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On the preimage of a sphere by a polynomial mapping

Received: 4 November 2019 / Accepted: 11 August 2020 /

Published online: 17 August 2020

Abstract. Let X be an irreducible complex affine variety of dimension greater than one and let $f : X \rightarrow \mathbb{C}^m$ be a polynomial mapping. Let $|\cdot|$ be a semialgebraic norm on $\mathbb{C}^m$. Then for R large enough the sets $f^{-1}(B_R)$, $f^{-1}(S_R)$, $X \setminus f^{-1}(B_R)$ are all connected, where $B_R = \{z \in \mathbb{C}^m : |z| \leq R\}$ and $S_R = \{z \in \mathbb{C}^m : |z| = R\}$. As an application we show that if F is a counterexample to the Jacobian Conjecture, then the non-properness set of F has a non-trivial link at infinity.

1. Introduction

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Despite of the fact that polynomial mappings are one of the most basic tools in real and complex analysis, their topological properties are not well understood. Let $|\cdot|$ be a semi-algebraic norm in $\mathbb{K}^m$ (e.g., $|x| = \sqrt{\sum_{i=1}^m |x_i|^2}$ or $|x| = \sup_{i=1, \dots, m} \{|x_i|\}$). Let S_R be the sphere of radius R and B_R the ball of radius R in $\mathbb{K}^m$, i.e., $B_R = \{z \in \mathbb{K}^m : |z| \leq R\}$ and $S_R = \{z \in \mathbb{K}^m : |z| = R\}$. It is well known (see e.g. [7, 9]) that if a polynomial mapping $f : \mathbb{K}^n \rightarrow \mathbb{K}^m$ ($n > 1$) is proper, then the set $f^{-1}(S_R)$ for large R is connected. This is not true for an arbitrary real polynomial mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$. In particular it is easy to see that for $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $f = (x_1^2 x_2 \dots x_n, x_1 x_2^2 \dots x_n, \dots, x_1 x_2 \dots x_n^2)$ the set $f^{-1}(S_R)$ has at least 2^n connected components. Indeed, $f^{-1}(S_R) \subset \mathbb{R}^n \setminus f^{-1}(0) = \mathbb{R}^n \setminus \{x_1 x_2 \dots x_n = 0\}$. The set $\mathbb{R}^n \setminus \{x_1 x_2 \dots x_n = 0\}$ has 2^n connected components. Now in each of these components there is a point $(\epsilon_1 r, \dots, \epsilon_n r)$, where $r^{n+1} = R$ and $\epsilon_k = \pm 1$, which belongs to $f^{-1}(S_R)$. We have assumed here that on $\mathbb{R}^n$ we have the sup norm, but of course we can easily modify the proof for other norms.

We show however that the complex case is different and for every polynomial mapping $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ the preimage of a sufficiently large sphere is connected. In fact we prove more:

Author is partially supported by the grant of Narodowe Centrum Nauki Number 2019/33/B/ST1/00755

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Mathematics Subject Classification: 14R99 · 14P10 · 30C10

Theorem 1.1. *Let X be an irreducible complex affine variety of dimension greater than one and let $f : X \rightarrow \mathbb{C}^m$ be a polynomial mapping. Let $|\ast|$ be a semi-algebraic norm on $\mathbb{C}^m$. Then for R large enough the sets $f^{-1}(B_R)$, $f^{-1}(S_R)$, $X \setminus f^{-1}(B_R)$ are all connected.*

In particular we have:

Corollary 1.2. *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ be a polynomial mapping. Let $|\ast|$ be a semi-algebraic norm on $\mathbb{C}^m$. Then for R large enough the sets $f^{-1}(B_R)$, $f^{-1}(S_R)$, $\mathbb{C}^n \setminus f^{-1}(B_R)$ are all connected.*

Proof. Note that for $n = 1$ the mapping f is either finite or constant and we can use [7,9]. Alternatively, we can note that for $X = \mathbb{C}$ Theorem 2.1 is still true. $\square$

As mentioned before, the last corollary is not true for an arbitrary real polynomial mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$. However, we believe that the following conjecture is true:

Conjecture. *Let $f = (f_1, \dots, f_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a polynomial dominant mapping (i.e., $\overline{f(\mathbb{R}^n)} = \mathbb{R}^n$) and $B(f)$ be the bifurcation set of f . Assume that:*

- (1) $\text{codim } B(f) \geq 2$,
- (2) $\text{codim } f^{-1}(B(f)) \geq 2$.

Then for R large enough the sets $f^{-1}(B_R)$, $f^{-1}(S_R)$, $\mathbb{R}^n \setminus f^{-1}(B_R)$ are all connected.

Let us recall that $B(f) = f(C(f)) \cup S_f$, where $C(f)$ denotes the set of critical points of f and S_f denotes the set of points at which f is not proper (for the definition of S_f see e.g., [3,4]).

Remark 1.3. It is not difficult to prove, that for every real polynomial mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, the set $f^{-1}(B_R)$ is connected for R large enough. Hence the difficult part of our Conjecture is to prove that the set $f^{-1}(S_R)$ is connected.

2. Proof of Theorem 1.1

First assume that X is a normal irreducible variety. Let X' be a projective closure of X and consider $\text{graph}(f) \subset X' \times Y$, where $Y = \overline{f(X)}$. Denote by Γ the normalization of the closure of $\text{graph}(f)$ in $X' \times Y$. We have the canonical proper projection $\pi : \Gamma \rightarrow Y$. We can assume that $X \subset \Gamma$ and $f = \pi|_X$, hence we can treat the mapping π as an extension of f to Γ . In further we will identify π with f . We will need the following important theorem (for a proof see [2, pp. 141–142]):

Stein Factorization Theorem. *Let $f : \Gamma \rightarrow Y$ be a proper dominating morphism of irreducible algebraic varieties. If Γ is normal, then there exists a normal variety W , a finite morphism $\mu : W \rightarrow Y$, and a surjective morphism $g : \Gamma \rightarrow W$ with connected fibers such that $f = \mu \circ g$.*

Note that in our case the variety Y is affine, so W is affine, too. Consider the mapping $F = \iota \circ \mu : W \rightarrow Y \subset \mathbb{C}^n$, where $\iota : Y \rightarrow \mathbb{C}^n$ is the inclusion. Let $|\ast|$ be a semi-algebraic norm on $\mathbb{C}^n$. We show that for R large enough the sets $F^{-1}(B_R)$, $F^{-1}(S_R)$, $W \setminus F^{-1}(B_R)$ are all connected. Put $V(R) := F^{-1}(B_R)$. Let us examine the behavior of the sets $V(R)$. To do this we introduce the function $\psi : W \ni y \mapsto |F(y)| \in \mathbb{R}_+$ (here $\mathbb{R}_+ = \{t \in \mathbb{R} : t \geq 0\}$). Note that the set W is connected and semialgebraic. We have $V(R) = \psi^{-1}([0, R])$. Since ψ is semialgebraic, it has only a finite number of bifurcation values $P_0 = 0 < P_1 < \dots < P_r \in \mathbb{R}_+$ and outside them ψ is a locally trivial fibration by the Thom Isotopy Lemma (see [1, Theorem 2.7.1]). For technical reasons we put additionally $P_{r+1} = +\infty$.

Note that the number of connected components of $V(R)$ can change only if ψ passes through a bifurcation value. Indeed, if $R_2 > R_1$ and $[R_1, R_2] \subset (P_i, P_{i+1})$, then $V(R_2) \setminus V(R_1) \cong (R_1, R_2] \times F^i$, where F^i is a generic fiber of ψ over $I_i = (P_i, P_{i+1})$. In particular $V(R_1)$ is a deformation retract of $V(R_2)$ and they have the same number of connected components. Hence after we pass through the last critical value, the number of connected component of $V(R)$ remains constant.

Fix $R_0 > P_r$. The set $W \setminus V(R_0)$ is homeomorphic to the product $(R_0, \infty) \times F_r$. In particular $V(R_0)$ is a deformation retract of W . Hence $V(R_0)$ has only one component.

Now we examine the behavior of $W \setminus V(R)$. We need the following result which follows from [8, Theorem 6.3]:

Theorem 2.1. *Let X be a connected normal Stein variety of dimension greater than one and let $K \subset X$ be a compact set. Then the set $X \setminus K$ has only one unbounded component.*

Take $R > R_0$. Let us note that $V(R)$ is a compact set and $W \setminus V(R) \cong (R, \infty) \times F_r$. Hence all connected components of $W \setminus V(R)$ are unbounded. This means by Theorem 2.1 that there is only one such component, i.e., $W \setminus F^{-1}(B_R)$ is connected. But $W \setminus F^{-1}(B_R) \cong (R, +\infty) \times F^{-1}(S_R)$, so $F^{-1}(S_R)$ is connected, too.

Because the mapping g is surjective and has connected fibers we conclude that the sets $\pi^{-1}(B_R)$, $\pi^{-1}(S_R)$, $\Gamma \setminus \pi^{-1}(B_R)$ are connected, too. The following statement is true (see [6, chapter VI]):

Theorem 2.2. *Let X be a normal analytic space and $G \subset X$ be an open connected set. Assume that $Q \subset G$ is a nowhere-dense analytic subset of G . Then $G \setminus Q$ is connected.*

Denote $Q = \Gamma \setminus X$. Since the set $\Gamma \setminus \pi^{-1}(B_R)$ is connected and the space Γ is normal, the set $(\Gamma \setminus \pi^{-1}(B_R)) \setminus Q$ is also connected (see Theorem 2.2). This means that $X \setminus f^{-1}(B_R)$ is connected.

We can repeat our previous argument for the function $\phi : X \ni x \mapsto |f(x)| \in \mathbb{R}_+$ to prove that the sets $f^{-1}(S_R)$ and $f^{-1}(B_R)$ are connected, for R large enough.

Finally, let X be an arbitrary irreducible affine variety and $f : X \rightarrow \mathbb{C}^m$ be a polynomial mapping. Let $q : X' \rightarrow X$ be the normalization and take $F = f \circ q$. Note that X' is also an affine irreducible variety. By our previous result the sets

$F^{-1}(B_R)$, $F^{-1}(S_R)$, $X' \setminus F^{-1}(B_R)$ are connected for R large enough. Since q is surjective, the sets $f^{-1}(B_R)$, $f^{-1}(S_R)$, $X \setminus f^{-1}(B_R)$ are also connected.

3. Application

In this section we apply our result to the Jacobian Conjecture. Let us recall famous:

Jacobian Conjecture. *Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial mapping with a nowhere vanishing jacobian. Then F is a biholomorphism.*

If F is a counterexample to the Jacobian Conjecture, then F is non-proper and its non-properness set S_F is non-empty and hence it is a hypersurface (see e.g., [3–5]). Let us recall that $S_F = \{y \in \mathbb{C}^n : \text{there is a sequence } x_n \rightarrow \infty \text{ such that } F(x_n) \rightarrow y\}$ (see e.g. [3,4]). We show here that the set S_F has a non-trivial link at infinity.

Definition 3.1. Let $X \subset \mathbb{C}^n$ be an algebraic hypersurface. For R large enough the topology of the pair $(S_R, S_R \cap X)$ does not depend on R (see section 2). We call the pair $(S_R, S_R \cap X)$ the link at infinity of X . We say that this link is trivial if $\pi_1(S_R \setminus S_R \cap X) = \mathbb{Z}$.

From Theorem 1.1 it follows that the link of irreducible variety is connected if $n \geq 3$. It is easy to see, that it is also connected if $n = 2$ and X is a parametric curve. Of course every trivial link has to be connected. We have:

Lemma 3.2. *Assume that the affine hypersurface $X \subset \mathbb{C}^n$ has trivial link at infinity. Then for sufficiently large R , for a smooth point $a \in S_R \cap X$ and a neighborhood $U_a \subset S_R$ there is a loop $\gamma \subset U_a$ which generates $H_1(S_R \setminus X)$.*

Proof. Since $\pi_1(S_R \setminus S_R \cap X) = \mathbb{Z}$ we have $H_1(S_R \setminus S_R \cap X) = \mathbb{Z}$. We show that the latter group is generated by a small loop around $S_R \cap X$.

Indeed, consider the function $\psi : \mathbb{C}^n \ni y \mapsto |y| \in \mathbb{R}_+$. The pair $(\mathbb{C}^n, X)$ has a finite Whitney algebraic stratification $\mathcal{S} = \{\mathbb{C}^n \setminus X, S_1, \dots, S_k\}$, where $X = \bigcup_{i=1}^r S_i$. Since ψ is semialgebraic, it has only a finite number of bifurcation values $P_0 = 0 < P_1 < \dots < P_r \in \mathbb{R}_+$ on $\mathcal{S}$ and outside them ψ is a locally trivial fibration which preserves strata (see [1, Theorem 2.7.1]). In particular the pair $(S_R, S_R \cap X)$ is a deformation retract of $(\mathbb{C}^n \setminus \text{Int}(B_R), X \setminus \text{Int}(B_R))$ for R large enough. Let us denote this retraction by r .

Since X is an irreducible hypersurface (it has a trivial link!), we have $H_1(\mathbb{C}^n \setminus X) = \mathbb{Z}$. Let α be a small loop around a which is a generator of $H_1(\mathbb{C}^n \setminus X)$ (for details see [10], Lemma 2.3, a). We can choose α so small, that $r(\alpha) := \gamma \subset U_a$. Since r induces the isomorphism

$$r_* : H_1((\mathbb{C}^n \setminus X) \setminus \text{Int}(B_R)) \rightarrow H_1(S_R \setminus S_R \cap X)$$

we see that α is homologous to γ in $\mathbb{C}^n \setminus X$. Now consider the mapping

$$\mathbb{Z} = H_1(S_R \setminus S_R \cap X) \rightarrow H_1(\mathbb{C}^n \setminus X) = \mathbb{Z}.$$

Since γ goes to the generator of $H_1(\mathbb{C}^n \setminus X)$, it has to be generator of $H_1(S_R \setminus S_R \cap X)$. $\square$

Theorem 3.3. *Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ has a nowhere zero Jacobian. Then the non-properness set S_F of F can not have a trivial link at infinity.*

Proof. Put $S_F := X$. Since the mapping F has finite fibers it is dominant, i.e., $\overline{F(\mathbb{C}^n)} = \mathbb{C}^n$. Moreover, if $X = \{x \in \mathbb{C}^n : h(x) = 0\}$, then $h \circ F \neq \text{const}$, and consequently $F^{-1}(X)$ is a hypersurface. Since F is open, it implies that the set $F(\mathbb{C}^n) \cap X$ is dense in X . In particular, we can choose a point $a \in S_R \cap X$ in such a way that $a = F(b)$, where $b \in F^{-1}(S_R) := Y$, and additionally a is a smooth point of X . Let U_a be a small neighborhood of a in S_R such that on U_a there exists a well defined branch ψ of a mapping F^{-1} such that $\psi(a) = b$. By Lemma 3.2 we can find a small loop $\gamma \subset U_a$, which generates $H_1(S_R \setminus S_R \cap X) = \mathbb{Z}$. Take a point $c \in \gamma$ and let $d = \psi(c)$. Note that the group $\pi_1(S_R \setminus S_R \cap X, c)$ is also generated by the loop γ .

By our first result the manifold Y is connected and the mapping F induces a covering $p : Y \setminus F^{-1}(X) \rightarrow S_R \setminus X$. Since γ is the image of the loop $\gamma' = \psi(\gamma)$ we have that the mapping $p_* : \pi_1(Y \setminus F^{-1}(X), d) \rightarrow \pi_1(S_R \setminus S_R \cap X, c)$ is an isomorphism. In particular the covering p has topological degree

$$d = [\pi_1(S_R \setminus S_R \cap X, c) : p_*\pi_1(Y \setminus F^{-1}(X), d)] = 1.$$

Thus the mapping F induces trivial covering $Y \setminus F^{-1}(X) \rightarrow S_R \setminus X$. This implies that F has topological degree one, consequently it is an isomorphism, a contradiction.

□

Acknowledgements The author is grateful to the anonymous referee for his helpful remarks.

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