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Topological computation of Stokes matrices of some weighted projective lines

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Abstract. By mirror symmetry, the quantum connection of a weighted projective line is closely related to the localized Fourier–Laplace transform of some Gauß–Manin system. Following an article of D’Agnolo, Hien, Morando, and Sabbah, we compute the Stokes matrices for the latter at ∞ for the cases $\mathbb{P}(1, 3)$ and $\mathbb{P}(2, 2)$ by purely topological methods. We compare them to the Gram matrix of the Euler–Poincaré pairing on $D^b(\mathrm{Coh}(\mathbb{P}(1, 3)))$ and $D^b(\mathrm{Coh}(\mathbb{P}(2, 2)))$, respectively. This article is based on the doctoral thesis of the author.

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Introduction

In [6], D’Agnolo et al. describe how to compute the Stokes matrices of the enhanced Fourier–Sato transform of a perverse sheaf on the affine line by purely topological methods. To a regular singular holonomic $\mathcal{D}$ -module $\mathcal{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_{\mathbb{A}^1})$ on the affine line, one associates a perverse sheaf via the regular Riemann–Hilbert correspondence

$$R\mathrm{Hom}_{\mathcal{D}_{\mathbb{A}^1}^{\mathrm{an}}}((\bullet)^{\mathrm{an}}, \mathcal{O}_{\mathbb{A}^1}^{\mathrm{an}})[1]: \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_{\mathbb{A}^1}) \xrightarrow{\sim} \mathrm{Perv}(\mathbb{C}_{\mathbb{A}^1}).$$

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Let $\Sigma \subset \mathbb{A}^1$ denote the set of singularities of $\mathcal{M}$. Following [6, Sect. 4.2], after suitably choosing a total order on Σ , the resulting perverse sheaf $F \in \text{Perv}_\Sigma(\mathbb{C}_{\mathbb{A}^1})$ can be described by linear algebra data, namely its quiver

$$(\Psi(F), \Phi_\sigma(F), u_\sigma, v_\sigma)_{\sigma \in \Sigma},$$

where $\Psi(F)$ and $\Phi_\sigma(F)$ are finite dimensional $\mathbb{C}$ -vector spaces and $u_\sigma : \Psi(F) \rightarrow \Phi_\sigma(F)$ and $v_\sigma : \Phi_\sigma(F) \rightarrow \Psi(F)$ are linear maps such that $1 - u_\sigma v_\sigma$ is invertible for any σ . The main result in [6] is a determination of the Stokes matrices of the enhanced Fourier–Sato transform of F and therefore of the Fourier–Laplace transform of $\mathcal{M}$ in terms of the quiver of F . This result builds on the irregular Riemann–Hilbert correspondence of D’Agnolo and Kashiwara [7], which provides a topological description of holonomic $\mathcal{D}$ -modules. As proven by Kashiwara and Schapira [14], this correspondence intertwines the Fourier–Laplace with the (enhanced) Fourier–Sato transform.

Mirror symmetry connects the weighted projective line $\mathbb{P}(1, 3)$ with the Landau–Ginzburg model

$$(\mathbb{G}_m, f = x + x^{-3}).$$

The quantum connection of $\mathbb{P}(1, 3)$ is closely related to the Fourier–Laplace transform of the Gauß–Manin system $H^0(\int_f \mathcal{O})$ of f . We compute that

$$F := \text{R}f_* \mathbb{C}[1] \in \text{Perv}_\Sigma(\mathbb{C}_{\mathbb{A}^1}),$$

where Σ denotes the set of singular values of f , is the perverse sheaf associated to $H^0(\int_f \mathcal{O})$ by the Riemann–Hilbert correspondence. In Sect. 1, we compute the localized Fourier–Laplace transform of $H^0(\int_f \mathcal{O})$. In Sect. 2, analogous to the examples in [6, Sect. 7], we carry out the topological computation of the Stokes matrices of the Fourier–Laplace transform of $H^0(\int_f \mathcal{O})$. In Sect. 3, we compare the Stokes matrix S_β , that we obtained from our topological computations, to the Gram matrix of the Euler–Poincaré pairing on $D^b(\text{Coh}(\mathbb{P}(1, 3)))$ with respect to a suitable full exceptional collection. Following Dubrovin’s conjecture about the Stokes matrix of the quantum connection, proven for the weighted projective space $\mathbb{P}(\omega_0, \dots, \omega_n)$ by Tanabé and Ueda in [19] and by Cruz Morales and van der Put in [5], they are known to be equivalent after appropriate modifications. We give the explicit braid of the braid group B_4 that deforms the Gram matrix into the Stokes matrix S_β . Section 4 tackles the computations for the case of non-coprime parameters. In comparison to the case of coprime parameters, this requires a slightly modified approach. We compute the Stokes matrices of the Fourier–Laplace transform of the Gauß–Manin system of the Landau–Ginzburg model of $\mathbb{P}(2, 2)$ and set it into relation with the Gram matrix of the Euler–Poincaré pairing on $D^b(\text{Coh}(\mathbb{P}(2, 2)))$.

This article is based on the doctoral thesis [18] of the author. The figures in Sects. 2 and 4 were mainly produced in SAGE. In the online version of this article, the figures are provided in color.

1. Gauß–Manin system and its Fourier–Laplace transform

Let X be affine and f a regular function $f: X \rightarrow \mathbb{A}^1$ on X . Denote by $\int_f(\bullet)$ the direct image in the category of $\mathcal{D}$ -modules and by $M := H^0(\int_f \mathcal{O}_X) \in \text{Mod}_{\text{rh}}(\mathcal{D}_{\mathbb{A}^1})$ the zeroth cohomology of the Gauß–Manin system of f . Following [9, Sect. 2.c], it is given by

$$M = \Omega^n(X) [\partial_t] / (d - \partial_t df \wedge) \Omega^{n-1}(X) [\partial_t].$$

Denote by $G := \widehat{M}[\tau^{-1}]$ the Fourier–Laplace transform of M , localized at $\tau = 0$. It is given by

$$G = \Omega^n(X) [\tau, \tau^{-1}] / (d - \tau df \wedge) \Omega^{n-1}(X) [\tau, \tau^{-1}].$$

G is a free $\mathbb{C}[\tau, \tau^{-1}]$ -module of finite rank. Rewriting in the variable $\theta = \tau^{-1}$ gives the $\mathbb{C}[\theta, \theta^{-1}]$ -module

$$G = \Omega^n(X) [\theta, \theta^{-1}] / (\theta d - df \wedge) \Omega^{n-1}(X) [\theta, \theta^{-1}].$$

G is endowed with a flat connection given as follows. For $\gamma = [\sum_{k \in \mathbb{Z}} \omega_k \theta^k] \in G$, where $\Omega^n(X) \ni \omega_k = 0$ for almost all k , the connection is given by (cf. [12, Definition 2.3.1]):

$$\theta^2 \nabla_{\frac{\partial}{\partial \theta}} (\gamma) = \left[\sum_k f \omega_k \theta^k + \sum_k k \omega_k \theta^{k+1} \right].$$

It is known that (G, ∇) has a regular singularity at $\theta = \infty$ and possibly an irregular one at $\theta = 0$.

We now consider the Laurent polynomial $f = x + x^{-3} \in \mathbb{C}[x, x^{-1}]$, being a regular function on the multiplicative group $\mathbb{G}_m$. For our computations we pass to the variable $\theta = \tau^{-1}$. We compute that for the given f , G is given by the free $\mathbb{C}[\theta, \theta^{-1}]$ -module

$$G = \mathbb{C}[x, x^{-1}] dx [\theta, \theta^{-1}] / (\theta d - (dx - 3x^{-4} dx) \wedge) \mathbb{C}[x, x^{-1}] [\theta, \theta^{-1}]$$

with basis over $\mathbb{C}[\theta, \theta^{-1}]$ given by $[\frac{dx}{x}]$, $[\frac{dx}{x^2}]$, $[\frac{dx}{x^3}]$, $[\frac{dx}{x^4}]$. In this basis, the connection is given by

$$\theta \nabla_{\frac{\partial}{\partial \theta}} = \theta \partial_{\theta} + \begin{pmatrix} 0 & \frac{4}{3\theta} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{4}{3\theta} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{4}{3\theta} \\ \frac{4}{\theta} & 0 & 0 & 1 \end{pmatrix}. \quad (1)$$

Via the cyclic vector $m = (1, 0, 0, 0)^t$, we compute the relation

$$\nabla_{\theta \partial_{\theta}}^4 m + 4 \nabla_{\theta \partial_{\theta}}^3 m + \frac{32}{9} \nabla_{\theta \partial_{\theta}}^2 m - \frac{256}{27\theta^4} m = 0$$

Fig. 1. Newton polygon of P

and therefore associate the differential operator

$$P = (\theta \partial_\theta)^4 + 4(\theta \partial_\theta)^3 + \frac{32}{9}(\theta \partial_\theta)^2 - \frac{256}{27\theta^4} \in \mathbb{C}[\theta, \theta^{-1}] \langle \partial_\theta \rangle = \mathcal{D}_{\mathbb{G}_m}.$$

As it is well known, one can read the type of the singularities at 0 and ∞ from the Newton polygon in the sense of Ramis (cf. [15, Chapter V]). The Newton polygon in Fig. 1 confirms that P —and therefore system (1)—has the nonzero slope 1 and therefore is irregular singular at $\theta = 0$ and regular singular at $\theta = \infty$.

2. Topological computation of the Stokes matrices

We consider the Laurent polynomial $f = x + x^{-3}: \mathbb{G}_m \rightarrow \mathbb{A}^1$. Its critical points are given by $\{\pm \sqrt[4]{3}, \pm \sqrt[4]{3}i\}$. The critical values of f are given by

$$\Sigma = \left\{ \pm \frac{4}{\sqrt[4]{27}}, \pm \frac{4i}{\sqrt[4]{27}} \right\} \subset \mathbb{A}^1.$$

The preimages of

- $\frac{4}{\sqrt[4]{27}}$ are $\sqrt[4]{3}$ (double), $\frac{-1-\sqrt{2}i}{\sqrt[4]{27}}$ and $\frac{-1+\sqrt{2}i}{\sqrt[4]{27}}$,
- $-\frac{4}{\sqrt[4]{27}}$ are $-\sqrt[4]{3}$ (double), $\frac{1-\sqrt{2}i}{\sqrt[4]{27}}$ and $\frac{1+\sqrt{2}i}{\sqrt[4]{27}}$,
- $\frac{4i}{\sqrt[4]{27}}$ are $\sqrt[4]{3}i$ (double), $\frac{-\sqrt{2}-i}{\sqrt[4]{27}}$ and $\frac{\sqrt{2}-i}{\sqrt[4]{27}}$,
- $-\frac{4i}{\sqrt[4]{27}}$ are $-\sqrt[4]{3}i$ (double), $\frac{\sqrt{2}+i}{\sqrt[4]{27}}$ and $\frac{-\sqrt{2}+i}{\sqrt[4]{27}}$.

Since f is proper, we compute by the adjunction formula that

$$R\mathrm{Hom}_{\mathcal{D}^{\mathrm{an}}} \left(\left(\int_f \mathcal{O} \right)^{\mathrm{an}}, \mathcal{O}^{\mathrm{an}} \right) \simeq Rf_*^{\mathrm{an}} R\mathrm{Hom}_{\mathcal{D}^{\mathrm{an}}} \left(\mathcal{O}^{\mathrm{an}}, f^{\dagger} \mathcal{O}^{\mathrm{an}} \right) \simeq Rf_*^{\mathrm{an}} \mathbb{C}.$$

Fig. 2. LHS: $\{x \mid \Re(f(x)) \geq 0\}$, RHS: $\{x \mid \Im(f(x)) \geq 0\}$

Since f is semismall, $Rf_*\mathbb{C}[1] \in \text{Perv}(\mathbb{C}_{\mathbb{A}^1})$ is a perverse sheaf (cf. [8]). Outside of Σ , f is a covering of degree 4, therefore $Rf_*\mathbb{C}[1] \in \text{Perv}_{\Sigma}(\mathbb{C}_{\mathbb{A}^1})$. By the regular Riemann–Hilbert correspondence

$$\text{Sol}(\bullet)[\dim X] := R\mathcal{H}om_{\mathcal{D}_X^{\text{an}}}((\bullet)^{\text{an}}, \mathcal{O}_X^{\text{an}})[\dim X] : \text{Mod}_{\text{rh}}(\mathcal{D}_X) \xrightarrow{\sim} \text{Perv}(\mathbb{C}_{X^{\text{an}}}),$$

we associate to $H^0(\int_f \mathcal{O})$ the perverse sheaf $F := Rf_*\mathbb{C}[1]$.

We fix $\alpha = e^{\frac{\pi i}{8}} \in \mathbb{A}^1$, $\beta = e^{\frac{3\pi i}{8}} \in (\mathbb{A}^1)^{\vee}$, such that $\Re(\langle \alpha, \beta \rangle) = 0$, $\Im(\langle \alpha, \beta \rangle) = 1$. This induces the following order on Σ (cf. [6, Sect. 4]):

$$\sigma_1 := \frac{4i}{\sqrt{27}} <_{\beta} \sigma_2 := -\frac{4}{\sqrt{27}} <_{\beta} \sigma_3 := \frac{4}{\sqrt{27}} <_{\beta} \sigma_4 := -\frac{4i}{\sqrt{27}}.$$

In Fig. 4, the σ_i are depicted in the following colors:

• σ_1 : green, • σ_2 : red, • σ_3 : purple, • σ_4 : orange.

The blue area in Fig. 2 shows where f has real (resp. imaginary) part greater than or equal to 0. In Fig. 3, the preimage of the imaginary (resp. real) axis under f is plotted in blue (resp. red) color. We consider lines passing through the singular values with phase $\frac{\pi}{8}$, as depicted in Fig. 4. The preimages of these lines are plotted in Fig. 5. We fix a base point e with $\Re(e) > \Re(\sigma_i)$ for all i and denote its preimages by e_1, e_2, e_3, e_4 , as depicted in Fig. 6. In the following, we adopt the notation of [6, Sect. 4]. The nearby and global nearby cycles of F are given by

$$\Psi_{\sigma_i}(F) := R\Gamma_c(\mathbb{A}^1; \mathbb{C}_{\ell_{\sigma_i}^{\times}} \otimes F) \simeq H^0 R\Gamma_c(\ell_{\sigma_i}^{\times}; F) \cong \bigoplus_{e_j \in f^{-1}(e)} \mathbb{C}_{e_j} \cong \mathbb{C}^4,$$

$$\Psi(F) := R\Gamma_c(\mathbb{A}^1; \mathbb{C}_{\mathbb{A}^1 \setminus \ell_{\Sigma}} \otimes F)[1] \simeq \Psi_{\sigma_i}(F) \cong \mathbb{C}^4.$$

Furthermore, we fix isomorphisms $i_{\sigma_i}^{-1} F[-1] \cong \bigoplus_{\sigma_i^j \in f^{-1}(\sigma_i)} \mathbb{C}_{\sigma_i^j} \cong \mathbb{C}^3$.

Fig. 3. Preimage of the imaginary (resp. real) axis in blue (resp. red) color

Fig. 4. Lines passing through σ_i with phase $\frac{\pi}{8}$

The exponential components at ∞ of the Fourier–Laplace transform of $H^0(\int_f \mathcal{O})$ are known to be of linear type with coefficients given by the $\sigma_i \in \Sigma$. The Stokes rays are therefore given by

$$\left\{ 0, \pm \frac{\pi}{4}, \pm \frac{\pi}{2}, \pm \frac{3\pi}{4}, \pi \right\}.$$

We consider loops γ_{σ_i} , starting at e and running around the singular value σ_i in counterclockwise orientation,¹ as depicted in Fig. 6. We denote by $\gamma_{\sigma_i}^j$ the preimage of γ_{σ_i} starting at e_j , $j = 1, 2, 3, 4$. The figure constitutes a rough drawing of the preimages of γ_{σ_i} . By taking into account the preimages of the different segments of the axes and the intersections of γ_{σ_i} with them, one recovers the $\gamma_{\sigma_i}^j$ as depicted in

¹ Counterclockwise orientation since the imaginary part of $\langle \alpha, \beta \rangle$ is positive.

Fig. 5. Preimages under f

the figure. From Fig. 6 we read, in the ordered basis e_1, e_2, e_3, e_4 , the monodromies

$$T_{\sigma_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_{\sigma_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T_{\sigma_3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad T_{\sigma_4} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In order to obtain the maps b_{σ_i} , we consider the half-lines $\ell_{\sigma_i} := \sigma_i + \alpha \mathbb{R}_{\geq 0}$. We denote their preimages under f by $\{\ell_{\sigma_i}^j\}_{j=1,2,3,4}$, depending on which $\gamma_{\sigma_i}^j$ they intersect. We label the preimages of σ_i by $\sigma_i^1, \sigma_i^2, \sigma_i^3$, as depicted in Fig. 7. By the derivation of the short exact sequence of quivers [6, (7.1.3)] and passing to Borel–Moore homology as described in [6, Lemma 5.3.1.(i)], b_{σ_i} is induced from the corresponding boundary value map from ℓ_{σ_i} to its origin σ_i . Therefore, b_{σ_i} encodes which lift of ℓ_{σ_i} starts at which preimage of σ_i . Namely, from Fig. 7 we read the following:

$$\sigma_1: \ell_{\sigma_1}^1 \mapsto \sigma_1^1, \ell_{\sigma_1}^2 \mapsto \sigma_1^1, \ell_{\sigma_1}^3 \mapsto \sigma_1^2, \ell_{\sigma_1}^4 \mapsto \sigma_1^3.$$

$$\text{Therefore, } b_{\sigma_1} \text{ is the transpose of } \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\sigma_2: \ell_{\sigma_2}^1 \mapsto \sigma_2^3, \ell_{\sigma_2}^2 \mapsto \sigma_2^1, \ell_{\sigma_2}^3 \mapsto \sigma_2^1, \ell_{\sigma_2}^4 \mapsto \sigma_2^2.$$

$$\text{Therefore, } b_{\sigma_2} \text{ is the transpose of } \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

$$\sigma_3: \ell_{\sigma_3}^1 \mapsto \sigma_3^1, \ell_{\sigma_3}^2 \mapsto \sigma_3^2, \ell_{\sigma_3}^3 \mapsto \sigma_3^3, \ell_{\sigma_3}^4 \mapsto \sigma_3^1.$$

Therefore, b_{σ_3} is the transpose of $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

$$\sigma_4: \ell_{\sigma_4}^1 \mapsto \sigma_4^1, \ell_{\sigma_4}^2 \mapsto \sigma_4^3, \ell_{\sigma_4}^3 \mapsto \sigma_4^1, \ell_{\sigma_4}^4 \mapsto \sigma_4^2.$$

Therefore, b_{σ_4} is the transpose of $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

We obtain, in the ordered bases $\sigma_i^1, \sigma_i^2, \sigma_i^3$ and $\ell_{\sigma_i}^1, \ell_{\sigma_i}^2, \ell_{\sigma_i}^3, \ell_{\sigma_i}^4$ each:

$$\begin{aligned} b_{\sigma_1} &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & b_{\sigma_2} &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ b_{\sigma_3} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, & b_{\sigma_4} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

Denote by $u_i := u_{\sigma_i}$, $v_i := v_{\sigma_i}$, $T_i := T_{\sigma_i}$ and $\Phi_i := \Phi_{\sigma_i}$. As described in [6, Sect. 7], we obtain $\Phi_i(F) \xrightleftharpoons[u_i]{v_i} \Psi(F)$ as the cokernels of the following diagrams:

$$\begin{array}{ccc} i_{\sigma_i}^{-1} F[-1] & \xrightarrow{b_{\sigma_i}} & \Psi(F) \\ \updownarrow & & \updownarrow 1-T_i \\ 0 & \longrightarrow & \Psi(F) \end{array}$$

We identify the cokernels of b_{σ_i} in the following way:

$$\begin{aligned} \bullet \text{ coker}(b_{\sigma_1}) &\simeq \mathbb{C} \text{ via } \begin{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} a_1 - a_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{bmatrix}, \\ \bullet \text{ coker}(b_{\sigma_2}) &\simeq \mathbb{C} \text{ via } \begin{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 0 \\ a_2 - a_3 \\ 0 \\ 0 \end{pmatrix} \end{bmatrix}, \\ \bullet \text{ coker}(b_{\sigma_3}) &\simeq \mathbb{C} \text{ via } \begin{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} a_1 - a_4 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{bmatrix}, \end{aligned}$$

$$\bullet \text{ coker}(b_{\sigma_4}) \simeq \mathbb{C} \text{ via } \left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \right] = \left[\begin{pmatrix} a_1 - a_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right].$$

We obtain that $(\Phi_i(F) \xrightarrow[u_i]{v_i} \Psi(F)) \simeq (\mathbb{C} \xrightarrow[u_i]{v_i} \mathbb{C}^4)$, where

$$u_1 = \begin{pmatrix} 1 & -1 & 0 & 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}, \\ u_3 = \begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 1 & 0 & -1 & 0 \end{pmatrix},$$

and $v_i = u_i^t$. By [6, Theorem 5.2.2], we obtain the following

Theorem. *Under the choices made, the Stokes matrices of the Fourier–Laplace transform of $H^0(\int_f \mathcal{O})$ at ∞ are given as*

$$S_\beta = \begin{pmatrix} 1 & u_1 v_2 & u_1 v_3 & u_1 v_4 \\ 0 & 1 & u_2 v_3 & u_2 v_4 \\ 0 & 0 & 1 & u_3 v_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ S_{-\beta} = \begin{pmatrix} \mathbb{T}_1 & 0 & 0 & 0 \\ -u_2 v_1 & \mathbb{T}_2 & 0 & 0 \\ -u_3 v_1 & -u_3 v_2 & \mathbb{T}_3 & 0 \\ -u_4 v_1 & -u_4 v_2 & -u_4 v_3 & \mathbb{T}_4 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & -1 & -1 & -1 \end{pmatrix} = -S_\beta^t,$$

where $\mathbb{T}_i := 1 - u_i v_i$. $S_{\pm\beta}$ describes crossing $h_{\pm\beta}$ from H_α to $H_{-\alpha}$, where

$$H_\alpha = \left\{ w \mid \arg(w) \in \left[-\frac{5\pi}{8}, \frac{3\pi}{8} \right] \right\}, \\ H_{-\alpha} = \left\{ w \mid \arg(w) \in \left[\frac{3\pi}{8}, \frac{11\pi}{8} \right] \right\} \subset (\mathbb{A}^1)^\vee$$

denote the closed sectors at ∞ and $h_{\pm\beta} = \pm\mathbb{R}_{>0}\beta \subset (\mathbb{A}^1)^\vee$, such that $H_\alpha \cap H_{-\alpha} = h_\beta \cup h_{-\beta}$. $\square$

3. Quantum connection and Dubrovin's conjecture

3.1. Quantum connection

The quantum connection of a Fano variety (resp. an orbifold) X is a connection on the trivial vector bundle over $\mathbb{P}^1$ with fiber $H^*(X, \mathbb{C})$ (resp. $H_{\text{orb}}^*(X, \mathbb{C})$), where z denotes the standard inhomogeneous coordinate at ∞ . By [11, (2.2.1)], the quantum connection is the connection given by

$$\nabla_{z\partial_z} = z \frac{\partial}{\partial z} - \frac{1}{z} (-K_X \circ) + \mu,$$

Fig. 6. l_{σ_i} and their preimages under f

Fig. 7. l_{σ_i} and their preimages under f

where the first term on the right hand side is ordinary differentiation, the second one is pointwise quantum multiplication by $(-K_X)$, and the third one is the grading operator

$$\mu(a) := \left(\frac{i}{2} - \frac{\dim X}{2} \right) a \quad \text{for } a \in H^i(X, \mathbb{C}).$$

The quantum connection is regular singular at $z = \infty$ and irregular singular at $z = 0$.

For the weighted projective line $\mathbb{P}(a, b)$, the orbifold cohomology ring is given by (cf. [16, Example 3.20])

$$H_{\text{orb}}^*(\mathbb{P}(a, b), \mathbb{C}) = \mathbb{C}[x, y, \xi] / \langle xy, ax^{\frac{a}{d}} - by^{\frac{b}{d}}\xi^{n-m}, \xi^d - 1 \rangle,$$

where $d = \gcd(a, b)$ and $m, n \in \mathbb{Z}$ such that $am + bn = d$. The grading is given as follows (cf. [1, Sect. 9]): $\deg x = \frac{1}{A}$, $\deg y = \frac{1}{B}$, $\deg \xi = 0$, where $A = \frac{a}{d}$, $B = \frac{b}{d}$. Quantum multiplication² is computed in

$$QH_{\text{orb}}^* = \mathbb{C}[x, y, \xi] / \langle xy - 1, ax^{\frac{a}{d}} - by^{\frac{b}{d}}\xi^{n-m}, \xi^d - 1 \rangle.$$

For $\gcd(a, b) = 1$, $-K_{\mathbb{P}(a,b)}$ is given by the element $[x^a + y^b] \in H_{\text{orb}}^1(\mathbb{P}(a, b), \mathbb{C})$. Taking into account that the grading is scaled by 2, the grading operator is defined by $\mu(a) = (i - \frac{\dim X}{2})a$ for $a \in H_{\text{orb}}^i(X, \mathbb{C})$.

We obtain the quantum connection of $\mathbb{P}(1, 3)$ as follows.

$$H_{\text{orb}}^*(\mathbb{P}(1, 3), \mathbb{C}) = \mathbb{C}[x, y] / \langle xy, x - 3y^3 \rangle$$

with grading given by $\deg x = 1$, $\deg y = \frac{1}{3}$. A basis over $\mathbb{C}$ is given by $1, y, y^2, y^3$. Quantum multiplication by $-K_{\mathbb{P}(1,3)} = [x + y^3] = [4y^3]$ in this basis is given by the matrix

$$\begin{pmatrix} 0 & \frac{4}{3} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & \frac{4}{3} \\ 4 & 0 & 0 & 0 \end{pmatrix}.$$

The grading μ is given by the matrix

$$\begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Therefore, the quantum connection of $\mathbb{P}(1, 3)$ is given by

$$\nabla_{z\partial_z} = z\partial_z - \frac{1}{z} \begin{pmatrix} 0 & \frac{4}{3} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & \frac{4}{3} \\ 4 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}. \quad (2)$$

² We always consider the case $q = 1$.

Observation. By the gauge transformation $h = \text{diag}(\theta^{-\frac{1}{2}}, \theta^{-\frac{1}{2}}, \theta^{-\frac{1}{2}}, \theta^{-\frac{1}{2}})$, which subtracts $\frac{1}{2}$ on the diagonal entries, and passing to $-\theta$, connection (1) arising from the Landau–Ginzburg model is exactly the quantum connection (2) of $\mathbb{P}(1, 3)$, as predicted by mirror symmetry.

3.2. Dubrovin’s conjecture

Let X be a Fano variety (or an orbifold), such that the bounded derived category $D^b(\text{Coh}(X))$ of coherent sheaves on X admits a full exceptional collection $\langle E_1, \dots, E_n \rangle$, where the collection $\langle E_1, \dots, E_n \rangle$ is called

- *exceptional* if $R\mathcal{H}om(E_i, E_i) = \mathbb{C}$ for all i and $R\mathcal{H}om(E_i, E_j) = 0$ for $i > j$,
- *full* if $D^b(\text{Coh}(X))$ is the smallest full triangulated subcategory of $D^b(\text{Coh}(X))$ containing $E_1, \dots, E_n$.

In [10], Dubrovin conjectured that, under appropriate choices, the Stokes matrix of the quantum connection of X equals the Gram matrix of the Euler–Poincaré pairing with respect to some full exceptional collection—modulo some action of the braid group, sign changes and permutations (cf. [4, Sect. 2.3]). Then the second Stokes matrix is the transpose of the first one. The Euler–Poincaré pairing is given by the bilinear form

$$\chi(E, F) := \sum_k (-1)^k \dim_{\mathbb{C}} \text{Ext}^k(E, F), \quad E, F \in D^b(\text{Coh}(X)).$$

The Gram matrix of χ with respect to a full exceptional collection is upper triangular with ones on the diagonal.

For $\mathbb{P}(a, b)$, $\langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(a+b-1) \rangle$ is a full exceptional collection of $D^b(\text{Coh}(\mathbb{P}(a, b)))$ (cf. [2, Theorem 2.12]). Following [3, Theorem 4.1], the cohomology of the twisting sheaves for $k \in \mathbb{Z}$ is given by

- $H^0(\mathbb{P}(a, b), \mathcal{O}(k)) = \bigoplus_{(m,n) \in I_0} \mathbb{C} x^m y^n$, where

$$I_0 = \{(m, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \mid am + bn = k\},$$

- $H^1(\mathbb{P}(a, b), \mathcal{O}(k)) = \bigoplus_{(m,n) \in I_1} \mathbb{C} x^m y^n$, where

$$I_1 = \{(m, n) \in \mathbb{Z}_{< 0} \times \mathbb{Z}_{< 0} \mid am + bn = k\},$$

- $H^i(\mathbb{P}(a, b), \mathcal{O}(k)) = 0$ for all $i \geq 2$.

We only need to compute $\text{Ext}^k(\mathcal{O}(i), \mathcal{O}(j))$ for $i < j$, which is given by $H^k(\mathcal{O}(j-i))$ (cf. [17, Lemma 4.5]). Therefore, the zeroth cohomologies of the twisting sheaves $\mathcal{O}(j-i)$ are the only ones that contribute to the Gram matrix of χ . For $\mathbb{P}(1, 3)$ we obtain the cohomology groups

$$H^0(\mathcal{O}(1)) \cong \mathbb{C}, \quad H^0(\mathcal{O}(2)) \cong \mathbb{C}, \quad H^0(\mathcal{O}(3)) \cong \mathbb{C}^2$$

and therefore the Gram matrix of the Euler–Poincaré pairing on $D^b(\mathrm{Coh}(\mathbb{P}(1, 3)))$ with respect to the full exceptional collection $\mathcal{E} := \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3) \rangle$ is given by

$$S_{\mathrm{Gram}} = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3)$$

3.3. Comparison of the Gram and Stokes matrix

Mirror symmetry relates the Laurent polynomial $f = x + x^{-3}$ to the weighted projective line $\mathbb{P}(1, 3)$. The pair $(\mathbb{G}_m, f = x + x^{-3})$ is a Landau–Ginzburg model of the weighted projective line $\mathbb{P}(1, 3)$. According to Dubrovin’s conjecture, the Stokes matrix of the quantum connection of $\mathbb{P}(1, 3)$ equals the Gram matrix of the Euler–Poincaré pairing with respect to some full exceptional collection of $D^b(\mathrm{Coh}(\mathbb{P}(1, 3)))$. Note that there is a natural action of the braid group on the Stokes matrix reflecting variations in the choices involved to determine the Stokes matrix (cf. [13]). In our case, we have to consider the braid group on four strands, namely

$$B_4 = \langle \beta_1, \beta_2, \beta_3 \mid \beta_1\beta_3\beta_1 = \beta_3\beta_1\beta_3, \beta_1\beta_2\beta_1 = \beta_2\beta_1\beta_2, \beta_2\beta_3\beta_2 = \beta_3\beta_2\beta_3 \rangle.$$

Proposition. S_{Gram} and S_β correspond to each other under the action of the elementary braid $\beta_1 \in B_4$.

Proof. We computed that the Gram matrix of χ with respect to the full exceptional collection $\mathcal{E}$ is given by (3). Following [13, Sect. 6], the braid β_1 acts on the Gram matrix as

$$S_{\mathrm{Gram}} \mapsto S_{\mathrm{Gram}}^{\beta_1} := A^{\beta_1}(S_{\mathrm{Gram}}) \cdot S_{\mathrm{Gram}} \cdot (A^{\beta_1}(S_{\mathrm{Gram}}))^t,$$

where $A^{\beta_1}(S_{\mathrm{Gram}})$ is given by

$$A^{\beta_1}(S_{\mathrm{Gram}}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We obtain that

$$S_{\mathrm{Gram}}^{\beta_1} = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = S_\beta.$$

□

Remark. $S_{\mathrm{Gram}}^{\beta_1} = S_\beta$ is the Gram matrix of the Euler–Poincaré pairing with respect to the right mutation $\mathbb{R}_1\mathcal{E}$ of the full exceptional collection $\mathcal{E}$ (cf. [4, Proposition 13.1]). In our topological computations, the action of the braid $\beta_1 \in B_4$ should correspond to a counterclockwise rotation of β .

Fig. 8. LHS: $\{x \mid \Re(f_1(x)) \geq 0\}$, RHS: $\{x \mid \Im(f_1(x)) \geq 0\}$

4. Non-coprime parameters

In this section, we consider the weighted projective line $\mathbb{P}(2, 2)$ as an example for the case of non-coprime parameters. The topological computation of the Stokes matrices of the quantum connection at ∞ requires some adaptations.

A Landau–Ginzburg model of $\mathbb{P}(2, 2)$ is given by the curve $\{x^2y^2 = 1\} \subset \mathbb{G}_m^2$ together with the potential $f = x + y$. This splits into two disjoint components $U_1 := \{xy + 1 = 0\}$ and $U_2 := \{xy - 1 = 0\}$. f restricts to $f_1 = x - x^{-1}$ on U_1 and to $f_2 = x + x^{-1}$ on U_2 , where we identified $y = -x^{-1}$ and $y = x^{-1}$, respectively. The blue area in Fig. 8 shows where f_1 has real (resp. imaginary) part greater than or equal to 0. The blue area in Fig. 9 shows where f_2 has real (resp. imaginary) part greater than or equal to 0. In Fig. 10, the preimages of the real (resp. imaginary) axis under f_1 and f_2 are plotted.

f has singular fibers at $\Sigma := \{\pm 2i, \pm 2\}$. For our topological computations, we consider the perverse sheaf $F = Rf_*\mathbb{C}[1] \in \text{Perv}_\Sigma(\mathbb{A}^1)$. The exponential components at ∞ of the Fourier–Sato transform of F are of linear type, with coefficients given by the $\sigma_i \in \Sigma$. The Stokes rays are therefore given by $\{0, \pm \frac{\pi}{4}, \pm \frac{\pi}{2}, \pm \frac{3\pi}{4}, \pi\}$.

- $f^{-1}(2) = \{(1, 1) \in U_2, (1 - \sqrt{2}, 1 + \sqrt{2}) \in U_1, (1 + \sqrt{2}, 1 - \sqrt{2}) \in U_1\}$, $(1, 1)$ being the double inverse image,
- $f^{-1}(-2) = \{(-1, -1) \in U_2, (-1 - \sqrt{2}, -1 + \sqrt{2}) \in U_1, (-1 + \sqrt{2}, -1 - \sqrt{2}) \in U_1\}$, $(-1, -1)$ being the double inverse image,
- $f^{-1}(2i) = \{(i, i) \in U_1, (i + \sqrt{2}i, i - \sqrt{2}i) \in U_2, (i - \sqrt{2}i, i + \sqrt{2}i) \in U_2\}$, (i, i) being the double inverse image,
- $f^{-1}(-2i) = \{(-i, -i) \in U_1, (-i + \sqrt{2}i, -i - \sqrt{2}i) \in U_2, (-i - \sqrt{2}i, -i + \sqrt{2}i) \in U_2\}$, $(-i, -i)$ being the double inverse image.

We choose $\alpha = e^{3\pi i/8}$, $\beta = e^{9\pi i/8}$. This induces the following order on Σ :

$$\sigma_1 := 2 <_\beta \sigma_2 := -2i <_\beta \sigma_3 := 2i <_\beta \sigma_4 := -2.$$

Denote by $\ell_{\sigma_i} = \sigma_i + \mathbb{R}_{\geq 0}\alpha$. Their preimages are depicted in Figs. 11 and 12.

Fig. 9. LHS: $\{x \mid \Re(f_2(x)) \geq 0\}$, RHS: $\{x \mid \Im(f_2(x)) \geq 0\}$

Fig. 10. Preimage of the real (resp. imaginary) axis in blue (resp. red) color under f_1 (LHS) and f_2 (RHS)

Fig. 11. Preimages under f_1 of lines passing through σ_2 and σ_3 with phase $3\pi/8$

Fig. 12. Preimages under f_2 of lines passing through σ_1 and σ_4 with phase $3\pi/8$

Fig. 13. Preimages of γ_{σ_i} under f_1 (LHS) and f_2 (RHS)

Fig. 14. Preimages of ℓ_{σ_i} under f_1 (LHS) and f_2 (RHS)

As in the previous example, only the lifts of γ_{σ_i} and ℓ_{σ_i} around the double preimages of σ_i , which we denote by σ_i^1 , contribute to the monodromy and the cokernel of b_{σ_i} . Therefore, in our figures, we restricted to this information. From Fig. 13 we read the monodromies in the ordered basis e_1, e_2, e_3, e_4 to be

$$T_{\sigma_1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_{\sigma_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$T_{\sigma_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad T_{\sigma_4} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Taking into account Fig. 14, we identify the cokernel of

- b_{σ_1} with $\mathbb{C}$ via $[(a_1, a_2, a_3, a_4)]^t = [(a_1 - a_3, 0, 0, 0)]^t$,
- b_{σ_2} with $\mathbb{C}$ via $[(a_1, a_2, a_3, a_4)]^t = [(0, a_2 - a_4, 0, 0)]^t$,
- b_{σ_3} with $\mathbb{C}$ via $[(a_1, a_2, a_3, a_4)]^t = [(0, a_2 - a_4, 0, 0)]^t$,
- b_{σ_4} with $\mathbb{C}$ via $[(a_1, a_2, a_3, a_4)]^t = [(a_1 - a_3, 0, 0, 0)]^t$.

We therefore obtain

$$\begin{aligned} u_{\sigma_1} &= (1 \quad 0 \quad -1 \quad 0) = u_{\sigma_4}, \\ u_{\sigma_2} &= (0 \quad 1 \quad 0 \quad -1) = u_{\sigma_3}, \end{aligned}$$

and $v_{\sigma_i} = u_{\sigma_i}^t$. In summary, we obtain the following

Theorem. *The Stokes matrices of the Fourier–Laplace transform of $H^0(\int_f \mathcal{O})$ in the chosen bases are given by*

$$S_\beta = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_{-\beta} = -S_\beta^t. \quad (4)$$

$S_{\pm\beta}$ describes passing $\pm\beta\mathbb{R}_{>0} \subset (\mathbb{A}^1)^\vee \setminus \{0\}$ from H_α to $H_{-\alpha}$, where

$$\begin{aligned} H_\alpha &= \left\{ w \mid \arg(w) \in \left[-\frac{7\pi}{8}, \frac{\pi}{8} \right] \right\}, \\ H_{-\alpha} &= \left\{ w \mid \arg(w) \in \left[\frac{\pi}{8}, \frac{9\pi}{8} \right] \right\} \subset (\mathbb{A}^1)^\vee \setminus \{0\}. \end{aligned}$$

□

In the non-coprime case $\gcd(a, b) \neq 1$, the computation of the orbifold cohomology of $\mathbb{P}(a, b)$ is more subtle. We refer to [16] for precise formulae and the correspondence of the quantum connection and the Fourier–Laplace transform of the Gauß–Manin connection of the Landau–Ginzburg model.

For $\mathbb{P}(2, 2)$ we get the cohomology groups

$$H^0(\mathcal{O}(1)) \cong H^0(\mathcal{O}(3)) \cong 0, \quad H^0(\mathcal{O}(2)) \cong \mathbb{C}^2$$

and therefore the Gram matrix of the Euler–Poincaré pairing on $D^b(\text{Coh}(\mathbb{P}(2, 2)))$ with respect to $\mathcal{E} = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3) \rangle$ is given by

$$S_{\text{Gram}} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (5)$$

Proposition. S_{Gram} and S_β correspond to each other under the action of S_4 .

Proof. By the permutation

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in S_4,$$

acting on the Gram matrix S_{Gram} as $P \cdot S_{\text{Gram}} \cdot P^{-1}$ (cf. [13, Sect. 6.c]), we find that the Gram matrix S_{Gram} (5) is transformed into the topologically computed Stokes matrix S_β (4). $\square$

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