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From cracked polytopes to Fano threefolds

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Abstract. We construct Fano threefolds with very ample anti-canonical bundle and Picard rank greater than one from *cracked polytopes*—polytopes whose intersection with a complete fan forms a set of unimodular polytopes—using Laurent inversion; a method developed jointly with Coates–Kasprzyk. We also give constructions of rank one Fano threefolds from cracked polytopes, following work of Christophersen–Ilten and Galkin. We explore the problem of classifying polytopes cracked along a given fan in three dimensions, and classify the unimodular polytopes which can occur as ‘pieces’ of a cracked polytope.

1. Introduction

We explain how to construct an extensible database of Fano manifolds in each dimension. In particular, we develop a combinatorial framework, based on the notion of *cracked polytopes* introduced in [44]. We show that this framework is flexible enough to obtain every Fano threefold with $-K_X$ very ample and $b_2 \geq 2$, famously classified by Mori–Mukai [34–38]. We show how one may extend these constructions to the rank one case – adapting work of Christophersen–Ilten [11, 12]—and to cases for which $-K_X$ is not very ample.

To implement our method we first fix a unimodular rational fan Σ of dimension n containing r rays. The *ray map* of Σ sends the i th element of the standard basis of \mathbb{Z}^r to the primitive generator of the i th ray. The transpose of this map is an embedding of lattices and, tensoring with \mathbb{C}^* , defines an embedding of affine spaces. The fan Σ also determines an embedded degeneration of $(\mathbb{C}^*)^n$ to a union of toric strata of \mathbb{C}^r . The co-ordinate ring of the central fibre of this degeneration is given by a *Stanley–Reisner ring* associated to the fan. Our prototypical example is the fan for \mathbb{P}^n , which determines the embedded degeneration $\{x_1 \cdots x_{n+1} = t\} \subset \mathbb{C}^{n+1}$ obtained as $t \rightarrow 0$. Given such a fan Σ , our general procedure consists of two steps.

- (i) Intersecting the fan Σ with a lattice polytope P , we describe how the embedding of $(\mathbb{C}^*)^n$ determined by Σ may be compactified to an embedding of the toric variety X_P in a non-singular toric variety Y . This is based on [44] and joint work [17] with Coates and Kasprzyk.

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- (ii) The embedding of affine spaces determined by Σ admits various possible deformations, and we explicitly construct embedded deformations in Y by homogenizing the co-ordinate rings of such families.

In this article the fans Σ we consider are simple enough that we can deform the corresponding embeddings explicitly. However, in Sect. 6 we outline a potentially sweeping generalisation using the work of Gross–Hacking–Keel [25] and Gross–Hacking–Siebert [26]. In particular, the authors construct mirror families to log Calabi–Yau varieties which deform the embeddings of the affine spaces and vertex varieties described above. In work in progress with Barrott and Kasprzyk we determine precisely when these families admit a fibrewise compactification in Y in the two-dimensional setting.

The connection between mirror log Calabi–Yau families and Fano threefolds is also currently being investigated by Corti–Hacking–Petracci in [18]. When fully established such work would guarantee the existence of a smooth Fano associated to each mirror *Minkowski polynomial*, see [1, 16]. In this context the current work forms a bridge between these (log) deformation theoretic constructions and the constructions of Mori–Mukai by providing explicit toric degenerations—embedded in a toric ambient space—from which one may deduce a birational description of general fibres.

The current work fits into another program of research, directed toward a novel approach to Fano classification. In [16] Coates–Corti–Galkin–Kasprzyk identify (a number of) *mirror Laurent polynomials* for each family of Fano threefolds. These constructions rely on the computation of the *quantum period* (part of the small J -function) of each Fano threefold, which in turn relies on the existence of good models of these Fano varieties; either as toric complete intersections, or via representation theoretic constructions. We make heavy use of these constructions, noting that these constructions are usually compatible with Laurent inversion. We note that the connection between toric degenerations and mirror symmetry is further explored by Ilten–Lewis–Przyjalkowski [29]. The Laurent polynomials discussed above are superpotentials for certain Landau–Ginzburg models. We refer to work of Clarke [13] for a duality construction for toric Landau–Ginzburg models which generalises that of Givental/Hori–Vafa [23, 28], and uses similar ideas to those appearing in the constructions we present below.

Fixing a complete (generalised) fan—which we refer to as the *shape*—we say a polytope is cracked along Σ if its intersection with each maximal cone of Σ is unimodular, see Definition 2.2. In [17] we show that embeddings of X_P into toric varieties, compactifying the embedding of affine varieties described above, are described by *scaffoldings*. Moreover, in [44] we show that embeddings of X_P into non-singular toric ambient spaces Y correspond to the combinatorial condition that the scaffolding is *full*, see Definition 2.7 and Theorem 2.8.

Theorem 1.1. *Every smooth Fano threefold with a very ample anti-canonical bundle and $b_2 \geq 2$ can be obtained by smoothing a Gorenstein toric Fano variety. In particular these can be constructed as deformations of toric embeddings provided by Laurent inversion, applied to a cracked polytope together with a full scaffolding S . Moreover, we may assume that the shape of the scaffolding S appears in Table 1.*

Table 1. The shape varieties used to construct Fano threefolds

Z	$\rho(Z)$	Z	$\rho(Z)$
pt	0	dP_7	3
\mathbb{P}^1	1	dP_6	4
\mathbb{P}^2	1	$Z_{10} = dP_7 \times \mathbb{P}^1$	4
\mathbb{P}^3	1	$dP_6 \times \mathbb{P}^1$	5
$\mathbb{P}^1 \times \mathbb{P}^1$	2	Z_{12}	5
$\mathbb{P}^2 \times \mathbb{P}^1$	2	$dP'_5 \times \mathbb{P}^1$	6
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	3		

We note that a related result on the existence of toric degenerations of Fano threefolds has recently appeared in work of Kasprzyk–Katzarkov–Przyjalkowski–Sakovics [31].

We recall that the ideal of X_P in the homogeneous co-ordinate ring of the toric ambient space Y is determined by the choice of shape Σ : for example, if $\text{TV}(\Sigma)$ is a product of projective spaces, a full scaffolding with this shape realises X_P as a toric complete intersection. Extending the list of shapes given in Table 1 to include the varieties Z_{2g-2} for $g \in \{2, 8, 9, 10, 12\}$ defined in Sect. 3, we obtain members of every family of Fano threefolds with very ample anti-canonical bundle from a cracked polytope and full scaffolding. We consider the Fano threefolds for which $-K_X$ is not very ample in Sect. 4.2.

We suggest that four-dimensional cracked polytopes form classes of polytopes from which it is natural to algorithmically construct Fano fourfolds. We note, by way of example, that each of the 738 families of Fano fourfolds which appear in [14] can be constructed from a polytope cracked along the fan determined by a product of projective spaces via a full scaffolding.

Conventions Throughout this article N will refer to an 3-dimensional lattice, and $M := \text{hom}(N, \mathbb{Z})$ will refer to the dual lattice. Given a ring R we write $N_R := N \otimes_{\mathbb{Z}} R$ and $M_R := M \otimes_{\mathbb{Z}} R$. For brevity we let $[k]$ denote the set $\{1, \dots, k\}$ for each $k \in \mathbb{Z}_{\geq 1}$. We work over the field \mathbb{C} of complex numbers throughout this article. Given a reflexive polytope $P \subset N_{\mathbb{R}}$, we assume throughout that X_P is the toric variety associated to the fan of cones over faces of P . Cracked polytopes will always be contained in $M_{\mathbb{R}}$; in particular if Q is a polytope cracked along a (generalised) fan Σ , Σ is a (generalised) fan in $M_{\mathbb{R}}$. Given a variety Y , and an identification $\text{Pic}(Y) \cong \mathbb{Z}^r$, we write $\mathcal{O}(a_1, \dots, a_r)$ for the line bundle of (multi) degree $a = (a_1, \dots, a_r) \in \mathbb{Z}^r$.

2. Cracked polytopes and Laurent inversion

The method *Laurent inversion*—introduced in [17]—was developed to construct models of Fano manifolds embedded in toric varieties. To describe this method we first fix a splitting $N = \bar{N} \oplus N_U$ of N . We fix a Fano polytope $P \subset N_{\mathbb{R}}$ and a smooth toric variety Z (the *shape*), such that \bar{N} is the character lattice of the dense

torus in Z . The central definition in the Laurent inversion construction is that of *scaffolding*. Loosely, a scaffolding is a collection of polytopes associated to nef divisors on Z whose convex hull is equal to P ; we define the notion of scaffolding precisely in Definition 2.3. From a scaffolding S we construct a polytope Q_S which projects to P° . The toric variety X_P embeds into the toric variety Y_S associated to the normal fan of Q_S . Moreover, the corresponding ideal in the homogeneous co-ordinate ring of Y_S is determined by Z . We then test explicit deformations of the equations cutting out X_P in Y_S to attempt to construct an embedded smoothing.

In this article we often work with *generalised fans* Σ , that is, fans whose cones are not necessarily strictly convex. In particular, we do not assume that the minimal cone of Σ is zero-dimensional.

Definition 2.1. We say that a (not necessarily strictly convex) rational polyhedral cone σ is *unimodular* if the quotient $\bar{\sigma}$ of σ by the maximal linear subspace contained in σ is a unimodular cone; that is, if the primitive ray generators of $\bar{\sigma}$ extend to an integral basis. We say that a generalised fan Σ is *unimodular* if all of its cones are unimodular.

For general choices of S , the variety Y_S may be highly singular: for example Y_S need not be \mathbb{Q} -Gorenstein. In [44] we explored the (restrictive) conditions on S which ensure that Y_S is non-singular, and introduced the following notion.

Definition 2.2. [44, Definition 2.1] Fix a convex polyhedron $P \subset M_{\mathbb{R}}$ containing the origin in its interior, and a unimodular generalised fan Σ . We say P is *cracked along* Σ if every tangent cone of $P \cap C$ is unimodular for every maximal cone C of Σ .

The shape Z is the toric variety associated to the quotient $\bar{\Sigma}$ of Σ by its minimal cone, and we will often find it convenient to say that P is cracked along the fan $\bar{\Sigma}$. It follows from [44, Proposition 2.5] that any cracked polytope is reflexive. In three dimensions the converse holds, in the sense that any reflexive polytope is cracked along *some* complete unimodular fan. Indeed, consider the fan Σ defined by taking the cone over every face of a maximal triangulation of the boundary of P ; the polytopes obtained by intersecting maximal cones of Σ with P are all standard simplices. Some examples of cracked polytopes are displayed in Fig. 1. The polytope shown in the left-hand image of Fig. 1 is cracked along the product of \mathbb{R}^2 with the fan determined by \mathbb{P}^1 , while the polytope shown in the right-hand image is cracked along the fan determined by $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Definition 2.3. [17, Definition 3.1] Fix a smooth projective toric variety Z with character lattice \bar{N} . A *scaffolding* of a polytope P is a set S of pairs (D, χ) —where D is a nef divisor on Z and χ is an element of N_U —such that

$$P = \text{conv} \left(P_D + \chi \mid (D, \chi) \in S \right).$$

We refer to Z as the *shape* of the scaffolding, and elements $(D, \chi) \in S$ as *struts*. We also assume that there is a unique $s = (D, \chi)$ such that $v \in P_D + \chi$ for every vertex $v \in \text{verts}(P)$.

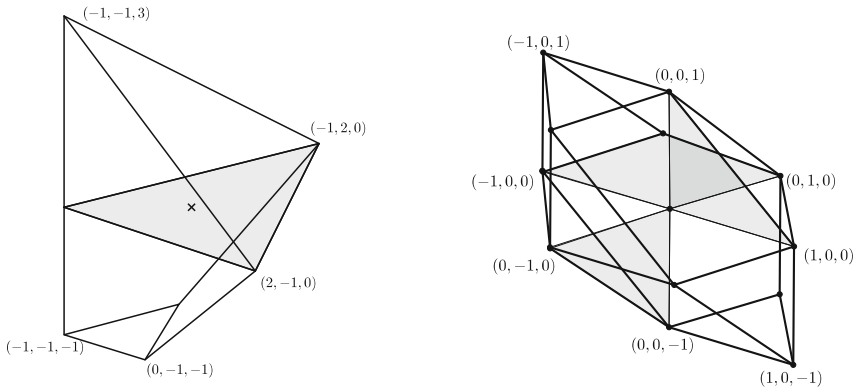


Fig. 1. Examples of cracked polytopes

As described in [17], scaffolding can be regarded as a generalisation of the notion of *nef partition*. These were introduced by Borisov in [6] and famously used to construct mirror partners of Calabi–Yau complete intersections by Batyrev–Borisov [5].

Scaffolding a polytope P determines an embedding of X_P into an ambient space Y_S . This is the main result of [17]; see also the treatment given in [44, §3]. We recall that given a toric variety X , which contains the complex torus T as a dense open set, the group of torus invariant divisors is denoted $\text{Div}_T(X)$. Moreover, if X determines a fan with l rays, $\text{Div}_T(X)$ is canonically isomorphic to \mathbb{Z}^l after ordering these rays.

Definition 2.4. [17, Definition A.1] Given a scaffolding S of P we define a toric variety Y_S , associated to the normal fan Σ_S of the polytope $Q_S \subset \tilde{M}_{\mathbb{R}} := (\text{Div}_{T_{\tilde{M}}} Z \oplus M_U) \otimes_{\mathbb{Z}} \mathbb{R}$, itself defined by the inequalities

$$\begin{cases} \langle (-D, \chi), - \rangle \geq -1 & \text{for all } (D, \chi) \in S \\ \langle (0, e_i), - \rangle \geq 0 & \text{for } i \in [\ell], \end{cases}$$

where e_i denotes the standard basis of $\text{Div}_{T_{\tilde{M}}} Z \cong \mathbb{Z}^{\ell}$.

We let ρ denote the ray map of the fan $\tilde{\Sigma}$ determined by Z , and set $\rho_s := (-D, \chi)$ for each $s = (D, \chi) \in S$. We also define a map of lattices,

$$\begin{array}{ccc} \theta := \rho^* \oplus \text{Id}: \tilde{N} \oplus N_U & \longrightarrow & \text{Div}_{T_{\tilde{M}}}(Z) \oplus N_U \\ \parallel & & \parallel \\ N & & \tilde{N}. \end{array}$$

The map $\rho^*: \tilde{N} \rightarrow \text{Div}_{T_{\tilde{M}}}(Z)$ is the *character-to-divisor* map for Z , and we recall that this map also plays a key role in Clarke’s mirror constructions [13].

Theorem 2.5. [17, Theorem 5.5] *A scaffolding S of a polytope P determines a toric variety Y_S and an embedding $X_P \rightarrow Y_S$. This map is induced by the map θ on the corresponding lattices of one-parameter subgroups.*

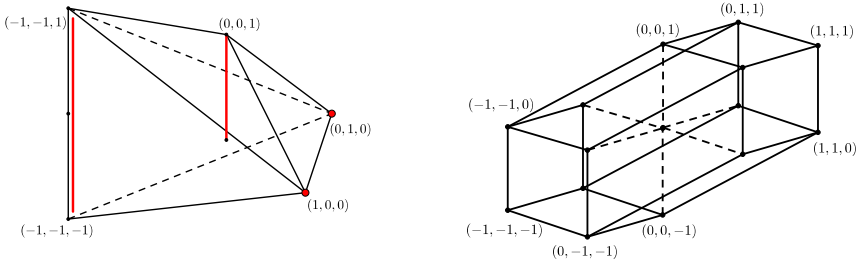


Fig. 2. Scaffolding the polytopes dual to those in Fig. 1

Remark 2.6. We can provide an explicit generating set for the ideal of X_P in the homogeneous co-ordinate ring of Y_S using the map θ . In particular, a hyperplane containing the image of θ defines a function h on the set of ray generators of Σ_S . X_P then satisfies the equation

$$\prod_{\{v:h(v)\geq 0\}} z_v^{h(v)} - \prod_{\{v:h(v)< 0\}} z_v^{h(v)} = 0,$$

where products are taken over the ray generators of Σ_S , and z_v is the homogeneous co-ordinate on Y_S corresponding to the ray generated by v .

Recall that each facet F of P° is dual to a vertex F^* of P , contained in a cone σ of Σ . Taking σ is minimal among such cones, σ corresponds to a non-singular toric stratum $Z(\sigma)$ of the toric variety $\text{TV}(\bar{\Sigma})$. It is shown in [44, Proposition 2.8] that the facet F of P° is a *Cayley sum* $P_{D_1} \star \cdots \star P_{D_k}$, where $\{D_i : 1 \leq i \leq k\}$ is a set of nef divisors on $Z(\sigma)$, and $k = \dim(\bar{\sigma}) + 1$. We call a face of P° *vertical* if it is contained in a factor P_{D_i} of some facet $F = P_{D_1} \star \cdots \star P_{D_k}$ and some $i \in \{1, \dots, k\}$, see [44, Definition 2.10].

Definition 2.7. [44, Definition 4.1] Given a Fano polytope $P \subset N_{\mathbb{R}}$ cracked along a generalised fan Σ in $M_{\mathbb{R}}$ we say a scaffolding S of P with shape $Z := \text{TV}(\bar{\Sigma})$ is *full* if every vertical face of P is contained in a polytope $P_D + \chi$ for a unique element $(D, \chi) \in S$.

We show in [44] that full scaffoldings on cracked polytopes give rise to embeddings $X_P \rightarrow Y_S$ where Y_S is smooth in a neighbourhood of X_P . Full scaffoldings of the polytopes dual to those shown in Fig. 1 are illustrated in Fig. 2. The scaffolding shown in the left hand image in Fig. 2 consists of a pair of line segments and a pair of points, while the scaffolding shown in right hand image consists of a pair of cubes which intersect at the origin.

Theorem 2.8. [44, Theorem 1.1] *Fix a polytope $P \subset M_{\mathbb{R}}$, and a rational generalised fan Σ in $M_{\mathbb{R}}$ such that the toric variety $Z := \text{TV}(\bar{\Sigma})$ is smooth and projective. Given a scaffolding S of P with shape Z , we have that the target of the corresponding embedding is smooth in a neighbourhood of the image of X_P if and only if P is cracked along Σ and S is full.*

2.1. Torus quotients

Every n -dimensional toric variety X (over \mathbb{C}) may be described as the quotient of a Zariski open set of affine space \mathbb{C}^{n+r} by a complex torus $\mathbb{T} := (\mathbb{C}^*)^r$. Recalling that, if X is determined by a fan in N whose rays generators ν_1, \dots, ν_{n+r} form a spanning set of N , we have an exact sequence

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^{n+r} \xrightarrow{\nu} N \longrightarrow 0$$

where $\nu: e_i \rightarrow \nu_i$ for each $i \in \{1, \dots, n+r\}$. The character lattice \mathbb{L}^* of \mathbb{T} fits into the dual sequence,

$$0 \longrightarrow M \longrightarrow (\mathbb{Z}^{n+r})^* \xrightarrow{R} \mathbb{L}^* \longrightarrow 0.$$

Moreover we recall that if X is smooth there is a canonical identification $\mathbb{L}^* \cong \text{Pic}(X)$, while if X is \mathbb{Q} -factorial there is a canonical identification of $\mathbb{L}^*_\mathbb{R} := \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{R} \cong \text{Pic}(X)_\mathbb{R}$. The map $R: (\mathbb{Z}^{n+r})^* \rightarrow \mathbb{L}^*$ is called the *weight data* for the toric variety. Recall that the possible fans in N , with rays generated by a subset of $\{\nu_1, \dots, \nu_{n+r}\}$, and such that the associated toric variety is projective, are indexed by the cones of a fan contained in the effective cone $\text{Eff}(X) \subset \text{Pic}(X)_\mathbb{R}$. This fan is called the *secondary fan* or *GKZ decomposition*.

Fixing a maximal cone (or *chamber*) σ in the secondary fan, the corresponding toric variety can be described as the torus quotient

$$X_\sigma = (\mathbb{C}^{n+r} \setminus Z(\sigma)) / \mathbb{T},$$

where $\mathbb{T} := (\mathbb{L}^* \otimes_{\mathbb{Z}} \mathbb{C}^*)$, the weights of the torus action are given by R , and the $Z(\sigma)$ is the *irrelevant locus*. Choosing a point (or *stability condition*) ω in the interior of σ , the irrelevant locus is defined by setting

$$Z(\sigma) := V(x_{i_1} \cdots x_{i_r} : \omega \in \langle R_{i_1}, \dots, R_{i_r} \rangle),$$

where $R_i = R(e_i)$ for each standard basis vector $e_i, i \in [n+r]$. Some of the constructions described in Sect. 4 make use of stability conditions contained in a codimension one cone (or *wall*) in the secondary fan.

Under an additional condition, we can use the GIT presentation of a toric variety to streamline the construction of the variety Y_S from a scaffolding S .

Assumption 2.9. There is a basis $B = \{b_i \in N_U : i \in [\dim N_U]\}$ such that

$$\{(0, b) : b \in B\} \subseteq S.$$

We assume for the remainder of this section that every scaffolding satisfies Assumption 2.9. With this condition, the cone generated by

$$B \cup \{e_i : i \in [\dim \text{Div}_{T_M}(Z)]\},$$

where the vectors e_i form the standard basis in the based lattice $\text{Div}_{T_M}(Z)$, defines a smooth torus invariant point in Y_S . We next explain how to form a weight matrix and stability condition which determine the variety Y_S directly from the scaffolding S . This construction follows [17, Algorithm 5.1].

Construction 2.10. Given a scaffolding S with shape Z of a polytope P , index the elements of S by $[s]$, and let (D_i, χ_i) denote the i^{th} element of S . It follows from our assumptions on S that the ray matrix of Σ_S is in echelon form

$$\left(\begin{array}{c|ccc} I_n & -D_1 & \cdots & -D_r \\ \hline & \chi_1 & \cdots & \chi_r \end{array} \right),$$

where $[s] \setminus [r]$ indexes the elements $(D_i, \chi_i) \in S$ of the form $(0, b_i)$, for a basis $\{b_i : i \in [\dim N_U]\}$, and $n = \dim \tilde{N}$. Thus R , the transpose of the kernel matrix, is given by

$$R = \left(\begin{array}{c|cc} & -\chi_1 & D_1 \\ \hline I_r & \vdots & \vdots \\ & -\chi_r & D_r \end{array} \right).$$

The variety Y_S is defined using the a polarising torus invariant divisor given by the sum of all rays corresponding to elements of S . The (multi) degree of this divisor is given by the sum of the first s columns of R . That is, the stability condition used to define Y_S is given by the sum of $(1, \dots, 1)^T$ with the columns of the matrix $(\chi_1, \dots, \chi_r)^T$

If Z is a product of c projective spaces, there is a partition of the columns of R containing the vectors $D_i \in \text{Div}_{T_M}(Z)$. In particular, the standard basis in $\text{Div}_{T_M}(Z)$ partitions into c sets C_1, \dots, C_c , such that C_i consists of divisors pulled back from the standard projection to the i^{th} projective space factor. For each $i \in [c]$ the degree of the line bundle L_i cutting out X_P in Y_S is given by the sum of the columns in C_i . In particular, there is a distinguished binomial $z^{m_1} - z^{m_2}$ in L_i , where m_1 is the sum of standard basis vectors in $(\mathbb{Z}^{n+r})^*$ corresponding to the columns of C_i , and m_2 is the unique lift of $L_i \in \mathbb{L}^*$ to $(\mathbb{Z}^r)^*$: the subspace of $(\mathbb{Z}^{n+r})^*$ corresponding to the first r columns of R . It is shown in [17], see also [44, §3], that X_P is the vanishing locus of these c binomials.

Example 2.11. Fix a 3-dimensional reflexive polytope P , and let Z be a crepant resolution of the toric variety determined by the normal fan of P . In particular, $\tilde{N} = N$ and $N_U = \{0\}$. Let $S := \{(D, 0)\}$, where $D \in |-K_Z|$ is the toric boundary of Z . Hence $P = P_D$, and the corresponding $1 \times n$ weight matrix R is equal to $(1 \ 1 \ \cdots \ 1)$, where $n = 1 + \dim \text{Div}_{T_M}(Z)$ columns. The stability condition is equal to $1 \in \mathbb{L}^* \cong \mathbb{Z}$, and hence $Y_S \cong \mathbb{P}^{n-1}$. This is nothing but the anti-canonical embedding of X_P into projective space.

Example 2.12. In [17, Example 3.5] we consider two distinct scaffoldings for the polygon P associated with the toric del Pezzo surface of degree six. One of these is illustrated in Fig. 3. The scaffolding illustrated in Fig. 3 has shape $Z = \mathbb{P}^1 \times \mathbb{P}^1$ and—letting $D_{i,a}$ denote the pullback of $\{a\} \subset \mathbb{P}^1$ along the i^{th} projection for each $i \in \{1, 2\}$ and $a \in [0, \infty]$ —we define

$$S := \{\{D_{1,0} + D_{2,0}\}, \{D_{1,\infty} + D_{2,\infty}\}\}.$$

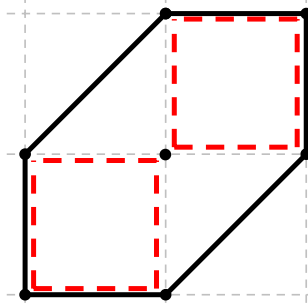


Fig. 3. The scaffolding corresponding to the embedding of dP_6 in $\mathbb{P}^2 \times \mathbb{P}^2$

Applying Construction 2.10 to S we obtain the weight matrix

$$\begin{pmatrix} & 1 & 0 & 1 & 0 \\ I_2 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

and stability condition $\omega = (1, 1)$. This is a GIT presentation of the toric variety $\mathbb{P}^2 \times \mathbb{P}^2$. The variety $X_P \cong dP_6$ is the vanishing locus of the binomials $x_1 y_1 = x_0 y_0$ and $x_2 y_2 = x_0 y_0$, where x_i and y_i denote homogeneous co-ordinates on the \mathbb{P}^2 factors.

3. Rank one Fano threefolds

Toric degenerations of rank one Fano manifolds have been obtained by Ilten and Christophersen [11, 12], using the deformation theory of Stanley–Reisner rings developed by Altmann–Christophersen [2, 3]. Using these results—and the work of Galkin [22] on small toric degenerations—we obtain cracked polytopes P corresponding to each of the 15 rank one Fano threefolds X with very ample anti-canonical bundle. In particular, we describe degenerations of these 15 Fano threefolds X to the toric varieties X_P . We remark that, since the toric degenerations in this case occur in the anti-canonical embedding, the use of cracked polytopes in this context is rather trivial; see Example 2.11.

Remark 3.1. We refer to the enumeration of three dimensional reflexive polytopes used in this article as ‘PALP ID’ in acknowledgement of the original work of Kreuzer–Skarke [33]. We note however that, as well as its implementation in PALP (‘a Package for Analyzing Lattice Polytopes’), this database has been implemented in SageMath [46] and Magma [7], and that polytopes with a given ID may be obtained from the Graded Ring Database [9].

To specify the toric varieties Z_{2n} for $n \in \{6, 7, 8, 9, 11\}$ which appear in Table 2, we set $Z_{10} := dP_7 \times \mathbb{P}^1$, and let $\ell_1^a, \dots, \ell_5^a$ denote the torus invariant divisors of $dP_7 \times \{a\} \subset Z_{10}$ for each $a \in \{0, \infty\}$.

- Z_{12} is the blow up of $Z_{10} := dP_7 \times \mathbb{P}^1$ in a toric invariant line $\ell_1^0 \subset Z_{10}$.

Table 2. Rank one Fano threefolds

Fano	PALP ID	Equations	Shape	Fano	PALP ID	Equations	Shape
\mathbb{P}^3	1	—	pt	V_8	4251	$\begin{cases} x_1x_2 - x_0^2 \\ x_3x_4 - x_0^2 \\ x_5x_6 - x_0^2 \end{cases}$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
Q^3	2	$x_1x_2 - x_0^2$	\mathbb{P}^1	V_{10}	4143	Section 3.2	$dP_7 \times \mathbb{P}^1$
B_5	246	Section 3.1	dP_7	V_{12}	3869	Section 3.2	Z_{12}
B_4	434	$\begin{cases} x_1x_2 - x_0^2 \\ x_3x_4 - x_0^2 \end{cases}$	$\mathbb{P}^1 \times \mathbb{P}^1$	V_{14}	3298	Section 3.2	Z_{14}
B_3	742	$x_1x_2x_3 - x_0^3$	\mathbb{P}^2	V_{16}	3034	Section 3.2	Z_{16}
B_2	428	Section 3.3	Z_2	V_{18}	2703	Section 3.2	Z_{18}
V_4	4312	$x_1x_2x_3x_4 - x_0^3$	\mathbb{P}^3	V_{22}	1943	Section 3.2	Z_{22}
V_6	4287	$\begin{cases} x_1x_2 - x_0^2 \\ x_3x_4x_5 - x_0^2 \end{cases}$	$\mathbb{P}^1 \times \mathbb{P}^2$.	

- Z_{14} is the blow up of Z_{12} in the strict transform (and pre-image) of $\ell_2^\infty \subset Z_{10}$.
- Z_{16} is the blow up of Z_{14} in the strict transform of the line $\ell_5^0 \subset Z_{10}$.
- Z_{18} is the blow up of Z_{16} in the strict transform of the line $\ell_3^\infty \subset Z_{10}$.

The fans determined these varieties define triangulations of the sphere via radial projection. The sequence of blow up maps described induces the *starring* operations on these triangulations described in [12]. We define the variety Z_{22} to be a crepant resolution of the toric variety determined by the normal fan of the reflexive polytope with ID 1942. Similarly, we define the variety Z_2 to be a crepant resolution of the toric variety determined by the normal fan of the (self-dual) reflexive polytope with ID 428.

The Fano variety \mathbb{P}^3 is toric, while Q^3 , B_3 , B_4 , V_4 , V_6 , and V_8 are well known to be toric complete intersections. These admit toric degenerations to the varieties defined by the equations given in Table 2. The variety B_2 is also a toric complete intersection (indeed, a hypersurface in $\mathbb{P}(1, 1, 1, 1, 2)$), but since this weighted projective space is not Gorenstein we treat this case separately in Sect. 3.3. To describe the scaffolding associated to each of these Fano threefolds, let d be the dimension of the shape variety Z , set $\tilde{N} := \mathbb{Z}^d$ and $N_U := \mathbb{Z}^{3-d}$. Letting $\{e_1, \dots, e_{3-d}\}$ denote the standard basis of N_U , we define

$$S := \{(0, e_1), \dots, (0, e_{3-d}), (D, \chi)\},$$

where $D \in |-K_Z|$ is the toric boundary of Z , and $\chi = (-1, \dots, -1) \in N_U$. This scaffolding is illustrated in the case B_3 in Fig. 25 (setting $a = 1$ and $b = 3$).

3.1. Pfaffian equations and B_5

The Fano threefold B_5 is a linear section of the Grassmannian $\text{Gr}(2, 5)$. We make heavy use of the fact that the ideal of the image of the Plücker embedding

$$\text{Gr}(2, n) \hookrightarrow \mathbb{P}^{\binom{n}{2}-1}$$

is generated by 4×4 Pfaffians of a skew-symmetric $n \times n$ matrix; entries of which are the Plücker co-ordinates of $\text{Gr}(2, n)$. Hyperplane sections can then be obtained by replacing entries with linear combinations of a subset of the Plücker co-ordinates. For example, B_5 can be described as the Pfaffians of the matrix

$$\begin{pmatrix} 0 & x_0 & x_1 & x_2 & x_0 \\ & 0 & tx_0 & x_3 & x_4 \\ & & 0 & x_0 & x_5 \\ & & & 0 & tx_0 \\ & & & & 0 \end{pmatrix}$$

for a fixed value of $t \neq 0$. Note that this since matrix is skew-symmetric we omit lower diagonal entries. Varying t defines a flat family, the central fibre of which is the projective cone over a toric variety with two ordinary double points, obtained from dP_5 by moving the four points at which \mathbb{P}^2 is blown up to two pairs of infinitely close points, and contracting the pair of resulting -2 curves in the central fibre. Setting $t = 0$ recovers five equations generating the ideal of a toric variety in \mathbb{P}^5 . This toric variety is isomorphic to X_P , where P denotes the toric variety with ID 742. The embedding $X_P \rightarrow \mathbb{P}^5$ is the embedding of X_P determined by the scaffolding $S = \{(0, 1), (D, 0)\}$, where $1 \in N_U \cong \mathbb{Z}$ and $D \in -K_Z$ (recalling that $Z = dP_7$) is the toric boundary of Z .

3.2. Higher genus Fano threefolds

The varieties V_{2n-2} for $n \in \{6, 7, 8, 9, 10, 12\}$ are linear sections of the *Mukai varieties* M_n [39]. Toric degenerations of these are related—by work of Ilten–Christoffersen [12]—to the *convex deltahedra* in the cases $n < 12$, while varieties in the family V_{22} admit a toric degeneration to a variety with ordinary double point singularities, see [22].

Given a Fano toric variety Z , let its *dual* Z^* be toric variety associated to the normal fan of the convex hull of the ray generators of the fan determined by Z .

Proposition 3.2. *The toric varieties V_{2n-2} admit toric degenerations to the Fano toric varieties Z_{2n-2}^* dual to Z_{2n-2} for each $n \in \{6, 7, 8, 9, 10, 12\}$.*

Proof. If $n < 12$ we recover the triangulations T_n of S^2 used in [12] to construct degenerations of Fano threefolds by removing the origin from $N_{\mathbb{R}} \cong \mathbb{R}^3$ and radially projecting the fan Σ_n determined by Z_{2n-2} . The result then follows immediately from [12, Proposition 2.3]. In the case $n = 12$ we observe that Z_{22}^* contains only ordinary double point singularities, and hence admits a smoothing. It is shown in [22] that the general fibre of this smoothing is a member of the family V_{22} . \square

In the cases $n \in \{6, 7, 8\}$ we can provide an explicit description of the toric degeneration.

- (i) V_{10} : varieties in this family can be described by the Pfaffians of a 5×5 skew-symmetric matrix, and one quadric equation. We can form a toric degeneration following Sect. 3.1.

- (ii) V_{12} : varieties in this family can be described via a system of 9 Pfaffian equations, and we refer to the treatment of 2–21 in Sect. 4 for a description of a toric degeneration using the same shape variety.
- (iii) V_{14} : varieties in this family can be described as the vanishing of the 4×4 Pfaffians of a 6×6 skew matrix. An explicit toric degeneration is given by the 4×4 Pfaffians of the matrix (1) below.

The vanishing 4×4 Pfaffians of the matrix

$$\begin{pmatrix} 0 & -x_1 & x_2 & tf_1 & x_3 & x_4 \\ & 0 & tg_1 & x_5 & x_1 & x_6 \\ & & 0 & x_7 & x_2 & x_0 \\ & & & 0 & x_0 & x_8 \\ & & & & 0 & th_1 \\ & & & & & 0 \end{pmatrix} \tag{1}$$

define a toric degeneration of V_{14} , a general linear section of $\text{Gr}(2, 6)$, where x_i are homogeneous co-ordinates on \mathbb{P}^9 and $f_1, g_1,$ and h_1 are general linear forms on \mathbb{P}^9 . The scaffolding S in each case is equal to the singleton set $\{(D, 0)\}$, where D is the toric boundary of Z .

3.3. The quartic hypersurface in $\mathbb{P}(1, 1, 1, 1, 2)$

Recall that the toric variety Y_S defined by a full scaffolding of a cracked polytope P is non-singular in a neighbourhood of the image of P . This excludes certain constructions of Fano manifolds as hypersurfaces as weighted projective spaces. In particular, consider the scaffolding (with shape \mathbb{P}^2) of the polytope P with ID 3313 illustrated in Fig. 25, setting $(a, b) = (1, 4)$. We have that $\tilde{N} \cong \mathbb{Z}^2, N_U \cong \mathbb{Z}$, and $S = \{(0, 1), (D_0 + D_1 + 2D_2, -1)\}$; where $D_i := \{x_i = 0\} \subset \mathbb{P}^2$. Computing the corresponding weight matrix we find

$$R = (I_r | \chi | D) = (1 | 1 | 1 | 1 \quad 2).$$

Thus X_P is the vanishing locus of a section of $\mathcal{O}(4)$ in $\mathbb{P}(1^4, 2) := \mathbb{P}(1, 1, 1, 1, 2)$. Notice that P° is *not* cracked along the fan of \mathbb{P}^2 . To obtain a construction from a *cracked* polytope we first embed $\mathbb{P}(1^4, 2)$ into \mathbb{P}^{10} via the linear system defined by sections of $\mathcal{O}(2)$. Sections of $\mathcal{O}(2)$ define the integral points of a polytope in \mathbb{Z}^4 given by the convex hull of the points given by the columns of the matrix

$$\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The quartic equation $x_0x_1x_2x_3 = y^2$ defines a projection of this polytope to the reflexive polytope P with ID 428. This polytope is self-dual, and we take the scaffolding of P with shape Z given by a crepant resolution of X_P , covering P with a single strut. This scaffolding corresponds to the anti-canonical embedding

of X_P into \mathbb{P}^{10} , which is the intersection of the image of the Veronese embedding of $\mathbb{P}(1^4, 2)$ with a (binomial) quadric. Deforming this quadric deforms X_P to a general quartic hypersurface in $\mathbb{P}(1^4, 2)$.

4. Constructions of Fano manifolds

There are 98 Fano threefolds with very ample anti-canonical bundle. Indeed, Fano threefolds which do not have very ample anti-canonical bundle are classified in [30]. These fall into five *hyperelliptic* cases, and two examples for which the anti-canonical system is not free. In the previous section we described constructions from cracked polytopes of the 15 of these which have Picard rank one. We now explain constructions in the remaining 83 cases. In particular, for each of these 83 Fano threefolds X , we exhibit a generalised fan Σ and polytope P cracked along Σ such that—for some full scaffolding of S with shape $Z := \text{TV}(\bar{\Sigma})$ —the toric variety X_P admits an embedded smoothing in Y_S to X .

Examples from ‘Quantum periods for 3-dimensional Fano manifolds’ Explicit constructions of Fano threefolds are provided in [16]. The authors use these constructions to compute (part of) the J -function of each Fano threefold using either the Quantum Lefschetz principle, or the Abelian-non Abelian correspondence. In particular, each Fano threefold X is exhibited either as a complete intersection in a weak Fano toric variety, or as the degeneracy locus of a map of homogeneous vector bundles.

Proposition 4.1. *Fix a Fano threefold X , such that the model of X in [16] describes X as the vanishing locus of a section of a split vector bundle $\Lambda = L_1 \oplus \dots \oplus L_c$ on a toric variety Y . In addition, we insist that the divisor*

$$-K_Y - L_1 - \dots - L_c$$

is ample. There is a reflexive polytope P , shape variety Z , and full scaffolding S of P such that $Y_S \cong Y$, and X_P admits an embedded smoothing to X in Y .

We note that the format described above covers many, though not all, the constructions of Fano threefolds which appear in [16]. The remaining examples either require the use of spaces with actions of non-abelian Lie groups, or require weakening the condition that $-K_Y - L_1 - \dots - L_c$ is ample.

Proof. Tables 3, 4 and 5 list binomial equations cutting out toric varieties to which Fano varieties in the various families satisfying our hypotheses degenerate. The leading monomial in each case is square-free and defines a subset of the columns C_i of the weight matrix listed in [16] for each $i \in \{1, \dots, c\}$. In every case the sets C_i are pairwise disjoint, and disjoint from a subset C of columns which define a basis of $\text{Pic}(Y)$. Reversing Construction 2.10, we can obtain a scaffolding from the weight matrices given in [16] and the binomial expressions listed in Tables 3, 4 and 5. The rank one complete intersection cases are listed in Table 2.

It follows from [44, Theorem 1.1], and smoothness of Y_S , that the polytope P° is cracked along the fan determined by Z , and S is full. □

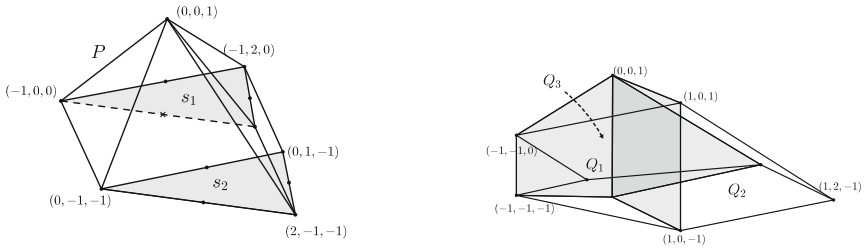


Fig. 4. Constructing 2–18 via Laurent inversion

Example 4.2. Consider a Fano threefold X in the family 2–18. X is a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$, branched in a divisor with bidegree $(2, 2)$. The construction in [16] describes this Fano threefold as a hypersurface in the projectivisation of a rank 3 split vector bundle on \mathbb{P}^2 .

Consider the scaffolding S with shape $Z = \mathbb{P}^2$ illustrated in the left hand image in Fig. 4. That is, $\bar{N} \cong \mathbb{Z}^2$, $N_U \cong \mathbb{Z}$, and $S = \{(D_1 + D_2, 0), (D_0 + D_2, -1)\}$, where $D_i = \{x_i = 0\}$ for homogeneous co-ordinates $(x_0 : x_1 : x_2)$ on \mathbb{P}^2 . This scaffolding exhibits X_P as the hypersurface given by the vanishing locus of the binomial $zy_2x_3 - y_1^2x_1^2$, in the toric variety with weight matrix

$$\begin{array}{cccccc} y_1 & x_1 & x_2 & x_3 & y_2 & z \\ \hline 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}$$

such that the class $(2, 1)$ is ample. Note that the weight matrix—up to a permutation of the columns—and stability condition $\omega = (2, 1)$ are identical to those appearing in [16, p. 40]. Thus the general member of the linear system $\mathcal{O}(2, 2)$ is a Fano threefold in the family 2–18.

Of the 83 Fano threefolds with very ample anti-canonical divisor and $b_2 \geq 2$, 68 of the constructions given in [16] coincide with constructions from full scaffoldings on cracked polytopes. Of these 67 constructions are summarised in Tables 3, 4 and 5. The construction given in [16, p. 58] expresses varieties in the remaining family, 3–2, as a hypersurface in a toric variety F which cannot be obtained using a full scaffolding of a cracked polytope. However, in the remarks on the construction given in [16, p. 59], the authors describe a second construction using a toric variety G . This toric variety does coincide with a toric ambient space obtained from a full scaffolding of a cracked polytope.

The column *Equations* in each table describes a generating set for the ideal in the homogeneous co-ordinate ring of the ambient variety Y described in [16]. The first monomial of each binomial is always square-free, and may be used to identify columns of the weight matrix defined by Y . If Y is a product of projective spaces the co-ordinates are not named in [16], and we name these x_0, \dots, x_m for the first projective space factor \mathbb{P}^m , y_0, \dots, y_n for the second, etc.

We now provide constructions from cracked polytopes of the 15 Fano threefolds whose construction in [16] is not directly related to a full scaffolding of a cracked

Table 3. Scaffolding constructions for Picard rank 2 Fano threefolds

Fano	Equations	Shape	Fano	Equations	Shape
2-4	$x_1y_1y_2y_3 - x_0y_0^3$	\mathbb{P}^3	2-23	$\begin{cases} xs_3s_4 - s_0x_5 \\ s_1s_2 - s_0^2 \end{cases}$	$\mathbb{P}^2 \times \mathbb{P}^1$
2-5	$s_1xx_3x_4 - x_2^3$	\mathbb{P}^3	2-24	$x_1y_1y_2 - x_0y_0^2$	\mathbb{P}^2
2-6	$x_1x_2y_1y_2 - x_0^2y_0^2$	\mathbb{P}^3	2-25	$x_1y_1y_2 - x_0y_0^2$	\mathbb{P}^2
2-7	$\begin{cases} x_1y_1y_2 - x_0y_0^2 \\ y_3y_4 - y_0^2 \end{cases}$	$\mathbb{P}^1 \times \mathbb{P}^2$	2-27	$x_iy_i - x_0y_0, i \in [2]$	$\mathbb{P}^1 \times \mathbb{P}^1$
2-9	$\begin{cases} x_1y_1 - x_0y_0 \\ x_2x_3y_2 - x_0^2y_0 \end{cases}$	$\mathbb{P}^1 \times \mathbb{P}^2$	2-28	$s_1s_2s_3x - s_0y$	\mathbb{P}^3
2-10	$\begin{cases} x_4x_5 - x_2^2 \\ xx_3s_1 - x_2^2 \end{cases}$	$\mathbb{P}^1 \times \mathbb{P}^2$	2-29	$x_3x_4 - x_2^2$	\mathbb{P}^1
2-11	$s_0s_1xx_4 - s_2x_3^2$	\mathbb{P}^3	2-30	$s_1s_2x - s_0x_4$	\mathbb{P}^2
2-12	$x_iy_i - x_0y_0, i \in [3]$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	2-31	$s_2x_4 - s_0x_3$	\mathbb{P}^1
2-13	$\begin{cases} x_1y_1 - x_0y_0 \\ x_2y_2 - x_0y_0 \\ y_3y_4 - y_0^2 \end{cases}$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	2-32	$x_1y_1 - x_0y_0$	\mathbb{P}^1
2-15	$s_0s_1s_2x - s_3^2x_4$	\mathbb{P}^3	2-33	—	pt
2-16	$\begin{cases} s_0s_1x - s_2x_3 \\ x_4x_5 - x_3^2 \end{cases}$	$\mathbb{P}^2 \times \mathbb{P}^1$	2-34	—	pt
2-18	$x_1y_1w - x_0^2y_0^2$	\mathbb{P}^2	2-35	—	pt
2-19	$\begin{cases} s_1x_5 - s_0x_4 \\ xs_2s_3 - s_0x_4 \end{cases}$	$\mathbb{P}^2 \times \mathbb{P}^1$	2-36	—	pt

Table 4. Scaffolding constructions for Picard rank 3 Fano threefolds

Fano	Equations	Shape	Fano	Equations	Shape
3-3	$x_1y_1z_1z_2 - x_0y_0z_0^2$	\mathbb{P}^3	3-19	$s_1xx_4 - x_3^2$	\mathbb{P}^2
3-6	$x_2^2y_0 - s_1xx_3y_1$	\mathbb{P}^3	3-20	$s_1t_3 - s_0t_2$	\mathbb{P}^1
3-7	$\begin{cases} x_1y_1z_1 - x_0y_0z_0 \\ y_2z_2 - y_0z_0 \end{cases}$	$\mathbb{P}^2 \times \mathbb{P}^1$	3-21	$y_1s - tx_0y_2^2$	\mathbb{P}^1
3-8	$s_1xy_1y_2 - x_2y_0^2$	\mathbb{P}^3	3-22	$x_1s - ty_0^2$	\mathbb{P}^1
3-9	$y_1xs_1s_2 - y_0^2$	\mathbb{P}^3	3-23	$s_2v - s_1x_0u$	\mathbb{P}^1
3-10	$s_1t_3xy - x_4^2$	\mathbb{P}^3	3-24	$x_2y_1 - s_0xy_0$	\mathbb{P}^1
3-11	$s_1s_2xy_1 - s_0x_3y_0$	\mathbb{P}^3	3-25	—	pt
3-12	$\begin{cases} s_3xy_1 - x_1y_0 \\ x_2y_2 - x_1y_0 \end{cases}$	$\mathbb{P}^2 \times \mathbb{P}^1$	3-26	—	pt
3-13	$\begin{cases} x_1y_1 - x_0y_0 \\ x_2z_1 - x_0z_0 \\ y_2z_2 - y_0z_0 \end{cases}$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	3-27	—	pt
3-15	$s_1s_2y - s_0t_3z$	\mathbb{P}^2	3-28	—	pt
3-17	$x_1y_1z_1 - x_0y_0z_0$	\mathbb{P}^2	3-29	—	pt
3-18	$s_1xx_3y_0 - s_0x_2y_1$	\mathbb{P}^3	3-30	—	pt
			3-31	—	pt

Table 5. Scaffolding constructions for Picard rank 4 Fano threefolds

Fano	Equations	Shape	Fano	Equations	Shape
4-1	$x_1y_1z_1w_1 - x_0y_0z_0w_0$	\mathbb{P}^3	4-7	$\begin{cases} y_1u_1 - x_0u_0 \\ z_1u_2 - x_0u_0 \end{cases}$	$\mathbb{P}^1 \times \mathbb{P}^1$
4-3	$y_0y_1 - s_0^2t_0^2x^2$	\mathbb{P}^1	4-8	$x_2y_2 - s_0xt_0y$	\mathbb{P}^1
4-4	$x_1y_1v - x_0y_0z_0u$	\mathbb{P}^2	4-9	$z_1u - x_0y_0v$	\mathbb{P}^1
4-5	$x_3y_4 - x_2^2y^2$	\mathbb{P}^1	4-10 to 4-13	–	pt

polytope. In six cases (2-14, 2-17, 2-20, 2-21, 2-22, 2-26) the corresponding construction in [16] does not describe the Fano threefold as a toric complete intersection. In the remaining nine cases (2-8, 3-1, 3-4, 3-5, 3-14, 3-16, 4-2, 4-6, 5-1) the construction given in [16] expresses the Fano threefold X as the vanishing locus of a section of split vector bundle $\Lambda = L_1 \oplus \dots \oplus L_c$ on a toric variety Y , such that $L := -K_Y - \sum_i L_i$ is nef but not ample. In the latter case the embedding cannot come from a scaffolding S , since Construction 2.10 uses L to polarise the ambient space.

Remark 4.3. Note that the numbering for the rank 4 Fano threefolds replicates that in [16], which differs from the original list of Mori–Mukai by the insertion of the family 4-2 which was omitted from the original classification (some lists instead append this family as 4-13).

Rank 2, number 8 Varieties in the family 2-8 are either,

- (i) the double cover of B_7 (the blow up of \mathbb{P}^3 at a point) with branch locus a member B of $|-K_{B_7}|$ such that $B \cap D$ is non-singular, where D is the exceptional divisor of the blow up $B_7 \rightarrow \mathbb{P}^3$, or;
- (ii) the specialisation of (i) where $B \cap D$ is reduced but singular.

We make use of the construction given in [16], which embeds Fano threefolds in the family 2-8 as hypersurfaces of bi-degree (2, 4) in the toric variety Y , defined by the weight matrix

$$\begin{array}{cccccc} y & x_0 & z & x_1 & x_2 & x_3 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 1 \end{array}$$

and a choice of stability condition in the chamber $\langle(0, 1), (1, 2)\rangle$. The coincidence of these two constructions is proved in [16, p. 31].

Consider the scaffolding $S = \{D\}$ of the reflexive polytope P with PALP ID 3263, with shape $Z = \mathbb{P}^1 \times dP'_5$. Here we take dP'_5 to be the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at three of its torus invariant points, and $D \in |-K_Z|$ is the toric boundary of Z .

This scaffolding corresponds to the anti-canonical embedding $X_P \rightarrow \mathbb{P}^9$, see Example 2.11. To exhibit an explicit smoothing in this embedding we consider another scaffolding of P —with shape $Z' = \mathbb{P}^2 \times \mathbb{P}^1$ —shown in Fig. 5. Note that

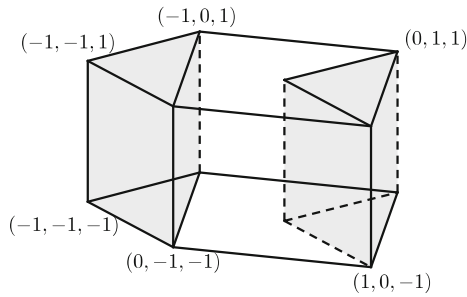


Fig. 5. The scaffolding used to construct 2–8

P° is not cracked along the fan determined by Z' . The scaffolding S' defines an embedding $X_P \rightarrow Y_{S'}$ where $Y_{S'}$ is the toric variety defined by weight matrix

$$\begin{array}{ccccccc} y & x_0 & z_0 & z_1 & x_1 & x_2 & x_3 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 1 & 1 \end{array}$$

and stability condition $\omega = (1, 2)$ —note that ω is contained in a wall. The toric variety X_P is the vanishing locus of a section of $E := \mathcal{O}(1, 2) \oplus \mathcal{O}(2, 4)$.

Lemma 4.4. *The vanishing locus of a general section of E is a Fano threefold 2–8.*

Proof. General sections of E do not vanish at the torus invariant point defined by the vanishing of all co-ordinates except z_1 . There is a projection from this point to the toric variety Y' , the toric variety defined by the same weight matrix as Y , but stability condition $\omega = (1, 2)$. The wall spanned by $(1, 2)$ is a flipping wall, and the birational transformation induced by crossing this wall is given by (the cone on) a Pachner move in the fan determined by Y . The intermediate variety has the non- \mathbb{Q} factorial point given by the vanishing of all homogeneous co-ordinates (labelled as for Y) except z . The image of the vanishing locus X of a general section of E in Y' misses this singularity. Hence the resolution of Y' induced by moving the stability condition from $(1, 2)$ into the chamber $\langle(1, 2), (0, 1)\rangle$ restricts to an isomorphism of X , and the result follows from [16, p. 31]. \square

Consider the embedding $\varphi_{\mathcal{O}(1,2)}: Y_{S'} \rightarrow \mathbb{P}(H^0(Y_{S'}, \mathcal{O}(1, 2)))^* = \mathbb{P}^{10}$. Composing $\varphi_{\mathcal{O}(1,2)}$ with the embedding $\iota: X_P \rightarrow Y_{S'}$, the pull-back of the line bundle $\mathcal{O}_{\mathbb{P}^{10}}(1)$ is the anti-canonical class on X_P by adjunction. Moreover X_P is the intersection of $\varphi_{\mathcal{O}(1,2)}(Y_{S'})$ with a quadric and a hyperplane in \mathbb{P}^{10} . In particular, restricting to this hyperplane, we obtain the anti-canonical embedding of X_P in \mathbb{P}^9 . Restricting to members of a general pencil of hyperplanes – and intersecting with a general pencil of quadrics—we see that X_P deforms in \mathbb{P}^9 to a variety in the family 2–8.

Rank 2, number 14 This example is the first of a sequence of examples—along with 2–20, 2–22, and 2–26—to make use of polytopes cracked along the fan of $Z := dP_7$. The corresponding embeddings are defined using the five 4×4 Pfaffians of a 5×5 matrix of polynomials in the homogeneous co-ordinate ring of a toric

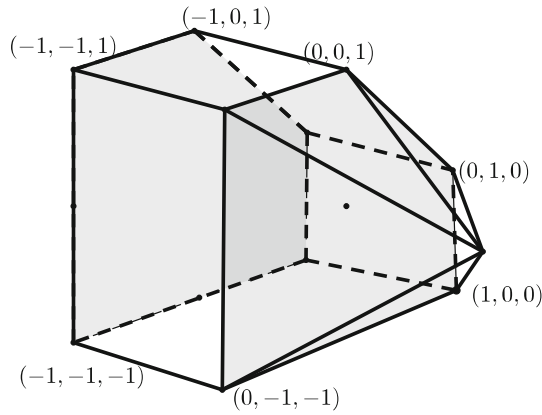


Fig. 6. The scaffolding used to construct 2–14

variety. Varieties in the family 2–14 are the blow up of B_5 (a three dimensional linear section of $\text{Gr}(2, 5)$) in an elliptic curve which is the intersection of two hyperplane sections.

Consider the polytope P with PALP ID 3028 together with the scaffolding with shape Z displayed in Fig. 6. We have that $\tilde{N} \cong \mathbb{Z}^2$, $N_U \cong \mathbb{Z}$, and $S = \{(0, 1), (D, 0), (D, -1)\}$, where D is the toric boundary of $Z = dP_7$.

The variety Y_S is determined by the weight matrix

$$\begin{array}{cccccccc} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & y \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

and stability condition $\omega = (2, 1)$. The corresponding secondary fan is shown in Fig. 7. The variety Y_S is consequently the blow up of \mathbb{P}^6 in a codimension 2 linear subspace. The ideal of X_P in Y_S is obtained by homogenizing the 4×4 Pfaffians of the skew-symmetric matrix

$$\begin{pmatrix} 0 & x_0y & x_1 & x_2 & x_0y \\ & 0 & 0 & x_3 & x_4 \\ & & 0 & x_0y & x_5 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}. \tag{2}$$

Consider the contraction $Y_S \rightarrow \mathbb{P}^6$, and observe that the intersection of the image of X_P with the centre $V := \{x_0 = x_1 = 0\}$ is a cycle of five (-1) -curves. Replacing the two 0 non-diagonal entries in (2) with general linear forms, this cycle of (-1) -curves becomes a (codimension 3) non-singular curve of genus one. Blowing up V produces a flat family deforming X_P to a Fano threefold in the family 2–14.

Rank 2 number 17 Varieties in the family 2–17 are the blow up of a quadric threefold in an elliptic curve of degree 5. We consider the polytope P with PALP ID 1528,

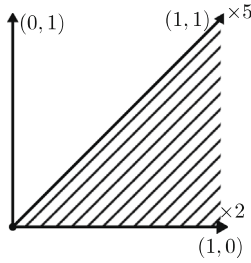


Fig. 7. Secondary fan for the variety Y_S used in the construction of 2–14

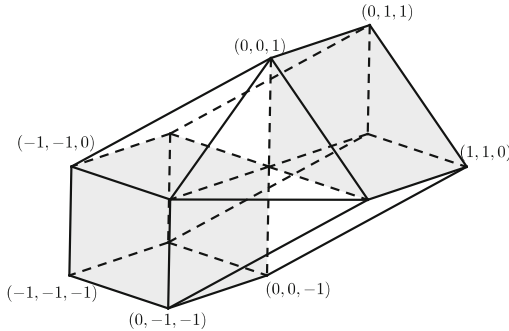


Fig. 8. The scaffolding used to construct 2–17

together with the scaffolding S shown in Fig. 8 using the shape variety $Z = \mathbb{P}^1 \times dP_7$.

The scaffolding S determines the toric variety $Y_S \cong \mathbb{P}^4 \times \mathbb{P}^3$. Letting x_0, \dots, x_4 and y_0, \dots, y_3 denote homogeneous co-ordinates on the respective projective space factors, X_P is the vanishing locus of the binomial $x_0y_0 = x_1y_1$, and the five 4×4 Pfaffians of the skew-symmetric matrix

$$\begin{pmatrix} 0 & y_0 & y_2 & y_3 & y_0 \\ & 0 & tf_1 & x_2 & x_4 \\ & & 0 & x_0 & x_3 \\ & & & 0 & tf_2 \\ & & & & 0 \end{pmatrix},$$

where $t = 0$ and f_i are general linear equations in x_0, \dots, x_4 . One of these five Pfaffians describes the threefold $x_2x_3 - x_0x_4 = 0$ in \mathbb{P}^4 , while the other 4 equations have bidegree $(1, 1)$. It is shown in [16, p. 38] that varieties in the family 2–17 may be obtained as the vanishing loci of general sections of the bundle

$$E := (S^* \boxtimes \mathcal{O}_{\mathbb{P}^3}(1)) \oplus (\det S^* \boxtimes \mathcal{O}_{\mathbb{P}^3}(1)) \oplus (\det S^* \boxtimes \mathcal{O}_{\mathbb{P}^3})$$

on the variety $\text{Gr}(2, 4) \times \mathbb{P}^3$. The Grassmannian $\text{Gr}(2, 4) \subset \mathbb{P}^5$ is a quadric fourfold, while sections of the line bundle $\det S^*$ define hyperplane sections in \mathbb{P}^5 . Moreover, the binomial $x_0y_0 = x_1y_1$ defines a section of the bundle obtained by pulling back

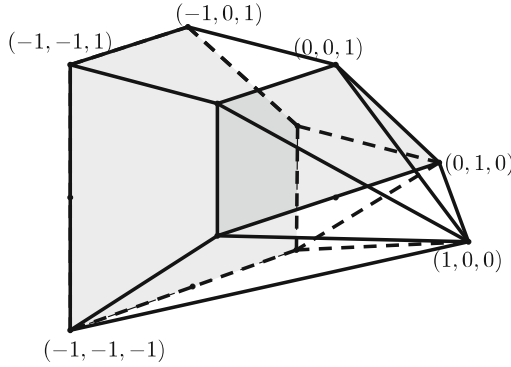


Fig. 9. The scaffolding used to construct 2–20

($\det S^* \boxtimes \mathcal{O}_{\mathbb{P}^3}(1)$) to the product of a hyperplane section in \mathbb{P}^5 with \mathbb{P}^3 . We claim that the remaining four Pfaffian equations define a section of the pull-back of ($S^* \boxtimes \mathcal{O}_{\mathbb{P}^3}(1)$) to this hyperplane section. Representing a point in $\text{Gr}(2, 4)$ as the row-space of a 2×4 matrix

$$M = \begin{pmatrix} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \\ y_{2,1} & y_{2,2} & y_{2,3} & y_{2,4} \end{pmatrix},$$

a section of the bundle S^* is determined by a vector $z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$, and this section vanishes precisely when z lies in the row space of M . This happens when the maximal minors of the matrix

$$\bar{M} = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \\ y_{2,1} & y_{2,2} & y_{2,3} & y_{2,4} \end{pmatrix}$$

vanish. Writing the 2×2 minors of M (the Plücker co-ordinates) as x_0, \dots, x_5 we have that sections of S^* are defined by four equations of degree 1 in the variables $\{x_i : i \in \{0, \dots, 5\}\}$ and constants $\{z_i : i \in \{1, \dots, 4\}\}$. Replacing each z_i with the homogeneous co-ordinate y_{i-1} we recover the 4 remaining Pfaffian equations found above, up to a linear relation eliminating x_5 . That is, X_P admits an embedded flat deformation to a variety in the family 2–17.

Rank 2 number 20 Varieties in the family 2–20 are the blow up of B_5 (a three dimensional linear section of $\text{Gr}(2, 5)$) in a twisted cubic. Consider the polytope P with PALP ID 1910 together with the scaffolding with shape $Z = dP_7$ displayed in Fig. 9.

The corresponding toric variety Y_S is isomorphic to $Bl_{\mathbb{P}^3}\mathbb{P}^6$. Moreover, the variety X_P is the blow up of the vanishing locus of the five 4×4 Pfaffians of

$$\begin{pmatrix} 0 & x_0 & x_1 & x_2 & x_0 \\ & 0 & 0 & x_3 & x_4 \\ & & 0 & x_0 & x_5 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}, \tag{3}$$

where x_0, \dots, x_6 are homogeneous co-ordinates on \mathbb{P}^6 , in the locus $\{x_0 = x_1 = x_6\}$. Note that the ideal $x_2x_4 = x_3x_5 = x_2x_5 = 0$ defines a (degenerate) twisted cubic. Replacing the two zero non-diagonal entries in (3) with general homogeneous forms of degree one we obtain a flat deformation of $X_P \hookrightarrow Y_S$ to the blow up of B_5 in a twisted cubic.

Rank 2 number 21 Varieties in the family 2–21 are the blow up of a quadric threefold in a rational curve of degree 4. These are shown in [16, p. 43] to be zero loci of sections of the vector bundle

$$E = (S^* \boxtimes \mathcal{O}_{\mathbb{P}^4}(1))^{\oplus 2} \oplus (\det S^* \boxtimes \mathcal{O}_{\mathbb{P}^4})$$

on $\text{Gr}(2, 4) \times \mathbb{P}^4$. Consider the polytope with PALP ID 703, with the scaffolding shown in Fig. 10. This scaffolding has shape $Z = Z_{12}$, the shape used in the construction of Fano threefolds in the family V_{12} . The ambient space Y_S defined by this scaffolding is isomorphic to $\mathbb{P}^4 \times \mathbb{P}^4$ with co-ordinates x_0, \dots, x_4 and y_0, \dots, y_4 respectively. The equations cutting out X_P in Y_S can be read off as relations between labelled lattice points in Fig. 11. In particular if $u_1 + v_1 = u_2 + v_2$, where u_i and v_i are lattice points labelled with variables z_i and w_i for each $i \in \{1, 2\}$, points in X_P satisfy the equation $z_1w_1 = z_2w_2$. There are nine such binomial equations, which can be written as the 4×4 Pfaffians of the following pair of matrices (setting $t = 0$),

$$\begin{pmatrix} 0 & y_0 & ty_3 & y_2 & y_1 \\ & 0 & x_2 & x_0 & x_3 \\ & & 0 & x_1 & x_0 \\ & & & 0 & tx_4 \\ & & & & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & y_0 & ty_1 & y_3 & y_4 \\ & 0 & x_4 & x_1 & x_0 \\ & & 0 & x_0 & x_3 \\ & & & 0 & -tx_2 \\ & & & & 0 \end{pmatrix}.$$

Note that these matrices both determine the Pfaffian equation $x_1x_3 - x_0^2 + tx_2x_4 = 0$, which defines a toric degeneration of a quadric threefold.

Following the treatment of the variety 2–17, we observe that each set of five Pfaffian equations defines a section of (the pullback to a hyperplane section of) $S^* \boxtimes \mathcal{O}_{\mathbb{P}^4}(1)$. Thus the general member of the family given by the set of 9 Pfaffian equations is isomorphic to a Fano threefold in the family 2–21.

Rank 2 number 22 Varieties in the family 2–22 are the blow up of B_5 in a conic. Consider the polytope P with PALP ID 1857, and the scaffolding with shape $Z = dP_7$ displayed in Fig. 12. The variety Y_S is the blow up of \mathbb{P}^6 in a plane; the toric variety determined by the weight matrix

$$\begin{array}{cccccccc} y & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

and stability condition $\omega = (1, 2)$. X_P is cut out by the five 4×4 Pfaffians of

$$\begin{pmatrix} 0 & x_0 & tf_{0,1} & x_2 & x_3 \\ & 0 & x_4 & x_5 & x_0y \\ & & 0 & x_0y & x_6 \\ & & & 0 & tf_{1,1} \\ & & & & 0 \end{pmatrix}$$

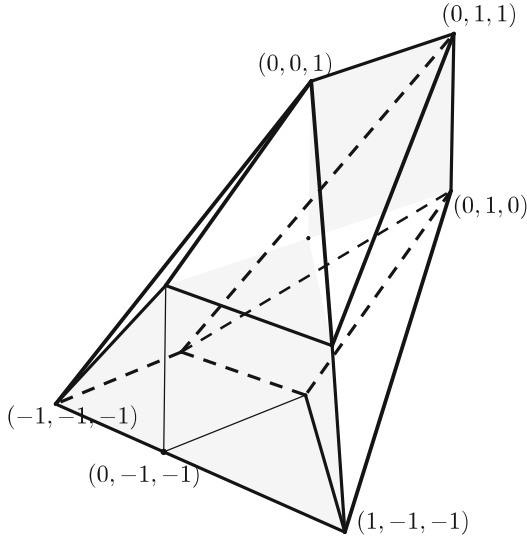


Fig. 10. The scaffolding used to construct 2–21

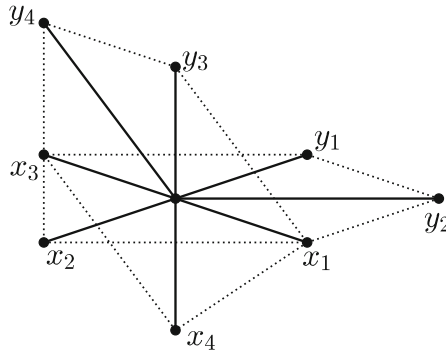


Fig. 11. Ray generators of Z_{12}

where $f_{i,j}$ is a generic polynomial of bi-degree (i, j) , and $t = 0$. The ambient variety Y_S is obtained from \mathbb{P}^6 with co-ordinates x_0, \dots, x_6 by blowing up the plane $\Pi := \{x_0 = x_1 = x_2 = x_3 = 0\}$. The Pfaffian equations defining X_P pull back to the single equation $x_5x_6 = tx_4f_{1,1}$ on this locus. Hence, for general values of t , the equations define the blow up of B_5 (cut out by 5 Pfaffian equations in \mathbb{P}^6) in a non-degenerate conic.

Rank 2 number 26 Varieties in the family 2–26 are the blow up of B_5 in a line. Consider the polytope P with PALP ID 1434 and scaffolding with shape $Z = dP_7$ displayed in Fig. 13. The variety Y_S is the blow up of \mathbb{P}^6 in the line with

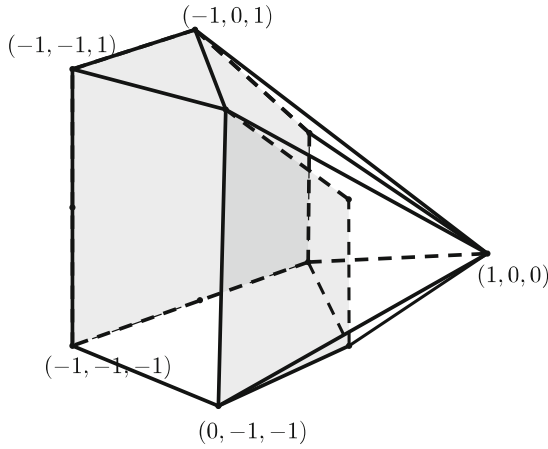


Fig. 12. The scaffolding used to construct 2–22

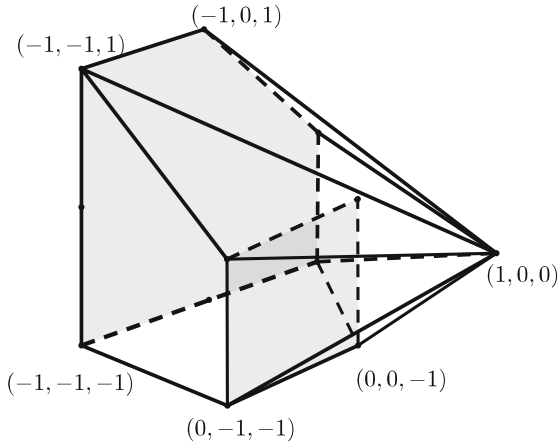


Fig. 13. The scaffolding used to construct 2–26

homogeneous co-ordinates $\{x_4, x_5\}$. Consider the one-parameter family

$$\begin{pmatrix} 0 & x_0 & x_1 & x_2 & x_0 \\ & 0 & t f_1 & x_3 & x_4 \\ & & 0 & x_0 & x_5 \\ & & & 0 & t g_1 \\ & & & & 0 \end{pmatrix},$$

where f_1 and g_1 are general linear forms with no terms in x_4 or x_5 . Varying t , this family contains the line with co-ordinates x_4 and x_5 for all values of t . Blowing up this line we obtain a flat family embedded in $Y_S \times \mathbb{A}_t^1$ with central fibre X_P , and general fibre a Fano threefold in the family 2–26.

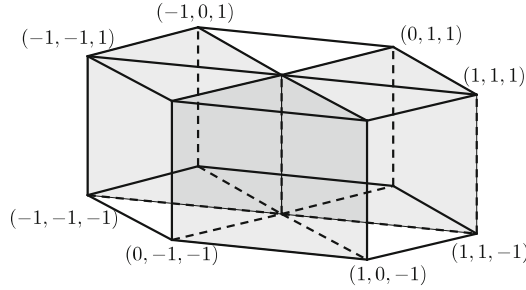


Fig. 14. The scaffolding used to construct 3–1

Rank 3 number 1 Varieties in the family 3–1 are double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched along a divisor of tri-degree $(2, 2, 2)$. Our treatment of this family is similar to that of 2–8. Consider the Fano polygon P with PALP ID 3875, illustrated in Fig. 14. We give P the ‘anti-canonical’ scaffolding; covering P with the polyhedron of sections of the toric boundary on the shape variety $Z = \mathbb{P}^1 \times dP_6$. This scaffolding reproduces the anti-canonical embedding $X_P \rightarrow \mathbb{P}^8$, see Example 2.11. We exhibit an explicit smoothing by factoring the anti-canonical embedding through a map to a toric variety obtained from a non-full scaffolding of P . Figure 14 shows a scaffolding S' of P with shape $Z' := \mathbb{P}^1 \times \mathbb{P}^2$. The scaffolding S' consists of three elements, and defines the toric variety $Y_{S'}$ with weight matrix

$$\begin{array}{cccccccc} x_0 & y_0 & z_0 & x_1 & y_1 & z_2 & w_0 & w_1 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{array}$$

and stability condition $\omega = (1, 1, 1)$. The hypersurface X_P is the vanishing locus of the binomial $w_0 w_1 = x_0^2 y_0^2 z_0^2$, a section of the line bundle L_1 with tri-degree $(2, 2, 2)$ —and $x_1 y_1 z_1 = x_0 y_0 z_0$ —a section of the line bundle L_2 tri-degree $(1, 1, 1)$. Note that the variety $Y_{S'}$ is not \mathbb{Q} -factorial along the line on which $x_0 = y_0 = z_0 = x_1 = y_1 = z_2 = 0$. General linear sections through this non-isolated singularity are isomorphic to the affine cone V over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, polarised by the line bundle of tri-degree $(1, 1, 1)$.

Consider a general section s of $E = L_1 \oplus L_2$, and its vanishing locus X . Projecting away from the point at which all co-ordinates except w_0 vanish, X is an isomorphism onto its image in a toric variety Y' . The variety F which appears in the construction in [16, p. 57] is obtained from the variety Y' by a making one of the three possible small resolutions of the singularity V . Since the variety X does not intersect the singular locus of Y' this resolution restricts to an isomorphism of X . The rest of the example follows our treatment of the family 2–8: the complete linear system determined by L_2 defines an embedding $Y_{S'} \rightarrow \mathbb{P}^9$ and varying a quadric section in the anti-canonical embedding of $Y_{S'}$ smooths X_P .

Rank 3 number 4 Fano threefolds in this family are obtained by blowing up the fibre of the projection map $X_{2-18} \rightarrow \mathbb{P}^2$, where X_{2-18} is a double cover of $\mathbb{P}^2 \times \mathbb{P}^1$ branched in a divisor of bidegree $(2, 2)$. In [16, p. 60] it is shown that varieties in

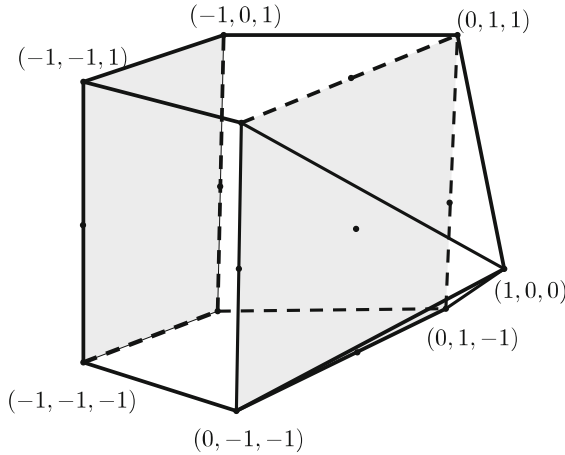


Fig. 15. The scaffolding used to construct 3–4

this family may be obtained as hypersurfaces of tri-degree $(2, 2, 2)$ contained in the toric variety Y defined by the weight matrix

$$\begin{array}{cccccc}
 x_0 & x_1 & y & z & t_0 & t_1 & w \\
 \hline
 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 & 1 & 1 & 1
 \end{array}$$

together with a stability condition ω in the chamber $\langle (1, 0, 0), (1, 1, 0), (1, 1, 1) \rangle$. We compare these toric hypersurfaces to the threefolds obtained by scaffolding the polytope P with PALP ID 2603 shown in Fig. 15. This scaffolding has shape $Z = \mathbb{P}^1 \times \mathbb{P}^1$, and hence defines a codimension 2 toric complete intersection in the toric variety Y_S with weights:

$$\begin{array}{cccccc}
 a_0 & a_1 & b_0 & b_1 & b_2 & c_0 & c_1 \\
 \hline
 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
 0 & 0 & 1 & 1 & 1 & 1 & 1
 \end{array}$$

and stability condition $\omega = (2, 1)$. Let X be the vanishing locus of a general section of the vector bundle $E := \mathcal{O}(1, 2) \oplus \mathcal{O}(2, 2)$. Note that the line bundle $\mathcal{O}(1, 2)$ is not nef on Y_S . We define a Segre type map $\phi: Y \rightarrow Y_S$, setting

$$\phi: (x_0, x_1, y, z, t_0, t_1, w) \mapsto (x_0, x_1, w, yt_0, yt_1, zt_0, zt_1).$$

It is easily verified that this map is homogeneous, and that $\phi^*: \text{Pic}(Y_S) \rightarrow \text{Pic}(Y)$ is given by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}^T$. Hence the stability condition $\omega = (2, 1)$, is mapped into the wall spanned by $(1, 0, 0)$ and $(1, 1, 1)$. Let Y' be the toric variety defined by weight matrix M and stability condition $(2, 1, 1)$.

Lemma 4.5. *Y is a small resolution a non-isolated singularity of Y' which is disjoint from the divisor $\{w = 0\}$.*

Proof. There is a morphism $\pi : Y_S \rightarrow \mathbb{P}^1$, expressing Y_S as a \mathbb{P}^4 bundle over \mathbb{P}^1 with co-ordinates $(a_0 : a_1)$. Similarly, Y and Y' admit projections to \mathbb{P}^1 with co-ordinates $(x_0 : x_1)$. The projection $Y' \rightarrow \mathbb{P}^1$ coincides with the composition of π with the inclusion $\iota : Y' \hookrightarrow Y_S$. Given a point $a \in \mathbb{P}^1$, the intersection $\pi^{-1}(a) \cap \iota(Y')$ is isomorphic to the projective closure of the (affine) ODP singularity in \mathbb{P}^4 with co-ordinates $(b_0 : b_1 : b_2 : c_0 : c_1)$. The (smooth) variety Y is obtained by making one of the two possible small resolutions of this line of conifold singularities. Note however that, for any fibre of π , the divisor $\{b_0 = 0\}$ is disjoint from the singular locus of Y' . Since $\iota^*b_0 = w$, the locus $w = 0$ is disjoint from the singular locus of Y' . \square

Note that Y' is a hypersurface in Y_S and determines an element of the divisor class associated to $\mathcal{O}(1, 2)$, cut out by $\det \begin{pmatrix} b_1 & c_0 \\ b_2 & c_1 \end{pmatrix}$. Moreover, we have that $\phi^*(2, 2) = (2, 2, 2)$; hence, by Lemma 4.5, any hypersurface cut out by a member of the linear system $(2, 2, 2)$ on Y is the vanishing locus of a section of E on Y_S . *Rank 3 number 5* It was shown in [16, p. 62] that varieties in the family 3–5 are codimension 2 complete intersections in the toric variety Y , determined by the weight matrix

$$\begin{array}{cccccccc} x_0 & x_1 & y_0 & y_1 & y_2 & z_0 & z_1 & t \\ \hline 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}$$

and a stability condition in the chamber $\langle (1, 0, 0), (0, 1, 0), (1, 1, 1) \rangle$. Varieties X in the family 3–5 are obtained as zero loci of sections of the bundle $\mathcal{O}(1, 2, 1)^{\oplus 2}$. The secondary fan for Y is illustrated in Fig. 16. Consider the scaffolding of the polytope P with PALP ID 1837 shown in Fig. 17. The variety Y_S is determined by the weight matrix

$$\begin{array}{ccccccc} a_0 & a_1 & b_0 & b_1 & b_2 & c_0 & c_1 \\ \hline 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}$$

and stability condition $(2, 1)$. The toric variety X_P is cut out of Y_S by a pair of binomial sections of $\mathcal{O}(1, 2)$. Observe that the linear system $(1, 2)$ is not nef on Y_S , and has base locus $B = \{b_0 = b_1 = b_2 = 0\}$. We claim that Y is obtained from Y_S by blowing up B . It is clear that the weight matrix defining Y is the same as the defining the toric variety $\text{Bl}_B Y_S$. Moreover, the map defined by setting

$$\phi : (x_0, x_1, y_0, y_1, y_2, z_0, z_1, t) \mapsto (x_0t, x_1t, y_0, y_1, y_2, z_1, z_2)$$

has pull-back defined by the matrix

$$[\phi^*] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

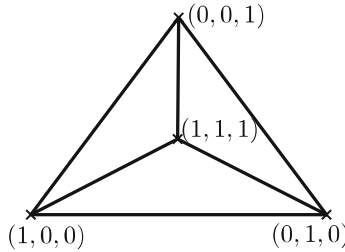


Fig. 16. Secondary fan of the toric variety Y , used to construct 3–5

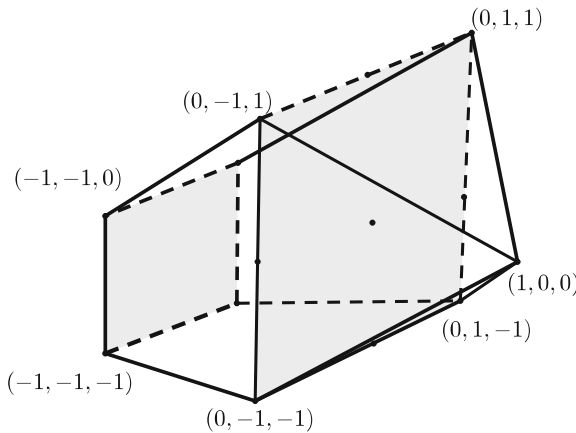


Fig. 17. The scaffolding used to construct 3–5

Hence, considering the ample class $\omega = (2, 1)$, $\phi^*\omega = (2, 1, 2)$, it remains to analyse the effect of crossing the wall in the secondary fan of Y generated by $(1, 0, 0)$ and $(1, 1, 1)$. We observe that moving the stability condition into this wall contracts the divisor $t = 0$ (defining the ray generated by $(0, 0, 1)$) to the locus $\{y_0 = y_1 = y_2 = 0\}$.

We claim that vanishing loci of general sections of $E := \mathcal{O}(2, 1)^{\oplus 2}$ are smooth. If so, the blow up of the base locus is an isomorphism on general sections, as the restriction of the base locus to a general fibre is a Cartier divisor. Smoothness follows directly from the Jacobian condition. Indeed, sections of E are of the form

$$c_0 f_1 + c_1 g_1 + a_0 f_2 + a_1 g_2,$$

where f_j and g_j are homogeneous polynomials of degree $j \in \{1, 2\}$ in b_0, b_1, b_2 . Taking two such sections the corresponding Jacobian matrix, evaluated at $b_0 = b_1 = b_2$ and—without loss of generality— $a_0 = c_0 = 1$, has the form $(0 \ L)$; a block matrix consisting of a 2×2 zero block and a 2×3 matrix L of linear forms in c_1 . Since the locus F in \mathbb{P}^5 where a 2×3 matrix drops rank has codimension 2, any projective line in this space which misses F determines a matrix L which does not drop rank.

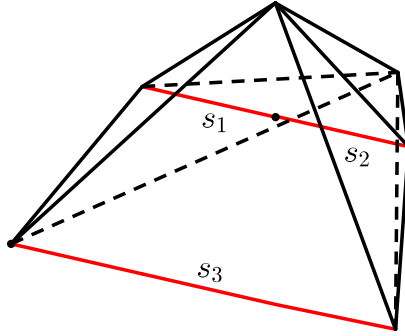


Fig. 18. The scaffolding used to construct 3–14

Rank 3 number 14 We consider the reflexive polytope P with PALP ID 143, together with the scaffolding S with shape $Z = \mathbb{P}^1$ shown in Fig. 18. This scaffolding expresses X_P as a hypersurface of tri-degree $(3, 1, 1)$ in the toric variety Y_S with weight matrix

$$\begin{array}{ccccccc} x_0 & x_1 & x_2 & y_0 & y_1 & z_0 & z_1 \\ \hline 1 & 1 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}$$

and stability condition $\omega = (3, 1, 1)$. Note that Y_S is not \mathbb{Q} -factorial at the point $w := \{x_0 = x_1 = x_2 = y_0 = z_0 = 0\}$. However, since the monomial $y_1 z_1$ defines a section of $\mathcal{O}(3, 1, 1)$ —and this does not vanish at w —a general hypersurface X with tri-degree $(3, 1, 1)$ misses this locus. Moving $\omega \in \text{Pic}(Y_S)_{\mathbb{R}}$ to $(4, 1, 1)$ induces a resolution of this singularity which restricts to an isomorphism of X , and recovers the ambient space considered in [16, p. 70]. Hence, by the argument given in [16, p. 70], the hypersurface X is isomorphic to a Fano variety in the family 3–14.

Remark 4.6. We could also construct varieties in this family using the scaffolding S' obtained by combining the two struts containing the origin in $N_{\mathbb{R}}$ into a single line segment of length two. This produces an embedding $X_P \rightarrow Y_S$, where Y_S is given by the weight matrix

$$\begin{array}{cccccc} s & x_0 & x_1 & x_2 & y & z \\ \hline 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{array}$$

and stability condition $\omega = (1, 3)$. X_P is the vanishing locus of the binomial $yz = s^2 x_0^3$.

Rank 3 number 16 Varieties in this family are obtained by blowing up $B_7 = \text{Bl}_{pt} \mathbb{P}^3$ with centre the strict transform of a twisted cubic passing through the centre of the blow up $B_7 \rightarrow \mathbb{P}^3$.

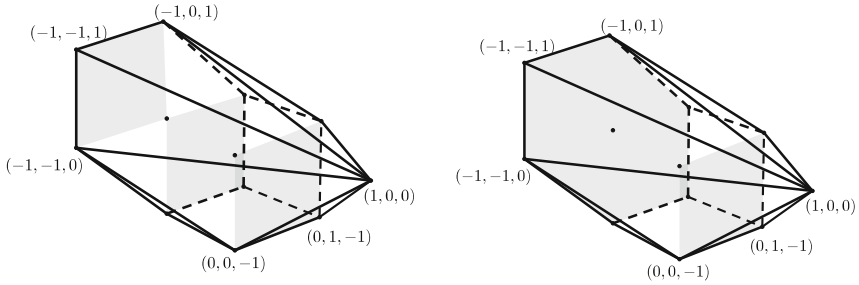


Fig. 19. The scaffolding used to construct 3–16

We can recover the construction used in [16, p. 71] using a scaffolding of a reflexive polytope. Indeed, consider the polytope P with PALP ID 1092, together with the scaffolding \mathcal{S} , with shape $Z = \mathbb{P}^1 \times \mathbb{P}^1$, displayed in Fig. 19. Note that this scaffolding is not full, and P° is not cracked along the fan defined by Z . The toric variety $Y_{\mathcal{S}}$ is determined by the weight matrix

$$\begin{array}{cccccccc}
 x_0 & x_1 & x_2 & y_0 & z & s & z_0 & z_1 \\
 \hline
 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1
 \end{array}$$

together with the stability condition $\omega = (2, 2, 1)$. The toric variety X_P is defined by the vanishing of a pair of binomial sections of $\mathcal{O}(1, 1, 1)$. A stability condition which lies in the cone spanned by $\langle (1, 0, 0), (1, 1, 0), (1, 1, 1) \rangle$ determines the toric variety $\hat{Y}_{\mathcal{S}}$ used in [16] to construct Fano varieties in 3–16. However ω lies in the wall spanned by $(1, 1, 0)$ and $(1, 1, 1)$. Moving ω into the chamber used in [16] resolves the singular locus $\{x_0 = x_1 = x_2 = y_0 = z_0 = z_1 = 0\}$. However general sections of $\mathcal{O}(1, 1, 1)$ do not vanish along this point, and hence the intersection of two general divisors of tri-degree $(1, 1, 1)$ are isomorphic to varieties in the family 3–16.

In order to provide a construction using a *cracked* polytope, we consider the scaffolding \mathcal{S}' of P with shape $Z = dP_6$, also shown in Fig. 19.

The scaffolding \mathcal{S}' defines the weight matrix

$$\begin{array}{cccccccc}
 x_0 & x_1 & x_2 & x_3 & y_0 & y_1 & y_2 & y_3 & y \\
 \hline
 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
 \end{array}$$

and stability condition $(2, 1)$. Let Y denote the toric variety determined by the weight matrix

$$\begin{array}{cccccccc}
 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1
 \end{array}$$

and stability condition $(2, 1)$. Note that general sections of $\mathcal{O}(1, 1)^{\oplus 2}$ define subvarieties of Y isomorphic to $Y_{\mathcal{S}'}$. There is a map $\theta: Y_{\mathcal{S}} \hookrightarrow Y$ —analogous to the

Segre embedding map $\mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^8$ – sending

$$(x_0, x_1, x_2, y_0, z, s, z_0, z_1) \\ \mapsto (x_0y_0, x_1y_0, x_2y_0, s, x_0z_0, x_0z_1, x_1z_0, x_1z_1, x_2z_0, x_2z_1, z).$$

We have that $\theta^*\mathcal{O}(1, 0) = \mathcal{O}(1, 1, 0)$, while $\theta^*\mathcal{O}(0, 1) = \mathcal{O}(0, 0, 1)$. Hence the ample line bundle $\mathcal{O}(2, 1)$ pulls back to $\mathcal{O}(2, 2, 1)$. This class is not ample on \hat{Y}_S and the image of the induced morphism $\hat{Y}_S \rightarrow Y$ factors through the contraction $\hat{Y}_S \rightarrow Y_S$. Indeed, we have the commutative diagram of embeddings

$$\begin{array}{ccccc} X_P & \longrightarrow & Y_S & \longrightarrow & Y \\ \parallel & & & \nearrow & \\ X_P & \longrightarrow & Y_{S'} & & \end{array}$$

We can deform X_P in Y_S by moving the section of $\mathcal{O}(1, 1, 1)^{\oplus 2}$ cutting out X_P . Alternatively, we obtain varieties in the family 3–16 as codimension 4 subvarieties of $Y_{S'}$ by fixing the embedding $Y_S \rightarrow Y$ and moving the sections used to cut out $Y_{S'}$ in Y .

Rank 4, number 2 Varieties in this family are obtained from $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by blowing up a curve of tri-degree $(1, 1, 3)$.

We consider the polytope with PALP ID 1081, together with the scaffolding shown in Fig. 20, with shape $Z = \mathbb{P}^2$. This scaffolding describes X_P as a hypersurface of tri-degree $(1, 1, 2)$ in the toric variety Y_S determined by the weight matrix

$$\begin{array}{cccccccc} x_0 & x_1 & y_0 & y_1 & z_0 & z_1 & z_2 & \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & \end{array}$$

and stability condition $\omega = (1, 2, 1)$. The variety Y_S is the projectivisation of the bundle $\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(0, 1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Note that the line bundle $\mathcal{O}(1, 1, 2)$ is not nef, and that its base locus is section of the projection $Y_S \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ defined by $z_0 = z_1 = 0$. Blowing up this base locus we obtain the variety F considered in [16, p. 82]. To check smoothness of general hypersurfaces in this linear system, note that general sections of L have the form

$$f = z_0^2 f_{1,1} + z_0 z_1 g_{1,1} + z_1^2 h_{1,1} + z_0 z_2 f_1 + z_1 z_2 g_1,$$

where $f_{1,1}$ and $g_{1,1}$ are polynomials of bidegree $(1, 1)$ in x_0, x_1, y_0, y_1 , while f_1, g_1 are linear polynomials in x_0, x_1 . Restricting the Jacobian to the locus $z_0 = z_1 = 0$, we see that the locus $\{f = 0\}$ is singular precisely when $f_1 = g_1 = 0$. However this locus is empty for general choices of f_1 and g_1 .

Since the restriction of the base locus of this linear system to a smooth member X is a Cartier divisor in X , its blow up is an isomorphism. Hence such hypersurfaces X are members of the family 4–2, and X_P is the central fibre of a toric degeneration in this family.

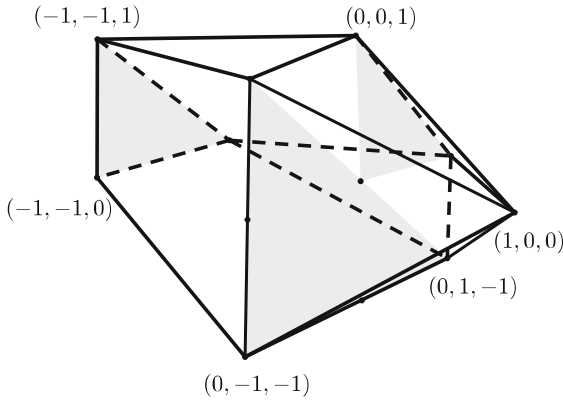


Fig. 20. The scaffolding used to construct 4–2

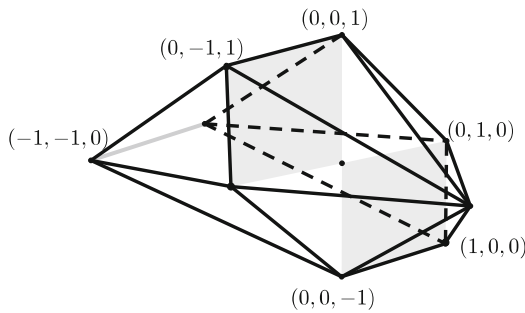


Fig. 21. The scaffolding used to construct 4–6

Rank 4 number 6 Varieties X in the family 4–6 are obtained by blowing up $\mathbb{P}^2 \times \mathbb{P}^1$ in curves of bidegree $(1, 2)$ and $(0, 1)$ respectively. Consider the polytope P with PALP ID 426, together with the scaffolding S with shape $\mathbb{P}^1 \times \mathbb{P}^1$ illustrated in Fig. 21.

The toric variety Y_S is defined by the weight matrix

$$\begin{array}{cccccccc}
 s_0 & s_1 & s_2 & y_0 & y_1 & x_0 & x_1 & x_2 \\
 \hline
 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0
 \end{array}$$

and stability condition $\omega = (1, 1, 2)$. The secondary fan of Y_S is illustrated in Fig. 22.

The variety Y_S is isomorphic to $\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^2}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1, 0))$; and the two chambers in the secondary fan correspond to isomorphic varieties—despite the presence of a non-trivial flopping locus. The projection $\pi : Y_S \rightarrow \mathbb{P}^2 \times \mathbb{P}^1$ corresponds to projecting out the variables s_i for all $i \in \{0, 1, 2\}$. The toric variety X_P is cut out

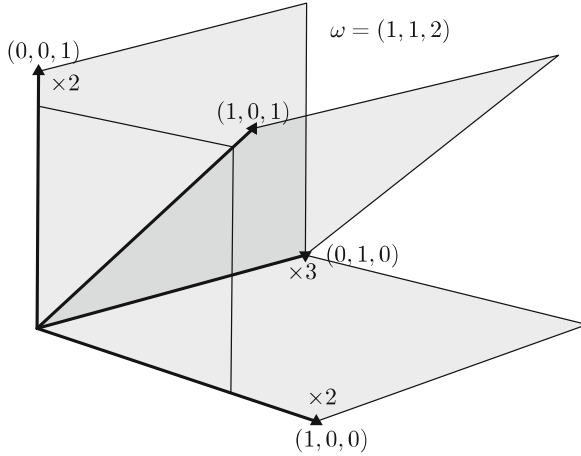


Fig. 22. Secondary fan of the toric variety Y_S

of Y_S by the binomial equations

$$s_2x_2 = s_0x_0y_0$$

$$s_1x_1 = s_0x_0.$$

These are sections of the line bundles L_1 and L_2 , with weights $(1, 1, 1)$ and $(1, 1, 0)$ respectively. Note that the line bundle L_1 is nef while L_2 is not.

Let X be the vanishing locus of a general section $s = l_1 + l_2$ of $E := L_1 \oplus L_2$. The section $l_1 \in \Gamma(Y_S, L_1)$ has the general form $s_0f_{1,1} + s_1g_{1,1} + s_2h_1$, where $f_{1,1}$ and $g_{1,1}$ have bi-degree $(1, 1)$ in x_0, x_1, x_2 and y_0, y_1 respectively; while h_1 has bi-degree $(1, 0)$. Similarly l_2 has the general form $s_0f_1 + s_1g_1$, where f_1 and g_1 have bi-degree $(1, 1)$.

Fibres of the restriction of π to X are given by the kernel of the matrix

$$\begin{pmatrix} f_{1,1} & g_{1,1} & h_1 \\ f_1 & g_1 & 0. \end{pmatrix}$$

That is, π is a graph away from the locus at which this matrix has rank ≤ 1 . This locus in $\mathbb{P}^2 \times \mathbb{P}^1$ has two connected components, one given by $h_1 = f_{1,1}g_1 - g_{1,1}f_1 = 0$, a curve of bidegree $(1, 2)$, and the other by $f_1 = g_1 = 0$, a curve of degree $(0, 1)$. Thus the morphism π exhibits X as a Fano threefold in the family 4–6.

Rank 5 number 1 Varieties in this family are obtained by first blowing up a quadric in a conic—obtaining a variety V in the family 2–29—and blowing up V in three exceptional lines. Consider the scaffolding S of the polytope with PALP ID 1083 with shape \mathbb{P}^2 , illustrated in Fig. 23.

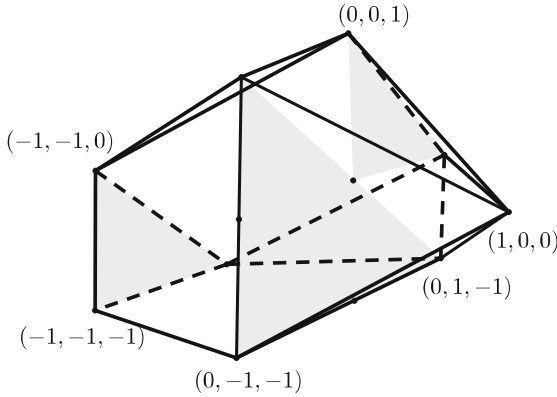


Fig. 23. The scaffolding used to construct 5–1

That is, we consider general hypersurfaces X of tri-degree $(1, 2, 1)$ in the toric variety Y_S defined by the weight matrix

$$\begin{array}{cccccccc} s_0 & s_1 & x_0 & x_1 & x_2 & y_0 & y_1 & \\ \hline 1 & 1 & 0 & 1 & 1 & 0 & -1 & \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \end{array}$$

and stability condition $(2, 1, 1)$. The variety Y_S admits a map to \mathbb{P}^1 (with coordinates $(s_0 : s_1)$), giving Y_S the structure of a $\mathbb{P}^2 \times \mathbb{P}^1$ fibre bundle. The variety X also admits a morphism to \mathbb{P}^1 , whose fibres are surfaces of bi-degree $(2, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^1$. Projecting $\mathbb{P}^2 \times \mathbb{P}^1$ to \mathbb{P}^2 we see that any such smooth fibre is the blow up of \mathbb{P}^2 in four (general) points; that is, isomorphic to the del Pezzo surface dP_5 .

Hypersurfaces of tri-degree $(1, 2, 1)$ have general form

$$y_0x_0(x_0f_1 + p_1) + y_1(x_0^2f_2 + x_0f_1q_1 + p_2),$$

where $p_i, q_i \in \mathbb{C}[x_1, x_2]$ and $f_i \in \mathbb{C}[s_0, s_1]$ are homogeneous polynomials of degree i for each $i \in \{1, 2\}$. Let X denote the vanishing locus of this polynomial. Note that X contains the surface $\{x_0 = y_1 = 0\}$. Fixing a point $(s_0, s_1) \in \mathbb{P}^1$, the dP_5 fibre of the projection $X \rightarrow \mathbb{P}^1$ is obtained by blowing up the intersection points of the conics $C_1 := \{x_0(x_0f_1 + p_1) = 0\}$ and $C_2 := \{(x_0^2f_2 + x_0f_1q_1 + p_2) = 0\}$ in \mathbb{P}^2 (with homogeneous co-ordinates $(x_0 : x_1 : x_2)$). First consider the case $x_0 = p_2 = 0$. Choosing a general p_2 , we find two distinct reduced points α_1, α_2 in $C_1 \cap C_2$; these are independent of the choice of $s = (s_0, s_1) \in \mathbb{P}^1$. The other two solutions depend on s , and lie in the line $(x_0f_1 + p_1) = 0$. Note that we may choose co-ordinates such that C_1 is defined by $\{x_0x_1 = 0\}$.

Hence we can construct four surfaces, each isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, contained in X : two surfaces— S_1 and S_2 —swept out by $\{\alpha_i\} \times \mathbb{P}^1_{(y_0:y_1)}$, the surface S_3 swept out by C_1 over $\mathbb{P}^1_{(s_0:s_1)}$, and the base locus $S_4 = \{x_0 = y_1 = 0\}$. Each of these surfaces

restrict to exceptional curves in the dP_5 fibres. Note that fibres of $X \rightarrow \mathbb{P}^1$ are not all smooth—there are two singular fibres—but they are smooth in a neighbourhood of $\bigcup_{i \in [4]} S_i$. Hence—applying a relative version of Castelnuovo’s criterion—we can have a morphism $X \rightarrow X'$ which contracts the disjoint surfaces $S_1, S_2,$ and S_3 to sections of the induced morphism $\pi : X' \rightarrow \mathbb{P}^1_{(s_0:s_1)}$. The smooth fibres of π are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, while singular fibres have a single nodal singularity; these are isomorphic to $\mathbb{P}(1, 1, 2)$. The surface S_4 is the strict transform of a surface S'_4 , which intersects every fibre F in a smooth section of $-\frac{1}{2}K_F$.

Letting $\rho(X)$ denote the Picard rank of X , we have that $\rho(X) = \rho(X') + 3$. Since X_P —and hence X —has degree 28, we can conclude from the classification of Fano 3-folds that if $\rho(X') \geq 2$, X is in the family 5–1. This is easily seen from the Leray spectral sequence

$$H^i(\mathbb{P}^1, R^j \pi_* \mathbb{Q}) \Rightarrow H^{i+j}(X', \mathbb{Q});$$

indeed—since $H^1(F, \mathbb{Q}) = 0$ for all fibres F of π —we have $b^2(X') = 1 + h^0(\mathbb{P}^1, R^2 \pi_* \mathbb{Q})$. However $h^0(\mathbb{P}^1, R^2 \pi_* \mathbb{Q}) \geq 1$ since the surface S'_4 defines a non-trivial class in $H^2(F, \mathbb{Q})$ for every fibre F .

Remark 4.7. Comparing our construction with that of Mori–Mukai [34], they first consider the blow up of a quadric threefold in a conic. Restricting the projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^1$ this blow up defines X' , a quadric surface bundle over \mathbb{P}^1 with two singular fibres (with singularities are disjoint from the exceptional locus). Note that the exceptional locus distinguishes a conic C in each fibre of π . To obtain varieties in 5–1 we then blow up X' in three exceptional lines. These lines are sections of the map $X' \rightarrow \mathbb{P}^1$ defined by a triple of points on the distinguished conic C in each fibre. That is, the surface S_4 is the strict transform of the exceptional locus obtained by the blow up of the quadric threefold; while $S_i, i \in \{1, 2, 3\}$ are obtained by blowing up exceptional lines.

4.1. Products

The remaining non-toric Fano threefolds X with $-K_X$ very ample are products of non-toric del Pezzo surfaces—that is, $dP_k \times \mathbb{P}^1$ for $k \in \{3, 4, 5\}$ —with \mathbb{P}^1 . We can easily construct toric degenerations of these from degenerations of dP_k for each k . Fix a reflexive polygon Q such that Q° is cracked along the fan of a shape variety Z' , together with a scaffolding S' of Q with shape Z' . We can produce a scaffolding S of $\text{conv}(Q, (0, 0, 1), (0, 0, -1))$ with shape $Z := Z' \times \mathbb{P}^1$ by setting $S = \{(\pi_1^*(D), \chi) : (D, \chi) \in S'\} \cup \{\pi_2^* D\}$ where D is the toric boundary of \mathbb{P}^1 , and π_i is the i^{th} projection from $Z' \times \mathbb{P}^1$. The example of $dP_3 \times \mathbb{P}^1$, together with a scaffolding with shape $Z = \mathbb{P}^2 \times \mathbb{P}^1$ is illustrated in Fig. 24, setting $a = 1$ and $b = 3$. We thus produce toric degenerations embedded in the following spaces:

- (i) $dP_3 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3 \times \mathbb{P}^2,$
- (ii) $dP_4 \times \mathbb{P}^1 \rightarrow \mathbb{P}^4 \times \mathbb{P}^2,$
- (iii) $dP_5 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2.$

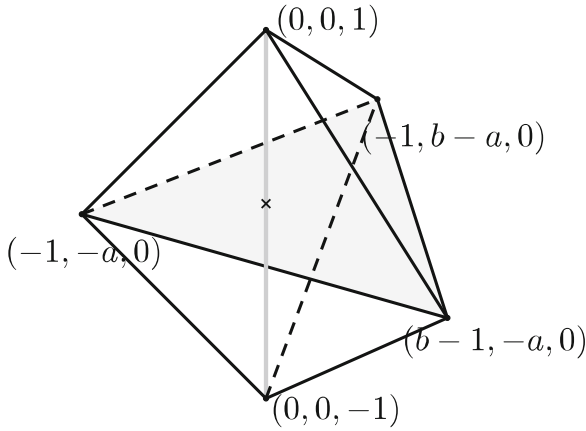


Fig. 24. The scaffolding used to construct $dP_n \times \mathbb{P}^1$ for $n \leq 3$

4.2. $-K_X$ not very ample

There are 7 families of Fano threefolds X for which $-K_X$ is not very ample. These fall into three distinct groups. We first consider the varieties

- (i) B_1 , a sextic in $\mathbb{P}(1, 1, 1, 2, 3)$; and,
- (ii) V_2 , a sextic in $\mathbb{P}(1, 1, 1, 1, 3)$.

Writing x_i for homogeneous co-ordinates of degree 1, and y, z for those of degree 2 and 3 respectively, B_1 degenerates to the toric hypersurface $x_2 y z = x_0^6$; while V_2 degenerates to the toric variety $x_1 x_2 x_3 z = x_0^6$. These toric varieties correspond to scaffoldings of non-reflexive toric varieties with shape \mathbb{P}^2 and \mathbb{P}^3 respectively. The scaffolding used to construct B_1 is illustrated in Fig. 25 in the case $(a, b) = (2, 6)$. The details of these constructions follow those described in Sect. 3.3.

The second group consists of the following three families of Picard rank 2 Fano threefolds.

- (i) 2-1, the blow up of B_1 is an elliptic curve formed by intersecting two members of $-\frac{1}{2}K_{B_1}$.
- (ii) 2-2, a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ branched along a divisor of bidegree $(2, 4)$.
- (iii) 2-3, the blow up of V_2 is an elliptic curve formed by intersecting two members of $-\frac{1}{2}K_{V_2}$.

In each case a toric complete intersection construction is given in [16], and each construction admits a toric degeneration to an embedding described by Laurent inversion. The corresponding scaffoldings have shapes $\mathbb{P}^2 \times \mathbb{P}^1$, \mathbb{P}^3 , and $\mathbb{P}^2 \times \mathbb{P}^1$ respectively. Letting $(x_0 : x_1 : x_2 : y : z)$ be homogeneous co-ordinates on $\mathbb{P}(1, 1, 1, 2, 3)$, and $(s_0 : s_1)$ be co-ordinates on \mathbb{P}^1 , varieties in the family 2-1 degenerate to the toric variety given by the binomial equations

$$\begin{cases} x_2 y z = x_0^6 \\ x_1 s_1 = x_0 s_0 \end{cases}$$

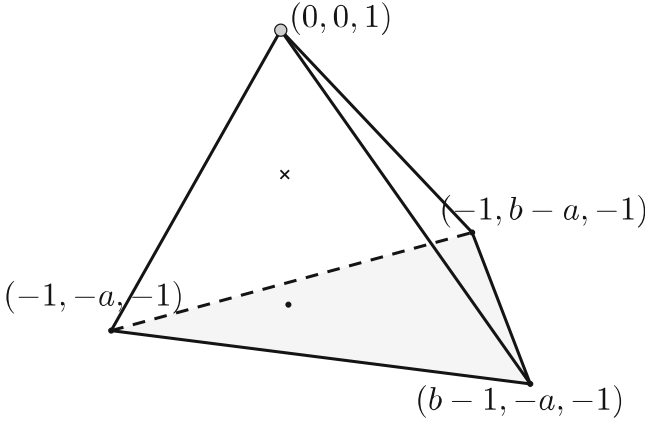


Fig. 25. The scaffolding used to construct B_i for each $i \in [3]$

in $\mathbb{P}(1, 1, 1, 2, 3) \times \mathbb{P}^1$. Varieties in the family 2–3 degenerate to the toric variety given by the binomial equations

$$\begin{cases} x_2x_3y = x_0^4 \\ x_1s_1 = x_0s_0 \end{cases},$$

where $(x_0 : x_1 : x_2 : x_3 : y)$ are homogeneous co-ordinates on $\mathbb{P}(1, 1, 1, 2)$. Finally, varieties in the family 2–2 degenerate to the hypersurface $x_1y_1y_2w = x_0^2y_0^4$ in the variety F described in [16, p. 25].

Finally, we have the following two families of products

- (i) $dP_2 \times \mathbb{P}^1$, recalling that dP_2 is a quartic in $\mathbb{P}(1, 1, 1, 2)$; and,
- (ii) $dP_1 \times \mathbb{P}^1$, recalling that dP_1 is a sextic in $\mathbb{P}(1, 1, 2, 3)$.

Let Q_1 and Q_2 denote the polygons associated to the toric varieties given by the binomials $\{x_1x_2y = x_0^4\}$ and $\{x_0^6 = x_1yz\}$ respectively. Q_1 and Q_2 are triangles and the corresponding scaffolding (with shape \mathbb{P}^2) covers each of these with a single strut. Hence we can scaffold $\text{conv}(Q_i, (0, 0, 1), (0, 0, -1))$ with a pair of struts—following the constructions made in Sect. 4.1—embedding $dP_2 \times \mathbb{P}^1 \rightarrow \mathbb{P}(1, 1, 1, 2) \times \mathbb{P}^2$ and $dP_1 \times \mathbb{P}^1 \rightarrow \mathbb{P}(1, 1, 2, 3) \times \mathbb{P}^2$. These scaffoldings are illustrated in Fig. 24, setting $(a, b) = (1, 4)$ and $(a, b) = (2, 6)$ respectively.

5. Classifying cracked 3-topes

We consider the combinatorial problem of classifying cracked polytopes, and present an algorithm to obtain such a classification in three dimensions.

5.1. One-dimensional shape variety

We refer to polytopes cracked along the fan of \mathbb{P}^1 as *cracked in half*, since their intersection with a pair of half spaces form unimodular polytopes.

Since polytopes cracked in half are reflexive [44, Proposition 2.5], we can proceed from the classification of reflexive 3-topes. Given a reflexive polytope $P \subset M_{\mathbb{R}}$, we define V_P to be the vector space spanned by the vertices $v \in P$ such that the tangent cone C_v to P at v is not unimodular. If P is cracked along \mathbb{P}^1 these must lie in a proper linear subspace of $M_{\mathbb{R}}$. Moreover, by [44, Proposition 2.8], no facet of P° contains an interior point. We use Magma to search for reflexive polytopes meeting both these conditions, and obtain a list of 91 reflexive 3-topes. In 73 cases V_P is two-dimensional, and hence unique determines the direction of the line segments used to scaffold P° . The remaining polytopes contain a square facet and admit two possible full scaffoldings.

Testing which of these 91 polytopes are cracked in half, we find there are 82 three dimensional polytopes cracked along the fan of \mathbb{P}^1 and we list these reflexive polytopes in Table 6. These polytopes are specified by their index in the Kreuzer–Skarke list of reflexive 3-topes. Note that, as elsewhere, we index this list from zero. The column *Fano* indicates the families Fano threefolds X for which there is a mirror Minkowski (as defined in [15, 16]) polynomial f such that $\text{Newt}(f)$ is isomorphic to the reflexive polytope with the indicated ID. Note that in each case there is at most one such family of Fano threefolds. Applying Laurent inversion to a full scaffolding on P with shape $Z = \mathbb{P}^1$, we obtain X_P as a Fano hypersurface. We expect to recover X by passing to a general hypersurface, although we have only partial results in this direction.

Proposition 5.1. [42] *For each P in Table 6 with no associated Fano threefold, X_P is not smoothable.*

Proof. The list of reflexive 3-topes with no associated Fano in Table 6 is a subset of the list of non-smoothable Fano threefolds which appears in work of Petracci [42, p. 10]. \square

Proposition 5.2. [22] *For each polytope P indexed in Table 6 such that each torus invariant point of X_P is either a smooth point, or an ordinary double point, X_P smooths to the indicated Fano manifold.*

Proof. By Namikawa’s results [40] all such toric varieties admit a smoothing. The invariants of the smoothed varieties were computed by Galkin in [22]. \square

Assuming the toric Fano varieties associated to the reflexive polyhedra listed in Table 6 all smooth as indicated, there are 22 non-toric Fano threefolds obtained from polytopes cracked along the fan of $Z = \mathbb{P}^1$; these are:

$$\begin{aligned} Q^3, & 2-29, 2-30, 2-31, 2-32, 3-14, \\ & 3-18, 3-19, 3-20, 3-21, 3-22, 3-23, \\ & 3-24, 4-3, 4-5, 4-6, 4-8, 4-9, \\ & 4-10, 4-11, 4-12, 4-13. \end{aligned}$$

Table 6. Reflexive polytopes cracked in two

PALP ID	Fano	PALP ID	Fano	PALP ID	Fano
2	Q^3	70	2–31	203	3–14
4	Q^3	72	2–29	205	3–23
14	2–30	73	3–26	207	3–24
15	–	74	3–25	208	3–20
16	–	75	3–19	212	3–18
18	3–27	76	3–22	214	3–21
19	2–29	77	3–23	215	4–10
20	3–31	78	3–24	216	4–8
21	2–31	79	3–24	217	4–9
22	2–32	80	3–20	218	4–8
23	2–32	81	3–28	289	3–9
24	2–34	131	2–29	341	3–18
34	2–28	143	3–14	344	3–9
46	2–31	171	2–29	346	4–9
52	3–28	178	3–23	354	3–9
55	2–28	180	4–10	374	3–9
57	3–19	181	4–12	393	3–18
58	–	184	3–21	404	3–20
59	–	186	3–14	408	5–2
60	4–13	190	4–12	409	4–6
61	–	191	4–11	426	4–6
62	4–11	192	–	427	4–5
63	3–18	193	–	683	4–5
64	3–22	194	5–2	684	4–3
65	–	195	5–3	727	4–5
66	–	196	4–5	728	4–3
67	4–9	197	–	735	4–3
69	2–28				

5.2. Classification algorithm

We present the general form of an algorithm which we can use to classify three-dimensional polytopes cracked along a given two-dimensional generalised fan Σ . Fixing a choice of Z , and letting Σ denote the corresponding fan, we first divide cases among possible *wrapping polyhedra*. In what follows we make use of the rays $\Sigma[1]$ of a generalised fan Σ , adopting the following conventions.

- (i) The set of rays of Σ coincides with that determined by the standard definition when Σ is a fan.
- (ii) The set of rays of Σ is the pair of rays (based at the origin) contained in the minimal cone of Σ when the minimal cone of Σ is one-dimensional.
- (iii) The set of rays is empty if the dimension of the minimal cone of Σ is greater than one.

Definition 5.3. Given a polytope P cracked along a generalised fan Σ , let C_v denote the tangent cone to P at a point $v \in P$. The *wrapping polyhedron* of P is the intersection of cones C_v as v varies over the primitive ray generators of Σ .

Note that the set of primitive ray generators is empty in the case $Z = \mathbb{P}^1$, and need not be a subset of the vertex set of P for any choice of shape Z . Given a ray ρ in a generalised fan Σ , we let Z_ρ denote the toric variety associated to ρ as follows. If Σ is a fan, Z_ρ is the usual codimension one subvariety associated to ρ . If ρ is contained in the one-dimensional minimal cone, we let Z_ρ be the toric variety associated to the fan $\bar{\Sigma}$; that is, the quotient of Σ by its minimal cone.

Lemma 5.4. Fix a shape variety Z determined by a generalised fan Σ in $M_{\mathbb{R}}$ and a ray $\rho \in \Sigma[1]$. Let Z_ρ denote the codimension one torus invariant subvariety of Z determined by ρ . There is a canonical inclusion, with bounded image, from the set wrapping polyhedra of reflexive polytopes P cracked along Σ to the set of lattice points in the cone

$$\prod_{\rho \in \Sigma[1]} \{ \overline{\text{Amp}}(Z_\rho) \times (M_{\mathbb{R}}/\mathbb{R}\rho) \}.$$

Proof. Fix a splitting $M \cong \mathbb{Z}v \oplus M_\rho$, and let Σ_ρ denote the fan in M_ρ determined by Z_ρ . The tangent cone at v to a wrapping polyhedron for Σ determines—and is determined by—a piecewise linear function $\theta: (M_\rho) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow M_{\mathbb{R}}$ which is linear on each cone of Σ_ρ , sends $0 \mapsto v$, and sends the cones of Σ_ρ into their corresponding cones in Σ . The connected component of the complement of the image of θ which contains the origin must be a convex set. Such maps θ are in bijection with points in $\overline{\text{Amp}}(Z_\rho) \times (M_{\mathbb{R}}/\mathbb{R}v) \subset \text{Div}_{T_{M_\rho}}(Z_\rho) \cong \mathbb{Z}^r$, for some $r \in \mathbb{Z}_{\geq 0}$. Hence the set of possible wrapping polyhedra is contained in the cone required.

To show this region is bounded, first note that each ray τ of Σ_ρ corresponds to a cone in Σ of dimension 2, generated by v and some $v' \in M$. Since v' must be in the same connected component as the origin of $M_{\mathbb{R}} \setminus \theta((M_\rho) \otimes_{\mathbb{Z}} \mathbb{R})$, the co-ordinate of θ —regarded as an element of \mathbb{Z}^r —corresponding to τ is bounded. Each pair (ρ, τ) , where $\rho \in \Sigma[1]$ and $\tau \in \Sigma_\rho[1]$ defines a linear inequality satisfied by any tuple of piecewise linear maps θ which define a wrapping polyhedron. The intersection of these half spaces with $\overline{\text{Amp}}(Z_\rho) \times (M_{\mathbb{R}}/\mathbb{R}\rho)$ defines a polytope, \mathcal{R}_Σ , which contains the image of each wrapping polyhedron. \square

Recall that a polytope is called *hollow* if it contains no lattice points in its interior.

Definition 5.5. Let Q be a unimodular hollow polytope in $M_{\mathbb{R}}$. We call Q a (*reflexive*) *piece* if $0 \in Q$ and, for any facet F of Q with primitive inner normal vector w , either $0 \in F$, or $w(F) = -1$.

The set of reflexive pieces has an obvious iterative structure: faces of reflexive pieces which contain the origin are themselves reflexive pieces. Thus the classification of reflexive pieces of dimension n makes use of the classification in dimensions $< n$. If Q is a 3-tope there are four cases, depending on the minimal dimension d of the face of Q containing the origin. In particular either

- (i) Q is a reflexive polytope;
- (ii) the origin is the unique interior point of a facet of Q ;
- (iii) the origin is the unique relative interior lattice point of an edge of Q , or;
- (iv) the origin is a vertex of Q , and every edge of Q containing v has lattice length 1.

Note that this generalises both the notion of reflexive polytope (the first case) and the notion of *top* [10] (the second case).

Assuming that the minimal face of Q containing the origin has dimension d , we say that a piece Q has *type* $3 - d$. Given a smooth cone σ with minimal face τ of dimension d , we call a reflexive piece Q' contained in a two-dimensional face of σ a *panel* if $Q' \cap \tau$ has dimension d . Fixing a function \mathbf{p} from two-dimensional faces of σ to panels contained in σ , we can consider the set of pieces P of type $3 - d$ such that every polygon in the image of \mathbf{p} is a facet of P . Let $\mathcal{P}(\mathbf{p})$ denote this set of pieces. Given an element $\varphi \in \mathcal{R}_\Sigma$, let $\mathcal{S}(\varphi)$ denote the set of functions from the collection of two dimensional cones τ of Σ to panels contained in τ which are contained in the wrapping polyhedron defined by φ , and have one dimensional intersection with the boundary of this polyhedron.

Algorithm 5.6. Fix a complete generalised fan Σ in N such that the dimension of the minimal cone of Σ is at most one.

- (i) Compute the integral points in the polytope \mathcal{R}_Σ .
- (ii) Exploit symmetries of Σ to obtain a minimal subset R of \mathcal{R}_Σ , containing a representative of every isomorphism class of cracked polytope in $N_{\mathbb{R}}$.
- (iii) Compute the set $\mathcal{S}(\varphi)$ for each point $\varphi \in R$, and iterate over this set of functions.
- (iv) For each $\varphi \in R$, $\mathbf{p} \in \mathcal{S}(\varphi)$, and maximal cone $\sigma \in \Sigma$, let \mathbf{p}_σ be the restriction of \mathbf{p} to the two dimensional faces of σ . There is a finite subset $\mathcal{A}(\varphi, \mathbf{p}, \sigma)$ of $\mathcal{P}(\mathbf{p}_\sigma)$ such that, for each polytope Q in this subset, $\langle w, v \rangle \geq -1$ for all inner normal vectors w to facets of Q which do not contain the origin, and vertices v of polygons in the image of \mathbf{p} (note that this image is a strict superset of \mathbf{p}_σ).
- (v) For each function from the set of maximal cones σ in Σ to $\coprod_\sigma \mathcal{A}(\varphi, \mathbf{p}, \sigma)$ such that the image of σ is contained in $\mathcal{A}(\varphi, \mathbf{p}, \sigma)$, test whether the union of the polytopes in the image is itself a convex reflexive and cracked polytope.

5.3. Classifying pieces

In order to implement Algorithm 5.6 in dimension n we require a database of pieces in dimension $< n$. We now treat the classification of pieces in dimension ≤ 3 . Note that the classification in dimension n divides into cases depending on the dimension k of the minimal face containing Q . The cases $k = n$ and $k = n - 1$ form known classes: indeed, if $k = n$, the corresponding pieces are polar dual to *smooth* polytopes, which have a well-known classification up to dimension 8 by Øbro [41]. If $k = n - 1$ the definition of reflexive piece coincides precisely with the

notion of a *top* [10,20] which is also a unimodular polytope; we call such polytopes *unimodular tops*.

In dimension one there are two possible cases, depending on the dimension k of the minimal face of P containing the origin:

- If $k = 1$, $P = \text{conv}(-1, 1)$ is a line segment of length two.
- If $k = 0$, $P = \text{conv}(0, 1)$.

It is well-known that hollow polytopes in dimension two are either Cayley polytopes or equal to $T := \text{conv}((0, 0), (2, 0), (0, 2))$ up to integral affine linear transformations. Hence we have three cases for pieces P in \mathbb{R}^2 , depending on the dimension k of the minimal face of P containing the origin:

- If $k = 2$, P is a reflexive polygon, of which five are unimodular.
- If $k = 1$, $P = T$ or a quadrilateral isomorphic to

$$\text{conv}((0, -1), (0, 1), (1, -1), (1, m)),$$

for some $m \in \mathbb{Z}_{\geq 0}$.

- If $k = 0$, P is isomorphic to

$$\text{conv}((0, 0), (0, 1), (0, 1), (1, m)),$$

for some $m \in \mathbb{Z}_{\geq 0}$.

In dimension three we have four possible cases depending on k . In the case $k = 3$, P is a unimodular reflexive polytope, of which there are 18. If $k = 2$, P is a *unimodular top*. We do not describe the classification of unimodular tops in dimension 3, as the algorithm given in Sect. 5.1 to treat the case $Z = \mathbb{P}^1$ does not rely on this classification. Moreover, this classification is contained in that of all three dimensional tops made by Bouchard–Skarke [8].

Assume next that $k = 1$; that is, assume that the origin lies in an edge E of the piece $P \subset \mathbb{R}^3$. Fixing a vertex $v \in E$, and making a change of co-ordinates, we can assume that the edges incident to v are parallel to the co-ordinate lines, E has direction e_3 , and $v = (0, 0, -1)$. Since E is itself a reflexive piece of dimension one, $(0, 0, 1)$ is a vertex of P . Let F_1 and F_2 denote the facets of P containing the origin. For each $i \in \{1, 2\}$, F_i contains an edge E_i incident to $(0, 0, 1)$ with direction vectors $(1, 0, \alpha_1)$ and $(0, 1, \alpha_2)$ respectively, such that, by the unimodularity of F_i , $\alpha_i \geq -1$. Assume without loss of generality that $\alpha_1 \geq \alpha_2$. Since F_i is a reflexive piece for each $i \in \{1, 2\}$, if $\alpha_i > -1$ we have that

$$F_i = \text{conv}(e_3, -e_3, e_i - e_3, e_i + (\alpha_i + 1)e_3);$$

while if $\alpha_i = -1$ we have that additional possibility that $F_i \cong T$. Let $\alpha := (\alpha_1, \alpha_2) \in \mathbb{Z}^2$, and, fixing a value of $l \in \mathbb{Z}_{\geq 0}$, define the Cayley polytopes $P(\alpha, l, 1)$ and $P(\alpha, l, 2)$ to be the convex hulls of the points given by the columns of the matrices

$$\begin{matrix} 0 & 0 & 1 & 0 & 1 & 0 & l & l \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & (\alpha_1 + 1) & (\alpha_2 + 1) & -1 & (\alpha_2 + l\alpha_1 + 1) \end{matrix}$$

and

$$\begin{array}{cccccccc}
 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
 0 & 0 & 0 & 1 & 0 & 1 & l & l \\
 1 & -1 & -1 & -1 & (\alpha_1 + 1) & (\alpha_2 + 1) & -1 & (\alpha_1 + l\alpha_2 + 1)
 \end{array}$$

respectively.

Lemma 5.7. *Let $P_i, i \in \{1, \dots, k\}$ be a collection of d -dimensional lattice polytopes in \mathbb{R}^d . If $P := P_1 \star \dots \star P_k \subset \mathbb{R}^{d+k}$ is a unimodular polytope, there is a non-singular projective toric variety Z such that P_i is the polyhedron of sections of an ample divisor D_i on Z for all $i \in \{1, \dots, k\}$.*

Proof. Since $P_{i_1} \star P_{i_2}$ is a face of P for any $i_1, i_2 \in \{1, \dots, k\}$, we assume without loss of generality that $k = 2$. Since P_1 is unimodular, its normal fan defines a non-singular projective toric variety Z . We claim that $P_2 = P_D$ for some ample divisor on Z .

Note that $\text{verts}(P) = \text{verts}(P_1) \amalg \text{verts}(P_2)$. Moreover, each vertex v of P_1 is contained in d edges of P_1 and $(d + 1)$ edges of P . Hence, fixing a facet F of P containing v and different from P_1 , $F \cap P_1$ is equal to a facet G of P_1 . G contains $(d - 1)$ edges of P_1 incident to v .

The normal fan of P consequently contains a ray for each facet of P_1 (or P_2), as well as rays ρ_1, ρ_2 dual to P_1 and P_2 respectively. Moreover, each vertex of P_1 is dual to a maximal cone, generated by ρ_1 and rays corresponding to facets of P_1 containing v . Since the same applies to vertices of P_2 , the toric variety associated to the normal fan of P has the structure of a fibre bundle over \mathbb{P}^1 , in particular the fibres over 0 and ∞ are isomorphic. □

Lemma 5.8. *If $\alpha_1, \alpha_2 > -1$, then P is isomorphic to $P(\alpha, l, j)$ for some $l \in \mathbb{Z}_{\geq 0}$ and $j \in \{1, 2\}$.*

Proof. The point $(1, 1, 0)$ cannot lie in the interior of P , and hence there is a $u \in N$ such that $\langle u, (1, 1, 0) \rangle \leq -1$, but $\langle u, p \rangle \geq -1$ for any point $p \in P$. In particular, writing $u = (u_1, u_2, u_3)$, and recalling that $(0, 0, \pm 1) \in P$, we have that $u_3 \in \{-1, 0, 1\}$. Similarly, $u_1 \geq -1 + u_3, u_2 \geq -1 + u_3, u_1 \geq -1 - \alpha_1 u_3$ and $u_2 \geq -1 - \alpha_2 u_3$. Hence, if $u_3 = 1, u_1 \geq 0$ and $u_2 \geq 0$, but no such points satisfy $u_1 + u_2 \leq -1$. If $u_3 = 0$, we have the solutions $(u_1, u_2) = (-1, -1), (-1, 0)$, or $(0, 1)$. These all define the Cayley sum of a pair of quadrilaterals, as T is not a panel of P by the assumption that $\alpha_i > -1$ for each $i \in \{1, 2\}$. Since the panels of P are Cayley polytopes (the sum of two line segments)—and P is unimodular— P is the Cayley sum of a pair of polyhedra of sections of ample divisors on a (fixed) Hirzebruch surface by Lemma 5.7. Such a polytope is isomorphic to $P(\alpha, l, j)$ for some α, l , and j .

In the case $u_3 = -1$ the bounds $\alpha_i > -1$ for each $i \in \{1, 2\}$, together with the inequalities $u_1 \geq -1 - \alpha_1 u_3$ and $u_2 \geq -1 - \alpha_2 u_3$, ensure that there are no further cases. □

Note that $P(\alpha, 0, 1) = P(\alpha, 0, 2)$ and $P(\alpha, -1, 1) = P(\alpha, -1, 2)$. Note also that whenever $\alpha_1 = \alpha_2, P(\alpha, l, 1) \cong P(\alpha, l, 2)$, although these polytopes are not

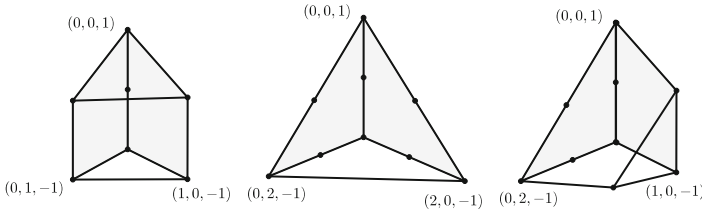


Fig. 26. Three exceptional pieces

equal. The remaining cases are $(\alpha_1, \alpha_2) = (0, -1)$ and $(\alpha_1, \alpha_2) = (-1, -1)$. In the latter case P is a sub-polytope of $\text{conv}(-e_3, e_3, 2e_1 - e_3, 2e_2 - e_3)$, and hence there are three possible polytopes, illustrated in Fig. 26. In the case $(\alpha_1, \alpha_2) = (0, -1)$, we introduce another infinite class of polytopes. Fixing a value of $l \in \mathbb{Z}_{\geq 1}$ define the ‘wedge’ polytope $W(l)$ to be the convex hull of the points given by the columns of the following matrix,

$$\begin{matrix} 0 & 0 & 0 & 1 & 1 & l & l & 2(l-1) \\ 0 & 0 & 2 & 0 & 0 & 1 & 1 & 2 \\ 1 & -1 & -1 & -1 & 1 & 0 & -1 & -1 \end{matrix} .$$

See Fig. 27 for an illustration of such a polytope. We also define

$$W'(l) := W(l) \cap \{x : \langle (-1, 1, 0), x \rangle \leq 1\}$$

for each l . There are additional cases which appear for small values of l ; in particular we define the polytopes $W_0(l)$ to be the convex hull of the points given by the columns of the following matrix,

$$\begin{matrix} 0 & 0 & 0 & 1 & 1 & 2l-1 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 1 & -1 & -1 & -1 & 1 & -1 \end{matrix} .$$

and $W'_0(l) := W_0(l) \cap \{x : \langle (-1, 1, 0), x \rangle \leq 1\}$ for each $l \in \{1, 2\}$.

Lemma 5.9. *If $\alpha = (0, -1)$, then P is isomorphic to one of*

- (i) $W(l)$, for some $l \in \mathbb{Z}_{\geq 2}$,
- (ii) $W'(l)$, for some $l \in \mathbb{Z}_{\geq 2}$,
- (iii) $W_0(l)$ for $l \in \{1, 2\}$,
- (iv) $W'_0(1)$; or,
- (v) $P(\alpha, l, 1)$ for some $l \in \mathbb{Z}_{\geq 0}$.

Proof. Since $\alpha = (0, -1)$ the polytope P is contained in the half-space $\{x : \langle (0, 1, 1), x \rangle \leq 1\}$. Moreover P is assumed to be contained in the translate of the positive orthant based at $(0, 0, -1)$; that is,

$$P \subset A := \mathbb{R}_{\geq 0} \times \text{conv}((0, \pm 1), (2, -1)) .$$

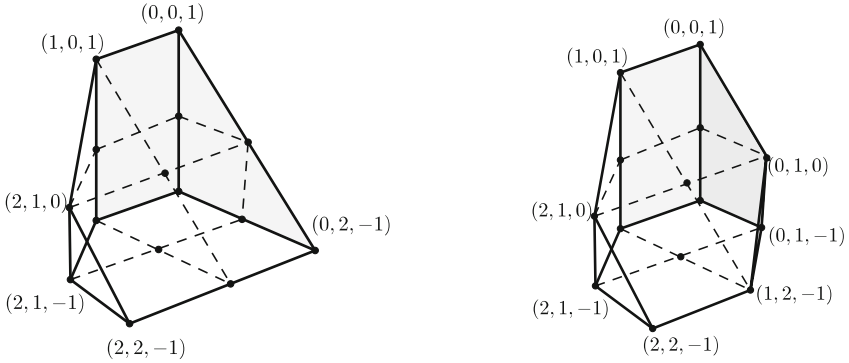


Fig. 27. Pieces $W(2)$ and $W'(2)$

We claim such pieces P are determined by the facet $F = P \cap \{u : \langle u, e_3^* \rangle = -1\}$. Indeed, fixing this polygon F it is easy to verify that

$$P = A \cap (F \times \mathbb{R}).$$

The possible polygons F are also easily classified. Choose co-ordinates on \mathbb{R}^2 such that the origin and $(1, 0)$ are vertices of F . If $F \cap \{y = 2\} = \emptyset$ both F and P are Cayley polytopes, and $P = P(\alpha, l, 1)$ for some $l \geq 0$. Otherwise F is a (possibly degenerate) hexagon with vertices given by the columns of

$$\begin{matrix} 0 & 1 & 1 & 2 & 2 & a \\ 0 & 0 & k_1 & k_2 & k_3 & 0, \end{matrix}$$

where $a \in \{1, 2\}$. Fix a value of $k_1 \geq 0$. By convexity and unimodularity of F at $(1, k_1)$, we have that $k_2 = 2(k_1 - 1)$; unless $k_1 \in \{1, 2\}$; which gives the additional cases $(k_1, k_2) = (1, 1)$ and $(k_1, k_2) = (2, 3)$. If $a = 2, k_3 = 0$ and $P = W(l)$ for some $l \in \mathbb{Z}_{\geq 0}$ or $W_0(l)$ for some $l \in \{0, 1\}$. Otherwise $a = 1$ and we have that $k_3 = 1$ (note $k_3 \neq 0$ as $(1, 0)$ is vertex of F) by unimodularity of F at the point $(2, k_3)$. In these cases $P = W'(l)$ for some $l \in \mathbb{Z}_{\geq 1}$ or $W'_0(2)$. Note that $W(1)$ and $W'_0(1)$ are not unimodular. Moreover, $P(\alpha, l, 1) = P(\alpha, l, 1)$ for $l \in \{0, 1\}$, while $P(\alpha, l, 2)$ is not unimodular if $l > 1$. □

We summarise the above calculations in the following proposition.

Proposition 5.10. *If P is a 3-dimensional piece and the origin is contained in the relative interior of an edge of P , then P belongs to one of the infinite families $P(\alpha, l, j)$, one of the three exceptional cases shown in Fig. 26, or one of the polytopes listed in Lemma 5.9.*

Finally, assume that $k = 0$. For each $l \in \mathbb{Z}_{\geq 0}$ and $j \in \{1, 2\}$, we define the Cayley polytopes $Q(\alpha, l, j)$ to be the intersection of $P(\alpha, l, j)$ with the half-space $\{u \in \mathbb{R}^3 : \langle e_3^*, u \rangle \geq 0\}$.

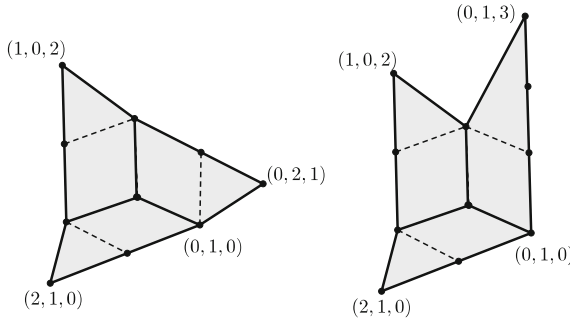


Fig. 28. Relative arrangement of panels

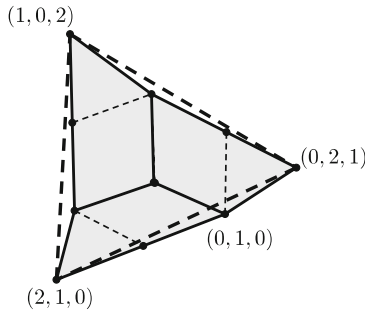


Fig. 29. An impossible arrangement of panels

Proposition 5.11. *If P is a 3-dimensional piece and the origin is a vertex of P , then P belongs to the infinite family $\mathcal{Q}(\alpha, l, j)$. The polytope $\mathcal{Q}(\alpha, l, j)$, where $\alpha = (\alpha_1, \alpha_2)$, is a reflexive piece if and only if one of the following holds.*

- (i) $\alpha_1 \geq 0, \alpha_2 \geq 0, j \in \{1, 2\}$, and $l \in \mathbb{Z}_{\geq 0}$.
- (ii) $\alpha_1 = 0, \alpha_2 = -1, j = 1$ and $l \in \mathbb{Z}_{\geq 0}$.
- (iii) $\alpha_1 \geq 0, \alpha_2 = -1, j = 2$ and $l = \alpha_1 + 1$.
- (iv) $\alpha_1 = -1, \alpha_2 = -1$.

Note that the only polytope which appears in the fourth case is the standard simplex.

Proof. The vertex set of a piece P contains the origin, and—in a suitable co-ordinate system—each of the three standard basis vectors. The polygon $F_i := \{e_i^* = 0\} \cap P$ is a two dimensional reflexive piece, which were classified above.

Thus we may assume that each polygon F_i is either a standard triangle or a Cayley sum of line segments. These polygons may be oriented relative to each other in two distinct ways, illustrated in Fig. 28. We show that the first case does not include any piece which is not a special case of the second. Polytopes in the first case contain vertices $(1, 0, k_1), (k_2, 1, 0)$, and $(0, k_3, 1)$. Note that we can assume that $k_i \geq 2$. If $k_i > 2$ for any $i \in \{1, 2, 3\}$, the lattice point $(1, 1, 1)$ is in the interior of the convex hull of the vertices of P , and hence $k_1 = k_2 = k_3 = 2$. However, as P is contained in the half space $\{u \in \mathbb{R}^3 : (1, 1, 1) \cdot u \leq 3\}$, P is a sub-polytope of

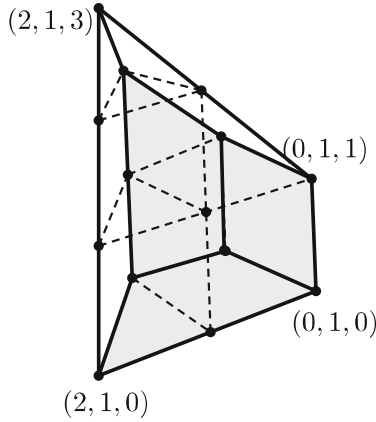


Fig. 30. The piece $Q((1, 0), 1, 1)$

the convex hull P' of the vertices shown in Fig. 29. Note that every vertex of this polytope is contained in a panel, and hence $P = P'$. Since P' is not unimodular it does not contribute to the list of pieces.

In the second case illustrated in Fig. 30, we observe that P is a Cayley polytope. Indeed, assuming that P contains the vertices $(1, 0, k_1)$, $(k_2, 1, 0)$, and $(0, 1, k_3)$, P is the Cayley sum of the facets contained in H_0 and H_1 , where $H_k := \{u : \langle e_2^*, u \rangle = k\}$. These are both 2-dimensional if $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$; and in this case it follows from Lemma 5.7 that P is of the form $Q(\alpha, l, j)$ for some $l \in \mathbb{Z}_{\geq 0}$ and $j \in \{1, 2\}$. The classification of the remaining possible pieces follows from a case-by-case analysis. The case $\alpha = (-1, -1)$ is trivial. If $\alpha = (0, -1)$, P is contained in the product of a standard simplex and a ray, and equal to some $Q(\alpha, 1, l)$. If $\alpha_1 > 0$ and $\alpha_2 = -1$ we note that the polytopes $Q(\alpha, 1, l)$ are not unimodular, while $Q(\alpha, 2, l)$ is a Cayley polytope $P_1 \star P_2$, such that P_1 is a standard simplex. P_2 is a dilate of a standard simplex by Lemma 5.7, and hence $l = \alpha_1 + 1$. \square

6. Connection to the Gross–Siebert program

The results and computations of this article fit into a larger program of research, directed toward a novel method of Fano classification. In particular, the authors of [16] construct a database of polytopes which support a mirror (Minkowski) Laurent polynomial to a given Fano threefold, see www.fanosearch.net. It is conjectured that this database describes precisely the toric varieties (associated to Minkowski polytopes) which smooth to a given Fano threefold.

In this article we have constructed degenerations proving part of this conjecture: every toric variety we obtain by degenerating a Fano threefold appears in the database generated in [16]. As discussed in the introduction, the Gross–Siebert program suggests a general approach to relate toric degeneration and mirror Laurent polynomials. Loosely, we first degenerate the toric variety X_P , associated to the Newton polytope P of a Minkowski polynomial, to a union of toric varieties. Using

methods from tropical and log geometry we can then (attempt to) generate both the smoothing of X_P to a Fano threefold and the Laurent polynomial mirror.

More specifically, we expect that our families are fibrewise compactifications of families mirror to certain log Calabi–Yau varieties, which may themselves be constructed from a scaffolding. The two-dimensional version of this program is current work in progress with Barrott and Kasprzyk [4], and we now outline the main features of the construction. As remarked in the introduction, if this program were complete in dimension three, the current work would relate the constructions of Mori–Mukai with the toric degeneration descriptions obtained via mirror symmetry considerations.

6.1. Compactifying families of log Calabi–Yau varieties

Fix a scaffolding S of a Fano polytope P with shape determined by the generalised fan Σ . Assume, for simplicity, that $N_U = \{0\}$, and hence $\tilde{N} = N \cong \mathbb{Z}^n$. The induced inclusion $\iota: X_P \hookrightarrow Y_S$ fits into the commutative diagram

$$\begin{array}{ccc} X_P & \hookrightarrow & Y_S \\ \uparrow & & \uparrow \\ (\mathbb{C}^*)^n & \hookrightarrow & \mathbb{C}^{\Sigma(1)}, \end{array}$$

where the horizontal and vertical arrows are closed and open embeddings respectively. Using standard methods, we can degenerate $(\mathbb{C}^*)^n$ into a union of copies of \mathbb{C}^n , determined by the cones of the unimodular generalised fan Σ . Moreover, there is a canonical embedding of this degeneration into $\mathbb{C}^{\Sigma(1)}$. We propose to consider the extension of this degeneration over the base of a family of log Calabi–Yau varieties considered by Gross–Hacking–Keel (in two dimensions) [25], and by Gross–Hacking–Siebert [26] in higher dimensions.

Assume for now that $n = 2$, and S is a full scaffolding of P . Let Z denote the shape variety of S , the toric variety associated to Σ . We construct a log Calabi–Yau variety U by blowing up points on the toric boundary of Z , and propose that the mirror family $\mathcal{V} \rightarrow T$ —constructed in [25]—fits into the following commutative diagram, where $\mathcal{X} \rightarrow T$ is projective and flat, and \mathcal{V} is an open subscheme of \mathcal{X} :

$$\begin{array}{ccc} \mathcal{X} & \hookrightarrow & Y_S \times T \\ \uparrow & & \uparrow \\ \mathcal{V} & \hookrightarrow & \mathbb{C}^{\Sigma(1)} \times T \end{array} \tag{4}$$

We construct the variety U using a notion of *mutability* for the scaffolding S . We recall from [1] that a mutation of a polygon is determined by a *weight vector* $w \in M \cong \mathbb{Z}^2$, and a *factor* $F \subset w^\perp$. We refer to [1] for the full definition of polytope mutation, but recall that a polytope P admits a mutation with respect to (w, F) if and only if each

$$P_a := P \cap \{x : \langle x, w \rangle = a\}$$

contains a translate of the polytope $\max(-a, 0)F$ whenever $P_a \neq \emptyset$. Fixing a convention for the orientation of w^\perp , a mutation in two dimensions is determined by the weight vector w ; taking F to be a unit length line segment in the one dimensional vector space w^\perp .

Definition 6.1. Given a pair (w, F) , we say that S admits a mutation in (w, F) if the polytope $P_D + \chi$ admits this mutation for each element $(D, \chi) \in S$.

Fix a scaffolding S with shape Z , where Z is a product of projective spaces. We recall from [17] that there is a standard choice of Laurent polynomials f_s , such that $\text{Newt}(f_s) = P_D + \chi$, where $s = (D, \chi) \in S$. Thus there is a standard choice of Laurent polynomial

$$f = \sum_{s \in S} f_s$$

such that $\text{Newt}(f) = P$. If S is mutable, the Laurent polynomial f admits an algebraic mutation [1] (also called a symplectomorphism of cluster type [32]). Hence we expect that f defines a global function on the variety U defined (in the 2 dimensional case) as follows.

Construction 6.2. Let v_ρ denote the ray generator of the ray ρ of Σ . Given a scaffolding S of P , suppose that S admits a mutation with weight vector v_ρ and factor F_ρ of lattice length ℓ_ρ . Let U be the complement of the strict transform of the toric boundary of Z under the blow-up π of Z with ℓ_ρ distinct reduced centres on the boundary divisor of Z corresponding to each ray ρ .

The following conjecture is the main result of [4].

Conjecture 6.3. *The mirror family \mathcal{V} to the log Calabi–Yau U constructed in [25] fits into the commutative diagram (4).*

Conjecture 6.3 offers a systematic way of constructing the deformations we build by hand throughout this article. The situation in higher dimensions is the subject of current and exciting research. We particularly refer here to ongoing work of Corti–Hacking–Petracci [18], which may be interpreted as an extension of Conjecture 6.3 to higher dimensions, in which the map $X_P \rightarrow Y_S$ is the anti-canonical embedding of the Gorenstein toric Fano variety X_P .

6.2. Example: A_2 cluster variety

We consider a particular case of the mirror family to a log Calabi–Yau in some detail. Let Z be the toric variety dP_7 , obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ in a single torus invariant point. Let \tilde{U} be the blow up of a (non-special) point on each of the pair of torus invariant curves C in Z such that $C^2 = 0$. Let U be the complement of the strict transform of the toric boundary of Z in \tilde{U} . It is well known, see for example [24], that U is the \mathcal{A} cluster variety associated to an A_2 quiver. The mirror family, described by [25] using θ -functions, is a family over $\text{Spec}(\mathbf{k}[\overline{\text{NE}}(\tilde{U})]) \cong \mathbb{A}_{\mathbf{k}}^5$, for

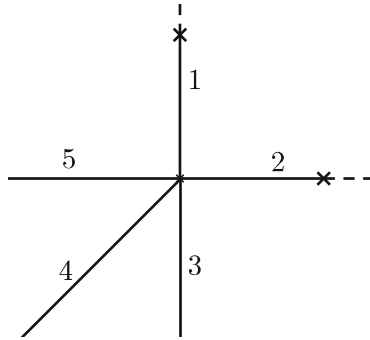


Fig. 31. Affine manifold associated to V

a choice of ground field \mathbf{k} . Specialising the parameters corresponding to $\overline{NE}(Z)$ to 1, we obtain the 2-parameter family defined by the 4×4 Pfaffians of the matrix

$$\begin{pmatrix} 0 & 1 & x_1 & x_2 & 1 \\ & 0 & t_1 & x_3 & x_4 \\ & & 0 & 1 & x_5 \\ & & & 0 & t_2 \\ & & & & 0 \end{pmatrix}, \tag{5}$$

where x_1, \dots, x_5 denote the theta functions corresponding to the five rays shown in Fig. 31. The parameters t_1 and t_2 correspond to the curve classes of the exceptional locus of the contraction $\tilde{U} \rightarrow Z$. We associate an integral affine manifold (with singularities) to U , illustrated in Fig. 31. The singular locus consists of a pair of *focus-focus* singularities. There is a monodromy operator conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ associated to each focus-focus singularity such that the subspace invariant under each operator is parallel to the ray containing the corresponding singular point.

Fix a Fano polygon P together with a scaffolding S which has shape Z . An example of such a scaffolding is shown in Fig. 33. Given an element $s = (D, 0) \in S$, let s_i denote the i th co-ordinate of $D \in \text{Div}_T(Z) = \mathbb{Z}^5$, using the ordering of the basis elements shown in Fig. 31. Recalling that the scaffolding S defines a toric embedding $X_P \rightarrow Y_S$, let x_i be the homogeneous coordinate corresponding to the i th basis element in $\text{Div}_T(Z)$ for each $i \in \{1, \dots, 5\}$.

The scaffolding S determines an embedding of X_P of codimension 3. Moreover, explicitly computing the ideal of this toric embedding, the image of X_P in Y_S is given by the 4×4 Pfaffians of the matrix

$$\begin{pmatrix} 0 & \prod_s y_s^{s_1+s_4-s_5} & x_1 & x_2 & \prod_s y_s^{s_2+s_4-s_3} \\ & 0 & 0 & x_3 & x_4 \\ & & 0 & \prod_s y_s^{s_3+s_5-s_4} & x_5 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}. \tag{6}$$

Note that each of the exponents of entries in the matrix appearing in (6) is non-negative as, writing $s = (D, 0)$, D is nef for any $s \in S$. In fact the nef cone of

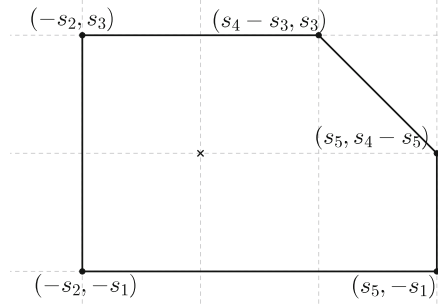


Fig. 32. Example of a strut with shape Z

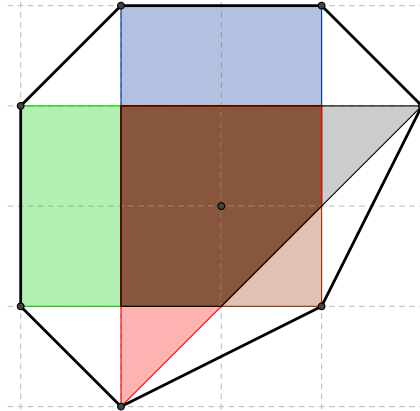


Fig. 33. Example scaffolding with shape Z

dP_7 is defined by the inequalities $s_3 + s_5 \geq s_4$, $s_2 + s_4 \geq s_3$, and $s_1 + s_4 \geq s_5$. This can easily be deduced from Fig. 32, which illustrates a general polygon with shape Z .

The variety X_P fits into the two-parameter family defined by the 4×4 Pfaffians of the matrix

$$\begin{pmatrix} 0 & \prod_s y_s^{s_1+s_4-s_5} & x_1 & x_2 & \prod_s y_s^{s_2+s_4-s_3} \\ & 0 & t_1 \prod_s y_s^{s_1+s_3-s_2} & x_3 & x_4 \\ & & 0 & \prod_s y_s^{s_3+s_5-s_4} & x_5 \\ & & & 0 & t_2 \prod_s y_s^{s_2+s_5-s_1} \\ & & & & 0 \end{pmatrix}$$

if and only if the exponents $s_1 + s_3 \geq s_2$ and $s_2 + s_5 \geq s_1$ are non-negative for each $s \in S$. However, this is immediately equivalent to the mutability S with respect to the weight vectors $(1, 0)$ and $(0, 1)$. Hence, mutability of the scaffolding is precisely the condition required for the mirror family to admit a compactification in Y_S .

For example, consider the polygon P , with scaffolding S shown in Fig. 33. This is evidently a mutable scaffolding, and indeed a general fibre of the family has equations identical to those used to construct the log del Pezzo surface $X_{5,5/3}$ in [19].

6.3. The Gross–Siebert algorithm

As well as the approach exploiting the results [25, 26] detailed above, we may attempt to make direct use of the *Gross–Siebert algorithm*, introduced in [27]. The existence of this algorithm entails a powerful smoothing result, namely that any *locally rigid, positive, pre-polarized tori log Calabi–Yau space* arises from a formal degeneration of log Calabi–Yau pairs. Moreover, the extension of these families to families over an analytic base with canonical co-ordinates is known (at least in the Calabi–Yau context) and we refer to the article [45] of Ruddat–Siebert for further details.

Therefore, if we can adapt our constructions to define such a toric log Calabi–Yau space we can define a smoothing using constructions in logarithmic geometry. The technical difficulties here are two-fold.

- (i) Local rigidity is a strong condition, and is restrictive even in three dimensions.
- (ii) Construction of a locally rigid toric log Calabi–Yau involves refining the triangulation of P , and we lose a reasonable ambient space for the resulting formal degeneration.

In [27] the authors explain how toric log Calabi–Yau spaces may be constructed from certain integral affine manifolds, together with additional discrete data (such as a polyhedral decomposition). Local rigidity is related to the notion of *simplicity* of the singularities of an integral affine manifold B associated to a toric log Calabi–Yau space. In three dimensions, an integral affine manifold with simple singularities is a topological manifold with an integral affine structure in the complement of a trivalent graph Δ , together with conditions on the monodromy of the integral affine structure around edges of Δ .

In our context B is the polytope P^* , dual to P , and we note that an integral affine manifold with simple singularities corresponding to each family of Fano threefolds was constructed in the author’s earlier work [43]. Constructing a *polyhedral decomposition* and *polarization* compatible with these integral affine manifolds allows us to describe a locally rigid toric log Calabi–Yau space. Indeed, adapting the constructions in [43], we expect that the Gross–Siebert algorithm can be used to construct all families of Fano 3-folds in this way. We note that the deformations of log Calabi–Yau spaces are still the subject of active research, and we hope that very recent work of Filip–Felton–Ruddat [21] will allow us to overcome some of the technical difficulties presented by the requirement of local rigidity in the Gross–Siebert algorithm.

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