponding to the Γ -admissible collection of polyhedra Σ_p consisting of the perfect cone decomposition $\Sigma_p(l)$ of $C(l)$ for every cusp Q_l of $\overline{\mathcal{F}_L(\Gamma)}^{BB}$.

Remark 2.9. Thanks to the criterion for projectivity stated in section IV.2.1 of [2] the perfect cone compactification is projective.

Definition 2.10. We denote by Δ the closure in $\overline{\mathcal{F}_L(\tilde{O}^+(L))}^p$ of the branch divisor of $D_L \rightarrow \mathcal{F}_L(\tilde{O}^+(L))$.

Recall that an element of finite order $r \in \text{GL}(n, \mathbb{C})$ is said to be a quasi-reflection if all but one of its eigenvalues are equal to 1 and the eigenspace associated to the eigenvalue different from 1 has dimension 1. It is said to be a reflection if it is a quasi-reflection with an eigenvalue equal to -1.

Lemma 2.11. *If $r \in N(l) \cap \tilde{O}^+(L)$ is a reflection, then its image $\bar{r} \in \overline{N(l)}$ is a reflection.*

Proof. Since r has order 2, it does not lie in the torsion free group $U(l) \cap \tilde{O}^+(L)$ and $\bar{r}$ has order 2 too. Let $H \subseteq L$ be the hyperplane pointwise fixed by r , then $\bar{r}$ fixes $(H \cap l)/l$ and is therefore a reflection. $\square$

Finally we can state our main theorem:

Theorem 2.12. *If every reflection $s \in \overline{N(l)}$ admits a lifting to a reflection in $N(l) \cap \tilde{O}^+(L)$, then the pair $(\overline{\mathcal{F}_L(\tilde{O}^+(L))}^p, \frac{1}{2}\Delta)$ has klt singularities.*

Kawamata log terminal (klt) singularities are mild singularities that appear in the context of the minimal model programme, we refer to Definition 4.4 for the precise definition. The proof of Theorem 2.12 relies on the results about singularities of quotients of toric varieties obtained in Sect. 4, in particular see Proposition 4.6.

Remark 2.13. In the next section we give a concrete example for which the theorem holds.

Proof. Thanks to [10], Proposition 4-4-4 the notion of klt singularities is local in the analytic category, therefore we may prove the statement for an open covering in the analytic topology.

Let $\Sigma_p(1)$ be the fan formed by the 1-dimensional cones of Σ_p . The associated partial compactification $\overline{\mathcal{F}_L(\tilde{O}^+(L))}^{\Sigma_p(1)}$ is an open subset of $\overline{\mathcal{F}_L(\tilde{O}^+(L))}^p$ containing all the points lying over the 1-cusps. Given a neat normal subgroup $\Gamma' \leq \tilde{O}^+(L)$ of finite index, then the partial compactification $\overline{\mathcal{F}_L(\Gamma')}^{\Sigma_p(1)}$ is smooth. The finite group $\tilde{O}^+(L)/\Gamma'$ acts on it with quotient $q: \overline{\mathcal{F}_L(\Gamma')}^{\Sigma_p(1)} \rightarrow \overline{\mathcal{F}_L(\tilde{O}^+(L))}^{\Sigma_p(1)}$. Thanks to Theorem 1.1 of [4] every component of the ramification divisor of q has ramification index equal to 1 and the branch divisor coincides with the restriction of Δ . Hence the pair $(\overline{\mathcal{F}_L(\tilde{O}^+(L))}^{\Sigma_p(1)}, \frac{1}{2}\Delta)$ has klt singularities thanks to Lemma 3.16 of [7].

We are left to show that any point lying over a 0-cusp has a neighbourhood U such that the pair $(U, \frac{1}{2}\Delta|_U)$ has klt singularities.

Given a point $\bar{x} \in \overline{\mathcal{F}_L(\tilde{O}^+(L))}^p$ lying over the 0-cusp in $\overline{\mathcal{F}_L(\tilde{O}^+(L))}^{BB}$ associated to the isotropic vector in $l \in L$. Let V denote the analytic open subset in the toric variety $X_{\Sigma_p(l)}$ giving the partial compactification of $D_L/(U(l) \cap \Gamma)$ whose quotient by $\overline{N(l)}$ is isomorphic to an open neighbourhood of $\bar{x}$. Let $x \in V$ be a lifting of $\bar{x}$ and $G := \text{Stab}_{\overline{N(l)}}(x)$ be its stabiliser, then there exists a G -invariant open neighbourhood $U_x \subseteq V$ of x such that its image in the quotient by $\overline{N(l)}$ is isomorphic to U_x/G . Let $\sigma \in \Sigma(l)$ be the cone in whose torus orbit x lies. We denote by T_σ the associated toric variety. It follows that $G \subseteq \text{Stab}_{\overline{N(l)}}(\sigma)$, that G acts on both $(U_x \cap T_\sigma)$ and T_σ and thus $(U_x \cap T_\sigma)/G \subseteq T_\sigma/G$ is open. Let R be the ramification divisor of the quotient morphism $\pi: T_\sigma \rightarrow T_\sigma/G$. Thanks to Proposition 2.3 the group $G \subseteq O(l^\perp/l)$ acts via group morphisms on T_σ , hence by

Proposition 4.6 it follows that the pair $((U_x \cap T_\sigma)/G, \frac{1}{2}\pi(R)|_{(U_x \cap T_\sigma)/G})$ has klt singularities.

Finally, thanks to Theorem 2.1 of [4], the divisor Δ is induced by reflections of $\tilde{O}^+(L)$. Hence, if $U_{\bar{x}} \subseteq \mathcal{F}_L(\tilde{O}^+(L))^p$ denotes the open neighbourhood of $\bar{x}$ isomorphic to $(U_x \cap T_\sigma)/G$, because of the bijection between the reflections in $\overline{N(l)}$ and $N(l) \cap \tilde{O}^+(L)$ we conclude that the pair $(U_{\bar{x}}, \frac{1}{2}\Delta|_{U_{\bar{x}}})$ has klt singularities and the claim is shown. $\square$

2.3. The moduli space of polarised K3 surfaces with ADE singularities

We recall here the construction of the moduli space of polarised K3 surfaces with ADE singularities and explain how Theorem 2.12 can be applied to it.

A complex K3 surface is a compact connected complex manifold of dimension 2 such that

$$\Omega_X^2 \cong \mathcal{O}_X \text{ and } H^1(X, \mathcal{O}_X) = 0.$$

The group $H^2(X, \mathbb{Z})$, equipped with the intersection form, is an even unimodular lattice of rank 22 of signature (3, 19) isomorphic to $\Lambda_{K3} := U^{\oplus 3} \oplus E_8^{\oplus 2}(-1)$ where U is the hyperbolic lattice and E_8 is the positive definite lattice associated to the E_8 Dynkin diagram (see Proposition 1.3.5 of [6]).

Let L be a line bundle on a K3 surface X . Recall that L is big and nef if and only if $(L, C) \geq 0$ for all closed curves $C \subset X$, and $L^2 > 0$ (see for example Proposition 2.61 of [8]). A quasi-polarised K3 surface is a pair (X, L) where X is a K3 surface and L is a primitive big and nef line bundle on it.

After we fixed an isomorphism of lattices $\varphi : H^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda_{K3}$, a polarisation of degree $2d$ (i.e. $L^2 = 2d$) gives an element $h = \varphi([L]) \in \Lambda_{K3}$ whose orthogonal complement $h^\perp$ is isomorphic to $\Lambda_{2d} := U^{\oplus 2} \oplus E_8^{\oplus 2}(-1) \oplus \langle -2d \rangle$. We consider the period domain:

$$\mathcal{D}_{2d} := \{x \in \mathbb{P}(\Lambda_{2d} \otimes \mathbb{C}) \mid (x, x) = 0 \text{ and } (x, \bar{x}) > 0\}^+$$

where the sign “+” denotes a connected component.

Denote by

$$O(\Lambda_{K3}, h) := \{g \in O(\Lambda_{K3}) \mid g(h) = h\} \leq O(\Lambda_{K3})$$

the subgroup of automorphisms of Λ_{K3} fixing h . The coarse moduli space of quasi-polarised K3 surfaces $\mathcal{F}_{2d}$ is given by the quotient of $\mathcal{D}_{2d}$ by the action of $\tilde{O}^+(\Lambda_{2d})$, i.e. the finite index subgroup of $O(\Lambda_{K3}, h)$ fixing the connected component $\mathcal{D}_{2d}$.

The disadvantage of working with the moduli functor of quasi-polarised K3 surfaces is that it is not separated; therefore it is more convenient to treat the moduli functor of polarised K3 surfaces with ADE singularities, which is separated and whose coarse moduli space can be identified with $\mathcal{F}_{2d}$ (for more details see Section 5.1.4 and Remark 6.4.5 of [6]).

The quotient $\mathcal{F}_{2d}$ is a quasi-projective variety of dimension 19. We can state the following:

Theorem 2.14. *Let $\overline{\mathcal{F}_{2d}}^p$ be the perfect cone compactification of the coarse moduli space of polarised K3 surfaces with ADE singularities and let Δ be the closure of the branch divisor of the morphism $\pi : D_{2d} \rightarrow \mathcal{F}_{2d}$. If d is square-free, the pair $(\overline{\mathcal{F}_{2d}}^p, \frac{1}{2}\Delta)$ has klt singularities.*

Proof. Thanks to Corollary 1.5.2 of [11], there is an isomorphism $O(\Lambda_{K3}, h) \simeq \tilde{O}(\Lambda_{2d})$. Thanks to [12] Theorem 4.0.1, if d is square free, there is one $\tilde{O}^+(\Lambda_{2d})$ orbit of primitive isotropic vectors. Thus, given $\ell \in \Lambda_{2d}$ primitive isotropic there exists an element $\ell' \in \Lambda_{2d}$ giving a splitting $\Lambda_{2d} \simeq U \oplus (\ell^\perp / \langle \mathbb{Z}\ell \rangle)$ with $\langle \ell, \ell' \rangle \simeq U$. Hence the assumptions of Theorem 2.12 are fulfilled and the claim follows from Theorem 2.12. $\square$

3. Canonical singularities

In this section we review some classes of singularities. We review firstly the definition of canonical singularities and we recall a characterisation of canonical singularities for toric varieties that will be used in our exposition. We are also interested in a particular class of singularities of pairs, namely the Kawamata log terminal singularities.

Definition 3.1. We say that a normal $\mathbb{Q}$ -Gorenstein variety X over $\mathbb{C}$ has canonical singularities if there exists a proper birational morphism $f : X' \rightarrow X$ from a smooth variety X' such that:

$$K_{X'} = f^* K_X + \sum_{i \in I} a_i E_i$$

with $a_i \in \mathbb{Q}_{\geq 0}$ where $\{E_i\}_{i \in I}$ is the set of all irreducible prime divisors lying in the exceptional locus of f .

If the condition $a_i \geq 0$ holds for a resolution, then for any proper birational morphism $g : V \rightarrow X$ with V smooth all the coefficients of the exceptional divisors in the ramification formula $K_V = g^* K_X + \sum_i b_i E_i$ are non-negative.

Toric varieties with canonical singularities can be characterised in terms of the cones defining them.

Let N be a finitely generated free abelian group and let T_N be the associated torus whose lattice of 1-parameter subgroups is given by N ; we denote by M the lattice of characters of T_N and by $\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}$ the natural pairing. Let $\sigma \subset N \otimes \mathbb{R}$ be a rational convex polyhedral cone and let U_σ be the associated affine toric variety. If $\rho \in \sigma(1)$ is a ray, we denote by u_ρ the unique non-trivial primitive vector in $\rho \cap N$. We consider the polytope $\Pi_\sigma := \text{convex-hull}(\{0\} \cup \{u_\rho | \rho \in \sigma(1)\}) \subset N \otimes \mathbb{R}$.

Proposition 3.2. (Proposition 11.4.12 of [3]) *With the above notation,*

- *The following are equivalent:*
 - *The variety U_σ is $\mathbb{Q}$ -Gorenstein;*
 - *There exists an $m \in M \otimes \mathbb{Q}$ such that $\forall \rho \in \sigma(1) \langle m, u_\rho \rangle = 1$;*
 - *The polytope Π_σ has a unique facet F not containing the origin.*

- If the variety U_σ is $\mathbb{Q}$ -Gorenstein, then the variety U_σ has canonical singularities if and only if the only non-zero lattice points of Π_σ lie in the facet not containing the origin (or, more concisely, $\Pi_\sigma \cap N \setminus \{0\} \subset F \cap N$).

Remark 3.3. If $\Sigma = \{\sigma_\alpha\}_\alpha$ is a fan and X_Σ is the associated toric variety, then X_Σ has canonical singularities if and only if for every cone $\sigma \in \Sigma$ the variety U_σ has canonical singularities.

We can further restrict ourselves to consider only maximal cones, in the sense that the variety X_Σ is $\mathbb{Q}$ -Gorenstein (respectively has canonical singularities) if and only if for every maximal cone $\sigma \in \Sigma$ the variety U_σ is $\mathbb{Q}$ -Gorenstein (respectively has canonical singularities).

We focus now on the toric varieties appearing in the definition of the perfect cone compactification. Let L be a finitely generated free abelian group and $C \subset L \otimes \mathbb{R}$ an open convex cone that is self-adjoint. We consider the set $K_p = \text{convex-hull}(\overline{C} \cap L \setminus \{0\})$ and we denote by Σ_p the fan given by taking cones over the faces of K_p and the trivial cone, that is:

$$\Sigma_p = \{\sigma \subset L_{\mathbb{R}} \text{ cone} \mid \text{there is a face } F \subset K_p \text{ such that } \sigma = \mathbb{R}_{\geq 0}F\}.$$

Theorem 3.4. *The toric variety X_{Σ_p} associated to the fan Σ_p is $\mathbb{Q}$ -Gorenstein and has canonical singularities.*

Proof. Thanks to Remark 3.3, in order to prove the theorem it is enough to show that for every $\sigma \in (\Sigma_p)_{\max}$ the variety T_σ is $\mathbb{Q}$ -Gorenstein and has canonical singularities. Let $\sigma = \mathbb{R}_{\geq 0}F$ be such a cone, where F is a facet of K_p . Let H denote the supporting hyperplane of F , i.e. $F = H \cap K_p$, and let H^- be the closed half-space such that $H^- \cap K_p = F$. The origin lies in the interior of H^- : since σ is of maximal dimension, the origin does not lie on H . On the other hand, if 0 is in the complement of H^- , then $\mathbb{R}_{>1}F \subseteq H^-$ and $\mathbb{R}_{>1}F \subseteq \mathbb{R}_{>1}K_p \subseteq K_p$, which is absurd.

Recall the definition of the polytope $\Pi_\sigma := \text{convex-hull}(\{0\} \cup \{u_\rho \mid \rho \in \sigma(1)\})$ and let $0 \neq v \in L \cap \Pi_\sigma \subseteq L \cap \sigma \cap H^-$; there exists an $n \in \mathbb{N}$ such that $nv \in K_p \subseteq \overline{C}$. Since $\overline{C}$ is a cone it follows that $v \in \overline{C}$; therefore $v \in L \cap \overline{C} \setminus \{0\} \subseteq K_p$. We conclude that $v \in K_p \cap H^- = F$ and, thanks to Proposition 3.2, that T_σ is $\mathbb{Q}$ -Gorenstein and has canonical singularities. $\square$

4. Singularities of quotients of toric varieties

In this section we analyse quotients of toric varieties with canonical singularities by the action of finite groups acting via group homomorphisms of the torus.

The following is essentially [4], Corollary 2.20. We include a proof to highlight the fact that the assumption of smoothness is not necessary.

Lemma 4.1. *Let N be a free abelian group and Σ a fan of strongly convex rational polyhedral cones contained in $N_{\mathbb{R}}$; let $T \hookrightarrow X_\Sigma$ be the corresponding toric variety. Let $G \leq \text{GL}(N)$ be a finite group acting on the variety X_Σ . Let D be a divisor which is pointwise fixed by a nontrivial element $1 \neq g \in G$, then D is not a torus-invariant divisor, i.e. $D \cap T \neq \emptyset$.*

Proof. Looking for a contradiction, we assume that D lies in the boundary.

Let M denote the dual of N . We can identify the torus T with the group homomorphisms $T = \text{Hom}(M, \mathbb{C}^*) \cong N \otimes \mathbb{C}^*$ and D corresponds to a 1-dimensional cone in Σ , say ρ ; in particular $D = \overline{O(\rho)}$ where $O(\rho) = \{u : \rho^\perp \cap M \rightarrow \mathbb{C}^* | u \text{ group homomorphism}\}$ is a torus orbit. Since g acts trivially on the divisor D , it acts trivially on $\rho^\perp \cap M \cong \frac{N}{\rho \cap N}$. Therefore, we get the following short exact sequence of $\langle g \rangle$ -equivariant group morphisms:

$$0 \rightarrow \mathbb{Z}(\rho \cap N) \rightarrow N \rightarrow \frac{N}{\rho \cap N} \rightarrow 0.$$

The action of g on $\mathbb{Z}(\rho \cap N)$ can be just $\pm id$; nonetheless, since g acts on $O(\rho)$, it acts on the toric affine variety U_ρ associated to the cone ρ , hence g acts as the identity on $\mathbb{Z}(\rho \cap N)$. It follows that g acts as the identity on N , which is a contradiction and the claim is shown. $\square$

Lemma 4.2. *Let T be a torus with associated 1-parameter subgroup lattice N . Let $T \hookrightarrow X$ be a toric variety acted on by a finite group $G < GL(N)$. Then the ramification divisor of the quotient morphism $\pi : X \rightarrow X/G$ is given by*

$$R(\pi) = \sum_{\substack{r \in G \\ \text{reflection}}} \text{Fix}(r)$$

where $\text{Fix}(r)$ denotes the fixed locus of r equipped with the reduced structure.

Proof. By Lemma 4.1, no component of the ramification divisor is torus-invariant, hence if $R(\pi)$ has support on a prime divisor D , then $D = \overline{D \cap T}$ and we are left to prove the statement for the restriction $\pi|_T : T \rightarrow T/G$.

For this, we assume that $X = T$ is the torus and we consider a point $x \in D$ for a prime divisor D fixed by an element $g \in G$. Multiplication by x induces a g -equivariant morphism between the tangent spaces $m_x : T_e T \rightarrow T_x T$. The tangent space to D at x is a g -invariant codimension 1 subspace $T_x D \subseteq T_x T$ and so is its preimage $m_x^{-1}(T_x D) \subseteq T_e T \simeq N \otimes \mathbb{C}$. In particular, g acts as the identity on $m_x^{-1}(T_x D)$, and is a quasi-reflection. Let λ be the unique non-trivial eigenvalue of g , since $g \in GL(N)$, its trace $\text{tr}(g) = n - 1 + \lambda$ is an integer, so g is a reflection and the claim is shown. $\square$

As the next example shows, the quotient of a toric variety with canonical singularities by a finite group acting via automorphisms of the one parameter subgroup lattice need not be $\mathbb{Q}$ -Gorenstein. In particular it seems that without more precise knowledge of the cones of the fan Σ it is not possible to conclude that the perfect cone compactification $\overline{\mathcal{F}_L(\widetilde{O}^+(L))}^p$ has canonical singularities.

Example 4.3. Let $\sigma \subset \mathbb{R}^3 = N_{\mathbb{R}}$ be the cone over the polytope

$$P := \text{convex-hull}((-1, -1, 1), (1, 1, 1), (0, 2, 1), (-1, 1, 1), (-1, -1, 1)).$$

The corresponding toric variety $T_N \hookrightarrow X_\sigma$ has, thanks to Proposition 3.2, canonical singularities. Let r be the reflection fixing the coordinate plane $\{(0, y, z) : y, z \in \mathbb{R}\}$. The cone σ is fixed by the action of r on $N_{\mathbb{R}}$, therefore the group $G = \langle r \rangle$ generated by the reflection r acts on the toric variety X_σ . The pull-back formula in section 2.1 of [14] reads:

$$K_{X_\sigma} = f^*(K_{X_\sigma/G} + \frac{1}{2}f(R))$$

where $R \in \text{Div}(X_\sigma)$ is the ramification divisor of the quotient morphism $f: X_\sigma \rightarrow X_\sigma/G$.

Thanks to Corollary 2.2 of [14] it follows that $K_{X_\sigma/G}$ is $\mathbb{Q}$ -Cartier if and only if R is. Thanks to Proposition 4.1.2 of [3] it can be computed that:

$$R = \text{div}(x - x^{-1}) - \sum_{\rho_i \neq (0,2,1)} D_{\rho_i}$$

where D_{ρ_i} is the torus-invariant divisor corresponding to the ray $\rho_i \subseteq \sigma$. In particular neither R nor $K_{X_\sigma/G}$ is $\mathbb{Q}$ -Cartier.

The same argument carries over to many other polytopes that are symmetric with respect to an axis.

Even if the quotient might not be $\mathbb{Q}$ -Gorenstein, we can prove a weaker statement about the singularities of quotients of toric varieties, for this we recall the notion of pair. We consider pairs (X, B) where X is a normal variety and $B := \sum d_i D_i$ is a boundary, i.e. a $\mathbb{Q}$ -divisor for which the coefficient of each prime divisor D_i satisfies $0 \leq d_i \leq 1$. If $K_X + B$ is $\mathbb{Q}$ -Cartier, its pullback along any morphism is well-defined and for a birational morphism $f: Y \rightarrow X$ with Y smooth, if E_i denote the exceptional divisors, we have the equality

$$K_Y = f^*(K_X + B) - f_*^{-1}(B) + \sum a_i E_i$$

for some rational numbers $a_i = a(E_i, X, B)$. The coefficient a_i is independent of the model Y , but it depends only on the centre of E_i ([10], Proposition-Definition 4-4-1), it is called discrepancy of (X, B) at E_i . We further define the discrepancy of (X, B) to be:

$$\text{disc}(X, B) := \inf\{a(E, X, B) : E \text{ corresponds to a discrete valuation of } k(X) \text{ such that } \text{Center}_X(E) \neq \emptyset \text{ and } \text{codim}_X(\text{Center}_X(E)) \geq 2\}.$$

Definition 4.4. The pair (X, B) with $B = \sum d_i D_i$ is said to have

- Canonical singularities if $K_X + B$ is $\mathbb{Q}$ -Cartier and $\text{disc}(X, B) \geq 0$;
- Kawamata log terminal (klt) singularities if $K_X + B$ is $\mathbb{Q}$ -Cartier, $d_i < 1$ for any i and $\text{disc}(X, B) > -1$.

Remark 4.5. The variety X has canonical singularities if and only if the pair $(X, 0)$ has canonical singularities.

Proposition 4.6. *Let T be a torus with associated 1-parameter subgroup lattice N . Let $T \hookrightarrow X$ be a toric variety with canonical singularities acted on by a finite group $G < GL(N)$. If $\pi: X \rightarrow X/G$ denotes the quotient morphism and R the ramification divisor, then the pair $(X/G, \frac{1}{2}\pi(R))$ has klt singularities.*

Proof. Thanks to Lemma 4.2 the ramification divisor equals $\sum_r D_r$, where the sum runs over all the reflections of G and D_r denotes the divisor pointwise fixed by the reflection r . The claim now follows from Lemma 3.16 of [7] or from generalising word by word the proof of Corollary 2.2 of [14] to the case of klt singularities. $\square$

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Declarations

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