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Homotopy nilpotency of some homogeneous spaces

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Abstract. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the field of reals or complex numbers and $\mathbb{H}$, the skew $\mathbb{R}$ -algebra of quaternions. We study the homotopy nilpotency of the loop spaces $\Omega(G_{n,m}(\mathbb{K}))$, $\Omega(F_{n;n_1,\dots,n_k}(\mathbb{K}))$, and $\Omega(V_{n,m}(\mathbb{K}))$ of Grassmann $G_{n,m}(\mathbb{K})$, flag $F_{n;n_1,\dots,n_k}(\mathbb{K})$ and Stiefel $V_{n,m}(\mathbb{K})$ manifolds. Additionally, homotopy nilpotency classes of p -localized $\Omega(G_{n,m}^+(\mathbb{K})_{(p)})$ and $\Omega(V_{n,m}(\mathbb{K})_{(p)})$ for certain primes p are estimated, where $G_{n,m}^+(\mathbb{K})_{(p)}$ is the oriented Grassmann manifolds. Further, the homotopy nilpotency classes of loop spaces of localized homogeneous spaces given as quotients of exceptional Lie groups are investigated as well.

Introduction

The homotopy nilpotency classes $\text{nil } X$ of associative H -spaces X has been extensively studied in addition to their homotopy commutativity. In particular, Hopkins [8] made great progress by giving (co)homological criteria for homotopy associative finite H -spaces to be homotopy nilpotent. For example, he showed that if a homotopy associative finite H -space has no torsion in the integral homology, then it is homotopy nilpotent. Later, Rao [19, 20] showed that the converse of the above criterion is true in the case of groups $\text{Spin}(n)$ and $SO(n)$ and a connected compact Lie group is homotopy nilpotent if and only if it has no torsion in homology. Eventually, Yagita [26] proved that, when G is a compact, simply connected Lie group, its p -localization $G_{(p)}$ is homotopy nilpotent if and only if it has no torsion in the integral homology. Although many results on the homotopy nilpotency are known, precise homotopy nilpotence classes have not been determined in most cases.

Let $\mathbb{K}P^n$ be the projective n -space for $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the field of reals or complex numbers and $\mathbb{H}$, the skew $\mathbb{R}$ -algebra of quaternions. The homotopy nilpotency of the loop spaces $\Omega(\mathbb{K}P^n)$ has been first studied by Ganea [5], Snaith [22] and then their p -localization $\Omega((\mathbb{K}P^n)_{(p)})$ by Meier [15].

For the James reduced product $J(X)$ of a space X , Cohen and Wu [4] have asked:

QUESTION. Is the Cohen group $[J(X), \Omega(Y)]$ nilpotent for any spaces X, Y ?

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We were first inspired by the homotopy nilpotency of $\Omega(\mathbb{H}P^n)$ for any $n \geq 1$ which does not appear in the literature known to the author and then by the question above as well. The paper grew out of our desire to develop techniques in calculating homotopy nilpotency classes of loop spaces of homogeneous spaces. Because no answer is known to the question above, we present some homogeneous spaces Y such that the group $[J(X), \Omega(Y)]$ is nilpotent for any space X .

In Section 1, we set stages for developments to come. This introductory section is devoted to a general discussion and establishes notations on homotopy nilpotency of associative H -spaces used in the rest of the paper.

Section 2, based on results [27, Chapter II] by Zabrodsky, takes up the systematic study of the homotopy nilpotency of loop spaces $\Omega(G_{n,m}(\mathbb{K}))$ and $\Omega(V_{n,m}(\mathbb{K}))$ of Grassmann $G_{n,m}(\mathbb{K})$ and Stiefel $V_{n,m}(\mathbb{K})$ manifolds. First, we make use of [27, Lemma 2.6.6] to derive:

Proposition 2.2. *If G is a compact Lie group and $K < G$ its closed subgroup with $\text{nil } K < \infty$ then $\text{nil } \Omega(G/K) < \infty$.*

Then, we obtain the following results:

Proposition 2.7. *If $1 \leq m \leq n \leq \infty$ then:*

- (1) $\text{nil } \Omega(G_{n,m}^+(\mathbb{R})_{(p)}) < \infty$ for $p \geq 3$;
- (2) $\text{nil } \Omega(G_{n,m}(\mathbb{K})) < \infty$ for $\mathbb{K} = \mathbb{C}, \mathbb{H}$.

In particular, $\text{nil } \Omega(\mathbb{H}P^n) < \infty$.

Certainly, for (oriented) flag $F_{n;n_1,\dots,n_k}(\mathbb{K})$ manifolds, Proposition 2.2 leads to:

- (1) $\text{nil } \Omega(F_{n;n_1,\dots,n_k}^+(\mathbb{R})_{(p)}) < \infty$ and $\text{nil } \Omega(F_{n;n_1,\dots,n_k}(\mathbb{R})_{(p)}) < \infty$ for $p \geq 3$;
- (2) $\text{nil } \Omega(F_{n;n_1,\dots,n_k}(\mathbb{K})) < \infty$ for $\mathbb{K} = \mathbb{C}, \mathbb{H}$.

Proposition 2.17. *If $1 \leq m \leq n$ then:*

- (1) $\text{nil } \Omega((V_{n,m})_{(p)}) < \infty$ for $p \geq 3$;
- (2) $\text{nil } \Omega(W_{n,m}) < \infty$;
- (3) $\text{nil } \Omega(X_{n,m}) < \infty$.

Corollaries 2.4, 2.9 and 2.18 answer Cohen-Wu question [4] for some homogeneous spaces given as quotients of classical Lie groups. In particular, for Grassmann, flag and Stiefel manifolds.

Next, we make use of results [10, 11] by Kaji and Kishimoto, [12] by Kishimoto and [14] by McGibbon on p -regular classical groups to estimate the homotopy nilpotency classes of the p -localized spaces $\Omega(G_{n,m}(\mathbb{K})_{(p)})$ and $\Omega(V_{n,m}(\mathbb{K})_{(p)})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the field of reals or complex numbers and $\mathbb{H}$, the skew $\mathbb{R}$ -algebra of quaternions.

Section 3 is devoted to the homotopy nilpotency of the loop spaces $\Omega(\mathbb{O}P^2)$ of the Cayley plane $\mathbb{O}P^2$ and of other homogeneous spaces being quotients of exceptional Lie groups. We also make use results [24, Theorem 12.1] by Theriault on the homotopy nilpotency classes of quasi- p -regular exceptional Lie groups to estimate the homotopy nilpotency classes of p -localized loop spaces of those homogeneous spaces.

1. Preliminaries

All spaces and maps in this note are assumed to be connected, based and of the homotopy type of CW -complexes. We also do not distinguish notationally between a continuous map and its homotopy class. We write $\Omega(X)$ (resp. $\Sigma(X)$) for the loop (resp. suspension) space on a space X and $[Y, X]$ for the set of homotopy classes of maps $Y \rightarrow X$.

Given a space X , we use the customary notations $X \vee X$ and $X \wedge X$ for the *wedge* and the *smash product* of X , respectively.

Recall that an H -space is a pair (X, μ) , where X is a space and $\mu : X \times X \rightarrow X$ is a map such that the diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{\mu} & X \\ \uparrow & \nearrow \nabla & \\ X \vee X & & \end{array}$$

commutes up to homotopy, where $\nabla : X \vee X \rightarrow X$ is the codiagonal map.

We call μ a *multiplication* or an H -structure for X . Two examples of H -spaces come in mind: topological groups and the spaces $\Omega(X)$ of loops on X . In the sequel, we identify an H -space (X, μ) with the space X .

An H -space X is called a *group-like space* if X satisfies all the axioms of groups up to homotopy. Recall that a homotopy associative H -space always has a homotopy inverse. More precisely, according to [27, 1.3.2. Corollary] (see also [1, Proposition 8.4.4]), we have:

Proposition 1.1. *If X is a homotopy associative H -space then X is a group-like space.*

If X is a homotopy associative H -space, then the functor $[-, X]$ takes its values in the category of groups. One may then ask when this functor takes its values in various subcategories of groups.

For example, X is homotopy commutative if and only if $[Y, X]$ is abelian for all Y .

Given a group-like space X , we write $\gamma_{X,1} = \text{id}_X : X \rightarrow X$ and $\bar{\gamma}_{X,2} : X \times X \rightarrow X$ for the commutator map of X . Observe that the restriction of $\bar{\gamma}_{X,2}$ to the wedge $X \vee X$ is null homotopic so $\bar{\gamma}_{X,2}$ extends to a map

$$\gamma_{X,2} : X \wedge X \rightarrow X.$$

For an integer $n \geq 1$, let $X^{\wedge(n+1)}$ be the $(n+1)$ -fold smash power of X . Define the iterated Samelson product

$$\gamma_{X,n+1} : X^{\wedge(n+1)} \rightarrow X$$

inductively by $\gamma_{X,n+1} = \gamma_{X,2} \circ (\gamma_{X,1} \wedge \gamma_{X,n})$ for $n \geq 2$. Notice that $\gamma_{X,n}$ has a universal property: any Samelson product of length n on X factors through $\gamma_{X,n}$.

Lemma 1.2. *If X is a homotopy associative H -space and Y a finite dimensional CW -complex with $\dim Y = n$ then the group $[Y, X]$ is nilpotent with the nilpotency class at most n .*

Proof. First, recall that given a homotopy associative H -space X , in view of [9], all its m -th Postnikov stages $P_m X$ are also a homotopy associative H -space and the canonical map $X \rightarrow P_m X$ is an H -map. Hence, for a CW -complex Y with $\dim Y = n$, there is an isomorphism $[Y, X] \approx [Y, P_n X]$ determined by the canonical map $X \rightarrow P_n X$. Then, the map $\gamma_{P_n X, n+1}(f_1 \wedge \cdots \wedge f_{n+1}) : Y^{\wedge(n+1)} \rightarrow P_n X$ is homotopy trivial for any maps $f_1, \dots, f_{n+1} : Y \rightarrow P_n X$ since the space $Y^{\wedge(n+1)}$ is n -connected. Consequently, $\text{nil}[Y, X] = \text{nil}[Y, P_n X] \leq n$ and the proof follows. $\square$

We point out that the result above has been stated in [8, 20] but for a finite CW -complex and a finite homotopy associative H -space X , only. Next, any CW -complex Y can be expressed as

$$Y = \varinjlim Y_\alpha$$

with Y_α finite. This leads to the short exact sequence

$$1 \rightarrow \varprojlim^1 [\Sigma Y_\alpha, X] \longrightarrow [Y, X] \longrightarrow \varprojlim_\alpha [Y_\alpha, X] \rightarrow 1$$

for any connected homotopy associative H -space X .

In view of [8, Proposition 1.2], we have that the intersection $\Gamma_d \cap (\varprojlim^1 [\Sigma Y_\alpha, X]) = 0$ for any finite connected, homotopy associative H -space X with $\dim X = d < \infty$, where Γ_d stands for the d -th member of the lower central series of the group $[Y, X]$. Then, Lemma 1.2 and [8, Proposition 1.1] yield:

Proposition 1.3. *If X is a finite connected, homotopy associative H -space then the group $[Y, X]$ is pro-nilpotent for any CW -complex Y .*

One might ask if there is an upper bound for the nilpotency class of $[Y, X]$ that is independent of Y . The *homotopy nilpotency class* of X is the least n such that $\gamma_{X, n+1} \simeq *$ and $\gamma_{X, n} \not\simeq *$. In this case, we write $\text{nil } X = n$ and call the homotopy associative H -space X *homotopy nilpotent*.

Note that $\text{nil } X = 1$ if and only if X is homotopy commutative.

By [2, Theorem 2.7], one has:

$$\text{nil } X = \sup_m \text{nil}[X^m, X] = \sup_Y \text{nil}[Y, X]$$

for the m -th Cartesian power X^m of X .

Given a space X , the number $\text{nil } \Omega(X)$ (if any) is called the *homotopy nilpotency class* of X .

Further, according to Zabrodsky [27, Lemma 2.6.1], we have:

Proposition 1.4. *A connected associative H -space X is homotopy nilpotent if and only if the functor $[-, X]$ on the category of all pointed spaces is nilpotent group valued.*

It is obvious that for homotopy nilpotent group-like spaces $X_1, \dots, X_m$, we have

$$\text{nil}(X_1 \times \cdots \times X_m) = \max\{\text{nil } X_1, \dots, \text{nil } X_m\}.$$

The first major advance was made by Hopkins [8, Theorem 2.1] (completed by Rao [20, Theorem 0.2]). He showed that a finite H -space X is homotopy nilpotent if and only if for sufficiently large n , $\gamma_{X,n}$'s induce trivial homomorphism in complex bordism. This is same as asking that $\gamma_{X,n}$'s induce trivial homomorphisms in all Morava K -theories.

Then, in [8, Corollary 2.2], it was deduced:

Corollary 1.5. *If X is a finite associative H -space and the homology $H_*(X, \mathbb{Z})$ is torsion free then X is homotopy nilpotent.*

This corollary implies:

$$\text{nil } U(n) < \infty \text{ and } \text{nil } Sp(n) < \infty.$$

But, the homotopy nilpotency does not imply the nilpotency of a topological group.

Remark 1.6. Since the commutators $[SO(3), SO(3)] = SO(3)$ and $[SU(2), SU(2)] = SU(2)$ and: $SO(3) \subseteq SO(n) \subseteq O(n)$ for $n \geq 3$, $SU(2) \subseteq SU(n) \subseteq U(n)$ for $n \geq 2$ and $SU(2) = Sp(1) \subseteq Sp(n)$ for $n \geq 1$, we derive that the groups:

- (1) $SO(n)$ and $O(n)$ are not nilpotent for $n \geq 3$;
- (2) $U(n)$ and $SU(n)$ are not nilpotent for $n \geq 2$;
- (3) $Sp(n)$ is not nilpotent for $n \geq 1$.

Later, Rao [19] showed that the converse of the criterion from Corollary 1.5 is true in the case of $\text{Spin}(n)$ and $SO(n)$ by showing that $\text{Spin}(n)$, $SO(n)$, $n \geq 7$ and $SO(3)$, $SO(4)$ are not homotopy nilpotent.

Because of H -homotopy equivalences $O(n) \simeq SO(n) \times \mathbb{Z}_2$ and $U(n) \simeq SU(n) \times \mathbb{S}^1$, we derive:

$$\text{nil } SO(n) = \text{nil } O(n) \text{ and } \text{nil } SU(n) = \text{nil } U(n) < \infty.$$

Next, write $O = \varinjlim O(n)$, $U = \varinjlim U(n)$ and $Sp = \varinjlim Sp(n)$. Then, notice that by Bott periodicity theorem: $\Omega^8(O) \simeq O$, $\Omega^2(U) \simeq U$ and $\Omega^8(Sp) \simeq Sp$, we get

$$\text{nil } O = \text{nil } U = \text{nil } Sp = 1.$$

Let now $G_{(p)}$ stand for the p -localization in the sense of [3] of a compact Lie group G . Then, Yagita [26, Theorem] has shown:

Theorem 1.7. *Let G be a simply connected Lie group. Then for each prime p , the p -localization $G_{(p)}$ is homotopy nilpotent if and only if the integral cohomology $H^*(G, \mathbb{Z})$ has no p -torsion.*

Next, Rao [20, Theorem 0.2] has generalized Theorem 1.7 as follows:

Theorem 1.8. *Let G be a compact connected Lie group and let p be a prime. Then $G_{(p)}$ is homotopy nilpotent if and only if $H_*(G, \mathbb{Z}_{(p)})$ is torsion-free.*

Now, write $J(X)$ for the James reduced product on a space X and BG for the classifying space of a topological group G . In view of [23, Theorem 8.6], there is an H -map $G \rightarrow \Omega(BG)$ which is a weak homotopy equivalence. Consequently, for a Lie group G there is an H -homotopy equivalence $\Omega(BG) \simeq G$. Then, for the question [4] by Cohen and Wu, we conclude:

Corollary 1.9. *If $n \geq 1$ and X is a space then:*

- (1) $\text{nil}[J(X), \Omega(BO(n)_{(p)})] = \text{nil}[J(X), \Omega(BSO(n)_{(p)})] < \infty$ for any prime $p \geq 3$;
- (2) $\text{nil}[J(X), \Omega(BU(n))] = \text{nil}[J(X), \Omega(BSU(n))] < \infty$;
- (3) $\text{nil}[J(X), \Omega(BSp(n))] < \infty$.

2. Main results

Let $f : X \rightarrow Y$ be an H -map of homotopy associative H -spaces. Recall from [27, Chapter II] that:

- (1) it is said $\text{nil } f \leq n$ if $f\gamma_{X,n} \simeq *$;
- (2) f is called *central* if $\gamma_{Y,2}(f \wedge \text{id}_Y) \simeq *$.

Notice that $\text{nil } f \leq \min\{\text{nil } X, \text{nil } Y\}$.

Then, in view of [27, Lemma 2.6.6], we have:

Lemma 2.1. *Let $F \xrightarrow{i} E \xrightarrow{q} B$ be an H -fibration, i.e., $F \xrightarrow{i} E \xrightarrow{q} B$ is a fibration, F , E and B are H -spaces and the maps $i : F \rightarrow E$, and $q : E \rightarrow B$ are H -maps.*

- (1) *If $\text{nil } q \leq n$ and $i : F \rightarrow E$ is central then $\text{nil } E \leq n + 1$;*
- (2) *if $\Omega(Y) \xrightarrow{i} E \xrightarrow{q} X$ is the induced H -fibration by an H -map $f : X \rightarrow Y$ then the map $i : \Omega(Y) \rightarrow E$ is central.*

If a topological group G acts freely on a paracompact spaces X then there is a homeomorphism

$$X/G \approx X \times_G EG,$$

where $G \rightarrow EG \rightarrow BG$ is the universal principal G -bundle.

Since the connecting map $\partial_X : \Omega(X/G) \rightarrow G$, in view of [23, Theorem 8.6] (see also [6, Corollary 3.4]), is an H -map, the fibration $X \rightarrow X/G \rightarrow EG/G \approx BG$ leads to the H -fibration

$$\Omega(X) \longrightarrow \Omega(X/G) \xrightarrow{\partial_X} G$$

Now, let G be a compact Lie group and $K < G$ its closed subgroup. Then, the quotient space G/K is a manifold and the quotient map $q : G \rightarrow G/K$ is a submersion. Hence, $q : G \rightarrow G/K$ has a local section at the point $q(e) = K$ for the unit $e \in G$. This certainly implies that the map $q : G \rightarrow G/K$ has a local section at

any point $q(g)$ for any $g \in G$. Consequently, the quotient map $q : G \rightarrow G/K$ is a fiber bundle with the fiber K as a principal K -bundle. Thus, we have a K -fibration

$$\Omega(G) \longrightarrow \Omega(G/K) \xrightarrow{\partial_G} K.$$

Since this fibration is induced by the inclusion map $K \hookrightarrow G$ which is certainly a K -map, Lemma 2.1 yields:

Proposition 2.2. *If G is a compact Lie group and $K < G$ its closed subgroup with $\text{nil } K < \infty$ then $\text{nil } \Omega(G/K) < \infty$.*

Because of the inclusion maps $U(n) \hookrightarrow SO(2n)$, $U(n) \hookrightarrow Sp(n)$ and $Sp(n) \hookrightarrow SU(2n) \hookrightarrow SO(4n)$, we get the H -fibrations:

- (1) $\Omega(SO(2n)) \rightarrow \Omega(SO(2n)/U(n)) \rightarrow U(n)$;
- (2) $\Omega(Sp(n)) \rightarrow \Omega(Sp(n)/U(n)) \rightarrow U(n)$;
- (3) $\Omega(SU(2n)) \rightarrow \Omega(SU(2n)/Sp(n)) \rightarrow Sp(n)$;
- (4) $\Omega(SO(4n)) \rightarrow \Omega(SO(4n)/Sp(n)) \rightarrow Sp(n)$.

Then, the fibrations above, Corollary 1.5 and Proposition 2.2 lead to:

Proposition 2.3. *If $n \geq 1$ then:*

- (1) $\text{nil } \Omega(SO(2n)/U(n)) < \infty$;
- (2) $\text{nil } \Omega(Sp(n)/U(n)) < \infty$;
- (3) $\text{nil } \Omega(SU(2n)/Sp(n)) < \infty$;
- (4) $\text{nil } \Omega(SO(4n)/Sp(n)) < \infty$.

For the question [4] by Cohen and Wu, we conclude:

Corollary 2.4. *If $n \geq 1$ then:*

- (1) $\text{nil}[J(X), \Omega(SO(2n)/U(n))] < \infty$;
- (2) $\text{nil}[J(X), \Omega(Sp(n)/U(n))] < \infty$;
- (3) $\text{nil}[J(X), \Omega(SU(2n)/Sp(n))] < \infty$;
- (4) $[J(X), \Omega(SO(4n)/Sp(n))] < \infty$

for any space X .

2.1. Grassmannians

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ be the field of reals or complex numbers and $\mathbb{H}$, the skew $\mathbb{R}$ -algebra of quaternions. Then, we set:

$$= \begin{cases} U_{\mathbb{K}}(n) \\ O(n) \text{ if } \mathbb{K} = \mathbb{R}; \\ U(n) \\ Sp(n) \end{cases} \quad \text{if } \mathbb{K} = \mathbb{C}; \text{ and } SU_{\mathbb{K}}(n) = \begin{cases} SO(n) \text{ if } \mathbb{K} = \mathbb{R}; \\ SU(n) \\ Sp(n) \end{cases} \quad \text{if } \mathbb{K} = \mathbb{H}.$$

Write $G_{n,m}(\mathbb{K})$ (resp. $G_{n,m}^+(\mathbb{K})$) for the (resp. oriented) Grassmannian of m -dimensional subspaces in the n -dimensional $\mathbb{K}$ -vector space. For example, the set of lines $G_{n+1,1}(\mathbb{K}) = \mathbb{K}P^n$, the projective n -space over $\mathbb{K}$.

It is well known that $G_{n,m}(\mathbb{K})$ (resp. $G_{n,m}^+(\mathbb{K})$) are smooth manifolds with diffeomorphisms

$$\begin{aligned} G_{n,m}(\mathbb{K}) &\approx (U_{\mathbb{K}}(n)/U_{\mathbb{K}}(m) \times U_{\mathbb{K}}(n-m)) \text{ and } G_{n,m}^+(\mathbb{K}) \\ &\approx (SU_{\mathbb{K}}(n)/SU_{\mathbb{K}}(m) \times SU_{\mathbb{K}}(n-m)) \end{aligned}$$

for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Since the homomorphism $\pi_1(SU(m)_{\mathbb{K}}) \rightarrow \pi_1(SU(n)_{\mathbb{K}})$ of fundamental groups determined by the inclusion map $SU(m)_{\mathbb{K}} \hookrightarrow SU(n)_{\mathbb{K}}$ for $2 \leq m \leq n$ is an epimorphism, we derive that the spaces $G_{n,m}^+(\mathbb{K})$ are simply connected for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Next, there is the universal covering map

$$\mathbb{Z}_2 \longrightarrow G_{n,m}^+(\mathbb{R}) \longrightarrow G_{n,m}(\mathbb{R})$$

and a fibre bundle

$$\mathbb{S}^1 \longrightarrow G_{n,m}^+(\mathbb{C}) \longrightarrow G_{n,m}(\mathbb{C}).$$

Since $G_{n+1,1}(\mathbb{K}) = \mathbb{K}P^n$, projective spaces are special cases of Grassmannians. Recall that by Ganea [5, Propositions 1.3-1.5] and Snaith [22, Corollary 3.3], we have:

- Proposition 2.5.** (1) $\text{nil } \Omega(\mathbb{R}P^{2n+1}) = \begin{cases} \leq 2 & \text{for } n \geq 0 \\ 1 & \text{if and only if } n = 0, 1 \text{ or } 3. \end{cases}$
- (2) $\text{nil } \Omega(\mathbb{R}P^{2n}) = \infty$ for $n \geq 1$.
- (3) $\text{nil } \Omega(\mathbb{C}P^{2n+1}) = \begin{cases} \leq 2 & \text{for any odd } n \geq 1; \\ 1 & \text{if and only if } n = 1. \end{cases}$
- (4) $3 \leq \text{nil } \Omega(\mathbb{C}P^{2n}) \leq 6$ for $n \geq 1$.
- (5) $3 \leq \text{nil } \Omega(\mathbb{H}P^n)$ for any $n \geq 1$.
- (6) $\text{nil } \Omega(\mathbb{H}P^n) = 3$ if $n \equiv -1 \pmod{24}$.

Then, Meier [15, Theorem 5.4] has shown some results on the homotopy nilpotency of p -localized projective spaces:

Theorem 2.6. *Let p be an odd prime and $n \geq 2$ a natural number. Then:*

- (1) $\text{nil } \Omega(\mathbb{C}P_{(p)}^n) = 1$.
- (2) $\text{nil } \Omega(\mathbb{H}P_{(p)}^n) = 1$ if $p > 3$.
- (3) $3 \leq \text{nil } \Omega(\mathbb{H}P_{(3)}^n) \leq 4$.
- (4) $\text{nil } \Omega(\mathbb{H}P_{(3)}^n) = 3$ if $n \equiv 2 \pmod{3}$.

Since the space $\mathbb{R}P^{2n+1}$ is simple, there is its localization $\mathbb{R}P_{(p)}^{2n+1}$ for any prime $p \geq 2$. It is also easy to see that $\text{nil } \Omega(\mathbb{R}P_{(p)}^{2n+1}) = 1$ for any odd prime p and $n \geq 0$. But, the nilpotency $\text{nil } \Omega(\mathbb{H}P^n)$ for any $n \geq 2$ does not appear in the literature known to the author. Next, recall that the classifying

$$BU_{\mathbb{K}}(m) = \lim_{\rightarrow} G_{n,m}(\mathbb{K}) = G_{\infty,m}(\mathbb{K}).$$

Since the cohomology $H^*(SO(n), \mathbb{Z})$ has only 2-torsions and $H^*(U(n), \mathbb{Z})$, $H^*(Sp(n), \mathbb{Z})$ are torsion free, the fibration

$$\Omega(SU_{\mathbb{K}}(n)) \longrightarrow \Omega(G_{n,m}^+(\mathbb{K})) \longrightarrow SU_{\mathbb{K}}(m) \times SU_{\mathbb{K}}(n-m),$$

Corollary 1.5, Theorem 1.8 and Proposition 2.2 lead to:

Proposition 2.7. *If $1 \leq m < n \leq \infty$ then:*

- (1) $\text{nil } \Omega(G_{n,m}^+(\mathbb{R})_{(p)}) < \infty$ for $p \geq 3$;
- (2) $\text{nil } \Omega(G_{n,m}(\mathbb{K})) < \infty$ and $\text{nil } \Omega(G_{n,m}^+(\mathbb{K})) < \infty$ for $\mathbb{K} = \mathbb{C}, \mathbb{H}$.
In particular, $\text{nil } \Omega(\mathbb{H}P^n) < \infty$.

We do not mention above any result on the p -localization of $G_{n,m}(\mathbb{R})$ because we are not sure on its existence.

Remark 2.8. The (resp. oriented) flag manifold $F_{n;n_1,\dots,n_k}(\mathbb{K})$ (resp. $F_{n;n_1,\dots,n_k}^+(\mathbb{K})$) with $1 \leq n_1 < \dots < n_k \leq n-1$ in the n -dimensional $\mathbb{K}$ -vector space is smooth with a diffeomorphism

$$F_{n;n_1,\dots,n_k}(\mathbb{K}) \approx (U_{\mathbb{K}}(n)/U_{\mathbb{K}}(n_1) \times U_{\mathbb{K}}(n_1-n_2) \times \dots \times U_{\mathbb{K}}(n_{k-1}-n_k) \times U_{\mathbb{K}}(n-n_k))$$

$$\text{and } F_{n;n_1,\dots,n_k}^+(\mathbb{K}) \approx (SU_{\mathbb{K}}(n)/SU_{\mathbb{K}}(n_1) \times SU_{\mathbb{K}}(n_1-n_2) \times \dots \times SU_{\mathbb{K}}(n_{k-1}-n_k) \times SU_{\mathbb{K}}(n-n_k)).$$

Further, $F_{n;n_1,\dots,n_k}(\mathbb{K}) \approx F_{n;n_1,\dots,n_k}^+(\mathbb{K})$ for $\mathbb{K} = \mathbb{C}, \mathbb{H}$.

Since the homomorphism $\pi_1(SU(m)_{\mathbb{K}}) \rightarrow \pi_1(SU(n)_{\mathbb{K}})$ determined by the inclusion map $SU(m)_{\mathbb{K}} \hookrightarrow SU(n)_{\mathbb{K}}$ for $2 \leq m \leq n$ of fundamental groups is an epimorphism, we derive that the spaces $F_{n;n_1,\dots,n_k}^+(\mathbb{K})$ are simply connected for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Furthermore, there is the universal covering map

$$(\mathbb{Z}_2)^k \rightarrow F_{n;n_1,\dots,n_k}^+(\mathbb{R}) \rightarrow F_{n;n_1,\dots,n_k}(\mathbb{R})$$

and a fibre bundle

$$(\mathbb{S}^1)^k \rightarrow F_{n;n_1,\dots,n_k}^+(\mathbb{C}) \rightarrow F_{n;n_1,\dots,n_k}(\mathbb{C}).$$

Then, in view of the fibration

$$\Omega(SU_{\mathbb{K}}(n)) \rightarrow \Omega(F_{n;n_1,\dots,n_k}^+(\mathbb{K})) \rightarrow SU_{\mathbb{K}}(n_1) \times SU_{\mathbb{K}}(n_1-n_2) \times \dots \times SU_{\mathbb{K}}(n_{k-1}-n_k) \times SU_{\mathbb{K}}(n-n_k),$$

Corollary 1.5, Theorem 1.8 and Proposition 2.2 lead to:

- (1) $\text{nil } \Omega(F_{n;n_1,\dots,n_k}^+(\mathbb{R})_{(p)}) < \infty$ for $p > 2$;
- (2) $\text{nil } \Omega(F_{n;n_1,\dots,n_k}(\mathbb{K})) < \infty$ and $\text{nil } \Omega(F_{n;n_1,\dots,n_k}^+(\mathbb{K})) < \infty$ for $\mathbb{K} = \mathbb{C}, \mathbb{H}$.

As above, we do not mention above any result on the p -localization of $F_{n;n_1,\dots,n_k}(\mathbb{R})$ because we are not sure on its existence.

For the question [4] by Cohen and Wu, we conclude:

Corollary 2.9. *For any space X and $m < n \leq \infty$:*

- (1) $\text{nil } [J(X), \Omega(G_{n,m}^+(\mathbb{R})_{(p)})] < \infty$ for $p \geq 3$;
- (2) $\text{nil } [J(X), \Omega(G_{n,m}(\mathbb{K}))] < \infty$ for $\mathbb{K} = \mathbb{C}, \mathbb{H}$;
- (3) $\text{nil } [J(X), \Omega(F_{n;n_1,\dots,n_k}^+(\mathbb{R})_{(p)})] < \infty$ for $p \geq 3$;
- (4) $\text{nil } [J(X), \Omega(F_{n;n_1,\dots,n_k}(\mathbb{K}))] < \infty$ for $\mathbb{K} = \mathbb{C}, \mathbb{H}$ and $1 \leq n_1 < \dots < n_k \leq n-1$.

Next, recall the following result of Hopf [7]. Let X be a connected H -space with $\dim H^*(X; \mathbb{Q}) < \infty$ for the field $\mathbb{Q}$ of rationals. Then, one has a homotopy equivalence

$$X_{(0)} \simeq \mathbb{S}_{(0)}^{2n_1-1} \times \cdots \times \mathbb{S}_{(0)}^{2n_l-1},$$

where $-(0)$ means the rationalization.

The sequence $(n_1, \dots, n_l)$ with $n_1 \leq \dots \leq n_l$ is called the *type* of G . Wilkerson [25] showed that a simply connected compact group G is not p -regular for $p < n_l$. Then, by combining with Kumpel's result [13] above, G is p -regular if and only if $p \geq n_l$.

The types of the connected compact simple Lie groups have been summarized in [18, Chapter IV] as follows:

		group	type	
A_n	$(n \geq 1)$	$SU(n+1)$	$(2, 3, \dots, n+1)$	(2.10)
B_n	$(n \geq 2)$	$SO(2n+1), \text{Spin}(2n+1)$	$(2, 4, \dots, 2n)$	
C_n	$(n \geq 3)$	$Sp(n)$	$(2, 4, \dots, 2n)$	
D_n	$(n \geq 4)$	$SO(2n), \text{Spin}(2n)$	$(3, 7, \dots, 2n)$	
G_2			$(2, 6)$	
F_4			$(2, 6, 8, 12)$	
E_6			$(2, 5, 6, 8, 9, 12)$	
E_7			$(2, 6, 8, 10, 12, 14, 18)$	
E_8			$(2, 8, 12, 14, 18, 20, 24, 30)$	

Let G be a compact, connected Lie group of type $(n_1, \dots, n_l)$. By Serre [21], the group G is p -regular for an odd prime p if there is a p -local (or p -complete) homotopy equivalence

$$G \simeq \mathbb{S}^{2n_1-1} \times \cdots \times \mathbb{S}^{2n_l-1}.$$

Then, by McGibbon [14, Theorem 2], we have:

Theorem 2.11. *Let G be a compact connected simple Lie group of type $(n_1, \dots, n_l)$. Then we have:*

- (1) $G_{(p)}$ is homotopy commutative if $p > 2n_l$;
- (2) $G_{(p)}$ is not homotopy commutative if $p < 2n_l$ except for $(G, p) = (Sp(2), 3), (G_2, 5)$.

Notice that Table (2.10) and Theorem 2.11 yield:

- (1) $\text{nil } SO(2n)_{(p)} = \text{nil } SO(2n+1)_{(p)} = 1$ if $p > 4n$;
- (2) $\text{nil } SU(n)_{(p)} = 1$ if $p > 2n$;
- (3) $\text{nil } Sp(n)_{(p)} = 1$ $p > 4n$ or $p = 3$ and $n = 2$.

The McGibbon's result on homotopy commutativity has been generalized by Kaji and Kishimoto [11, Theorem 2] to homotopy nilpotency in the following cases:

Theorem 2.12. *Let G be a compact, simply connected, simple Lie group of type $(n_1, \dots, n_l)$. If G is p -regular then $G_{(p)}$ is homotopy nilpotent with:*

- (1) $\text{nil } G_{(p)} = 2$ if $\frac{3}{2}n_l < p < 2n_l$;
 (2) $\text{nil } G_{(p)} = 3$ if $n_l \leq p \leq \frac{3}{2}n_l$ except for the cases that $(G, p) = (F_4, 17), (E_6, 17), (E_8, 41), (E_8, 43)$ or $\text{rank } G = 1$ with $p = 2$;
 (3) $\text{nil } G_{(p)} = 2$ in the above exceptional cases.

Then, Table (2.10) and Theorem 2.12 lead to:

- Corollary 2.13.** (1) $\text{nil } SO(2n)_{(p)} = \text{nil } SO(2n+1)_{(p)} = \begin{cases} 2 & \text{for } 3n < p < 4n, \\ 3 & \text{for } 4n \leq p \leq 3n; \end{cases}$
 (2) $\text{nil } SU(n)_{(p)} = \begin{cases} 2 & \text{for } \frac{3}{2}n < p < 2n, \\ 3 & \text{for } n \leq p \leq \frac{3}{2}n \end{cases}$ when p is odd, and $\text{nil } SU(2)_{(2)} = \text{nil } \mathbb{S}_{(2)}^3 = 2$;
 (3) $\text{nil } Sp(n)_{(p)} = \begin{cases} 2 & \text{for } 3n < p < 4n, \\ 3 & \text{for } 2n \leq p \leq 3n. \end{cases}$

Now, recall that according to Mimura-Toda [17], a compact, connected Lie group G is *quasi- p -regular* for odd prime p if there is a p -local (or p -complete) homotopy equivalence $G \simeq \prod_{i=1}^l B_i$, where each B_i is either a sphere or a sphere bundle over a sphere. Recall that by [17, Theorem 4.2], we have:

Theorem 2.14. A Lie group G is *quasi- p -regular* if and only if:

- (1) $p > n$ for $G = Sp(n)$;
 (2) $p > \frac{n}{2}$ for $G = SU(n)$;
 (3) $p > \frac{n-1}{2}$ for $G = Spin(n)$;
 (4) $p \geq 5$ for $G = G_2, F_4, E_6$;
 (5) $p \geq 11$ for $G = E_7, E_8$.

Kishimoto [12, Theorem 1.1] determined the homotopy nilpotency class of all quasi- p -regular cases of $SU(n)$. The result is:

Theorem 2.15. Let $p > 5$ be a prime. Then, we have:

$$\text{nil } SU(n)_{(p)} = \begin{cases} 3 & \text{for } p = n + 1 \text{ or } \frac{n}{2} < p \leq \frac{2n+1}{3}; \\ 2 & \text{for } \frac{2n+1}{3} < p \leq n - 2. \end{cases}$$

Making use of the above, Lemma 2.1 and the fibration

$$\Omega(U_{\mathbb{K}}(n)_{(p)}) \longrightarrow \Omega(G_{n,m}(\mathbb{K})_{(p)}) \longrightarrow (U_{\mathbb{K}}(m))_{(p)} \times (U_{\mathbb{K}}(n-m))_{(p)}$$

for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and a prime p , we may state:

Proposition 2.16. If $1 \leq m \leq n$ then:

- (1) $\text{nil } \Omega(G_{n,m}(\mathbb{R})_{(p)}) \leq 2$ with $p > 4k$ for $n = 2k + 1$ or $n = 2k$;
 (2) $\text{nil } \Omega(G_{n,m}(\mathbb{C})_{(p)}) \leq \begin{cases} 2 & \text{for } p > 2n; \\ 3 & \text{for } \frac{3}{2}n < p < 2(n-m); \\ 4 & \text{for } n \leq p \leq \frac{3}{2}(n-m); \\ 3 & \text{for } \frac{2n+1}{3} < p \leq n-m-2. \end{cases}$
 (3) $\text{nil } \Omega(G_{n,m}(\mathbb{H})_{(p)}) \leq \begin{cases} 2 & \text{for } p > 4n; \\ 3 & \text{for } 3n < p < 4(n-m); \\ 4 & \text{for } 2n \leq p \leq 3(n-m). \end{cases}$

2.2. Stiefel manifolds

The Stiefel manifold $V_{n,m}(\mathbb{K})$ is the set of all orthonormal m -frames in the vector space $\mathbb{K}^n$. That is, it is the set of ordered orthonormal m -tuples of vectors in $\mathbb{K}^n$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

It is well known that $V_{n,m}(\mathbb{K})$ is a smooth manifold and there are diffeomorphisms:

- (1) $V_{n,m}(\mathbb{R}) = V_{n,m} \approx O(n)/O(n-m) \approx SO(n)/SO(n-m)$;
- (2) $V_{n,m}(\mathbb{C}) = W_{n,m} \approx U(n)/U(n-m) \approx SU(n)/SU(n-m)$;
- (3) $V_{n,m}(\mathbb{H}) = X_{n,m} \approx Sp(n)/Sp(n-m)$.

Since the homomorphism $\pi_1(SU(m)\mathbb{K}) \rightarrow \pi_1(SU(n)\mathbb{K})$ of fundamental groups determined by the inclusion map $SU(m)\mathbb{K} \hookrightarrow SU(n)\mathbb{K}$ for $2 \leq m \leq n$ is an epimorphism, we derive that the spaces $V_{n,m}(\mathbb{K})$ are simply connected for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Then, Corollary 1.5 and Proposition 2.2 lead to:

Proposition 2.17. *If $1 \leq m \leq n$ then:*

- (1) $\text{nil } \Omega(V_{n,m})_{(p)} < \infty$ for $p \geq 3$;
- (2) $\text{nil } \Omega(W_{n,m}) < \infty$;
- (3) $\text{nil } \Omega(X_{n,m}) < \infty$.

Then, for the question [4] by Cohen and Wu, we conclude:

Corollary 2.18. *If $1 \leq m \leq n$ and X is a space then:*

- (1) $\text{nil } [J(X), \Omega(V_{n,m})_{(p)}] < \infty$ for $p \geq 3$;
- (2) $\text{nil } [J(X), \Omega(W_{n,m})] < \infty$;
- (3) $\text{nil } [J(X), \Omega(X_{n,m})] < \infty$.

Making use of Lemma 2.1, Corollary 2.13, Theorem 2.15 and the fibration

$$\Omega(U_{\mathbb{K}}(n))_{(p)} \longrightarrow \Omega(V_{n,m}(\mathbb{K}))_{(p)} \longrightarrow (U_{\mathbb{K}}(n-m))_{(p)}$$

for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and a prime p , we may state:

Corollary 2.19. *If $1 \leq m \leq n$ then:*

- (1) $\text{nil } \Omega(V_{n,m})_{(p)} \leq 2$ with $p > k$ for $n-m = 2k+1$ or $n-m = 2k$;
- (2) $\text{nil } \Omega(W_{n,m})_{(p)} \leq \begin{cases} 2 & \text{for } p > 2(n-m); \\ 3 & \text{for } \frac{3}{2}(n-m) < p < 2(n-m); \\ 4 & \text{for } n-m \leq p \leq \frac{3}{2}(n-m); \\ 4 & \text{for } p = n-m+1 \text{ or } \frac{n-m}{2} < p \leq \frac{2(n-m)+1}{3}; \\ 3 & \text{for } \frac{2(n-m)+1}{3} < p \leq n-m-2. \end{cases}$
- (3) $\text{nil } \Omega(X_{n,m})_{(p)} \leq \begin{cases} 2 & \text{for } p > 4(n-m); \\ 3 & \text{for } 3(n-m) < p < 4(n-m); \\ 4 & \text{for } 2(n-m) \leq p \leq 3(n-m). \end{cases}$

Since the spaces $SO(2n)/U(n)$, $Sp(n)/U(n)$ and $SU(2n)/Sp(n)$ are simply connected, Lemma 2.1 and the results above yield:

Proposition 2.20. *If $n \geq 1$ then:*

$$(1) \operatorname{nil} \Omega(SO(2n)/U(n))_{(p)}, \operatorname{nil} \Omega(Sp(n)/U(n))_{(p)} \leq \begin{cases} 2 \text{ for } p > 2n, \\ 3 \text{ for } \frac{3}{2}n < p < 2n, \\ 4 \text{ for } n \leq p \leq \frac{3}{2}n; \end{cases}$$

$$(2) \operatorname{nil} \Omega(SU(2n)/Sp(n))_{(p)} \leq \begin{cases} 2 \text{ for } p > 4n, \\ 3 \text{ for } 2n < p < 4n \text{ or } \frac{2n+1}{3} < p \leq n-2 \text{ with } p \geq 5, \\ 4 \text{ for } 2n \leq p \leq 3n, \text{ or } p = n+1 \text{ or, } \frac{n}{2} < p \leq \frac{2n+1}{3} \text{ with } p \geq 5. \end{cases}$$

3. The Cayley plane $\mathbb{O}P^2$ and other homogeneous spaces

The real Cayley plane $\mathbb{O}P^2 \approx F_4/\operatorname{Spin}(9)$, where F_4 is a compact form of an exceptional Lie group and $\operatorname{Spin}(9)$ is the spin group of nine-dimensional Euclidean space (realized in F_4).

Since, by means of [19], the group $\operatorname{Spin}(9)$ is not homotopy nilpotent we cannot deduce from the fibration $\Omega(F_4) \rightarrow \Omega(\mathbb{O}P^2) \rightarrow \operatorname{Spin}(9)$ the homotopy nilpotency of $\mathbb{O}P^2$.

Recall that in view of [18], the following is a list of the compact, simply connected, simple groups with p -torsion:

p	group
2	$G_2, F_4, E_6, E_7, E_8, \operatorname{Spin}(n) \text{ for } n \geq 7$
3	F_4, E_6, E_7, E_8
5	E_8

(3.1)

Now, since $\mathbb{O}P^2$ is simply connected, Theorem 1.8, Lemma 2.1, Theorem 2.12 and the fibration

$$\Omega(F_4)_{(p)} \rightarrow \Omega(\mathbb{O}P^2_{(p)}) \rightarrow \operatorname{Spin}(9)_{(p)}$$

for a prime p yield:

- Proposition 3.2.** (1) *If $p \geq 3$ then $\operatorname{nil} \mathbb{O}P^2_{(p)} < \infty$;*
 (2) *if $p > 16$ then $\operatorname{nil} \mathbb{O}P^2_{(p)} \leq 2$;*
 (3) $\operatorname{nil} \mathbb{O}P^2_{(p)} \leq \begin{cases} 4 \text{ for } p = 11, \\ 3 \text{ for } p = 13. \end{cases}$

Next, the inclusions

$$G_2 \hookrightarrow F_4 \hookrightarrow E_6 \hookrightarrow E_7 \hookrightarrow E_8$$

of exceptional simply connected simple Lie groups lead to fibration:

- (1) $\Omega(E_8) \rightarrow \Omega(E_8/E_n) \rightarrow E_n$ for $n = 6, 7$;
- (2) $\Omega(E_7) \rightarrow \Omega(E_7/E_6) \rightarrow E_6$;
- (3) $\Omega(E_7) \rightarrow \Omega(E_7/F_4) \rightarrow F_4$;
- (4) $\Omega(E_8) \rightarrow \Omega(E_8/F_4) \rightarrow F_4$;

- (5) $\Omega(E_6) \rightarrow \Omega(E_6/F_4) \rightarrow F_4$;
- (6) $\Omega(E_8) \rightarrow \Omega(E_8/G_2) \rightarrow G_2$;
- (7) $\Omega(E_7) \rightarrow \Omega(E_n/G_2) \rightarrow G_2$ for $n = 6, 7$;
- (8) $\Omega(F_4) \rightarrow \Omega(F_4/G_2) \rightarrow G_2$.

Recall that methods developed by Theriault [24, Theorem 12.1] were sufficiently powerful to allow for explicit calculations of the homotopy nilpotency classes of quasi- p -regular exceptional Lie groups.

Consequently, the result [24, Theorem 12.1], Theorem 1.8, Lemma 2.1, Theorem 2.12, Table (3.1) and localized fibrations (1)–(8) above lead to:

Proposition 3.3. *If p is a prime then:*

- (1) $\text{nil } \Omega(E_8/E_7)_{(p)} < \begin{cases} \infty & \text{for } p \geq 5, \\ 3 & \text{for } p > 36, \\ 4 & \text{for } p = 11, 13, 17, 29, 31 \\ 5 & \text{for } p = 19, 23; \end{cases}$
- (2) $\text{nil } \Omega(E_8/E_6)_{(p)}, \text{nil } \Omega(E_7/E_6)_{(p)} < \begin{cases} \infty & \text{for } p \geq 5; \\ 3 & \text{for } p > 24, \\ 4 & \text{for } p = 7, 11, 19, 23, \\ 5 & \text{for } p = 13, 17; \end{cases}$
- (3) $\text{nil } \Omega(E_8/F_4)_{(p)}, \text{nil } \Omega(E_7/F_4)_{(p)} < \begin{cases} \infty & \text{for } p \geq 5, \\ 3 & \text{for } p > 24, \\ 4 & \text{for } p = 7, 11, 19, 23, \\ 5 & \text{for } p = 13, 17; \end{cases}$
- (4) $\text{nil } \Omega(E_8/G_2)_{(p)}, \text{nil } \Omega(E_7/G_2)_{(p)}, \text{nil } \Omega(E_6/G_2)_{(p)}, \text{nil } \Omega(F_4/G_2)_{(p)} < \begin{cases} \infty & \text{for } p \geq 3, \\ 3 & \text{for } p > 12, \\ 4 & \text{for } p = 11, \\ 5 & \text{for } p = 7. \end{cases}$

We close the paper with the following suggested by Stephen Theriault. The homotopy nilpotency class $\Omega(G/K)$ can be sometimes determined precisely when the group G is p -regular or quasi- p -regular. To aim that we need some prerequisites.

Given an H -space (X, μ) , consider the map $t : X \times X \rightarrow X \times X$ given by $t(x, x') = (x', x)$ for $x, x' \in X$ and write $d_X = \mu - \mu t : X \times X \rightarrow X$. Then, (X, μ) is homotopy commutative if and only if d_X is null homotopic.

Lemma 3.4. (1) *If an H -map $f : X \rightarrow Y$ has a right homotopy inverse $s : Y \rightarrow X$ and X is homotopy commutative then Y is homotopy commutative as well.*

(2) *If $F \xrightarrow{i} E \xrightarrow{q} B$ is a fibration and $i : F \rightarrow E$ has a left homotopy inverse then the map $\Omega(q) : \Omega(E) \rightarrow \Omega(B)$ has a right homotopy inverse.*

Proof. (1): First notice that $d_Y \circ (f \times f) \simeq f \circ d_X$. Since, X is homotopy commutative we derive that $d_Y(f \times f) \simeq *$. Then, the right homotopy inverse $s : Y \rightarrow X$ of $f : X \rightarrow Y$ leads to $d_Y \simeq *$.

(2): Since $i : F \rightarrow E$ has a left homotopy, the fibration $F \xrightarrow{i} E \xrightarrow{q} B$ leads to the short exact sequence

$$1 \rightarrow [\Omega(B), \Omega(F)] \xrightarrow{\Omega(i)*} [\Omega(B), \Omega(E)] \xrightarrow{\Omega(q)*} [\Omega(B), \Omega(B)] \rightarrow 1$$

which yields a map $s : \Omega(B) \rightarrow \Omega(E)$ such that $\Omega(q) \circ s \simeq \text{id}_{\Omega(B)}$ and the proof is complete. $\square$

In particular, if G be a compact Lie group, $K < G$ its closed subgroup and $q : K \rightarrow G/K$ the quotient map then the fibration

$$\Omega(K_{(p)}) \xrightarrow{\Omega(i)} \Omega(G_{(p)}) \xrightarrow{\Omega(q)} \Omega((G/K)_{(p)})$$

leads to a right homotopy inverse of the loop map $\Omega(q) : \Omega(G_{(p)}) \rightarrow \Omega(G/K)_{(p)}$ provided G is p -regular and the canonical map $K_{(p)} \rightarrow G_{(p)}$ has a left homotopy inverse. Since the localized odd sphere $\mathbb{S}_{(p)}^{2n+1}$ is an associative H -space for $p > 3$, Lemma 3.4 implies that space $\Omega((G/K)_{(p)})$ is homotopy commutative or equivalently $\text{nil } \Omega((G/K)_{(p)}) = 1$.

For example, the p -local homotopy decompositions of Mimura, Nishida and Toda [16] imply that if $K < G$ is one of:

- (1) $SU(m) < SU(n)$ if $m < n$ and $p \geq n$;
- (2) $Sp(m) < Sp(n)$ if $m < n$ and $p > 2n$;
- (3) $\text{Spin}(m) < \text{Spin}(n)$ if $m < n$ and $p > 2[\frac{n}{2}]$;
- (4) $G_2 < F_4$ or $G_2 < E_6$ if $p = 5$ or $p > 11$;
- (5) $F_4 < E_6$ if $p > 3$;
- (6) $F_4 < E_7$ or $G_2 < E_7$ if $p > 17$

then the canonical map $K_{(p)} \rightarrow G_{(p)}$ has a left homotopy inverse.

Note that the $p = 5$ case in part (4) is quasi- p -regular, and the $p = 5, 7, 11$ cases in part (5) are quasi- p -regular. The others are all p -regular.

Then, we derive:

Corollary 3.5. *The following spaces:*

- (1) $\Omega((W_{n,m})_{(p)}) = \Omega((SU(n)/SU(m))_{(p)})$ if $m < n$ and $p \geq n$;
- (2) $\Omega((X_{n,m})_{(p)}) = \Omega((Sp(n)/Sp(m))_{(p)})$ if $m < n$ and $p > 2n$;
- (3) $\Omega((Spin(n)/Spin(m))_{(p)})$ if $m < n$ and $p > 2[\frac{n}{2}]$;
- (4) $\Omega((F_4/G_2)_{(p)})$ and $\Omega((E_6/G_2)_{(p)})$ if $p = 5$ or $p > 11$;
- (5) $\Omega((E_6/F_4)_{(p)})$ if $p > 3$;
- (6) $\Omega((E_7/F_4)_{(p)})$ if $p > 17$

are homotopy commutative.

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