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Thomas Bauer · Grzegorz Malara · Tomasz Szemberg · Justyna Szpond

Quartic unexpected curves and surfaces

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Abstract. Our research is motivated by recent work of Cook II, Harbourne, Migliore, and Nagel on configurations of points in the projective plane with properties that are unexpected from the point of view of the postulation theory. In this note, we revisit the basic configuration of nine points appearing in work of Di Gennaro/Ilardi/Vallès and Harbourne, and we exhibit some additional new properties of this configuration. We then pass to projective three-space $\mathbb{P}^3$ and exhibit a surface with unexpected postulation properties there. Such higher dimensional phenomena have not been observed so far.

1. Introduction

Let $P_1, \dots, P_s$ be a set of $s \geq 1$ generic points in $\mathbb{P}^N(\mathbb{C})$ and let $m_1, \dots, m_s$ be positive integers. It is a classical problem in algebraic geometry to study the linear systems $\mathcal{L}_N(d; m_1, \dots, m_s)$ of hypersurfaces of degree d passing through each of the points P_i with multiplicity at least m_i for $i = 1, \dots, s$. This linear system is viewed as the projectivization of the vector space of homogeneous polynomials of degree d vanishing at points $P_1, \dots, P_s$ to order $m_1, \dots, m_s$ respectively. Determining its projective dimension is one of the fundamental questions in the area.

Problem 1. Determine $\dim \mathcal{L}_N(d; m_1, \dots, m_s)$.

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T. Bauer: Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Hans-Meerwein-Straße, 35032 Marburg, Germany.
e-mail: tbauer@mathematik.uni-marburg.de

G. Malara · T. Szemberg (✉) · J. Szpond: Department of Mathematics, Pedagogical University of Cracow, Podchorążych 2, 30-084 Kraków, Poland.
e-mail: tomasz.szemberg@gmail.com

G. Malara: e-mail: grzegorzmalara@gmail.com

J. Szpond: e-mail: szpond@gmail.com

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The *expected dimension* of $\mathcal{L}_N(d; m_1, \ldots, m_s)$ is the number given by the naive conditions count

$$\text{edim}\, \mathcal{L}_N(d; m_1, \ldots, m_s) = \max\left\{-1, \binom{N+d}{N} - \sum_{i=1}^{s}\binom{N+m_i-1}{N} - 1\right\}. \tag{1}$$

We have always

$$\dim \mathcal{L}_N(d; m_1, \ldots, m_s) \geq \text{edim}\, \mathcal{L}_N(d; m_1, \ldots, m_s)\,. \tag{2}$$

If equality holds in (2), then we say that the system $\mathcal{L}_N(d; m_1, \ldots, m_s)$ is *non-special*. Otherwise it is called *special*.

Remark 1. It is well known that a single point with arbitrary multiplicity (that is, $s = 1$ and m_1 arbitrary) is non-special for any d and N. Similarly, generic points with multiplicity 1 (that is $m_1 = \cdots = m_s = 1$) impose always independent conditions on forms of arbitrary degree in projective spaces of arbitrary dimension.

Special linear systems with multiplicities $m_1 = \cdots = m_s = 2$ have been completely classified by Alexander and Hirschowitz [1]. The Segre–Harbourne–Gimigliano–Hirschowitz Conjecture governs the speciality of planar systems with points of arbitrary multiplicity, see [3] for a very nice survey. In $\mathbb{P}^3$ the special linear systems $\mathcal{L}_3(d; m_1, \ldots, m_s)$ are the subject of a conjecture due to Laface and Ugaglia [11]. In higher dimensions there are some partial results due to Alexander and Hirschowitz [2] and scattered partial conjectures due to various authors, see e.g. [8]. The complete picture remains however rather obscure.

In the groundbreaking article [4], Cook II, Harbourne, Migliore and Nagel opened a new path of research. They propose to study systems

$$\mathcal{L}_N(d; Z, m_1, \ldots, m_s), \tag{3}$$

where Z is a finite set of points (with multiplicity 1) and $P_1, \ldots, P_s$ are generic fat points, i.e., $m_1, \ldots, m_s \geq 2$. Thus the classical problem outlined above is the case $Z = \emptyset$. In [4] the authors focus on the case $N = 2$ and $s = 1$. They show that, somewhat unexpectedly in the view of Remark 1, there exist special linear systems in this situation. They relate the existence of such systems to properties of line arrangements determined by lines dual to points in Z. This leads to a very nice geometric explanation of the existence of special curves.

In the present note we exhibit a new phenomenon: The existence of special linear systems of type (3) in a higher dimensional projective space, namely in $\mathbb{P}^3$. In the subsequent paper [14], the last named author will show how our example can be generalized to higher dimensional projective spaces.

Our main result is Theorem 1. Conjecture 1 proposes a geometric explanation for the existence of the special surfaces in Theorem 1. Our research has been accompanied by Singular [5] experiments.

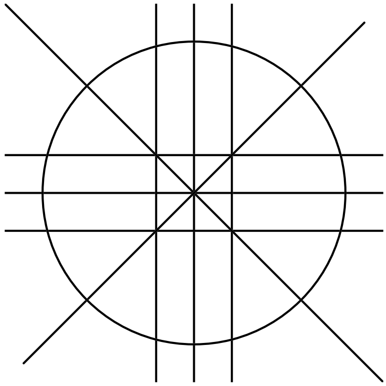

Fig. 1. The $B3$ arrangement of lines

2. Plane quartics with nine base points and a general triple point

We begin with a Coxeter arrangement of lines classically denoted by B3 (or $A(9, 1)$ in Grünbaum's notation). This arrangement is given by the linear factors of the polynomial

$$f = xyz(x+y)(x-y)(x+z)(x-z)(y+z)(y-z)\,.$$

It is depicted in Fig. 1, with the convention that the line at infinity $z = 0$ is indicated by the circle.

We use the following convention for the duality between lines and points: A point $(A : B : C)$ corresponds to the line $Ax + By + Cz = 0$ and vice versa. Thus the linear factors of f correspond to the points

$$\begin{array}{lll} P_1 = (1:0:0), & P_2 = (0:1:0), & P_3 = (0:0:1), \\ P_4 = (1:1:0), & P_5 = (1:-1:0), & P_6 = (1:0:1), \\ P_7 = (1:0:-1), & P_8 = (0:1:1), & P_9 = (0:1:-1). \end{array}$$

It was observed in [4] that these points impose independent conditions on quartics in $\mathbb{P}^2$.

Remark 2. There is some ambiguity in the choice of coordinates of the points $P_1, \ldots, P_9$. Harbourne introduced in [10, Example 4.1.10] some coordinates which due to their lack of symmetry are not as convenient to work with as those used by Dimca in [7, Example 3.6]. Therefore we prefer to work with Dimca's coordinates.

Let $V \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))$ be the linear subspace of all quartics vanishing at $P_1, \ldots, P_9$. As we have $\dim V = 6$ by [4], one expects that the system becomes empty when 6 further conditions are imposed. Unexpectedly, however, there exists for any choice of an additional point $R = (a : b : c)$ a quartic $Q_R \in V$ with a triple point at R. This quartic can be written down explicitly as

$$Q_R(x:y:z) = 3a(b^2-c^2)\cdot x^2yz + 3b(c^2-a^2)\cdot xy^2z + 3c(a^2-b^2)\cdot xyz^2 + a^3\cdot y^3z - a^3\cdot yz^3 + b^3\cdot xz^3 - b^3\cdot x^3z + c^3\cdot x^3y - c^3\cdot xy^3. \tag{4}$$

It is elementary to check that Q_R indeed vanishes at the points $P_1,\dots,P_9$ and vanishes at R to order 3.

The Eq. (4) can be viewed also as a cubic equation in the variables a, b, c with parameter $S=(x:y:z)$. Taking this dual point of view, it becomes

$$Q_S(a:b:c) = yz(y^2-z^2)\cdot a^3 + xz(z^2-x^2)\cdot b^3 + xy(x^2-y^2)\cdot c^3 + 3x^2yz\cdot ab^2 - 3xy^2z\cdot a^2b + 3xyz^2\cdot a^2c - 3x^2yz\cdot ac^2 + 3xy^2z\cdot bc^2 - 3xyz^2\cdot b^2c\,. \tag{5}$$

(Of course the Eqs. (4) and (5) are the same, just treated from a different perspective.) Surprisingly, each of the cubic curves (5) has a triple point at $S=(x:y:z)$. All the cubics therefore split in a product of three lines – except for those with parameter values $(x:y:z)$ that correspond to the points $P_1,\dots,P_9$, in which cases the right hand side of Eq. (5) vanishes identically. Currently we do not have a theoretical explanation for this property.

The family of cubics Q_S parameterized by $S=(x:y:z)\in\mathbb{P}^2$ has no additional base points. This can be easily verified for specific and sufficiently general values of $(x:y:z)$.

Remark 3. Recently, Farnik, Galuppi, Sodomaco and Trok have announced an interesting result to the effect that the quartic curve discussed in this section is, up to projective equivalence, the only unexpected curve of this degree, see [9].

3. Quartic surfaces with 31 base points and a general triple point

Let F be the Fermat-type ideal in $\mathbb{C}[x,y,z,w]$ generated by

$$x^3-y^3,\ y^3-z^3,\ z^3-w^3.$$

Its zero locus Z consists of the 27 points

$$P_{(\alpha,\beta,\gamma)} = (1:\varepsilon^\alpha:\varepsilon^\beta:\varepsilon^\gamma)$$

where ε is a primitive root of unity of order 3 and $1\le\alpha,\beta,\gamma\le 3$. Let I be the ideal of the union W of Z with the 4 coordinate points, that is,

$$I = F\cap(x,y,z)\cap(x,y,w)\cap(x,z,w)\cap(y,z,w).$$

Lemma 1. *The ideal I is generated by the following* 8 *binomials of degree* 4*:*

$$x(y^3-z^3),\ x(z^3-w^3),\ y(x^3-z^3),\ y(z^3-w^3),$$
$$z(x^3-y^3),\ z(y^3-w^3),\ w(x^3-y^3),\ w(y^3-z^3)\,.$$

Proof. Let J be the ideal generated by the 8 binomials. We show first the containment $J \subset I$. Since I is a radical ideal by definition, it is enough to check that $\mathrm{Zeroes}(I) \subset \mathrm{Zeroes}(J)$. To this end it is enough to verify that every binomial generating J vanishes along W. This is obvious because the vanishing in the coordinate points is guaranteed by the fact that every binomial involves 3 different variables and vanishing along Z is provided by the cubic term in brackets.

For the reverse containment let f be an element of I. In particular, f vanishes at all points of Z, so we may write f in the following way

$$f = g_y \cdot (x^3 - y^3) + g_z \cdot (x^3 - z^3) + g_w \cdot (x^3 - w^3)\,, \tag{6}$$

with homogeneous polynomials g_y, g_z, g_w. From now on we work modulo J. We want to show that $f = 0$. Since $z(x^3 - y^3), w(x^3 - y^3) \in J$, we may assume that g_y depends only on x and y. By the same token, we may assume that g_z depends only on x and z, and g_w respectively on x and w.

We have

$$xy(x^3 - y^3) = -y \cdot x(y^3 - z^3) + x \cdot y(x^3 - z^3)\,,$$

so that $xy(x^3 - y^3) \in J$. Thus $g_y = a_y x^d + b_y y^d$ for some $a_y, b_y \in \mathbb{C}$ and $d \geq 0$. Similarly $g_z = a_z x^d + b_z z^d$ and $g_w = a_w x^d + b_w w^d$.

Evaluating (6) at $(0:1:0:0)$ we obtain

$$0 = f(0:1:0:0) = g_y(0:1:0:0) = b_y.$$

Similarly, $b_z = b_w = 0$. Thus

$$f = x^d \left(a_y(x^3 - y^3) + a_z(x^3 - z^3) + a_w(x^3 - w^3)\right)\,. \tag{7}$$

If $d = 0$, then evaluating again in the coordinate points $(0:1:0:0)$, $(0:0:1:0)$ and $(0:0:0:1)$ we obtain $a_y = a_z = a_w = 0$ and we are done.

If $d > 0$, then evaluating at $(1:0:0:0)$ we get from (7)

$$a_y + a_z + a_w = 0\,. \tag{8}$$

Since $xy^3 = xw^3$ and $xz^3 = xw^3$ modulo J, we get from (7) and (8)

$$f = x^{d-1}\left(a_y x(x^3 - w^3) + a_z x(x^3 - w^3) + a_w x(x^3 - w^3)\right) = 0$$

and we are done. □

Let $V \subset H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))$ be the linear space of quartics vanishing along W. It follows from Lemma 1 that $\dim(V) = 8$. As vanishing to order 3 at a point in $\mathbb{P}^3$ imposes 10 conditions on forms of arbitrary degree, we do not expect that for a general point $R = (a:b:c:d)$ there exists a quartic $Q_R \in V$ vanishing to order three at R. However this is the case, as we now show:

Theorem 1. *The system $\mathcal{L}_3(4; W, 3)$ is special, i.e., for any point $R = (a : b : c : d)$ in $\mathbb{P}^3 \setminus W$ there exists a quartic Q_R vanishing to order 3 at R and vanishing at all points from the set W. Moreover, the quartic Q_R has 4 additional singularities at*

$$R_1 = (-2a : b : c : d)\,, \; R_2 = (a : -2b : c : d)\,,$$
$$R_3 = (a : b : -2c : d)\,, \; R_4 = (a : b : c : -2d)\,.$$

These singularities are double points.

Proof. Since we know from Lemma 1 that V is of dimension 8, it is enough to prove the existence of a quartic as claimed. It can be checked by elementary calculations that the quartic

$$\begin{aligned} Q_R(x : y : z : w) &= b^2(c^3 - d^3) \cdot x^3 y + a^2(d^3 - c^3) \cdot xy^3 + c^2(d^3 - b^3) \cdot x^3 z \\ &+ c^2(a^3 - d^3) \cdot y^3 z + a^2(b^3 - d^3) \cdot xz^3 + b^2(d^3 - a^3) \cdot yz^3 \\ &+ d^2(b^3 - c^3) \cdot x^3 w + d^2(c^3 - a^3) \cdot y^3 w + d^2(a^3 - b^3) \cdot z^3 w \\ &+ a^2(c^3 - b^3) \cdot xw^3 + b^2(a^3 - c^3) \cdot yw^3 + c^2(b^3 - a^3) \cdot zw^3 \end{aligned} \tag{9}$$

satisfies the assertion. $\square$

As before, the equation in (9) can be viewed as a quintic polynomial Q_S in variables a, b, s, d. Let $S = (x : y : z : w)$, then we have

$$\begin{aligned} Q_S(a : b : c : d) &= y(w^3 - z^3) \cdot a^3 b^2 + x(z^3 - w^3) \cdot a^2 b^3 + z(y^3 - w^3) \cdot a^3 c^2 \\ &+ z(w^3 - x^3) \cdot b^3 c^2 + x(w^3 - y^3) \cdot a^2 c^3 + y(x^3 - w^3) \cdot b^2 c^3 \\ &+ w(z^3 - y^3) \cdot a^3 d^2 + w(x^3 - z^3) \cdot b^3 d^2 + w(y^3 - x^3) \cdot c^3 d^2 \\ &+ x(y^3 - z^3) \cdot a^2 d^3 + y(z^3 - x^3) \cdot b^2 d^3 + z(x^3 - y^3) \cdot c^2 d^3. \end{aligned}$$

It can be checked by elementary computation that Q_S has a triple point at S.

4. Geometry of the unexpected quartic

It is well known that a quartic surface X in $\mathbb{P}^3$ with a triple point is rational. (This is easily seen by projecting X from the triple point.) More importantly, X is the image of $\mathbb{P}^2$ under the rational mapping φ defined by the linear system of plane quartics vanishing along a complete intersection 0-dimensional subscheme U of length 12, see [12]. In this section, for $X = Q_R$, we identify the subscheme U and the mappings explicitly.

We begin with the projection. Let $\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be the projection from $R = (a : b : c : d)$ onto the plane $\mathbb{P}^2$ with coordinates $(p : q : r)$, which is defined by

$$\pi : (x : y : z : w) \mapsto (p : q : r) = (dz - cw : cy - bz : bx - ay).$$

Let L_i be the line joining R and R_i for $i = 1, \dots, 4$ (where the R_i are the points from Theorem 1). By Bezout's theorem, these lines are contained in Q_R. They get contracted under π onto points F_i respectively, where

$$F_1 = (0:0:1),\ F_2 = (0:-c:a),\ F_3 = (-d:b:0),\ F_4 = (1:0:0).$$

Taking some generic sections of Q_R and their images in $\mathbb{P}^2$ we determine 4 points

$$\begin{aligned}&B_1 = (dc : -cb : ab),\ B_2 = (dc : 0 : -ab),\\ &B_3 = (a^2b^2(c^3-d^3) : a^2d^2(b^3-c^3) : c^2d^2(a^3-b^3)),\ B_4 = (0:1:0).\end{aligned}$$

which will be additional assigned base points of the linear series of quartics that we will consider. Let Γ_R be the plane cubic given by the equation

$$\begin{aligned}\Gamma_R(p:q:r) &= bc^2d(a^3-b^3)\cdot p^2q + c^2d^2(a^3-b^3)\cdot pq^2\\ &+ a^2bd(c^3-b^3)\cdot p^2r + 2a^2d^2(c^3-b^3)\cdot pqr\\ &+ a^2b^2(c^3-d^3)\cdot q^2r + acd^2(c^3-b^3)\cdot pr^2 + ab^2c(c^3-d^3)\cdot qr^2\end{aligned}$$

and let Δ be the plane quartic defined by the equation

$$\begin{aligned}\Delta_R(p:q:r) &= a^2bc^2(a^3-b^3)(c^3-d^3)\cdot pq^2r\\ &+ a^2c^2d(a^3-b^3)(c^3-d^3)\cdot q^3r\\ &- ab^2d^2(a^3-c^3)(b^3-c^3)\cdot p^2r^2 + b^3cd(a^3-c^3)(c^3-d^3)\cdot qr^3\\ &+ ad(c^3-d^3)(a^3b^3+a^3c^3-2b^3c^3)\cdot q^2r^2 - bcd^3(a^3-c^3)(b^3-c^3)\cdot pr^3\\ &- ab(b^3c^6-a^3c^6+2a^3b^3d^3-a^3c^3d^3-3b^3c^3d^3+2c^6d^3)\cdot pqr^2.\end{aligned}$$

Let U_R be the scheme-theoretic complete intersection of Γ_R and Δ_R. Then U has length 12 and it is supported on the 8 points $B_1, \dots, B_4$ and $F_1, \dots, F_4$. The points B_i for $i = 1, \dots, 4$ are reduced in U. The points F_i for $i = 1, \dots, 4$ support each a structure of length 2. In these points the curves Γ_R and Δ_R are tangent to each other.

The linear system $|\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{I}_U|$ has (projective) dimension 3. It is spanned by $f_0 = p\Gamma_R$, $f_1 = q\Gamma_R$, $f_2 = r\Gamma_R$, and $f_3 = \Delta_R$. In order to recover the original coordinates of all points, it is however necessary to define $\varphi = (g_0 : g_1 : g_2 : g_3) : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ in the following basis:

$$\begin{aligned}g_0 &= abc^2(a^3-b^3)\ f_0 + 2ac^2d(a^3-b^3)\ f_1 + d(a^3b^3+2a^3c^3-3b^3c^3)\ f_2 - ab^2\ f_3,\\ g_1 &= b^2c^2(a^3-b^3)\ f_0 + 2bc^2d(a^3-b^3)\ f_1 + a^2bd(b^3-c^3)\ f_2 - b^3\ f_3,\\ g_2 &= bc^3(a^3-b^3)\ f_0 - c3d(a^3-b^3)\ f_1 + a^2cd(b^3-c^3)\ f_2 - b^2c\ f_3,\\ g_3 &= -2bc^2d(a^3-b^3)\ f_0 - c^2d^2\ f_1 + a^2d^2(b^3-c^3)\ f_2 - b^2d\ f_3.\end{aligned}$$

Note that the mapping φ contracts the cubic curve Γ to the triple point R.

All claims in this section can be in principle checked by tedious hand calculations. In order to allow a more convenient verification, we provide a Singular code in [13].

5. General geometric considerations

Cook et al. establish in [4, Prop. 5.10] a method that allows to determine unexpected curves from syzygies. We propose here a conjecture that generalizes their idea to the surface case. To set it up, we need to generalize the notions of *multiplicity index* and *speciality index* that were introduced in [4] for the case of $\mathbb{P}^2$.

Definition 1. Let Z be a reduced 0-dimensional subscheme of $\mathbb{P}^n$. The *multiplicity index* of Z is the number

$$m_Z = \min\left\{ j \in \mathbb{Z} \,\middle|\, \dim[I_{Z+jP}]_{j+1} > 0 \right\}$$

where P is a general point in $\mathbb{P}^n$.

The *speciality index* u_Z of Z is the least integer j such that, for a general point $P \in \mathbb{P}^n$, the scheme $Z + jP$ imposes independent conditions on the system $|\mathcal{O}_{\mathbb{P}^n}(j+1)|$, i.e., the smallest j such that

$$\dim[I_{Z+jP}]_{j+1} = \binom{j+1+n}{n} - \binom{n-1+j}{n} - |Z| \,.$$

Consider now a reduced scheme $Z \subset \mathbb{P}^3$ of d points P_i. For each point P_i let $\ell_i \in \mathbb{K}[x, y, z, w]$ be a linear form defining the plane dual to P_i, and set $f = \ell_1 \cdot \ldots \cdot \ell_d$. Further, let ℓ be a linear form defining a general plane in $\mathbb{P}^3$.

Conjecture 1. Assume that the characteristic of $\mathbb{K}$ does not divide $|Z|$ and that $m_Z \leq u_Z$. Let

$$\begin{aligned} s_0 f_x + s_1 f_y + s_2 f_z + s_3 f_w + s_4 \ell &= 0 \\ s_0' f_x + s_1' f_y + s_2' f_z + s_3' f_w + s_4' \ell &= 0 \end{aligned}$$

be linearly independent syzygies of least degree of the ideal

$$\mathrm{Jac}(f) + (\ell) = (f_x, f_y, f_z, f_w, \ell),$$

and consider the rational maps $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ given as

$$\sigma = (s_0 : s_1 : s_2 : s_3) \quad \text{and} \quad \sigma' = (s_0' : s_1' : s_2' : s_3') \,.$$

Further, consider the rational map

$$\begin{aligned} \Phi : \mathbb{P}^3 &\dashrightarrow (\mathbb{P}^3)^* \\ Q &\mapsto \text{the plane through } Q, \sigma(Q), \sigma'(Q) \end{aligned}$$

Then the image of the restriction of Φ to ℓ is an unexpected surface for Z.

Remark 4. The conjecture above is supported by the following considerations:

(1) *For each i, all points on the line $\ell \cap \ell_i$ are mapped to the point P_i (i.e., the line goes through Z, as desired).*

Proof. Let Q be a point on $\ell \cap \ell_i$. One shows first that $\sigma(Q) \in \ell_i$ (this works as in [4, Prop. 5.10]). But we also have $\sigma'(Q) \in \ell_i$ by the same argument. Thus, the three points $Q, \sigma(Q), \sigma'(Q)$ all lie on ℓ_i. By definition of Φ, this implies $\Phi(Q) = \ell_i$. □

(2) *The points in* $\ell \cap \sigma(\ell) \cap \sigma'(\ell)$ *map to the general point* P.

Proof. Let Q be a point in $\ell \cap \sigma(\ell) \cap \sigma'(\ell)$. As above, we have then $\sigma(Q) \in \ell$ and $\sigma'(Q) \in \ell$. By definition of Φ it follows that $\Phi(Q) = \ell$. □

To prove the conjecture, one would need to show:

(a) The image of Φ is a surface.
(b) Φ is undefined only in certain points of the lines $\ell \cap \ell_i$. (This would follow from the following condition: For every point Q on ℓ, the points $\sigma(Q)$ and $\sigma'(Q)$ are not collinear. In other words, the syzygy vectors (s_0, s_1, s_2) and (s'_0, s'_2, s'_2) are linearly independent in all points of ℓ.)
(c) The multiplicity of the point P in (2) is high enough.

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