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Erratum

Erratum to: On the cohomology of the moduli space of parabolic connections

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In the proof of Theorem 1.1 in [4], there are two mistakes that are explained in Remark 4, and therefore the vanishing of the first and the second cohomologies in Theorem 1.1 is still open. We mention that all other results in [4] still hold. We should restate Theorem 1.1 as follows. We recall that the coarse moduli space $M(d)$ of $\boldsymbol{\nu}$-$\mathfrak{sl}_2$-parabolic connections on $(\mathbb{P}^1, t_1 + \cdots + t_5)$ of degree d has a stratification

$$M(d) = M(d)^0 \cup M(d)^1,$$

where $M(d)^k$ denotes the subvariety defined in [4, 2.4]. For simplicity, let us denote $Z = M(d)^1$, and $M(d)^0 = M(d) \setminus Z$.

Theorem 1. 1. *We have*

$$H^i(M(d), \mathcal{O}_{M(d)}) = \begin{cases} \mathbb{C}, & i = 0, \\ 0, & i > 2. \end{cases}$$

2. *If the following connecting homomorphism of cohomology groups*

$$\delta : H^1(M(d)^0, \mathcal{O}_{M(d)^0}) \longrightarrow H^2_Z(M(d), \mathcal{O}_{M(d)})$$

is an isomorphism, we have $H^i(M(d), \mathcal{O}_{M(d)}) = 0$ for $i = 1, 2$, and coversely.

In the process of the proof of Theorem 1, we have the following

Proposition 2. *We have*

$$H^i(M(d)^0, \mathcal{O}_{M(d)^0}) = \begin{cases} \mathbb{C}, & i = 0, \\ 0, & i > 1. \end{cases}$$

Remark 3. In general, $\dim H^1(M(d)^0, \mathcal{O}_{M(d)^0}) \neq 0$.

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Proof of Theorem 1. By the local cohomology theory, there is a long exact sequence

$$\begin{aligned}0 &\to H^0_Z(M(d),\mathcal{O}) \to H^0(M(d),\mathcal{O}) \to H^0(M(d)^0,\mathcal{O})\\ &\to H^1_Z(M(d),\mathcal{O}) \to H^1(M(d),\mathcal{O}) \to H^1(M(d)^0,\mathcal{O})\\ &\xrightarrow{\delta} H^2_Z(M(d),\mathcal{O}) \to H^2(M(d),\mathcal{O}) \to H^2(M(d)^0,\mathcal{O})\\ &\to H^3_Z(M(d),\mathcal{O}) \to H^3(M(d),\mathcal{O}) \to H^3(M(d)^0,\mathcal{O})\\ &\to H^4_Z(M(d),\mathcal{O}) \to H^4(M(d),\mathcal{O}) \to H^4(M(d)^0,\mathcal{O})\\ &\to 0. \end{aligned} \tag{1}$$

By [1, Theorem 1(ii)], we have that $H^i(M(d),\mathcal{O}) = 0$ for $i > 2$. Since Z is a locally complete intersection in $M(d)$ and $\operatorname{codim}_{M(d)}(Z) = 2$, we have that $\operatorname{depth}_{I_Z}(\mathcal{O}) \geq 2$, where I_Z is the ideal sheaf of Z. Therefore, we get $H^1_Z(M(d),\mathcal{O}) = 0$ by [3, Theorem 3.8]. Now the theorem follows from Proposition 2. □

Remark 4. In the proof of Theorem 1.1, there are two mistakes.

1. We found that the action of $\mathfrak{S}_2$ is not commutative with Leray's spectral sequence. Therefore, the action of $\mathfrak{S}_2$ on higher cohomologies is highly non-trivial, and a detailed calculation is needed.
2. The subscheme $Z = M(d)^1$ has codimension 2 in $M(d)$ and it is isomorphic to $\mathbb{A}^2$. Therefore, we ignored the contribution of the local cohomology $H^2_Z(M(d),\mathcal{O})$. However, in general, $\dim H^2_Z(M(d),\mathcal{O}_{M(d)}) \neq 0$. It is explained as follows. From [3, Theorem 2.8], we have that

$$\varinjlim_k \operatorname{Ext}^i(\mathcal{O}/\mathcal{I}^k_Z,\mathcal{O}) \xrightarrow{\sim} H^i_Z(M(d),\mathcal{O}),$$

   where $\mathcal{I}_Z$ is the ideal sheaf of Z. Furthermore, there is a spectral sequence

$$E_2^{p,q} = \varinjlim_k H^p(M(d),\mathcal{E}xt^q(\mathcal{O}/\mathcal{I}^k_Z,\mathcal{O})) \Rightarrow \varinjlim_k \operatorname{Ext}^i(\mathcal{O}/\mathcal{I}^k_Z,\mathcal{O}).$$

   Let us consider $H^2_Z(M(d),\mathcal{O})$. From the above discussion, we have that

$$H^2_Z(M(d),\mathcal{O}) \simeq \varinjlim_k \operatorname{Ext}^2(\mathcal{O}/\mathcal{I}^k_Z,\mathcal{O}) = \bigoplus_{p+q=2} H^p(M(d),\mathcal{E}xt^q(\mathcal{O}/\mathcal{I}^k_Z,\mathcal{O})).$$

   Moreover, since $\operatorname{supp}(\mathcal{E}xt^q(\mathcal{O}/\mathcal{I}^k_Z,\mathcal{O})) = Z$ and $Z \simeq \mathbb{A}^2$, we have

$$H^2_Z(M(d),\mathcal{O}) \simeq \varinjlim_k H^0(Z,\mathcal{E}xt^2(\mathcal{O}/\mathcal{I}^k_Z,\mathcal{O})).$$

   Therefore, $H^2_Z(M(d),\mathcal{O}) = 0$ if and only if $\mathcal{E}xt^2(\mathcal{O}/\mathcal{I}^k_Z,\mathcal{O}) = 0$ for $k \gg 0$. We have not known yet whether this is zero or not.

All other results in [4] still hold.

Remark 5. We do not know whether the connecting homomorphism δ in the sequence (1) is an isomorphism or not. Therefore, the vanishing of the cohomologies $H^i(M(d),\mathcal{O}_{M(d)})$ for $i = 1, 2$ is still an open problem.

Keeping the notation in [4], we prove the Proposition 2. We can suppose that $d = -1$. Denote by W the product of the surfaces $\widetilde{\mathbb{F}}_3 \times \widetilde{\mathbb{F}}_3$, by T the divisor $D \times \widetilde{\mathbb{F}}_3 + \widetilde{\mathbb{F}}_3 \times D + \widetilde{\mathbb{F}}_3 \times_{\mathbb{P}^1} \widetilde{\mathbb{F}}_3$ on W, and by Δ the diagonal of W. Also, denote by $\varphi : \widetilde{W} \to W$ the blowing-up along Δ, and by E the exceptional divisor. We denote by $\widetilde{T}$ the strict transform of T. Let us compute the cohomology of $U := \widetilde{W} \setminus \widetilde{T}$.

By the local cohomology theory, we have a long exact sequence

$$\begin{aligned} 0 &\to H^0_{\widetilde{T}}(\widetilde{W}, \mathcal{O}_{\widetilde{W}}) \to H^0(\widetilde{W}, \mathcal{O}_{\widetilde{W}}) \to H^0(U, \mathcal{O}_U) \\ &\to H^1_{\widetilde{T}}(\widetilde{W}, \mathcal{O}_{\widetilde{W}}) \to H^1(\widetilde{W}, \mathcal{O}_{\widetilde{W}}) \to H^1(U, \mathcal{O}_U) \\ &\to H^2_{\widetilde{T}}(\widetilde{W}, \mathcal{O}_{\widetilde{W}}) \to H^2(\widetilde{W}, \mathcal{O}_{\widetilde{W}}) \to H^2(U, \mathcal{O}_U) \\ &\to H^3_{\widetilde{T}}(\widetilde{W}, \mathcal{O}_{\widetilde{W}}) \to H^3(\widetilde{W}, \mathcal{O}_{\widetilde{W}}) \to H^3(U, \mathcal{O}_U) \\ &\to H^4_{\widetilde{T}}(\widetilde{W}, \mathcal{O}_{\widetilde{W}}) \to H^4(\widetilde{W}, \mathcal{O}_{\widetilde{W}}) \to H^4(U, \mathcal{O}_U) \\ &\to 0. \end{aligned} \tag{2}$$

Since $\widetilde{W}$ is a nonsingular projective rational variety, we have

$$H^i(\widetilde{W}, \mathcal{O}_{\widetilde{W}}) = \begin{cases} \mathbb{C}, & i = 0, \\ 0, & i > 0. \end{cases}$$

Again by the local cohomology theory, we have $H^i_{\widetilde{T}}(\widetilde{W}) = \varinjlim H^{i-1}(\widetilde{T}, N_{n\widetilde{T}})$. There is a short exact sequence

$$0 \to \mathcal{O}_{\widetilde{W}} \to \mathcal{O}_{\widetilde{W}}(n\widetilde{T}) \to N_{n\widetilde{T}} \to 0,$$

which induces the following exact sequence

$$\begin{aligned} 0 &\to H^0(\widetilde{W}, \mathcal{O}_{\widetilde{W}}) \to H^0(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(n\widetilde{T})) \to H^0(\widetilde{T}, N_{n\widetilde{T}}) \\ &\to H^1(\widetilde{W}, \mathcal{O}_{\widetilde{W}}) \to H^1(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(n\widetilde{T})) \to H^1(\widetilde{T}, N_{n\widetilde{T}}) \\ &\to H^2(\widetilde{W}, \mathcal{O}_{\widetilde{W}}) \to H^2(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(n\widetilde{T})) \to H^2(\widetilde{T}, N_{n\widetilde{T}}) \\ &\to H^3(\widetilde{W}, \mathcal{O}_{\widetilde{W}}) \to H^3(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(n\widetilde{T})) \to H^3(\widetilde{T}, N_{n\widetilde{T}}) \\ &\to H^4(\widetilde{W}, \mathcal{O}_{\widetilde{W}}) \to H^4(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(n\widetilde{T})) \to 0. \end{aligned} \tag{3}$$

So, it suffices to compute the cohomology

$$H^i(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(n\widetilde{T})) = H^i(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(n\widetilde{T}') \otimes \mathcal{O}_{\widetilde{W}}(-nE)),$$

where $\widetilde{T}' := \widetilde{T} + E = \varphi^*(T)$. There is a short exact sequence

$$0 \to \mathcal{O}_{\widetilde{W}}(n\widetilde{T}') \otimes \mathcal{O}_{\widetilde{W}}(-nE) \to \mathcal{O}_{\widetilde{W}}(n\widetilde{T}') \to \mathcal{O}_{\widetilde{W}}(n\widetilde{T}') \otimes \mathcal{O}_{\widetilde{W}}/\mathcal{O}_{\widetilde{W}}(-nE) \to 0$$

which induces the following cohomology sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_{\widetilde{W}}(n\widetilde{T}') \otimes \mathcal{O}_{\widetilde{W}}(-nE)) \to H^0(\mathcal{O}_{\widetilde{W}}(n\widetilde{T}')) \\ &\to H^0(\mathcal{O}_{\widetilde{W}}(n\widetilde{T}') \otimes \mathcal{O}/\mathcal{O}(-nE)) \\ &\to H^1(\mathcal{O}_{\widetilde{W}}(n\widetilde{T}') \otimes \mathcal{O}_{\widetilde{W}}(-nE)) \to H^1(\mathcal{O}_{\widetilde{W}}(n\widetilde{T}')) \end{aligned}$$

$$
\begin{aligned}
&\to H^1(\mathcal{O}_{\widetilde{W}}(n\widetilde{T}') \otimes \mathcal{O}/\mathcal{O}(-nE)) \\
&\to H^2(\mathcal{O}_{\widetilde{W}}(n\widetilde{T}') \otimes \mathcal{O}_{\widetilde{W}}(-nE)) \to H^2(\mathcal{O}_{\widetilde{W}}(n\widetilde{T}')) \\
&\to H^2(\mathcal{O}_{\widetilde{W}}(n\widetilde{T}') \otimes \mathcal{O}/\mathcal{O}(-nE)) \\
&\to H^3(\mathcal{O}_{\widetilde{W}}(n\widetilde{T}') \otimes \mathcal{O}_{\widetilde{W}}(-nE)) \to H^3(\mathcal{O}_{\widetilde{W}}(n\widetilde{T}')) \\
&\to H^3(\mathcal{O}_{\widetilde{W}}(n\widetilde{T}') \otimes \mathcal{O}/\mathcal{O}(-nE)) \\
&\to H^4(\mathcal{O}_{\widetilde{W}}(n\widetilde{T}') \otimes \mathcal{O}_{\widetilde{W}}(-nE)) \to H^4(\mathcal{O}_{\widetilde{W}}(n\widetilde{T}')) \to 0. \qquad (4)
\end{aligned}
$$

To apply the Künneth formula, considering the push forward of these cohomologies via $\varphi : \widetilde{W} \to W$, we have the following cohomology sequence

$$
\begin{aligned}
0 &\to H^0(W, \mathcal{O}_W(nT) \otimes I_\Delta^n) \to H^0(W, \mathcal{O}_W(nT)) \to H^0(W, \mathcal{O}_W(nT) \otimes \mathcal{O}/I_\Delta^n) \\
&\to H^1(W, \mathcal{O}_W(nT) \otimes I_\Delta^n) \to H^1(W, \mathcal{O}_W(nT)) \to H^1(W, \mathcal{O}_W(nT) \otimes \mathcal{O}/I_\Delta^n) \\
&\to H^2(W, \mathcal{O}_W(nT) \otimes I_\Delta^n) \to H^2(W, \mathcal{O}_W(nT)) \to H^2(W, \mathcal{O}_W(nT) \otimes \mathcal{O}/I_\Delta^n) \\
&\to H^3(W, \mathcal{O}_W(nT) \otimes I_\Delta^n) \to H^3(W, \mathcal{O}_W(nT)) \to H^3(W, \mathcal{O}_W(nT) \otimes \mathcal{O}/I_\Delta^n) \\
&\to H^4(W, \mathcal{O}_W(nT) \otimes I_\Delta^n) \to H^4(W, \mathcal{O}_W(nT)) \to 0, \qquad (5)
\end{aligned}
$$

where I_Δ is the ideal sheaf of Δ.

Proposition 6. $H^i(W, \mathcal{O}_W(nT)) = 0$ *for* $i > 0$.

Proof. By the Künneth formula, we have

$$H^i(W, \mathcal{O}_W(nT)) = \bigoplus_{p+q=i} H^p(\widetilde{\mathbb{F}}_3, \mathcal{O}(nD')) \otimes H^q(\widetilde{\mathbb{F}}_3, \mathcal{O}(nD')),$$

where $D' = D + F$ and F is a generic fiber. Therefore, it suffices to show that $H^p(\widetilde{\mathbb{F}}_3, \mathcal{O}(nD')) = 0$ for $p > 0$. It is clear that $H^2(\widetilde{\mathbb{F}}_3, \mathcal{O}(nD')) = 0$ by Serre duality. So, it remains to show that $H^1(\widetilde{\mathbb{F}}_3, \mathcal{O}(nD')) = 0$. Firstly, suppose that $n = 1$. The divisor D' is linearly equivalent to $2\widetilde{s}_0 - \sum E_i^\pm$, where $\widetilde{s}_0$ is the strict transform of the 0-section s_0 of $\mathbb{F}_3$. Then, we have a short exact sequence

$$0 \to \mathcal{O}_{\widetilde{\mathbb{F}}_3}(D') \to \mathcal{O}_{\widetilde{\mathbb{F}}_3}(2\widetilde{s}_0) \to \mathcal{O}_{\sum E_i^\pm} \to 0,$$

which induces the following cohomology sequence

$$
\begin{aligned}
0 &\to H^0(\widetilde{\mathbb{F}}_3, \mathcal{O}(D')) \to H^0(\widetilde{\mathbb{F}}_3, \mathcal{O}(2\widetilde{s}_0)) \xrightarrow{\psi} \bigoplus H^0(E_i^\pm, \mathcal{O}_{E_i^\pm}) \\
&\to H^1(\widetilde{\mathbb{F}}_3, \mathcal{O}(D')) \to 0.
\end{aligned}
$$

Let us show that ψ is surjective. Let $f \in H^0(\widetilde{\mathbb{F}}_3, \mathcal{O}(2\widetilde{s}_0))$ be a global section of $\mathcal{O}(2\widetilde{s}_0)$. On some local chart, we can write $f(x, y) = a_{0,2}y^2 + \sum_{i=0}^3 a_{i,1}x^i y + \sum_{i=0}^6 a_{i,0}x^i$. If $f(x, y) \in \mathrm{Ker}(\psi)$, then $f(x, y)$ satisfies $f(t_j, \nu_j^\pm) = 0$ for $j = 1, \ldots, 5$. Then we have the system

$$
\begin{pmatrix}
1 & t_1 & \ldots & t_1^6 & \nu_1 & \ldots & t_1^3\nu_1 & \nu_1^2 \\
1 & t_1 & \ldots & t_1^6 & -\nu_1 & \ldots & -t_1^3\nu_1 & \nu_1^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & t_5 & \ldots & t_5^6 & \nu_5 & \ldots & t_5^3\nu_5 & \nu_5^2 \\
1 & t_5 & \ldots & t_5^6 & 1-\nu_5 & \ldots & t_5^3(1-\nu_5) & (1-\nu_5)^2
\end{pmatrix}
\begin{pmatrix}
a_{0,0} \\ a_{1,0} \\ \vdots \\ a_{3,1} \\ a_{0,2}
\end{pmatrix} = 0.
$$

Since $t_i \neq t_j$ for $i \neq j$, we can see that the coefficient matrix is of full rank. Therefore, ψ is surjective.

Secondly, we have the short exact sequence

$$0 \to \mathcal{O}((n-1)D') \to \mathcal{O}(nD') \to N_{D'}^{\otimes n} \to 0,$$

which induces the following cohomology exact sequence

$$H^1(\widetilde{\mathbb{F}}_3, \mathcal{O}((n-1)D')) \to H^1(\widetilde{\mathbb{F}}_3, \mathcal{O}(nD')) \to H^1(D', N_{D'}^{\otimes n}) \to 0.$$

So, it suffices to show that $H^1(D', N_{D'}^{\otimes n}) = 0$ for $n > 1$. By the Riemann-Roch theorem, we have

$$\chi(\mathcal{O}_{D'}) = -\frac{D'.(D' + K_{\widetilde{\mathbb{F}}_3})}{2} = -1.$$

Since $(D')^2 = 2$, we have $\deg(N_{D'}^{\otimes n}) = 2n > \deg(\omega_{D'}) = 3$ for $n > 1$. Therefore, we get $H^1(D', N_{D'}^{\otimes n}) = 0$ for $n > 1$, and $H^1(\widetilde{\mathbb{F}}_3, \mathcal{O}(nD')) = 0$ for $n > 1$ by induction. □

Proposition 7. $H^i(W, \mathcal{O}_W(nT) \otimes \mathcal{O}_W/I_\Delta^n) = 0$ *for* $i > 0$.

Proof. We have exact sequences

$$\begin{aligned} 0 &\to \mathcal{O}_W(nT) \otimes I_\Delta^{n-m}/I_\Delta^{n-m+1} \to \mathcal{O}_W(nT) \otimes \mathcal{O}_W/I_\Delta^{n-m+1} \\ &\to \mathcal{O}_W(nT) \otimes \mathcal{O}_W/I_\Delta^{n-m} \to 0 \end{aligned}$$

for $m = 1, \ldots, n-1$. By the proof of Proposition 6, we have

$$H^i(W, \mathcal{O}_W(nT) \otimes \mathcal{O}_W/I_\Delta) = H^i(\widetilde{\mathbb{F}}_3, \mathcal{O}(2nD')) = 0$$

for $i > 0$. So, let us show that

$$H^i(W, \mathcal{O}_W(nT) \otimes I_\Delta^{n-m}/I_\Delta^{n-m+1}) \simeq H^i(\widetilde{\mathbb{F}}_3, \mathcal{O}(2nD') \otimes \mathrm{Sym}^{n-m}(\Omega^1_{\widetilde{\mathbb{F}}_3})) = 0$$

for $i > 0$. Consider the projective space bundle $\mathbb{P}(\Omega^1_{\widetilde{\mathbb{F}}_3})$ associated to $\Omega^1_{\widetilde{\mathbb{F}}_3}$, and denote it by P. There is a natural projection π from P to $\widetilde{\mathbb{F}}_3$. On P, we have the line bundle $\mathcal{O}_P(1)$ and denote it by L.

Lemma 8. *We have*

$$H^i(P, \pi^*(\mathcal{O}(2nD')) \otimes L^k) \simeq H^i(\widetilde{\mathbb{F}}_3, \mathcal{O}(2nD') \otimes \mathrm{Sym}^k(\Omega^1_{\widetilde{\mathbb{F}}_3}))$$

for $k > 0$.

Proof. By construction, we have that $R^1\pi_* L^k = 0$ and $\pi_* L^k \simeq \mathrm{Sym}^k(\Omega^1_{\widetilde{\mathbb{F}}_3})$. Therefore, the statement follows from Leray's spectral sequence and the projection formula. □

We have that

$$\begin{aligned} K_P &\simeq \pi^*(K_{\widetilde{\mathbb{F}}_3}) \otimes K_{P/\widetilde{\mathbb{F}}_3} \\ &\simeq \pi^*(K_{\widetilde{\mathbb{F}}_3}) \otimes \pi^*(\wedge^2 \Omega^1_{\widetilde{\mathbb{F}}_3}) \otimes L^{(-2)} \\ &\simeq \pi^*(2K_{\widetilde{\mathbb{F}}_3}) \otimes L^{(-2)}. \end{aligned}$$

Therefore, $H^i(P, \pi^*\mathcal{O}(2nD') \otimes L^k) \simeq H^i(P, \pi^*\mathcal{O}(2nD' - 2K_{\widetilde{\mathbb{F}}_3}) \otimes L^{k+2} \otimes K_P)$. So, if $\pi^*\mathcal{O}(2nD' - 2K_{\widetilde{\mathbb{F}}_3}) \otimes L^{k+2}$ is a nef and big line bundle, we get the result by the Kawamata-Viehweg vanishing theorem.

Firstly, let us check that this line bundle is nef. Let C be an irreducible curve in P. If $\pi(C) = p \in \widetilde{\mathbb{F}}_3$, C is linearly equivalent to the generic fiber of $\pi : P \to \widetilde{\mathbb{F}}_3$. In this case, we have that

$$\left(\pi^*\mathcal{O}(2nD' - 2K_{\widetilde{\mathbb{F}}_3}) \otimes L^{k+2}.C\right) = k + 2 > 0.$$

Next suppose that $\pi(C) = C'$ is a curve on $\widetilde{\mathbb{F}}_3$. Since $\mathrm{Pic}(\widetilde{\mathbb{F}}_3)$ is generated by $\widetilde{s_\infty}$, F, and $E_i^\pm$ $(i = 1, \ldots, 5)$, it is enough to check the case that C' is one of these. If $C' = \widetilde{s_\infty}$, then $(\pi_* L.\widetilde{s_\infty}) = \deg(\Omega^1_{\widetilde{\mathbb{F}}_3}|_{\widetilde{s_\infty}}) = (K_{\widetilde{\mathbb{F}}_3}.\widetilde{s_\infty}) = 1$, and $(2nD' - 2K_{\widetilde{\mathbb{F}}_3}.\widetilde{s_\infty}) = -2$. Therefore,

$$\left(\pi^*\mathcal{O}(2nD' - 2K_{\widetilde{\mathbb{F}}_3}) \otimes L^{k+2}.C\right) = rk > 0,$$

where r is some positive integer.

If $C' = F$, then $(\pi_* L.F) = \deg(\Omega^1_{\widetilde{\mathbb{F}}_3}|_F) = (K_{\widetilde{\mathbb{F}}_3}.F) = -2$, and $(2nD' - 2K_{\widetilde{\mathbb{F}}_3}.F) = 4n + 4$. Therefore,

$$\left(\pi^*\mathcal{O}(2nD' - 2K_{\widetilde{\mathbb{F}}_3}) \otimes L^{k+2}.C\right) = 2r'(2n - k) > 0,$$

where r' is some positive integer.

If $C' = E_i^\pm$, then $(\pi_* L.E_i^\pm) = \deg(\Omega^1_{\widetilde{\mathbb{F}}_3}|_{E_i^\pm}) = -1$, and $(2nD' - 2K_{\widetilde{\mathbb{F}}_3}.E_i^\pm) = 2n + 2$. Therefore,

$$\left(\pi^*\mathcal{O}(2nD' - 2K_{\widetilde{\mathbb{F}}_3}) \otimes L^{k+2}.E_i^\pm\right) = r''(2n - k) > 0,$$

where r'' is some positive integer. Thus $\pi^*\mathcal{O}(2nD' - 2K_{\widetilde{\mathbb{F}}_3}) \otimes L^{k+2}$ is a nef line bundle.

Secondly, let us check that this bundle is also a big line bundle. From the definition of Chern classes, we have

$$L^2 - \pi^* c_1(\Omega^1_{\widetilde{\mathbb{F}}_3}).L + \pi^* c_2(\Omega^1_{\widetilde{\mathbb{F}}_3}).f = 0,$$

where f is a generic fiber of π. Since $c_1(\Omega^1_{\widetilde{\mathbb{F}}_3})^2 = -2$ and $c_2(\Omega^1_{\widetilde{\mathbb{F}}_3}) = 14$, we have $L^3 = -16$, and

$$\left(\pi^*\mathcal{O}(2nD' - 2K_{\widetilde{\mathbb{F}}_3}) \otimes L^{k+2}\right)^3 = 24kn^2 + 48n^2 - 16k^3 - 84k^2 - 168k - 128.$$

This is positive if $n > 3$. Thus, $\pi^*\mathcal{O}(2nD' - 2K_{\widetilde{\mathbb{F}}_3}) \otimes L^{k+2}$ is a big line bundle, and this completes the proof. □

Proof of Proposition 2. We have that $H^0(M(-1)^0, \mathcal{O}) = \mathbb{C}$ by Corollary 4.7 and Lemma 5.2 in [4]. By using Proposition 6 and 7, we have that

$$H^i(W, \mathcal{O}_W(nT) \otimes I_\Delta^n) = 0$$

for $i > 1$ and $n > 3$.

From the cohomology sequences (2), (3), and (4), we get $H^i(U, \mathcal{O}_U) = 0$ for $i > 1$. Since $U/\mathfrak{S}_2 \simeq M(-1)^0$, we have $H^i(M(-1)^0, \mathcal{O}) = 0$ for $i > 1$. □

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