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The p-adic L-function for half-integral weight modular forms

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Abstract. The p-adic L-function for modular forms of integral weight is well-known. For certain weights the p-adic L-function for modular forms of half-integral weight is also known to exist, via a correspondence, established by Shimura, between them and forms of integral weight. However, we construct it here without any recourse to the Shimura correspondence, allowing us to establish its existence for all weights, including those exempt from the Shimura correspondence. We do this by employing the Rankin–Selberg method, and proving explicit p-adic congruences in the resultant Rankin–Selberg expression.

1. Introduction

The centrality of general p-adic L-functions to the Iwasawa main conjectures almost goes without saying, as they form the backbone of the analytic side of the conjecture. Given the diversity of L-functions and their, established or not, p-adic analogues, natural attempts can be made to formulate different versions of this. In its first form, one considers the GL_1-case in which the Kubota-Leopoldt L-function takes centre stage as the p-adic analogue of the Dirichlet L-function. Moving up to GL_2, we can consider modular forms and their p-adic L-functions. Should the modular form have an integral weight then the conjectures in toto are well formulated and in some cases even established (e.g. $k = 2$ see [21]). However, due to insufficiently developed theory the conjectures for when the modular forms are of half-integral weight are not possible to even state.

The main issue in the case where f is a modular form of half-integral weight is the difficulty of developing a 'Galois side', which forms the second and last backbone of the Iwasawa main conjectures. Such difficulty can be seen in Section 11 in the informal notes of Buzzard in [1]. Recent work of Weissman in [23] has made some serious progress in this regard by developing L-groups for metaplectic covers, the length and methods of which further underline the complications here. Nevertheless, the *analytic* p-adic theory of half-integral weight modular forms has been substantially developed in the thesis of [11]. So then it's germane to ask whether we can at least construct the p-adic L-function in this case.

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The p-adic L-function attached to a half-integral weight modular has been constructed before using an indirect method. The Shimura correspondence, established by Shimura himself in [15], goes between modular forms of half-integral weight $k = \frac{\kappa}{2}$ for odd $\kappa \in \mathbb{Z}$ and modular forms of integer weight $\kappa - 1$ which respects the action of the Hecke operators. For large enough κ this is a bijection, in which case the well-known interpolation of the L-function for modular forms of integer weight immediately yields that of the half-integral weight L-function. We go ahead anyway and provide in this paper a *direct* interpolation of the L-function for half-integer weights for reasons given in the rest of this introduction.

The method employed is the Rankin–Selberg method which has been a highly successful method for p-adic interpolation e.g. see [3,10,13]. In spite of all this the recording of the method in this setting is useful for a number of reasons. Primarily, an analogy can be drawn here with the construction of the p-adic L-functions for totally real fields, for which two constructions have proven to be equally useful. On the one hand Deligne and Ribet in [5] constructed this function using constant terms of Eisenstein series, and this particular construction is vital in the proof of the Iwasawa main conjecture by Wiles in [24]. Cassou-Noguès in [2], however, gave an entirely different construction of this p-adic L-function using the Shintani decomposition, this being apposite to our analogy in that it allowed Colmez to prove the p-adic residue formula, see [4]. In our situation here we note a potentially fruitful observation, in the very final section, regarding the integrality of the p-adic measure resulting from our particular construction when contrasted to the original one for integer weight Siegel modular forms. In addition, this paper gives some impetus to extend this to half-integral weight Siegel modular forms of higher degree $n > 1$ for which, crucially, there is no longer a Shimura correspondence. Such p-adic L-functions are currently not known to exist. Finally, in Sect. 3 we encounter a substantial deviation in this setting from the integer weight one, which involves the construction of the p-stabilisation of our modular form. This construction has not been done previously for half-integer weight forms.

Section 2 gives a well-known overview of the basic theory of half-integer weight modular forms. Some necessary results on the trace map and Hecke operators are given in Sect. 4, which allow us to reduce the level of the inner product in the integral expression of our L-function. After reducing the level of the inner product, the existence of the p-adic measure is proven by looking at the Fourier development inside the inner product. The paper is concluded in the final section by a comparison of the integer and half-integer weight p-adic measures in accordance with the Shimura correspondence, and a discussion on the integrality of this p-adic measure.

2. Half-integral weight modular forms

Much of the theory we use here to exhibit half-integral weight modular forms is taken from [15]. First, some preliminary notation. If $\alpha \in M_n$ is a matrix, then $|\alpha| = \det(\alpha)$. We denote by $GL_2^+(\mathbb{R})$ the space of all invertible 2×2 matrices with positive determinant. The upper half-plane is $\mathbb{H} = \{z \in \mathbb{C} \mid \mathfrak{Im}(z) > 0\}$. The fractional linear transformation action of $GL_2^+(\mathbb{R})$ on $\mathbb{H}$ is given by

$$\alpha \cdot z = \frac{az+b}{cz+d} \qquad \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}), z \in \mathbb{H}.$$

We let $\mathbb{T}$ denote the unit circle and let

$$\mathfrak{G} := \{(\alpha, \varphi) \mid \alpha \in GL_2^+(\mathbb{R}), \varphi \text{ holomorphic}, \varphi(z)^2 = t|\alpha|^{-\frac{1}{2}}(cz+d), t \in \mathbb{T}\}$$

be a group whose law of multiplication is given by

$$(\alpha, \varphi(z))(\beta, \psi(z)) = (\alpha\beta, \varphi(\beta \cdot z)\psi(z)). \tag{2.1}$$

We also have a natural projection $P : \mathfrak{G} \to GL_2^+(\mathbb{R})$ and $\ker(P) \cong \mathbb{T}$. Let $f : \mathbb{H} \to \mathbb{C}$ be a function, then for some $\kappa \in \mathbb{Z}$ we can define a weight κ action of $\mathfrak{G}$ on f by

$$(f|[\xi]_\kappa)(z) = f(\xi \cdot z)\,\varphi(z)^{-\kappa} = f(\alpha \cdot z)\,\varphi(z)^{-\kappa}$$

where $\xi = (\alpha, \varphi)$. By virtue of (2.1) we have $f|[\xi\eta]_\kappa = (f|[\xi]_\kappa)|[\eta]_\kappa$ for any two $\xi, \eta \in \mathfrak{G}$. Let $\mathfrak{G}_1 := \{\xi = (\alpha, \varphi) \in \mathfrak{G} \mid |\alpha| = 1\} \leq \mathfrak{G}$, then a Fuchsian subgroup $\Delta \leq \mathfrak{G}_1$ is a subgroup such that $P(\Delta)$ is a discrete subgroup of $SL_2(\mathbb{R})$, $P(\Delta)\backslash\mathbb{H}$ is of finite volume with respect to $d\mu = y^{-2}dxdy$, Δ contains no elements of the form $(1, t)$ for $1 \neq t \in \mathbb{T}$, and contains no elements of the form $(-1, t)$ for $1 \neq t \in \mathbb{T}$ should $-1 \in P(\Delta)$.

Definition 2.1. Let κ be an odd integer and $\Delta \leq \mathfrak{G}_1$ a Fuchsian subgroup. We say that a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight $k = \frac{\kappa}{2}$ with respect to Δ if

(i) $f|[\xi]_\kappa = f$ for all $\xi \in \Delta$;
(ii) f is holomorphic at all cusps of $P(\Delta)$.

The space of all such forms is denoted $\mathscr{M}_k(\Delta)$.

Condition (ii), that f be holomorphic at cusps, is made precise in [15, p. 444]. Much like integral weight modular forms any $f \in \mathscr{M}_k(\Delta)$ has a Fourier expansion of the form

$$f = \sum_{n=0}^{\infty} a_n q^n$$

where $q = e^{2\pi i z}$. Then the subspace of cusp forms, denoted $\mathscr{S}_k(\Delta)$, consists of all forms f the Fourier developments of which take the form $f|[\xi]_\kappa = \sum_{n=1}^{\infty} a_n q^n$ for all $\xi \in \mathfrak{G}$.

In this paper Δ will always be obtained by the following types of congruence subgroup of $SL_2(\mathbb{Z})$,

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \mid a \equiv d \equiv 1 \pmod{N} \right\},$$

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) \mid b \equiv 0 \pmod{N} \right\}.$$

Let $\theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2}$ be the theta series of weight $\frac{1}{2}$ with respect to $\Gamma_0(4)$ and fix the factor of automorphy

$$j(\gamma, z) = \frac{\theta(\gamma \cdot z)}{\theta(z)}$$

for $\gamma \in \Gamma_0(4)$. From [15, p. 447] for any $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$ this factor satisfies

$$j(\alpha, z) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) (cz + d)^{\frac{1}{2}} \tag{2.2}$$

$$j(\alpha, z)^2 = \left(\frac{-1}{d}\right) (cz + d) \tag{2.3}$$

where $\varepsilon_d = 1$ if $d \equiv 1 \pmod 4$ and $\varepsilon_d = i$ if $d \equiv 3 \pmod 4$. Then we have an embedding

$$\begin{aligned} \Gamma_0(4) &\hookrightarrow \mathfrak{G}_1 \\ \gamma &\mapsto \gamma^* := (\gamma, j(\gamma, z)) \end{aligned}$$

and if $4 \mid N$ then we denote by $\Delta_0(N)$, $\Delta_1(N)$, $\Delta(N)$ the respective images of $\Gamma_0(N)$, $\Gamma_1(N)$, $\Gamma(N)$ under the above embedding. For ease of notation we write $[\gamma]_\kappa = [\gamma^*]_\kappa$ for $\gamma \in \Gamma(N)$. Let N be divisible by 4, κ an odd integer with $k = \frac{\kappa}{2}$, and ψ a Dirichlet character modulo N such that $\psi(-1) = 1$, then put

$$\mathscr{M}_k(N, \psi) := \{f \in \mathscr{M}_k(\Delta_1(N)) \mid f|[\gamma]_\kappa = \psi(d) f \text{ if } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)\}$$

and let $\mathscr{S}_k(N, \psi)$ denote the subspace of cusp forms. Note that by [15, p. 447] we have $\mathscr{M}_k(N, \psi) = 0$ if $\psi(-1) = -1$.

We finish this section with a brief discussion on Hecke operators in this setting. Let $\Delta_i \leq \mathfrak{G}_1$ be Fuchsian and let $\xi \in \mathfrak{G}$ be such that Δ_1 and $\xi \Delta_2 \xi^{-1}$ are commensurable. Then we have a finite disjoint union

$$\Delta_1 \xi \Delta_2 = \bigsqcup_\nu \Delta_1 \xi_\nu.$$

If $f \in \mathscr{M}_k(\Delta_1)$ then we get a new function $f|[\Delta_1 \xi \Delta_2]_\kappa$ by putting

$$f|[\Delta_1 \xi \Delta_2]_\kappa = |\xi|^{\frac{\kappa}{4} - 1} \sum_\nu f|[\xi_\nu]_\kappa$$

and $f|[\Delta_1 \xi \Delta_2]_\kappa \in \mathscr{M}_k(\Delta_2)$. The above action is also defined for $\mathscr{S}_k(\Delta_i)$. The space of all formal finite sums generated by such double cosets forms an algebra. We focus on the case $\Delta_1 = \Delta_2 = \Delta \leq \mathfrak{G}_1$ is a Fuchsian subgroup. If $\Gamma = P(\Delta)$, $\alpha = P(\xi)$, and $P : \Delta \xi \Delta \to \Gamma \alpha \Gamma$ is a bijection, then equivalently $L(\alpha \gamma \alpha^{-1}) = \xi L(\gamma) \xi^{-1}$ for any lift $L : \Gamma \to \Delta$ and $\gamma \in \Gamma \cap \alpha^{-1} \Gamma \alpha$. In this case, the projection map respects the decomposition of the double coset, that is

$$\Gamma \alpha \Gamma = \bigsqcup_\nu \Gamma P(\xi_\nu)$$

if $\Delta \xi \Delta = \bigsqcup_\nu \Delta \xi_\nu$, and there's a workable theory of Hecke operators.

Therein lies the issue, the above condition of P being a bijection on double cosets is not always satisfied and causes some differences in the theory of Hecke operators, which means that we only really have operators $T(m)$ when m is a square number. Generally for $\gamma \in \Gamma \cap \alpha^{-1}\Gamma\alpha$ we at least have

$$L(\alpha\gamma\alpha^{-1}) = \xi L(\gamma)\xi^{-1}(1, t(\gamma))$$

for a homomorphism $t : \Gamma \cap \alpha^{-1}\Gamma\alpha \to \mathbb{T}$. Then Proposition 1.0 in [15] says that $f|[\Delta\xi\Delta]_\kappa = 0$ if $f \in \mathscr{M}_k(\Delta)$ and $t^\kappa \not\equiv 1$. Also found in [15, pp. 447–448] is the following proposition:

Proposition 2.2. *Take $L(\gamma) = \gamma^*$ for $\gamma \in \Gamma_0(4)$, and let $m, n \in \mathbb{Z}$. Let $\alpha = \left(\begin{smallmatrix} m & 0 \\ 0 & n \end{smallmatrix}\right)$ and $\xi = (\alpha, t(n/m)^{\frac{1}{4}})$ for any $t \in \mathbb{T}$. Then*

$$\xi\gamma^*\xi^{-1} = (\alpha\gamma\alpha^{-1})^* \cdot \left(1, \left(\frac{mn}{d}\right)\right)$$

if $\gamma \in \Gamma_0(4) \cap \alpha^{-1}\Gamma_0(4)\alpha$.

So in particular if we take $\Delta = \Delta_1(N)$ for some N divisible by 4, and a prime $p \nmid N$ then the pth Hecke operator is given by $[\Delta\xi\Delta]_\kappa$ where $\xi = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right), p^{\frac{1}{4}}\right)$, but then if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an appropriate element as per Proposition 2.2 with d not a quadratic residue modulo p we have $t(\gamma)^\kappa = \left(\frac{p}{d}\right)^\kappa = \left(\frac{d}{p}\right)^\kappa = -1$. So we get that $t^\kappa \not\equiv 1$ and $T(p)$ sends $\mathscr{M}_k(\Delta_1(N))$ to 0. Then $T(p)$ is of no interest here and we must generally consider $T(m)$ for m a square number. In the case that $m, n \in \mathbb{Z}$ are square then $T(m)$ and $T(n)$ commute and one can also write the explicit actions of $T(p^2)$ on the Fourier coefficients of f, see Proposition 1.6 and Theorem 1.7 in [15].

So we have seen that $T(p)$ is simply the zero operator should $p \nmid N$, but the question remains of whether it is of interest when $p \mid N$. In this case, actually $T(p)$ is much like it is in the integral weight setting, it shifts the coefficients of a form. In [15, p. 448] we have the following:

Proposition 2.3. *Let $\mathbb{Z} \ni m > 0$ such that the conductor of $\mathbb{Q}(m^{\frac{1}{2}})$ divides N and all the primes dividing m also divide N. Then let $\xi = (\left(\begin{smallmatrix} 1 & 0 \\ 0 & m \end{smallmatrix}\right), m^{\frac{1}{4}})$ and put $\Delta_1 = \Delta_1(N)$ for $4 \mid N$. If $f = \sum_{n=0}^\infty a_n q^n \in \mathscr{M}_k(N, \psi)$ then*

$$f|[\Delta_1\xi\Delta_1]_\kappa = \sum_{n=0}^\infty a_{mn}q^n \in \mathscr{M}_k(N, \psi')$$

for $\psi'(d) := \psi(d)\left(\frac{m}{d}\right)$.

In particular, if $p \mid N$ then the two conditions in the proposition are satisfied and $f|T(p) = \sum_{n=0}^\infty a_{pn}q^n$ is a non-trivial action.

3. Complex L-functions and their integral expressions

To start with, we define the complex L-function attached to half-integral weight modular forms. With the given data of a normalised eigenform $f \in \mathscr{S}_k(\Delta_1), k = \frac{\kappa}{2}$, $\kappa \in 2\mathbb{Z}+1$, $\Delta_1 = \Delta_1(N)$, and $4 \mid N \in \mathbb{Z}$ we can associate for any prime p, as seen in [18, p. 46], a Satake p-parameter $\lambda_p \in \mathbb{C}^\times$. Subsequently, the local factors are given as

$$L_p(t) := \begin{cases} (1 - p\lambda_p t) & \text{if } p \mid N \\ (1 - p\lambda_p t)(1 - p\lambda_p^{-1} t) & \text{if } p \nmid N \end{cases}$$

and, augmenting the given data with a character χ of some modulus, the actual L-function is

$$L(s, f, \chi) := \prod_p L_p(\chi(p)p^{-s})^{-1}$$

with $s \in \mathbb{C}$ and which product is absolutely convergent should $\mathfrak{Re}(s) > \frac{3n}{2} + 1$.

The $SL_2(\mathbb{Z})$-invariant differential $d\mu := y^{-2}dxdy$ on $\mathbb{H}$, along with the fundamental domain $B(\Gamma)$ of $\Gamma\backslash\mathbb{H}$ for some congruence subgroup $\Gamma \leq SL_2(\mathbb{Z})$, are used in defining the Petersson inner product of two modular forms. If ψ is an even Dirichlet character modulo N, and if $B(N) := B(\Gamma_0(N))$, then the Petersson inner product of $f,\ g\ \in\ \mathscr{M}_k(N, \psi)$ is defined by

$$\langle f, g\rangle_N := \int_{B(N)} f(z)\overline{g(z)}y^k d\mu$$

which integral is convergent whenever one of f, g is a cusp form.

With the complex L-function now defined the aim is to give a Rankin–Selberg expression of said L-function in terms of integrals of the form in the definition of the above inner product. To this end, we first define what our integrands shall be. One role is given by the eigenform f which we henceforth assume is a *newform* belonging to $\mathscr{S}_k(N) := \mathscr{S}_k(N, 1)$ and has Fourier coefficients a_n for $1 \leq n \in \mathbb{Z}$. There exists some minimal square-free positive integer t at which we have $a_t \neq 0$, and normalise the form so that $a_t = 1$. Such an integer exists by (i), (ii) of Corollary 1.8 in [15]. Furthermore by the strong multiplicity one theorem we can take t such that $p \nmid t$, since otherwise $a_q = 0$ for all $q \neq p$.

Let $\mathcal{P} := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \mid c = 0\}$ be the parabolic subgroup of $SL_2(\mathbb{Z})$, for any $1 \leq M \in \mathbb{Z}$ we put $\Gamma = \Gamma_0(M)$, and let η be a Dirichlet character modulo M. Then the (non-holomorphic) Eisenstein series we need is of integral weight $\ell \in \mathbb{Z}$, level M, character η^{-1}, and is expressed as the following sum in the two variables $z \in \mathbb{H}$, $s \in \mathbb{C}$:

$$E(z, s; \ell, \eta, M) := y^{\frac{s}{2}} \sum_{\gamma \in (\mathcal{P}\cap\Gamma)\backslash\Gamma} \eta(d)(cz+d)^{-k}|cz+d|^{-s}.$$

Taking $\nu \in \{0, 1\}$ such that $\eta(-1) = (-1)^{\lfloor k \rfloor + \nu}$, which character has conductor c_η, then we define the theta series by

$$\theta_\eta^{(\nu)}(tz) := \sum_{n\in\mathbb{Z}} \eta(n)n^\nu q^{tn^2}.$$

This is of weight $\nu+\frac{1}{2}$, character $\eta\rho_t$ (where ρ_t is the quadratic character associated to $\mathbb{Q}(i^{\frac{1}{2}}(2t)^{\frac{1}{2}})$), and level $4tc_\eta^2$, see Proposition 2.1 of [19]. The other role of the inner product is then played by a theta series multiplied by an Eisenstein series of the above kinds.

Now let χ be a Dirichlet character of p-power conductor $c_\chi = p^{m_\chi}$, and also put $N_\chi := Ntc_\chi^2$. Since $c_\chi \mid N_\chi$ we can (and in the following Eisenstein series do) view χ as a character of modulus N_χ in the natural way. Putting $n = 1$, $F = \mathbb{Q}$, and decoding the notation of [19, (4.1)] yields the following integral expression:

$$\begin{aligned} &2(4t\pi)^{-\frac{s+k-2}{2}}\Gamma\left(\tfrac{s+k-2}{2}\right)L(s, f, \chi) \\ &\quad = g_t^\chi(s)\int_{B(N_\chi)} f(z)\overline{\theta_\chi^{(\nu)}(tz)\mathscr{E}_\nu(z, \chi\rho_t, s, N_\chi)}d\mu \end{aligned} \tag{3.1}$$

where

$$g_t^\chi(s) := L_t(s,\chi)L(s,\chi)^{-1} = \prod_{q|t}(1-\chi(p)p^{-s})$$

$$\mathscr{E}_\nu(z, \chi\rho_t, s, N_\chi) := L_{N_\chi}(\bar{s}-\tfrac{1}{2}, \chi)E(z, s-k+\nu; k-\nu-\tfrac{1}{2}, \chi\rho_t, N_\chi).$$

Let $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}}_p$ denote the completion of an algebraic closure of the p-adic numbers, and extend the p-adic norm to this field. Let $\iota_p : \mathbb{Q}_p \hookrightarrow \mathbb{C}_p$ be a fixed embedding and we henceforth work under ι_p without explicitly denoting it.

We shall always assume that $p \neq 2$, that $p \nmid N$, and furthermore that f is p-ordinary—by which we mean that the eigenvalue of f at $T(p^2)$ is a unit at p. The Hecke operators $T(p^2)$ in [18] lack the normalising factor of $(p^2)^{\frac{k}{2}-1}$ that ours possess. As a result, if $f|[T(p^2)]_\kappa = \omega_p f$ then $\omega_p = p^{k-2}\lambda(p)$ where $\lambda(p)$ are the eigenvalues in the sense of [18]. By definition of the λ_p in (5.4a) of [18] we have

$$\omega_p = p^{k-2}\lambda(p) = p^{k-1}\lambda_p + p^{k-1}\lambda_p^{-1}$$

and so as ω_p is a unit at p we may assume that $\alpha_p := p^{k-1}\lambda_p$ is a unit at p. Put $\beta_p := p^{k-1}\lambda_p^{-1}$, then ω_p, α_p, and β_p satisfy

$$1-\omega_p X + p^{2k-2}X^2 = (1-\alpha_p X)(1-\beta_p X).$$

Let $N_1 := Np^2$ and let $[T(p; N_1)]_\kappa$ denote the (now non-zero) pth Hecke operator of level N_1, which just shifts the coefficients of a form along. When acting on forms of level N_1 the operator $[T(p; N_1)]_\kappa = [T(p)]_\kappa$ is just the usual pth Hecke operator, but the notation is used to emphasise the fact that this is the operator that shifts coefficients even on forms of level N – for example f – and is not equal to $[T(p)]_\kappa$ in that latter case. We have $[T(p^2; N_1)]_\kappa = [T(p; N_1)]_\kappa^2$, and upon viewing f as a form of level N_1 Proposition 2.3 gives that $f|[T(p; N_1)]_\kappa = \sum_{n\geq 1} a_{np}q^n$. If η is any Dirichlet character of modulus M then we define the twist of f by η to be

$$f_\eta(z) := \sum_{n=1}^{\infty}\eta(n)a_n q^n$$

which, by easy generalisation of Proposition 17 (b) in [9, pp. 127–128] to half-integral forms, is in $\mathscr{S}_k(NM^2, \eta^2)$. Set $\chi_p(n) = \left(\frac{n}{p}\right)$ which has modulus p and satisfies $\chi_p^2 = 1$.

Let $f_1(z) := f(z) - \left(\frac{-1}{p}\right)^{\lfloor k \rfloor} p^{-\lfloor k \rfloor} \beta_p f_{\chi_p}(z) - \beta_p f(p^2 z) \in \mathscr{S}_k(N_1)$, and we work mostly with f_1 whose primary benefit over f is that p now divides the level.

Lemma 3.1. *The form f_1 is an eigenform for all the Hecke operators of level N_1. Should $p \nmid m$ then f_1 and f share the same eigenvalues for $T(m)$, whereas*

$$f_1|[T(p; N_1)]_\kappa = \alpha_p f.$$

Proof. To prove this we note that

(a) $f_{\chi_p}|[T(p; N_1)]_\kappa = 0$;
(b) $f(p^2 z)|[T(p; N_1)]_\kappa^2 = f$;
(c) $f|[T(p; N_1)]_\kappa^2 = \omega_p f - \left(\frac{-1}{p}\right)^{\lfloor k \rfloor} p^{\lfloor k \rfloor - 1} f_{\chi_p} - p^{2k-2} f(p^2 z)$.

It is easy to see (a)–(b) by virtue of Proposition 2.3. To prove (c) we make use of Corollary 1.8 in [15] points (i) and (ii), which gives for $p^2 \nmid t$

(i) $\omega_p a_t = a_{tp^2} + \left(\frac{-1}{p}\right)^{\lfloor k \rfloor} p^{\lfloor k \rfloor - 1} \left(\frac{t}{p}\right) a_t$;
(ii) $a_{tp^{2m+2}} = \omega_p a_{tp^{2m}} - p^{2k-2} a_{tp^{2m-2}}$.

If $p^2 \mid n$ then write $n = p^{2m} t$ for $p^2 \nmid t$ and (ii) then gives $a_{np^2} = \omega_p a_n - p^{2k-2} a_{\frac{n}{p^2}}$. We get

$$\begin{aligned}
\omega_p f - p^{2k-2} f(p^2 z) &= \sum_{p^2 \nmid n} a_{np^2} q^n + \left(\frac{-1}{p}\right)^{\lfloor k \rfloor} p^{\lfloor k \rfloor - 1} \sum_{p^2 \nmid n} \left(\frac{n}{p}\right) a_n q^n \\
&\quad + \sum_{p^2 \mid n} \left[\omega_p a_n - p^{2k-2} a_{\frac{n}{p^2}}\right] q^n \\
&= f|[T(p; N_1)]_\kappa^2 + \left(\frac{-1}{p}\right)^{\lfloor k \rfloor} p^{\lfloor k \rfloor - 1} f_{\chi_p}
\end{aligned}$$

noting that $\left(\frac{n}{p}\right) = 0$ if $p^2 \mid n$ anyway. Now that we have proven (a)–(c) then the lemma follows since

$$\begin{aligned}
f_1|[T(p; N_1)]_\kappa^2 &= \omega_p f - \left(\frac{-1}{p}\right)^{\lfloor k \rfloor} p^{\lfloor k \rfloor - 1} f_{\chi_p} - p^{2k-2} f(p^2 z) - \beta_p f \\
&= \alpha_p f_1
\end{aligned}$$

using $\omega_p = \alpha_p + \beta_p$ and $p^{2k-2} = \alpha_p \beta_p$.

Let $q \neq p$, then to see that the eigenvalue $\omega_{q,1}$ of f_1 is equal to ω_q of f, we need the fact that $f(\ell z)|[T(q^2)]_\kappa = (f|[T(q^2)]_\kappa)(\ell z)$ for any $(\ell, q) = 1$. This can be seen easily by considering the coset decomposition, found in [15, p. 451], of

$\Delta_1(N_1)\begin{pmatrix} 1 & 0 \\ 0 & q^2 \end{pmatrix}^* \Delta_1(N_1)$. So $f(p^2z)$ and f share the same eigenvalue for $[T(q^2)]_\kappa$. To show that f_{χ_p} also has the same eigenvalue as f we make use of Theorem 1.7 from [15] to show that $f_{\chi_p}|[T(q^2)]_\kappa = (f|[T(q^2)]_\kappa)_{\chi_p}$. Let b_n be the Fourier coefficients of $f|[T(q^2)]_\kappa$, then these are given as

$$b_n = a_{nq^2} + \left(\frac{-1}{q}\right)^{\lfloor k \rfloor} q^{\lfloor k \rfloor - 1} \left(\frac{n}{q}\right) a_n + q^{2k-2} a_{\frac{n}{q^2}}$$

where we understand $a_{\frac{n}{q^2}} = 0$ if $q^2 \nmid n$. Using this same construction for f_{χ_p} we get the nth coefficient of $f_{\chi_p}|[T(q^2)]_\kappa$ to be

$$\left(\frac{nq^2}{p}\right) a_{nq^2} + \left(\frac{-1}{q}\right)^{\lfloor k \rfloor} q^{\lfloor k \rfloor - 1} \left(\frac{n}{p}\right)\left(\frac{n}{q}\right) a_n + q^{2k-2} \left(\frac{n/q^2}{p}\right) a_{\frac{n}{q^2}}$$

where we understand $\left(\frac{n/q^2}{p}\right) = 0$ if $q^2 \nmid n$. Since $\left(\frac{nq^2}{p}\right) = \left(\frac{n/q^2}{p}\right) = \left(\frac{n}{p}\right)$ we see that the nth Fourier coefficient of $f_{\chi_p}|[T(q^2)]_\kappa$ is $\left(\frac{n}{p}\right) b_n$, which is the nth Fourier coefficient of $(f|[T(q^2)]_\kappa)_{\chi_p}$. Thus f_{χ_p} and f share the same eigenvalue for $[T(q^2)]_\kappa$. □

Using the definitions of the L-function and f_1 we get the following relations:

Lemma 3.2. *$L(s, f_1) = (1 - p\lambda_p^{-1} p^{-s})L(s, f)$ and $L(s, f_1, \chi) = L(s, f, \chi)$.*

Proof. To see why this is true we really need to see how the numbers λ_p are obtained in [18, p. 46]. For any prime q the number λ_q satisfies

$$\sum_{m=0}^{\infty} \lambda(q^m)t^m = \begin{cases} (1 - q\lambda_q t)^{-1} & \text{if } q \mid N \\ \frac{1-qt^2}{(1-q\lambda_q t)(1-q\lambda_q^{-1} t)} & \text{if } q \nmid N \end{cases} \tag{3.2}$$

where we recall that $\omega_q = q^{k-2}\lambda(q)$, and let $\omega_{q,1}, \lambda_1(q), \lambda_{q,1}$ denote the corresponding numbers for f_1. Then our job is to compare λ_q with $\lambda_{q,1}$. Suppose that $q \neq p$, then $f|[T(q^{2m})]_\kappa = f_1|[T(q^{2m})]_\kappa$ by Lemma 3.1 and so $\lambda_q = \lambda_{q,1}$ in this case.

Assume now that $q = p$ and for ease of notation label $\delta_p := p\lambda_p$ and $\gamma_p := p\lambda_p^{-1}$. We claim that

$$\lambda(p^m) = \delta_p^m + \gamma_p \lambda(p^{m-1}) - p\delta_p^{m-2} \tag{3.3}$$

and we use induction to see why this is true. First multiply both sides of (3.2) by the denominator and compare coefficients of t^m for $m \geq 2$ to get

$$\lambda(p^m) = \lambda(p)\lambda(p^{m-1}) - p^2\lambda(p^{m-2}).$$

Using $\lambda(p) = \delta_p + \gamma_p$, $p^2 = \delta_p\gamma_p$, and the induction hypthesis, this gives

$$\begin{aligned} \lambda(p^m) &= \delta_p \left[\delta_p^{m-1} + \gamma_p\lambda(p^{m-2}) - p\delta_p^{m-3}\right] + \gamma_p\lambda(p^{m-1}) - p^2\lambda(p^{m-2}) \\ &= \delta_p^m + \gamma_p\lambda(p^{m-1}) - p\delta_p^{m-2} \end{aligned}$$

as desired, and the base case $m = 2$ is easily read off from (3.2).

The operator $[T(p^{2m})]_\kappa$ acts as $[T(p)]_\kappa^{2m}$ on f_1, so that $\omega_{p^m,1} = (p^{k-2})^m \lambda_1(p^m)$ and unwrapping all the notation we have

$$\omega_{p^m,1} = \alpha_p^m = (p^{k-1})^m \lambda_p^m = (p^{k-2})^m \delta_p^m$$

and so $\lambda_1(p^m) = \delta_p^m$. So we wish to find the numbers $\lambda_{p,1}$ that satisfy

$$\sum_{m\geq 0} \delta_p^m t^m = (1 - p\lambda_{p,1} t)^{-1}.$$

From the now established identity of (3.3) above, we have

$$\begin{aligned}\sum_{m=0}^{\infty} \delta_p^m t^m - p \sum_{m=2}^{\infty} \delta_p^{m-2} t^m &= \sum_{m=0}^{\infty} \lambda(p^m) t^m - p\lambda_p^{-1} \sum_{m=1}^{\infty} \lambda(p^{m-1}) t^m \\ (1 - pt^2) \sum_{m=0}^{\infty} \delta_p^m t^m &= (1 - p\lambda_p^{-1} t) \sum_{m=0}^{\infty} \lambda(p^m) t^m \\ &= (1 - pt^2)(1 - p\lambda_p t)^{-1}\end{aligned}$$

using (3.2) in the last line. Hence $\lambda_{p,1} = \lambda_p$. In the Euler factors this becomes

$$L_p(f_1, s) = (1 - p\lambda_p p^{-s})^{-1} = (1 - p\lambda_p^{-1} p^{-s}) L_p(f, s)$$

whence the lemma. □

Notice that the tth coefficient of f_1 is given by $a_t - (\frac{-1}{p})^{\lfloor k \rfloor} (\frac{t}{p}) p^{-\lfloor k \rfloor} \beta_p a_t$. Since α_p is a p-adic unit and $\alpha_p = \beta_p^{-1} p^{2k-2}$, we cannot have $\beta_p^{-1} = (\frac{-1}{p})^{\lfloor k \rfloor} (\frac{t}{p}) p^{-\lfloor k \rfloor}$. So the tth coefficient of f_1 is also non-zero, and we normalise f_1 by this value.

Proposition 3.3. *Assume χ has conductor $c_\chi = p^{m_\chi}$ for $1 \leq m_\chi \in \mathbb{Z}$ and let $\nu \in \{0, 1\}$ satisfy $\chi(-1) = (-1)^{\lfloor k \rfloor + \nu}$. Then*

$$2 \frac{\Gamma\left(\frac{s+k-2}{2}\right)}{(4t\pi)^{\frac{s+k-2}{2}} g_t^\chi(s)} L(s, f_1, \chi) = \int_{B(N_\chi)} f_1(z) \overline{\theta_\chi^{(\nu)}(tz) \mathscr{E}_\nu(z, \chi\rho_t, s, N_\chi)} d\mu. \quad (3.4)$$

Proof. Just re-use [[19], (4.1)] as we did before, noting that the only thing changing by replacing f with f_1 is the level by a factor of p, and since $p \mid c_\chi$ this has no real bearing on (4.1) in [19] as for example, $N_\chi = \mathrm{lcm}(N, 4c_\chi^2) = \mathrm{lcm}(N_1, 4c_\chi^2)$. □

4. The trace map

Letting

$$\beta_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$$

$$W(N) := (\beta_N, N^{\frac{1}{4}}(-iz)^{\frac{1}{2}})$$

then we state the following from [15, p. 448]:

Proposition 4.1. *If $4 \mid N$, then the operator $[W(N)]_\kappa^2$ is the identity on $\mathscr{M}_k(\Delta_1(N))$. Moreover $[W(N)]_\kappa$ sends $\mathscr{M}_k(N,\psi)$ and $\mathscr{S}_k(N,\psi)$, respectively, to $\mathscr{M}_k(N,\psi^*)$ and $\mathscr{S}_k(N,\psi^*)$, where $\psi^*(d) = \bar{\psi}(d)\left(\frac{N}{d}\right)$.*

For any two integers $M_1 \mid M_2$ the general trace map sends forms of level M_2 to forms of level M_1; denote the trace map $S_\chi : \mathscr{M}_k(\Delta_1(N_\chi)) \to \mathscr{M}_k(\Delta_1(N_1))$.

Our aim in this section is twofold. As $p \mid N_\chi$, the pth Hecke operator $[T(p)]_\kappa$ of level N_χ is non-zero on $\mathscr{M}_k(\Delta_1(N_\chi))$. We first wish to express S_χ in terms of $T(p)$, and secondly to find the adjoint of $T(p)$ of level N_1. This allows us to reduce the inner product in (3.4) to be over N_1 instead of N_χ.

Proposition 4.2. *Let κ be an odd integer. The trace map and $[T(p^2)]_\kappa$ are relatable as follows:*

$$[S_\chi W(N_1)]_\kappa = (p^{m_\chi - 1})^{2-k}\left[W(N_\chi)T(p^2)^{m_\chi - 1}\right]_\kappa.$$

Proof. Let $g \in \mathscr{M}_k(N_\chi)$, then by definition we have

$$g|[S_\chi]_\kappa = \sum_{e\in\mathbb{Z}/p^{2m_\chi-2}\mathbb{Z}} g\left|\left[\begin{pmatrix}1 & 0\\ N_1 e & 1\end{pmatrix}\right]_\kappa\right. .$$

Since $g|[\xi\eta]_\kappa = (g|[\xi]_\kappa)|[\eta]_\kappa$, to prove the proposition we can now work with multiplication in $\mathfrak{G}$. We claim that in $\mathfrak{G}$ we have

$$\begin{pmatrix}1 & 0\\ N_1 e & 1\end{pmatrix}^* W(N_1) = W(N_\chi)\left(\begin{pmatrix}p^{2-2m_\chi} & -ep^{2-2m_\chi}\\ 0 & 1\end{pmatrix}, p^{\frac{m_\chi - 1}{2}}\right). \tag{4.1}$$

Multiplication of the matrices is easy, using $N_1 = N_\chi p^{2-2m_\chi}$. So to prove (4.1) we only need show multiplication on the part of the functions. Using the rule (2.1) on the left-hand side of (4.1), as well as Eq. (2.2), nets us

$$j\left(\begin{pmatrix}1 & 0\\ N_1 e & 1\end{pmatrix}, -\frac{1}{N_1 z}\right) N_1^{\frac{1}{4}}(-iz)^{\frac{1}{2}} = N_1^{\frac{1}{4}}(-i(z-e))^{\frac{1}{2}},$$

and on the right-hand side of (4.1) the functions multiply to give precisely the same

$$N_\chi^{\frac{1}{4}}\left(-ip^{2-2m_\chi}(z-e)\right)^{\frac{1}{2}} p^{\frac{m_\chi - 1}{2}} = N_1^{\frac{1}{4}}(-i(z-e))^{\frac{1}{2}}.$$

So the claim is true and using it, and the fact that $[(p^{2-2m_\chi} I_2, 1)]_\kappa$ is the identity operator, we have

$$g|[S_\chi W(N_1)]_\kappa = \sum_{e\in\mathbb{Z}/p^{2m_\chi-2}\mathbb{Z}} (g|[W(N_\chi)]_\kappa)\left|\left[\left(\begin{pmatrix}1 & -e\\ 0 & p^{2m_\chi - 2}\end{pmatrix}, p^{\frac{m_\chi - 1}{2}}\right)\right]_\kappa\right. .$$

By routine calculation on the Fourier expansion it is easy to see that the operator $p^{\frac{\kappa}{4}-1}\sum_{e\in\mathbb{Z}/p\mathbb{Z}}[(\left(\begin{smallmatrix}1 & -e\\ 0 & p\end{smallmatrix}\right), p^{\frac{1}{4}})]_\kappa$ shifts the Fourier coefficients along by p, and is therefore, by Proposition 2.3, the pth Hecke operator. Extending this argument to powers of p yields the result. □

This concludes the first of our aims, and we now calculate the adjoint of $[T(p)]_\kappa$ of level N_1.

Lemma 4.3. *The set up for this lemma is a whole bunch of data. Firstly we let $\Gamma \leq SL_2(\mathbb{Z})$ is a congruence subgroup and $\Delta = \Gamma^*$. The crucial data are two elements $\xi = (\alpha, \varphi)$, and $\xi_0 = (\alpha_0, \varphi_0) \in \mathfrak{G}$ whose functions satisfy the definition of $\mathfrak{G}$ with respective unit circular elements t and t_0, and whose matrices are $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\alpha_0 := \det(\alpha)\alpha^{-1}$.*

(a) *If $\alpha^{-1}\Gamma\alpha \subseteq SL_2(\mathbb{Z})$ then for all $f \in \mathscr{S}_k(\Delta)$ and $g \in \mathscr{M}_k(\alpha^{-1}\Gamma\alpha)$ we have*

$$\langle f|[\xi]_\kappa, g\rangle_{\alpha^{-1}\Gamma\alpha} = \overline{tt_0}^{\frac{\kappa}{2}} \langle f, g|[\xi]_\kappa\rangle_\Gamma.$$

(b) *Suppose that $\Delta\xi_{(0)}\Delta = \bigsqcup_\ell \Delta\xi_{(0)\ell}$ where $\xi_{(0)\ell} = (\alpha_{(0)\ell}, \varphi_{(0)\ell})$ with $\varphi_{(0)\ell}$ satisfying the definition of $\mathfrak{G}$ with unit circular elements $t_{(0)\ell}$, then $t_\ell = t$ and $t_{0\ell} = t_0$ for all ℓ. For any $f \in \mathscr{S}_k(\Delta)$ and $g \in \mathscr{M}_k(\Delta)$ we have*

$$\langle f|[\Delta\xi\Delta]_\kappa, g\rangle = \overline{tt_0}^{\frac{\kappa}{2}} \langle f, g|[\Delta\xi_0\Delta]_\kappa\rangle.$$

Proof. **(a)** Expanding out the action in the product on the left-hand side gives

$$\begin{aligned} &\int_{B(\alpha^{-1}\Gamma\alpha)} \varphi(z)^{-\kappa} f(\alpha z)\overline{g(z)} y^{\frac{\kappa}{2}} \, d\mu(z) \\ &= \int_{B(\Gamma)} \varphi(\alpha_0 z)^{-\kappa} f(z)\overline{g(\alpha_0 z)}\, \mathfrak{Im}(\alpha_0 z)^{\frac{\kappa}{2}} \, d\mu(z) \end{aligned}$$

making the change of variables $z \mapsto \alpha_0 z$, noting that $f(\alpha\alpha_0 z) = f(z)$, and that $B(\alpha^{-1}\Gamma\alpha)$ maps to $B(\Gamma)$.

Notice that $\overline{\varphi_0(z)}^2 = t_0^{-1}|\alpha|^{-\frac{1}{2}}(a - c\bar{z}) = (t_0^{-1}\,\varphi_0(\bar{z}))^2$ and we get

$$\mathfrak{Im}(\alpha_0 z) = \frac{|\alpha|^{\frac{1}{2}}|\alpha|^{\frac{1}{2}}}{(a - cz)(a - c\bar{z})} y = \varphi_0(z)^{-2} \frac{t_0^2}{\varphi_0(\bar{z})^2} y = \varphi_0(z)^{-2}\overline{\varphi_0(z)}^{-2} y.$$

Plugging this back into the integral one has

$$\int_{B(\Gamma)} (\varphi(\alpha_0 z)\,\varphi_0(z))^{-\kappa} f(z)\overline{g|[\xi_0]_\kappa(z)} y^{\frac{\kappa}{2}} \, d\mu(z)$$

and to finish we claim that $\varphi(a_0 z)\,\varphi_0(z) = (tt_0)^{\frac{1}{2}}$. To do so denote the regular factor of automorphy $j^\star(\alpha, z) = (cz + d)$, $j^\star(\alpha_0, z) = (a - cz)$, then

$$(\varphi(\alpha_0 z)\,\varphi_0(z))^2 = tt_0|\alpha|^{-1} j^\star(\alpha, \alpha_0 z) j^\star(\alpha, z) = tt_0|\alpha|^{-1} j^\star(\alpha\alpha_0, z)$$

by the well-known cocyle relation. Then as $j^\star(\alpha\alpha_0, z) = j^\star(|\alpha| I_2, z) = |\alpha|$ we are done.

(b) As we are just taking $\Delta = \Gamma^*$ the projection P is a bijection between $\Delta\xi\Delta$ and $\Gamma\alpha\Gamma$, and so we have $\Gamma\alpha\Gamma = \bigsqcup_\ell \Gamma\alpha_\ell$. Let ℓ be arbitrary, then for any $\gamma_3 \in \Gamma$ there exist $\gamma_1, \gamma_2 \in \Gamma$ such that

$$\gamma_1\alpha\gamma_2 = \gamma_3\alpha_\ell \qquad \gamma_1^*\xi\gamma_2^* = \gamma_3^*\xi_\ell$$

and from the first we have $|\alpha_\ell| = |\alpha|$. By the law of multiplication in $\mathfrak{G}$ the second gives $j(\gamma_1, \xi\gamma_2 z)\,\varphi(\gamma_2 z)\, j(\gamma_2, z) = j(\gamma_3, \xi_\ell z)\,\varphi(z)$. Squaring both sides and using the known cocycle relation on $j^\star$ gives

$$t|\alpha|^{-\frac{1}{2}} j^\star(\gamma_1\xi\gamma_2, z) = t_\ell|\alpha_\ell|^{-\frac{1}{2}} j^\star(\gamma_3\xi_\ell, z)$$

and that $t_\ell = t$ follows because both $|\alpha| = |\alpha_\ell|$ and $\gamma_1\xi\gamma_2 = \gamma_3\xi_\ell$.

Now that we know $t = t_\ell$ and $t_0 = t_{0\ell}$ for all ℓ the result follows using part (a) on the decompositions of both $\Delta\xi\Delta$ and $\Delta\xi_0\Delta$. □

Proposition 4.4. *The adjoint of* $[T(p; N_1)]_\kappa = [\Delta_1(N_1)\xi\Delta_1(N_1)]_\kappa$ *with* $\xi = (\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right), p^{\frac{1}{4}})$ *is*

$$[W(N_1)T(p; N_1)W(N_1)]_\kappa.$$

Proof. First we claim that $W(N_1)$ normalises $\Delta_1(N_1)$. On the part of matrices this is the well-known and easy matrix multiplication $\beta_{N_1}^{-1}\gamma\beta_{N_1} \in \Gamma_1(N_1)$, if $\gamma \in \Gamma_1(N_1)$. It is shown in the proof of Proposition 1.4 in [15] that if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N_1)$ then

$$W(N_1)^{-1}\gamma^* W(N_1) = (\beta_{N_1}^{-1}\gamma\beta_{N_1})^* \left(1, \left(\frac{N_1}{d}\right)\right).$$

But note that $\left(\frac{N_1}{d}\right) = 1$ if $\gamma \in \Gamma_1(N_1)$ so that $\Delta_1(N_1) = \Gamma_1(N_1)^*$ gives the claim. Of particular use will be

$$W(N_1)\Delta_1(N_1) = \Delta_1(N_1)W(N_1) \tag{4.2}$$

$$\Delta_1(N_1)W(N_1)^{-1} = W(N_1)^{-1}\Delta_1(N_1). \tag{4.3}$$

To finish we use Lemma 4.3. In the notation of that Lemma we have $\alpha_0 = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\varphi_0(z)^2 = p^{-\frac{1}{2}}$, where we chose $t_0 = 1$. By (b) in Lemma 4.3 the adjoint is then $[\Delta_1(N_1)\xi_0\Delta_1(N_1)]_\kappa$ where $\xi_0 = (\alpha_0, \varphi_0)$. We claim that $\xi_0 = W(N_1)^{-1}\xi W(N_1)$, and the matrix multiplication is easy. To show that the functions match up we show $W(N_1)\xi_0 = \xi W(N_0)$, and the law of $\mathfrak{G}$-multiplication gives $p^{\frac{1}{4}} N_1(-iz)^{\frac{1}{2}}$ immediately from the right-hand side, whereas the left-hand side gives

$$N_1^{\frac{1}{4}}(-i(pz))^{\frac{1}{2}} p^{-\frac{1}{4}} = N_1^{\frac{1}{4}}(-iz)^{\frac{1}{2}} p^{\frac{1}{4}}$$

and so they do indeed match up.

So now the adjoint is $[\Delta_1(N_1)W(N_1)^{-1}\xi W(N_1)\Delta_1(N_1)]_\kappa$ and we just use (4.2) and (4.3) to finish, noting that $[W(N_1)^{-1}]_\kappa = [W(N_1)]_\kappa$. □

5. Fourier expansion of Eisenstein series

This section involves material all of which has long been well-known, however we include it here anyway for two reasons. The first is for further clarity in the calculation of Fourier coefficients in the next section, but secondly that it gives a nice motivation, outside of algebraicity, for the choice of special values to be interpolated. We calculate the explicit Fourier expansion of the non-holomorphic Fricke-involuted Eisenstein series of an integral weight $\ell \in \mathbb{Z}$, level $1 \leq M \in \mathbb{Z}$, and character χ^{-1}. For the variables $z \in \mathbb{H}$ and $s \in \mathbb{C}$ the Eisenstein series is defined as in Sect. 2. We can choose a set of representatives for $(\mathcal{P} \cap \Gamma)\backslash\Gamma$ as follows

$$E(z, s; \ell) = E(z, s; \ell, \chi, M) = y^{\frac{s}{2}} \sum_{\substack{(cM,d)=1 \\ d\in\mathbb{N}, c\in\mathbb{Z}}} \chi(d)(cMz + d)^{-\ell}|cMz + d|^{-s}.$$

Then

$$\begin{aligned}\mathscr{E}(z, s; \ell) &:= L(s + \ell, \chi)E(z, s; \ell) \\ &= y^{\frac{s}{2}} \sum_{\substack{(cM,d)\in\mathbb{Z}^2 \\ d>0}} \chi(d)(cMz + d)^{-\ell}|cMz + d|^{-s}\end{aligned}$$

and here we seek the Fourier expansion of

$$\mathscr{E}^*(z, s) := \mathscr{E}(z, s; \ell)|[W(M)]_{2\ell} = (-iz\sqrt{M})^{-\ell}\mathscr{E}\left(-\frac{1}{Mz}, s; \ell\right)$$

for certain values of s that we specify later. We replicate the calculation found in [16], which there is done for half-integral weight Eisenstein series. Using that $\mathfrak{Im}(-\frac{1}{Mz}) = y^{\frac{s}{2}}M^{-\frac{s}{2}}|z|^{-s}$ we get

$$\mathscr{E}\left(-\frac{1}{Mz}, s; \ell\right) = y^{\frac{s}{2}}M^{-\frac{s}{2}} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ d>0}} \chi(d)\left(-\frac{c}{z} + d\right)^{-\ell} |-c + dz|^{-s}.$$

Throwing the $z^{-\ell}$ from the action of $W(M)$ into the sum we get

$$i^{-\ell}M^{\frac{s+\ell}{2}}y^{-\frac{s}{2}}\mathscr{E}^*(z, s) = \sum_{\substack{(b,d)\in\mathbb{Z}^2 \\ d>0}} \chi(d)(b + dz)^{-\ell}|b + dz|^{-s} =: \mathscr{E}'(z, s)$$

so it just remains to figure out the Fourier expansion of $\mathscr{E}'(z, s)$. Writing $b = dj+m$ where $j \in \mathbb{Z}$ and $1 \leq m \leq d$, this then takes the form

$$\mathscr{E}'(z, s) = \sum_{d=1}^{\infty} \chi(d)d^{-\ell-s} \sum_{m=1}^{d} \sum_{j=-\infty}^{\infty} \left(z + \frac{m}{d} + j\right)^{-\ell-\frac{s}{2}} \left(\bar{z} + \frac{m}{d} + j\right)^{-\frac{s}{2}}$$

which is amenable to the following lemma:

Lemma 5.1. [16, p. 84] *If $\alpha, \beta \in \mathbb{C}$ with $\mathfrak{Re}(\alpha), \mathfrak{Re}(\beta) > 0$ and $\mathfrak{Re}(\alpha+\beta) > 1$ then for $w = x + iy \in \mathbb{H}$ we have*

$$\sum_{j=-\infty}^{\infty} (w+j)^{-\alpha}(\bar{z}+j)^{-\beta} = \sum_{n=-\infty}^{\infty} \tau_n(y,\alpha,\beta)e^{2\pi inx}$$

where

$$i^{\alpha-\beta}(2\pi)^{-\alpha-\beta}\Gamma(\alpha)\Gamma(\beta)\tau_n(y,\alpha,\beta) = \begin{cases} n^{\alpha+\beta-1}e^{-2\pi ny}\sigma(4\pi ny,\alpha,\beta) & \text{if } n > 0; \\ |n|^{\alpha+\beta-1}e^{-2\pi|n|y}\sigma(4\pi|n|y,\beta,\alpha) & \text{if } n < 0; \\ \Gamma(\alpha+\beta-1)(4\pi y)^{1-\alpha-\beta} & \text{if } n = 0; \end{cases}$$

and if $\mathfrak{Re}(\beta) > 0$ we have

$$\sigma(y,\alpha,\beta) := y^{-\beta}\int_0^{\infty}(1+y^{-1}t)^{\alpha-1}t^{\beta-1}e^{-t}dt$$

which we can continue analytically to the whole β-plane as in [16].

Define a divisor sum function by $\sigma'_{\ell,\chi}(n) = \sum_{d|n}\chi(n/d)d^{\ell}$. Applying the above lemma with $\alpha = \ell + \frac{s}{2}$, $\beta = \frac{s}{2}$, $w = z + \frac{m}{d} = x + \frac{m}{d} + iy$, we obtain

$$\mathscr{E}'(z,s) = \sum_{n=-\infty}^{\infty} \alpha(n,s)\tau_n(y,\ell+\tfrac{s}{2},\tfrac{s}{2})e^{2\pi inx}$$

where, if $n \neq 0$ we have

$$\begin{aligned}\alpha(n,s) &:= \sum_{d=1}^{\infty}\chi(d)d^{-\ell-s}\sum_{m=1}^{d}e^{2\pi in\frac{m}{d}} = \sum_{0<d|n}\chi(d)d^{1-\ell-s} \\ &= |n|^{1-s-\ell}\sigma'_{s+\ell-1,\chi}(|n|)\end{aligned}$$

and if $n = 0$ this is just $\alpha(0,s) = L(\ell+s-1,\chi)$. So now the Fourier expansion of $\mathscr{E}'(z,s)$ is

$$\begin{aligned}&L(\ell+s-1,\chi)\tau_0(y,\ell+\tfrac{s}{2},\tfrac{s}{2}) \\ &\quad+ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |n|^{1-s-\ell}\sigma'_{s+\ell-1,\chi}(|n|)\tau_n(y,\ell+\tfrac{s}{2},\tfrac{s}{2})e^{2\pi inx}.\end{aligned}$$

Explicit expressions of the τ_n for certain values of s are now deduced. Suppose that $s = m$ where m is a negative even integer, and $s > -\ell+1$. We claim that in this situation we have $\tau_n(y,\ell+\frac{s}{2},\frac{s}{2}) = 0$ if $n \leq 0$ and is non-zero for $n > 0$.

The easiest of these is if $n = 0$ in which case

$$\tau_0(y,\ell+\tfrac{s}{2},\tfrac{s}{2}) = \frac{i^{-\ell}(2\pi)^{s+\ell}}{\Gamma(\ell+\frac{s}{2})\Gamma(\frac{s}{2})}\Gamma(\ell+s-1)(4\pi y)^{1-s-\ell}$$

and since $\frac{s}{2}$ is a negative integer, there is a pole at $\Gamma(\frac{s}{2})$ which won't be cancelled in the numerator as $s > -\ell + 1$. So $\tau_0(y, \ell + \frac{s}{2}, \frac{s}{2}) = 0$.

Suppose that $n < 0$, then

$$\tau_n(y, \ell + \tfrac{s}{2}, \tfrac{s}{2}) = \frac{i^{-\ell}(2\pi)^{\ell+s}|n|^{\ell+s-1}}{\Gamma(\ell + \frac{s}{2})\Gamma(\frac{s}{2})} e^{-2\pi|n|y}\sigma(4\pi|n|y, \tfrac{s}{2}, \ell + \tfrac{s}{2}).$$

Now $\ell + \frac{s}{2} > 0$ is a positive integer, so that σ is here defined by the integral in Lemma 5.1 and is finite. The pole in the denominator at $\Gamma(\frac{s}{2})$ is then still not cancelled out and we again obtain $\tau_n(y, \ell + \frac{s}{2}, \frac{s}{2}) = 0$.

The difficulty is in showing that τ_n is non-zero for $n > 0$, and then actually finding its explicit expression. We have

$$\tau_n(y, \ell + \tfrac{s}{2}, \tfrac{s}{2}) = \frac{i^{-\ell}(2\pi)^{\ell+s}n^{\ell+s-1}}{\Gamma(\ell + \frac{s}{2})} e^{-2\pi ny}\Gamma(\tfrac{s}{2})^{-1}\sigma(4\pi ny, \ell + \tfrac{s}{2}, \tfrac{s}{2})$$

and since $\frac{s}{2} < 0$ we need to make use of the analytic continuation of σ to proceed, which essentially cancels out the pole in this case. This analytic continuation is given in [16, p. 83] as

$$\Gamma(\beta)^{-1}\sigma(y, \alpha, \beta) = \frac{e^{-\pi i\beta}}{2\pi i}\Gamma(1-\beta)y^{-\beta}\int_{\infty}^{(0+)}(1 + y^{-1}t)^{\alpha-1}t^{\beta-1}e^{-t}dt$$

where we are integrating over the key-hole contour going to $+\infty$ on the real axis and positively oriented about the origin. More specifically, recalling that $\frac{s}{2}$ is a negative integer and using the binomial theorem we have

$$\begin{aligned}
&\Gamma(\tfrac{s}{2})^{-1}\sigma(4\pi ny, \ell + \tfrac{s}{2}, \tfrac{s}{2}) \\
&\quad = \frac{(-1)^{\frac{s}{2}}}{2\pi i}\Gamma(1 - \tfrac{s}{2})(4\pi ny)^{-\frac{s}{2}}\sum_{j=1}^{\ell+\frac{s}{2}-1}\binom{\ell + \frac{s}{2} - 1}{j} \\
&\quad\quad \times (4\pi ny)^{-j}\int_{\infty}^{(0+)} t^{\frac{s}{2}+j-1}e^{-t}dt.
\end{aligned}$$

The Gamma function has a well-known meromorphic continuation, which is given by $(e^{2\pi is} - 1)\Gamma(s) = \int_{\infty}^{(0+)} t^{s-1}e^{-t}dt$, and which gives

$$\begin{aligned}
&\Gamma(\tfrac{s}{2})^{-1}\sigma(4\pi ny, \ell + \tfrac{s}{2}, \tfrac{s}{2}) \\
&= \frac{(-1)^{\frac{s}{2}}\Gamma(1 - \frac{s}{2})}{2\pi i}\sum_{j=1}^{\ell+\frac{s}{2}-1}\binom{\ell + \frac{s}{2} - 1}{j}(4\pi ny)^{-j-\frac{s}{2}}(e^{2\pi i(\frac{s}{2}+j)} - 1)\Gamma(\tfrac{s}{2} + j).
\end{aligned}$$

If $\frac{s}{2} + j > 0$ we have $(e^{2\pi i(\frac{s}{2}+j)} - 1)\Gamma(\frac{s}{2} + j) = 0$, whereas if $\frac{s}{2} + j \leq 0$ we get

$$(e^{2\pi i(\frac{s}{2}+j)} - 1)\Gamma(\tfrac{s}{2} + j) = \frac{(-1)^{-\frac{s}{2}-j}(2\pi i)}{(-\frac{s}{2} - j)!}.$$

By the choice of s we have $-\frac{s}{2} < \ell + \frac{s}{2} - 1$, which gives

$$\begin{aligned}
&\Gamma(\tfrac{s}{2})^{-1}\sigma(4\pi ny, \ell + \tfrac{s}{2}, \tfrac{s}{2}) \\
&\quad = \Gamma(1 - \tfrac{s}{2}) \sum_{j=0}^{-\frac{s}{2}} \binom{\ell + \frac{s}{2} - 1}{j} (4\pi ny)^{-j-\frac{s}{2}} \frac{(-1)^j}{\Gamma(1 - \frac{s}{2} - j)}.
\end{aligned}$$

Now we obtain the the following Fourier expansion for $\mathscr{E}'(z, s)$:

$$\begin{aligned}
&\frac{(2\pi)^{\ell+s}\Gamma(1 - \frac{s}{2})}{i^\ell \Gamma(\ell + \frac{s}{2})} \sum_{n=1}^{\infty} \left[\sum_{j=0}^{-\frac{s}{2}} \binom{\ell + \frac{s}{2} - 1}{j} \frac{(-1)^j}{\Gamma(1 - \frac{s}{2} - j)} (4\pi ny)^{-j-\frac{s}{2}} \right] \\
&\quad \sigma'_{s+\ell-1,\chi}(n) q^n.
\end{aligned}$$

Recalling that $\mathscr{E}^*(z, s) = i^\ell M^{-\frac{s+\ell}{2}} y^{\frac{s}{2}} \mathscr{E}'(z, s)$ then we obtain our final Fourier development of this, whenever $0 \geq s > -\ell + 1$ is an even integer, to be

$$\begin{aligned}
&\frac{(2\pi)^{\ell+\frac{s}{2}}\Gamma(1 - \frac{s}{2})}{2^{\frac{s}{2}} M^{\frac{s+\ell}{2}} \Gamma(\ell + \frac{s}{2})} \sum_{n=1}^{\infty} \left[\sum_{j=0}^{-\frac{s}{2}} \binom{\ell + \frac{s}{2} - 1}{j} \frac{(-1)^j n^{-j-\frac{s}{2}}}{\Gamma(1 - \frac{s}{2} - j)} (4\pi y)^{-j} \right] \\
&\quad \sigma'_{s+\ell-1,\chi}(n) q^n.
\end{aligned}$$

A function $f : \mathbb{H} \to \mathbb{C}$ is said to be *nearly holomorphic* of weight $k = \frac{\kappa}{2}$ for even or odd κ, and of level Γ, if it satisfies $f|[\xi]_\kappa = f$ for all $\xi \in \Gamma$ and if it has a Fourier expansion of the form

$$f = \sum_{j=0}^{r} (\pi y)^{-j} \sum_{n=0}^{\infty} a_n q^n$$

where $r \in \mathbb{Z}$. Denote this space by $\mathscr{N}_k(\Gamma)$ and we can analogously define the spaces $\mathscr{N}_k(N, \psi)$ if $4 \mid N \in \mathbb{Z}$ and ψ is a character modulo N.

By our calculation above we have $\mathscr{E}^*(z, s) \in \mathscr{N}_\ell(M, \chi^{-1})$ for any even $0 \geq s > -\ell + 1$.

We define the Shimura–Maass differential operators for any $\lambda \in \mathbb{R}$ and for any $0 \leq a \in \mathbb{Z}$ as in [17, p. 812] to be

$$\begin{aligned}
\delta(\lambda) &:= \frac{1}{2\pi i} \left(\frac{\lambda}{2iy} + \frac{\partial}{\partial z} \right) \\
\delta_k^a &:= \prod_{j=1}^{a} \delta(k + 2j - 2)
\end{aligned}$$

where the product is with respect to composition, and we then have $\delta_k^a \mathscr{N}_k \subseteq \mathscr{N}_{k+2a}$.

Lemma 5.2. [17, p. 813] *Any $g \in \mathscr{N}_k$ can be written uniquely as*

$$g = g_0 + \sum_{\nu=1}^{r} \delta_{k-2\mu}^{\mu} g_\mu$$

where $g_\mu \in \mathscr{M}_{k-2\mu}$ for some $r \leq \frac{k}{2}$, and g_0 is known as the holomorphic projection of g. Moreover, if $g \in \mathscr{N}_k(M, \psi)$ for an integer $4 \mid M$, then $\langle f, g\rangle = \langle f, g_0\rangle$ for any $f \in \mathscr{S}_k(M, \psi)$.

6. Interpolation

We are now in a position to construct our p-adic L-function, and this is achieved by constructing a p-adic measure. Here we are actually constructing two families of measures, one for each $\nu \in \{0, 1\}$.

Theorem 6.1. *Let $1 \leq N \in \mathbb{Z}$ be divisible by 4, $\nu \in \{0, 1\}$, and for any Dirichlet character χ set*

$$\delta_k^{(\nu)}(\chi) = \begin{cases} 1 \text{ if } \chi(-1)(-1)^{\lfloor k \rfloor} = (-1)^\nu \\ 0 \text{ otherwise.} \end{cases}$$

Let $m \in \frac{1}{2}\mathbb{Z} \backslash \mathbb{Z}$ be any half-integer satisfying $0 \geq m - k > -k + \frac{3}{2}$ and that $m - k + \nu \in 2\mathbb{Z}$. If $p \nmid N$ and $f \in \mathscr{M}_k(N)$ is a p-ordinary normalised eigenform, there exist unique measures $\mu_{f,m}^{(\nu)}$ on $\mathbb{Z}_p^\times$ such that if χ is a Dirichlet character whose conductor is $c_\chi = p^{m_\chi}$ for $1 \leq m_\chi \in \mathbb{Z}$, then

$$\int_{\mathbb{Z}_p^\times} \chi d\mu_{f,m}^{(\nu)} = \delta_k^\nu(\chi) D^{(\nu)} p^{2k-2m-1} p^{m_\chi(k+m-3)} \alpha_p^{-m_\chi} \frac{G(\bar{\chi}) L(m-\nu, f, \chi)}{g_t^\chi(m) \pi^{\frac{m+k-\nu-2}{2}} \langle f, f\rangle_N};$$

for the trivial character it gives

$$\int_{\mathbb{Z}_p^\times} d\mu_{f,m}^{(\nu)} = \delta_k^{(\nu)}(1) D^{(\nu)} p^{3-2m} (1 - \beta_p p^{m+\nu-k-1})(1 - \beta_p p^{2-k-m+\nu}) \frac{L(m-\nu, f)}{g_t^0(m) \pi^{\frac{m+k-\nu-2}{2}} \langle f, f\rangle_N}$$

where $g_t^0 = g_t^{\mathrm{id}}$, and the constant $D^{(\nu)} = D^{(\nu)}(k, m, N)$ will be given later.

To prove the above we first take our integral expression from Sect. 3 and manipulate the inner product using the results of Sect. 4, reducing it from level N_χ to N_1. Removing the dependence of the inner product on χ allows us to bring sums over Dirichlet characters inside the inner product, in particular to the second argument. The main theorem is then proved by giving the precise Fourier development of this sum given in Sect. 5, noting it is p-integral and rational, and then using finite dimensionality of such forms to deduce boundedness of the measure.

Proposition 6.2. *Let χ have conductor $c_\chi = p^{m_\chi}$ where $1 \le m_\chi \in \mathbb{Z}$. If $m_\chi \le r \in \mathbb{Z}$, then we have*

$$L(s, f_1, \chi) = \frac{1}{2}(4t\pi)^{\frac{s+k-2}{2}} \Gamma\left(\tfrac{s+k-2}{2}\right)^{-1} g_t^\chi(s) \alpha_p^{m_\chi - r} \langle f_1|[W(N_1)]_\kappa, H_\chi^{(\nu)}|[T(p)]_\kappa^{2r-2}\rangle_{N_1}$$

where

$$H_\chi^{(\nu)}(z, s) = H_\chi^{(\nu)}(z) := (p^{m_\chi - 1})^{2-k}(\theta_\chi^{(\nu)}(tz)\mathscr{E}_\nu(z, \chi\rho_t, s, N_\chi))|[W(N_\chi)]_\kappa.$$

Proof. By the integral expression (3.4) it is enough to show that

$$\langle f_1, \theta_\chi^{(\nu)}\mathscr{E}_\nu(z, \chi, s, N_\chi)\rangle_{N_\chi} = \alpha_p^{m_\chi - r}\langle f_1|[W(N_1)]_\kappa, H_\chi^{(\nu)}|[T(p)]_\kappa^{2r-2}\rangle_{N_1}.$$

Using Proposition 4.2 we obtain

$$\begin{aligned}\langle f_1, \theta_\chi^{(\nu)}\mathscr{E}_\nu(z, \chi, s, N_\chi)\rangle_{N_\chi} &= \langle f_1, \theta_\chi^{(\nu)}\mathscr{E}_\nu(z, \chi, s, N_\chi)|[S_\chi]_\kappa\rangle_{N_1} \\ &= \langle f_1|[W(N_1)]_\kappa, H_\chi^{(\nu)}|[T(p)]_\kappa^{2m_\chi - 2}\rangle_{N_1}.\end{aligned}$$

Then writing $f_1|[T(p)]_\kappa^{2(r-m_\chi)} = \alpha_p^{r-m_\chi} f_1$ and using that $[W(N_0)]_\kappa^2 = 1$ in the above one has

$$\alpha_p^{m_\chi - r}\langle f_1|[W(N_1)T(p)W(N_1)]_\kappa^{2(r-m_\chi)}[W(N_1)]_\kappa, H_\chi^{(\nu)}|[T(p)]_\kappa^{2m_\chi - 2}\rangle_{N_1}$$

and then Proposition 4.4 gives the result. □

The next task is to repeat the above for the trivial character, but recall the integral expression in (3.4) was confined to the non-trivial characters. It's pretty much the same in the case of the trivial character χ_0, but particular emphasis will be placed here in noting that the character in the Eisenstein series is the (naturally) lifted character modulo N_1, so call it $\chi_0^\star$. This is 1 everywhere except when $(n, N_1) \neq 1$ where it's zero. Then from [19] the integral expression we get is

$$2\frac{\Gamma\left(\frac{s+k-2}{2}\right)}{(4t\pi)^{\frac{s+k-2}{2}} g_t^0(s)} L(s, f_1, \chi_0) = \int_{B(N_1)} f_1(z)\overline{\theta_{\chi_0}^{(\nu)}(tz)\mathscr{E}_\nu(z, \chi_0^\star\rho_t, s, N_1)}d\mu \quad (6.1)$$

where, as usual, $\nu \in \{0, 1\}$ is such that $1 = \chi_0(-1) = (-1)^{\lfloor k\rfloor + \nu}$, and where

$$g_t^0(s) = \zeta_t(s)\zeta(s)^{-1} = \prod_{q|t}(1 - p^{-s}).$$

Proposition 6.3. *For any $1 \le r \in \mathbb{Z}$ we have*

$$\begin{aligned}&p^{3-2s}(1 - \beta_p p^{s+\nu-k-1})L(s, f_1, \chi_0) \\ &= \frac{1}{2}(4t\pi)^{\frac{s+k-2}{2}}\Gamma\left(\tfrac{s+k-2}{2}\right)^{-1} g_t^0(s)\alpha_p^{-r} \\ &\quad \times \langle f_1|[W(N_1)]_\kappa, \mathfrak{h}^{(\nu)}|[p^{3-2s}T(p)^{2r} - p^{k-s+\nu}T(p)^{2r-2}]_\kappa\rangle_{N_1}\end{aligned}$$

where

$$\mathfrak{h}^{(\nu)}(z) := \theta_{\chi_0}^{(\nu)}(tz)\mathscr{E}_\nu(z, \chi_0^\star\rho_t, s, N_1)|[W(N_1)]_\kappa.$$

Proof. Note that $p^{3-2s}(1-\beta_p p^{s+\nu-k-1}) = p^{3-2s} - \frac{p^{k-s+\nu}}{\alpha_p}$ via $\alpha_p\beta_p = p^{2k-2}$. Using the expression (6.1) then it is proven in precisely the same manner as Proposition 6.2, the minor differences occuring since there is no need to reduce the level of the product down, it has level N_1 in the first place. $\square$

Now let m be any value as specified in Theorem 6.1. Note that $m+k \equiv \nu \pmod 2$ so that $\Gamma(\frac{m+k-2}{2}) \in \pi^{\frac{\nu}{2}}\mathbb{Z}$, and so for some $\zeta \in \mathbb{Q}$ the expression in Propositions 6.2 and 6.3 respectively become

$$L(m, f_1, \chi) = t^{\frac{\nu}{2}}\pi^{\frac{m+k-\nu-2}{2}}\zeta g_t^{\chi}(m)\alpha_p^{m_\chi - r} \times \langle f_1|[W(N_1)]_\kappa, H_\chi^{(\nu)}|[T(p)]_\kappa^{2r-2}\rangle_{N_1} \tag{6.2}$$

$$L(m, f_1, \chi_0) = p^{2m-3}(1-\beta_p p^{m+\nu-k-1})^{-1} t^{\frac{\nu}{2}}\pi^{\frac{m+k-\nu-2}{2}}\zeta g_t^{0}(m)\alpha_p^{-r} \times \langle f_1|[W(N_1)]_\kappa, \mathfrak{h}^{(\nu)}|[p^{3-2m}T(p)^{2r} - p^{k-m+\nu}T(p)^{2r-2}]_\kappa\rangle_{N_1}. \tag{6.3}$$

For any $1 \le r \in \mathbb{Z}$ let Δ_r denote the set of *non-trivial* Dirichlet characters whose conductors divide p^r, $\Delta_r^{(\nu)} := \{\chi \in \Delta_r \mid \chi(-1) = (-1)^{\lfloor k\rfloor+\nu}\}$, and $C_r := \Delta_r \cup \{\chi_0\}$. By definition of $d\mu_f^{(\nu)}$ note that for any integer e prime to p we have

$$\sum_{\chi\in C_r}\chi(e^{-1})\int_{\mathbb{Z}_p^\times}\chi d\mu_{f,m}^{(\nu)} = \int_{\mathbb{Z}_p^\times}\sum_{\chi\in C_r}\chi(e^{-1}x)d\mu_{f,m}^{(\nu)}(x)$$

and then by orthogonality relations on characters the integrand is 0 unless we have $x \in e + p^r\mathbb{Z}_p$ at which point it is $\varphi(p^r)$. So we ultimately get

$$\sum_{\chi\in C_r}\chi(e^{-1})\int_{\mathbb{Z}_p^\times}\chi d\mu_{f,m}^{(\nu)} = \varphi(p^r)\mu_{f,m}^{(\nu)}(e+p^r\mathbb{Z}_p)$$

and, by using (6.2) and (6.3) above, the expression

$$\mu_{f,m}^{(\nu)}(e+p^r\mathbb{Z}_p) = t^{\frac{\nu}{2}}\zeta\alpha_p^{-r}\frac{\langle f_1|[W(N_1)]_\kappa, \mathfrak{R}_r^{(\nu)}\rangle_{N_1}}{\langle f, f\rangle_N},$$

where we define

$$\mathfrak{R}_r^{(\nu)} = \mathfrak{R}_r^{(\nu)}(z,m) = R_r^{(\nu)}|[T(p)]_\kappa^{2r-2} + D^{(\nu)}\mathfrak{h}_0^{(\nu)}|[p^{3-2m}T(p)^{2r} - T(p)^{2r-2}]_\kappa$$

if $(-1)^{\lfloor k\rfloor+\nu} = 1$, else we define it to be $R_r^{(\nu)}|[T(p)]_\kappa^{2r-2}$, and where

$$R_r^{(\nu)} := \frac{D^{(\nu)}p^{2k-2m-1}}{\varphi(p^r)}\sum_{\chi\in\Delta_r^{(\nu)}}\bar{\chi}(e)G(\bar{\chi})p^{m_\chi(k+m-3)}(H_\chi^{(\nu)})_0. \tag{6.4}$$

We have denoted by $(H_\chi^{(\nu)})_0$ and $\mathfrak{h}_0^{(\nu)}$ the respective holomorphic projections.

To finish the proof of Theorem 6.1 we must show that the above has bounded p-adic norm as e ranges over $(\mathbb{Z}/p^r\mathbb{Z})^\times$ and as $r \to \infty$. This becomes possible through the particular Fourier development of $\mathfrak{R}_r^{(\nu)}$.

For some character η we modify the divisor sum function σ' at p to give the p-modified power divisor sum

$$^{p}\sigma'_{\ell,\eta}(n) = \sum_{p\nmid d|n} \eta(nd^{-1})d^{\ell}.$$

In Chapter 7 of [7] this is p-adically interpolated, a fact that will essentially yield our interpolation.

Lemma 6.4. *For any integer ℓ, any Dirichlet character η, and positive integer n we have*

$$\sigma'_{\ell,\eta}(p^2n) - p^{2\ell}\sigma'_{\ell,\eta}(n) = \eta(p)(\eta(p) + p^{\ell})\left[{}^{p}\sigma'_{\ell,\eta}(n)\right].$$

In particular, if the modulus of η is divisible by p then $\sigma'_{\ell,\eta}(p^2n) = p^{2\ell}\sigma'_{\ell,\eta}(n)$.

Proof. By definition $\sigma'_{\ell,\eta}(pn) - p^{\ell}\sigma'_{\ell,\eta}(n) = {}^{p}\sigma'_{\ell,\eta}(pn) = \eta(p)\,{}^{p}\sigma'_{\ell,\eta}(n)$. We get

$$\begin{aligned}
\sigma'_{\ell,\eta}(p^2n) - p^{2\ell}\sigma'_{\ell,\eta}(n) &= \sum_{d|p^2n} \eta(p^2n/d)d^{\ell} - \sum_{d|n} \eta(n/d)(p^2d)^{\ell} \\
&= \sum_{d|p^2n} \eta(p^2n/d)d^{\ell} - \sum_{p^2|d|n} \eta(p^2n/d)d^{\ell} \\
&= {}^{p}\sigma'_{\ell,\eta}(p^2n) + \sum_{\substack{d|p^2n \\ \mathrm{ord}_p(d)=1}} \eta(p^2n/d)d^{\ell} \\
&= {}^{p}\sigma'_{\ell,\eta}(p^2n) + p^{\ell}\sigma'_{\ell,\eta}(pn) - p^{2\ell}\sigma'_{\ell,\eta}(n) \\
&= \eta(p)(\eta(p) + p^{\ell})\,{}^{p}\sigma'_{\ell,\eta}(n)
\end{aligned}$$

using that ${}^{p}\sigma'_{\ell,\eta}(p^2n) = \eta(p)^2\,{}^{p}\sigma'_{\ell,\eta}(n)$ in the end. □

Remark 6.5. In fact, the identity $\sigma'_{\ell,\eta}(p^2n) = p^{2\ell}\sigma'_{\ell,\eta}(n)$ for η of modulus divisible by p is pretty immediate via the definitions, and the above lemma is not strictly necessary for this conclusion.

Theorem 6.6. *If $\mathfrak{a}_n^{(\nu)}(r)$ are the Fourier coefficients of $\mathfrak{R}_r^{(\nu)}$ then for all $n \geq 1$ we have that $\mathfrak{a}_n^{(\nu)}(r) \in \mathbb{Z}_p \cap \mathbb{Q}$.*

Before going on to prove this theorem with a series of lemmas, first note that by the previous section we have $H_\chi^{(\nu)}$ is nearly holomorphic and so can be written in the form

$$H_\chi^{(\nu)} = \sum_{j=0}^{r}(4\pi y)^{-j}\sum_{n=0}^{\infty} c_{n,j}^{(\nu)}(\chi)q^n$$

for some r.

Lemma 6.7. *There exists a constant* $0 < \ell \in \mathbb{Z}$ *and linear forms* $F_n(X_0, \dots, X_r)$, *which belong to* $\mathbb{Z}[X_0, \dots, X_r]$ *and are dependent only on* ℓ *and* n, *such that*

$$\ell(H_\chi^{(\nu)})_0 = \sum_{n=0}^{\infty} F_n(c_{n,0}^{(\nu)}(\chi), \dots, c_{n,r}^{(\nu)}(\chi))q^n$$

and, crucially, $F_n(X_0, \dots, X_r) \equiv \ell X_0 \pmod{n}$.

Proof. We make use of Lemma 5.2 and the following identity

$$\delta_k^a = \sum_{j=0}^{a} \binom{a}{j}(-4\pi y)^{-j}\gamma_{k,j}^a\left(\frac{1}{2\pi i}\frac{\partial}{\partial z}\right)^{a-j}$$

where

$$\gamma_{k,j}^a := \begin{cases} (k+a-1)(k+a-2)\cdots(k+a-j) & \text{if } j \geq 1 \\ 1 & \text{if } j = 1. \end{cases}$$

This identity is easily checked using induction and the binomial theorem, as in the integral weight case, the major differences with the identity appearing in [13, p. 211] here occuring since factorials are not even defined for half-integers. Nevertheless with this identity the rest of the proof in [13] still follows through nicely. □

We now give the values of $c_{n,j}^{(\nu)}(\chi)$ whenever χ is a character of conductor p^{m_χ} and then when it is the trivial character χ_0.

In the final Fourier expansion of the previous section, we plug in $\ell = k - \nu - \frac{1}{2}$ and $s = m - k + \nu$ for our particular values of m.

$$\begin{aligned}
&\mathscr{E}_\nu^*(z, m, \chi\rho_t, N_\chi) \\
&\quad = L_{N_\chi}(m - \tfrac{1}{2}, \chi)E(z, m - k + \nu; k - \nu - \tfrac{1}{2}, \chi\rho_t, N_\chi)|[W_{N_\chi}]_{\kappa-2\nu-1} \\
&\quad = \frac{(2\pi)^{\frac{k-\nu+m-1}{2}}\Gamma(1 + \frac{k-m+\nu}{2})}{2^{\frac{m-k+\nu}{2}}N_\chi^{\frac{2m-1}{4}}\Gamma(\frac{k-\nu+m-1}{2})} \\
&\qquad \times \sum_{j=0}^{\frac{k-\nu-m}{2}}(4\pi y)^{-j}\sum_{n=1}^{\infty}\left[\binom{\frac{k-\nu+m-3}{2}}{j}\frac{(-1)^j n^{\frac{k-\nu-m}{2}-j}}{\Gamma(1+\frac{k-m+\nu}{2}-j)}\sigma'_{m-\frac{3}{2},\chi\rho_t}(n)\right]q^n.
\end{aligned}$$

Note that $\frac{k-\nu+m-i}{2} = k - \frac{i}{2} + \frac{m-k+\nu}{2} \in \mathbb{Z}$ for $i = 1, 3$ since $m - k + \nu$ is even, so that everything is well-defined, and the Γ values are integers.

As for the theta series, it is known from [15, p. 457] that

$$\theta_\chi^{(\nu)}\left(-\frac{1}{N_\chi tz}\right) = (-i)^\nu p^{-\frac{m_\chi}{2}}G(\chi)\left(-p^{m_\chi}\frac{iNtz}{2}\right)^{\nu+\frac{1}{2}}\theta_{\bar{\chi}}^{(\nu)}(tN'z)$$

where $N' = \frac{N}{4}$. From this we get

$$\theta_\chi^{(\nu)}(tz)|[W(N_\chi)]_{2\nu+1} = \frac{(-i)^\nu N_\chi^{\frac{2\nu+1}{4}}t^{\nu+\frac{1}{2}}}{2^\nu\sqrt{2}}p^{-m_\chi(\nu+1)}G(\chi)\sum_{n=0}^{\infty}2\bar{\chi}(n)n^\nu q^{tn^2N'z}$$

where $N' = \frac{N}{4}$.

Let $W_n := \{(n_1, n_2) \in \mathbb{N}^2 \mid tn_1^2 N' + n_2 = n,\ p \nmid n_1\}$. Then by multiplying the above expressions together along with the factor of $(p^{m_\chi - 1})^{2-k}$ occuring in $H_\chi^{(\nu)}$ we easily see that we have the following lemma.

Lemma 6.8. *There are constants* $\mathscr{C}_j^{(\nu)} = \mathscr{C}_j^{(\nu)}(k, m, N) \in (2tp\sqrt{N})^{\frac{1}{2}} \pi^{\frac{k-\nu+m-1}{2}} \mathbb{Q}^\times$ *for all* $j \in \{0, \ldots, \frac{k-m}{2}\}$ *such that*

$$c_{n,j}^{(\nu)}(\chi) = \mathscr{C}_j^{(\nu)} p^{m_\chi(2-k-m)} G(\chi) \sum_{(n_1,n_2)\in W_n} 2\bar{\chi}(n_1) n_1^\nu n_2^{\frac{k-\nu-m}{2} - j} \sigma'_{m-\frac{3}{2}, \chi\rho_t}(n_2) q^n.$$

Explicitly, we have

$$\mathscr{C}_j^{(\nu)} = \binom{\frac{k-\nu+m-3}{2}}{j} \frac{(-1)^{j+\nu} i^\nu \Gamma(1 + \frac{k-m+\nu}{2}) p^{2-k} 2^{k-2\nu-1} t^{\nu+\frac{1}{2}} \pi^{\frac{k-\nu+m-1}{2}}}{N^{\frac{m-\nu-1}{2}} \Gamma(\frac{k-\nu+m-1}{2}) \Gamma(1 + \frac{k-m+\nu}{2} - j)}.$$

Now let $V_n := \{(n_1, n_2) \in (\mathbb{Z}_{\geq 0})^2 \mid tn_1^2 N' p^2 + n_2 = n, n_2 \neq 0\}$, then repeat the above procedure for the trivial character χ_0. For our values of m the relevant Eisenstein series has expansion

$$\begin{aligned}
&\mathscr{E}^*(z, m, \chi_0^\star \rho_t, N_1) \\
&\quad = \frac{(2\pi)^{\frac{k-\nu+m-1}{2}} \Gamma(1 + \frac{k-m+\nu}{2})}{2^{\frac{m-k+\nu}{2}} N^{\frac{2m-1}{4}} p^{m-\frac{1}{2}} \Gamma(\frac{k-\nu+m-1}{2})} \\
&\quad \times \sum_{j=0}^{\frac{k-\nu-m}{2}} (4\pi y)^{-j} \sum_{n=1}^\infty \left[\binom{\frac{k-\nu+m-3}{2}}{j} \frac{(-1)^j n^{\frac{k-\nu-m}{2} - j}}{\Gamma(1 + \frac{k-m+\nu}{2} - j)} \sigma'_{m-\frac{3}{2}, \chi_0^\star \rho_t}(n) \right] q^n
\end{aligned}$$

and the theta series has expansion

$$\theta_{\chi_0}^{(\nu)}(tz) | [W(N_1)]_{2\nu+1} = \frac{(-i)^\nu N_1^{\frac{2\nu+1}{4}} t^{\nu+\frac{1}{2}}}{2^\nu \sqrt{2}} \left[1 + \sum_{n=1}^\infty 2n^\nu q^{t(np)^2 N'} \right]$$

so that if

$$\mathfrak{h}^{(\nu)}(z) = \sum_{j=0}^r (4\pi y)^{-j} \sum_{n=0}^\infty c_{n,j}^{(\nu)}(\chi_0) q^n$$

then we obtain:

Lemma 6.9. *For* $j \in \{0, \ldots, \frac{k-m}{2}\}$ *then the constants* $\mathscr{C}_j^{(\nu)}$ *in Lemma 6.8 also satisfy*

$$c_{n,j}^{(\nu)}(\chi_0) = \mathscr{C}_j^{(\nu)} p^{k-m-1} \sum_{(n_1,n_2)\in V_n} \delta_{n_1} n_1^\nu n_2^{\frac{k-\nu-m}{2} - j} \sigma_{m-\frac{3}{2}, \chi_0^\star \rho_t}(n_2)$$

where $\delta_{n_1} = 1$ *if* $n_1 = 0$ *but* $\delta_{n_1} = 2$ *otherwise.*

Lemma 6.10. *There's a global constant $C^{(\nu)} \in (2pt\sqrt{N})^{\frac{1}{2}}\pi^{\frac{k-\nu+m-1}{2}}\mathbb{Q}^\times$ so that if $n, r \in \mathbb{N}$, $e \in \mathbb{Z}$ with $(e, p) = 1$ are all arbitrary, and $\nu \in \{0, 1\}$ satisfies $(-1)^{\lfloor k \rfloor+\nu} = 1$, then we have the congruence*

$$\mathfrak{C}_1 := C^{(\nu)}\Bigg[p^{2k-2m-1}\sum_{\chi\in\Delta_r^{(\nu)}}\bar{\chi}(e)G(\bar{\chi})p^{m_\chi(k+m-3)}c^{(\nu)}_{np^{2r-2},0}(\chi)$$
$$+ p^{3-2m}c^{(\nu)}_{np^{2r},0}(\chi_0) - p^{k-m-\nu}c^{(\nu)}_{np^{2r-2},0}(\chi_0)\Bigg] \equiv 0 \pmod{p^r}$$

and actually we can put $C^{(\nu)} = (\mathscr{C}_0^{(\nu)})^{-1}$.

Furthermore, if $(H_\chi^{(\nu)})_0 = \sum_{n=0}^\infty c_n^{(\nu)}(\chi)q^n$ is the Fourier expansion of the holomorphic projection, $r \geq 2$, then putting $D^{(\nu)} = \ell C^{(\nu)}$ where ℓ is as in Lemma 6.7 we have

$$\mathfrak{C}_2 := D^{(\nu)}\Bigg[p^{2k-2m-1}\sum_{\chi\in\Delta_r^{(\nu)}}\bar{\chi}(e)G(\bar{\chi})p^{m_\chi(k+m-2)}c^{(\nu)}_{np^{2r-2}}(\chi)$$
$$+ p^{3-2m}c^{(\nu)}_{np^{2r}}(\chi_0) - p^{k-m+\nu}c^{(\nu)}_{np^{2r-2}}(\chi_0)\Bigg] \equiv 0 \pmod{p^r}.$$

Proof. Using $G(\chi)G(\bar{\chi}) = \chi(-1)p^{m_\chi}$ and that $C^{(\nu)}$ cancels $\mathscr{C}_0^{(\nu)}$ then we can write $\mathfrak{C}_1$ as

$$p^{2k-2m-1}\Bigg[\sum_{(n_1,n_2)\in W_{np^{2r-2}}}\sum_{\chi\in\Delta_r}2\bar{\chi}(-en_1)n_1^\nu n_2^{\frac{k-\nu-m}{2}}\sigma'_{m-\frac{3}{2},\chi\rho_t}(n_2)$$
$$+\sum_{(n_3,n_4)\in V_{np^{2r}}}\delta_{n_3}n_3^\nu(p^{-2}n_4)^{\frac{k-\nu-m}{2}}p^{3-2m}\sigma'_{m-\frac{3}{2},\chi_0^\star\rho_t}(n_4)$$
$$-\sum_{(n_5,n_6)\in V_{np^{2r-2}}}\delta_{n_5}(pn_5)^\nu n_6^{\frac{k-\nu-m}{2}}\sigma'_{m-\frac{3}{2},\chi_0^\star\rho_t}(n_6)\Bigg].$$

We have the following bijections

$$\begin{aligned} W_{np^{2r-2}} &\to \{(n_3, n_4) \in V_{np^{2r}} \mid p \nmid n_4, n_3 \neq 0\} \\ (n_1, n_2) &\mapsto (n_1, p^2 n_2) \end{aligned} \tag{6.5}$$

$$\begin{aligned} V_{np^{2r-2}} &\to \{(n_3, n_4) \in V_{np^{2r}} \mid p \mid n_4, n_3 \geq 0\} \\ (n_5, n_6) &\mapsto (pn_5, p^2 n_6). \end{aligned} \tag{6.6}$$

In accordance with these two bijections we can split up the $V_{np^{2r}}$ appearing in $c_{np^{2r},0}(\chi_0)$ and redistribute them. By Lemma 6.4 we have got that $p^{3-2m}\sigma_{m-\frac{3}{2},\chi_0^\star\rho_t}{}'(p^2n_2) = \sigma'_{m-\frac{3}{2},\chi_0^\star\rho_t}(n_2)$, giving

$$p^{2k-2m-1}\Bigg[\sum_{(n_1,n_2)\in W_{np^{2r-2}}}\sum_{\chi\in C_r} 2\bar{\chi}(-en_1)n_2^{\frac{k-\nu-m}{2}}\sigma'_{m-\frac{3}{2},\chi\rho_t}(n_2)$$
$$+\sum_{(n_5,n_6)\in V_{np^{2r-2}}}\delta_{n_5}(pn_5)^{\nu}n_6^{\frac{k-\nu-m}{2}}(p^{3-2m}\sigma'_{m-\frac{3}{2},\chi_0^{\star}\rho_t}(p^2n_6)-\sigma'_{m-\frac{3}{2},\chi_0^{\star}\rho_t}(n_6))\Bigg]$$

using also that $\delta_{pn_5}=\delta_{n_5}$. Since $p\nmid n_2$ we can replace the $\sigma'_{m-\frac{3}{2},\bar{\chi}\rho_t}(n_2)$ in the first sum by ${}^{p}\sigma'_{m-\frac{3}{2},\bar{\chi}\rho_t}(n_2)$. Now the second sum vanishes, again by Lemma 6.4. So we are left with

$$2p^{2k-2m-1}\sum_{(n_1,n_2)\in W_{np^{2r-2}}} n_2^{\frac{k-\nu-m}{2}}M_{r,m}(-en_1)$$

where

$$M_{r,m}(x):=\sum_{\chi\in C_r}\bar{\chi}(x)\left[{}^{p}\sigma'_{m-\frac{3}{2},\chi\rho_t}(n_2)\right]$$

which is $\equiv 0 \pmod{p^r}$ whenever $(x,p)=1$ by the properties of the known measure interpolating ${}^{p}\sigma'_{m-\frac{3}{2},\bar{\chi}\rho_t}(n_2)$, see Chapter 7 of [6]. This proves the first congruence.

The second congruence follows precisely as in [13] by using Lemma 6.7. More specifically, we have got $\ell c_n^{(\nu)}(\chi)=F_n(c_{n,0}^{(\nu)}(\chi),\ldots,c_{n,\frac{k-m}{2}}^{(\nu)}(\chi))$ and

$$F_n(c_{n,0}^{(\nu)}(\chi),\ldots,c_{n,\frac{k-m}{2}}^{(\nu)}(\chi))\equiv \ell c_{n,0}^{(\nu)}(\chi)\pmod{n\mathbb{Z}[c_{n,0}^{(\nu)}(\chi),\ldots,c_{n,\frac{k-m}{2}}^{(\nu)}(\chi)]}.$$

Also note that

$$(\mathscr{C}_0^{(\nu)})^{-1}\mathscr{C}_j^{(\nu)}=\binom{\frac{k-\nu+m-3}{2}}{j}\frac{(-1)^j\Gamma(1-\frac{k-m+\nu}{2})}{\Gamma(1-\frac{k-m+\nu}{2}-j)}\in\mathbb{Z}$$

so that replacing $c_{n,0}^{(\nu)}(\chi)$ with the $c_{n,j}^{(\nu)}(\chi)$ preserves integrality of the first congruence. We then have $\ell\mathfrak{C}_1\equiv\mathfrak{C}_2 \pmod{p^{2r-2}}$, so that for $r\geq 2$ we get the second congruence from the first. □

Proof of Theorem 6.6. Note that

$$\frac{D^{(\nu)}p^{2k-2m-1}}{\varphi(p^r)}\sum_{\chi\in\Delta_r^{(\nu)}}\bar{\chi}(e)G(\bar{\chi})p^{m_\chi(k+m-3)}c_{np^{2r-2},j}^{(\nu)}(\chi)$$
$$=\frac{\ell\mathscr{C}_j^{(\nu)}p^{2k-2m-1}}{\mathscr{C}_0^{(\nu)}\varphi(p^r)}\sum_{(n_1,n_2)\in W_{np^{2r-2}}}n_1^{\nu}n_2^{\frac{k-\nu-m}{2}-j}\sum_{\chi\in\Delta_r^{(\nu)}}\bar{\chi}(-en_1)\sigma'_{m-\frac{3}{2},\chi\rho_t}(n_2)$$

and this is in $\mathbb{Q}$. Indeed, ℓ, $(\mathscr{C}_0^{(\nu)})^{-1}\mathscr{C}_j^{(\nu)}\in\mathbb{Z}$, $k-\nu-m\in 2\mathbb{Z}$ and finally $\Delta_r^{(\nu)}$ consists solely of all even characters, or else it's all odd characters, and so separating the χ in the definition of $\sigma'_{m-\frac{3}{2},\chi}(n_2)$ and shunting the rest to the preceding sum, then we can use orthogonality relations to see that it is in fact rational. If we replace $c^{(\nu)}_{np^{2r-2},j}(\chi)$ in the above by the coefficients of the holomorphic projection $c^{(\nu)}_{np^{2r-2}}(\chi)$ then we obtain $\mathfrak{a}^{(\nu)}_n(r)$, in the case $(-1)^{\lfloor k\rfloor+\nu} \neq 1$, and by Lemma 6.7 this is still in $\mathbb{Q}$. The other case is very similar.

That they are in $\mathbb{Z}_p$ is immediate from Lemma 6.10 when $(-1)^{\lfloor k\rfloor+\nu} = 1$ since we just have the congruence $\mathfrak{C}_2$. If $(-1)^{\lfloor k\rfloor+\nu} = -1$ then we have that $C^{(\nu)}_r = \Delta^{(\nu)}_r$ does not contain the trivial character, and that the $c^{(\nu)}_{n,j}(\chi)$ satisfies the congruence

$$\mathfrak{C}_3 := D^{(\nu)} p^{2k-2m-1} \sum_{\chi\in\Delta^{(\nu)}_r} \bar{\chi}(e)G(\bar{\chi})p^{m_\chi(k+m-3)}c^{(\nu)}_{np^{2r-2}}(\chi) \equiv 0 \pmod{p^r}$$

is immediate via the logic of the proof of Lemma 6.10. This cancels out the power of p^r appearing in the denominator of $\mathfrak{R}_r$, and this ends the proof. □

The proof of Theorem 6.1 now follows. Let $F = \mathbb{Q}(f_1)$ and let $\mathscr{M}_k(N_0, F)$ be the F-space of modular forms of weight k with Fourier coefficients in F. There's an F-linear form

$$\begin{aligned} \ell_f : \mathscr{M}_k(N_1, F) &\to F \\ h &\mapsto \frac{\langle f_1|[W(N_1)]_\kappa, h\rangle_{N_1}}{\langle f_1|[W(N_1)]_\kappa, f_1^\rho\rangle_{N_1}}. \end{aligned}$$

Proposition 6.11. *Let $\varepsilon(f) = \pm 1$ satisfy $f|[W(N)]_\kappa = \varepsilon(f)f$, then there exists $X \in \mathbb{C}_p(x)$ whose coefficients have p-adic norm independent of m and that satisfies*

$$\langle f_1|[W(N_1)]_\kappa, f_1^\rho\rangle_{N_1} = \varepsilon(f)X(p^{-k})\langle f, f\rangle_N.$$

Proof. Since p and N are corprime we have that f_{χ_p} is once again a newform of level N_1, so let $\varepsilon(f_{\chi_p})$ satisfy $f_{\chi_p}|[W(N_1)]_\kappa = \varepsilon(f_{\chi_p})f_{\chi_p}$ putting $\varepsilon_\chi := \varepsilon(f_{\chi_p})\varepsilon(f)$. Since $W(N_1) = W(N)\begin{pmatrix} p^2 & 0 \\ 0 & 1\end{pmatrix}^*$ we obtain

$$f_1|[W(N_1)]_\kappa = \varepsilon(f)[p^{2k-2}f(p^2z) - \varepsilon_\chi \left(\frac{-1}{p}\right)^{\lfloor k\rfloor} p^{\lfloor k\rfloor-1}\alpha_p^{-1} f_{\chi_p} - \alpha_p^{-1} f]$$

and so writing out the definition of f_1 and expanding the inner product we have a linear combination of inner products which we now deal with separately. The easiest are $\langle f, f^\rho\rangle_{N_1} = p^2\langle f, f\rangle_N$ and $\langle f(p^2z), f^\rho(p^2z)\rangle_{N_1} = p^{-2}\langle f, f\rangle_N$, the latter being just a change of variables followed by a reduction of the level.

We exploit the identity

$$\begin{aligned} &\langle f - f(p^2z)|[W(N_1)]_\kappa, f - f(p^2z)\rangle \\ &\quad = \langle f - f(p^2z), f^\rho - f^\rho(p^2z)|[W(N_1)]_\kappa\rangle \end{aligned}$$

to get $\langle f(p^2z), f^\rho\rangle = \langle f, f^\rho(p^2z)\rangle$. Then make use of (c) in the proof of Lemma 3.1 to get

$$(p^{2k}+p^{2k-2})\langle f(p^2z), f^\rho\rangle_{N_1} = p^2\omega_p\langle f, f\rangle_N - \left(\tfrac{-1}{p}\right)^{\lfloor k\rfloor} p^{\lfloor k\rfloor-1}\langle f_{\chi_p}, f^\rho\rangle_{N_1}$$

using the fact that the adjoint of $[T(p)]_\kappa^2$ is $\left[\left(\begin{smallmatrix}p^2&0\\0&1\end{smallmatrix}\right)\right]_\kappa$ via Lemma 4.3 on the coset decompositions.

It remains just to calculate inner products involving f_{χ_p}. The easiest is the product $\langle f_{\chi_p}, f^\rho(p^2z)\rangle_{N_1} = p^{-2k}\langle f_{\chi_p}|[T(p)]_\kappa^2, f^\rho\rangle_{N_1} = 0$ since we know that $f_{\chi_p}|[T(p)]_\kappa = 0$. Note that $D(s, f_{\chi_p}, f_{\chi_p}) = D(s, f, f)$ since $\chi_p^2 = 1$, so by Proposition 22.2 (3) in [20, p. 178] we have $\langle f_{\chi_p}, f_{\chi_p}\rangle_{N_1} = \langle f, f\rangle_{N_1}$.

The hardest and final product to calculate is $\langle f_{\chi_p}, f^\rho\rangle_{N_1}$; by [20] we have the relation

$$\langle f_{\chi_p}, f^\rho\rangle_{N_1} = \left[\frac{\mathrm{Res}_{s=k}\, D(s, f_{\chi_p}, f^\rho)}{\mathrm{Res}_{s=k}\, D(s, f, f)}\right]\langle f, f\rangle_{N_1}.$$

Note that

$$D(s, f_{\chi_p}, f) = \sum_{n=1}^{\infty}\left(\tfrac{tn^2}{p}\right)a(tn^2)a(tn^2)n^{-s} = \left(\frac{t}{p}\right)\sum_{p\nmid n} a(tn^2)a(tn^2)n^{-s}$$

whose Euler product will be the same as $D(s, f, f)$ but with the Euler factors at p removed. So by taking $t = 1$ this inner product becomes $\langle f_{\chi_p}, f\rangle_{N_1} = X_p^\circ(p^{-k})^{-1}Y_p(p^{-k})\langle f, f\rangle_{N_1}$ where

$$\begin{aligned} Y_p(x) &= (1-\alpha_p\beta_p x)^2(1-\alpha_p^2x)(1-\beta_p^2x)\\ X_p^\circ(x) &= 1+\left[2(\alpha_p+\beta_p)\left(\frac{-1}{p}\right)^{\lfloor k\rfloor}\left(\frac{t}{p}\right)p^{\lfloor k\rfloor-1}\right]x\\ &\quad+[(\alpha_p+\beta_p)(\alpha_p^2+\beta_p^2+\alpha_p\beta_p)+(\alpha_p+\beta_p)^2p^{2\lfloor k\rfloor-2}-\alpha_p^2\beta_p^2]x^2. \end{aligned}$$

Collecting this all together gives the proposition. □

By the above lemma we are left with

$$\mu_{f,m}^{(\nu)}(e+p^r\mathbb{Z}_p) = t^{\frac{\nu}{2}}\zeta\alpha_p^{-r}\varepsilon(f)X(p^{-k})\ell_f(\mathfrak{R}_r^{(\nu)}).$$

Let $\mathscr{J}_k(N_1) = \mathscr{M}_k(N_1, \mathbb{Z}_p\cap\mathbb{Q})$ denote the $\mathbb{Z}$-module of forms in $\mathscr{M}_k(N_1)$ with rational and p-integral coefficients. By Theorem 6.6 we have $\mathfrak{R}_r^{(\nu)} \in \mathscr{J}_k(N_1)$. It is known that $\mathscr{J}_{\lceil k\rceil}(N_1)\otimes_{\mathbb{Z}}\mathbb{Z}_p$ is a finitely generated $\mathbb{Z}_p$-module, and thus so too is $\mathscr{J}_k(N_1)\otimes_{\mathbb{Z}}\mathbb{Z}_p$ via the embedding $f\mapsto\theta f$. In extending ℓ_f by $F_{\mathfrak{p}}$-linearity to $\mathscr{M}_k(N_1, F)\otimes_F F_{\mathfrak{p}}$ where, recall, $F = \mathbb{Q}(f_1)$ and $F_{\mathfrak{p}}$ is the completion of F at some prime $\mathfrak{p}$ above p, we are done, as then $\ell_f(\mathfrak{R}_m^{(\nu)})$ will always be bounded by the generator of $\mathscr{J}_k(N_1)\otimes_{\mathbb{Z}}\mathbb{Z}_p$ with the largest p-adic norm.

To now actually define the p-adic L-function we let ω denote the Teichmüller character and χ be any Dirichlet character of p-power conductor. Normalise each measure $d\mu_{f,m}^{(\nu)}$ by $(D^{(\nu)})^{-1}$.

Proposition 6.12. *We have the following relation between the measures*

$$d\mu^{(\nu)}_{f,m}(x) = x^{m-k+\nu} d\mu^{(\nu)}_{f,k-\nu}$$

Proof. As seen in the proofs of Lemma 6.10 and Theorem 6.6 we have see that our measures come down to an expression in $M_{r,m}(x)$ and so if we show that

$$M_{r,m}(x) \equiv x^{m-k+\nu} M_{r,k-\nu}(x) \pmod{p^{2r-1}}$$

then in the limit we obtain our desired relation on measures.

As already noted, there exists a p-adic measure interpolating ${}^p\sigma'_{m-\frac{3}{2},\chi\rho_t}(n)$, in particular at the end of Section 7.1 in [7] one sees that there exists a power series $B_\chi(n; T) \in \mathbb{Z}_p[[T]]$ such that $B_{\chi\rho_t}(n; \chi\rho_t\omega^k(u)u^{m-\frac{3}{2}} - 1) = {}^p\sigma'_{m-\frac{3}{2},\chi\rho_t}(n)$ where u is a topological generator of $\mathbb{Z}_p^\times$. Thus we obtain an identity similar to (3.20) in [14], which, in the proof of the analogous Lemma 3.9 in [14, pp. 617–618], is the one remaining dependence on the weights of the modular forms separating our settings. So our result follows now exactly as in [14]. □

Define our p-adic L-function by

$$\mathcal{L}_p(s, f, \chi) := \int_{\mathbb{Z}_p^\times} (\bar{\chi}\omega^{s-1})(z) z^{s-k+\nu} d\mu^{(\nu)}_f$$

where $d\mu^{(\nu)}_f = d\mu^{(\nu)}_{f,k-\nu}$.

Corollary 6.13. *Let χ be a Dirichlet character whose conductor is p^{m_χ} and satisfies $\chi(-1) = (-1)^{\lfloor k \rfloor + \nu}$. Let m be one of the values specified in Theorem 6.1.*

(i) *If $m_\chi \geq 1$ then*

$$\mathcal{L}_p(m-\nu, f, \chi) = \delta^\nu_k(\chi) p^{2k-2m-1} \alpha_p^{-m_\chi} p^{m_\chi(k+m-3)} \times \frac{G(\chi\bar{\omega}^{m-\nu-1}) L(m-\nu, f, \bar{\chi}\omega^{m-\nu-1})}{g^\chi_t(m)\pi^{\frac{m+k-\nu-2}{2}}\langle f, f\rangle_N};$$

(ii) *If $\chi = \chi_0$ then*

$$\mathcal{L}_p(m-\nu, f, \chi_0) = \delta^\nu_\chi(1) p^{3-2m}(1-\beta_p p^{m+\nu-k-1})(1-\beta_p p^{2-k-m+\nu}) \times \frac{L(m-\nu, f, \chi_0\omega^{m-\nu-1})}{g^0_t(m)\pi^{\frac{m+k-\nu-2}{2}}\langle f, f\rangle_N}.$$

7. A comparison with the integer weight measure

Given the Shimura correspondence, we expect our interpolation above to yield the interpolation of L-values of integer weight modular forms as well. So in this section we assume $\nu = 0$, state the interpolation in the integer weight case and see how the two match up at the special value $m = k$. If $f \in S_\ell(Np^r, L)$ is an ordinary p-stabilised newform, where now $\ell \in \mathbb{Z}$ and $L \subseteq \overline{\mathbb{Q}}_p$ is a finite extension of $\mathbb{Q}_p$, and if χ is a Dirichlet character of p-power conductor p^{m_χ} then, according to [22, p. 34], the p-adic L-function of f is given by

$$\mathcal{L}_p(m, f, \chi) = \alpha_p^{-m_\chi}\left(1 - p^m \alpha_p^{-1}\right)(-1)^\ell p^{mm_\chi} \frac{G(\chi\bar{\omega}^{-m})m! L_{\mathrm{sk}}(f, \bar{\chi}\omega^{-m}, m+1)}{(-2\pi i)^m \Omega_f^{(-1)^m}}$$

for integers $0 \leq m \leq k-2$. This measure works on the assumption that f has already been p-stabilised, so let's also assume this for our half-integral form too. For emphasis on the different normalisation of the L-function in [22, p. 34] we put L_{sk}. Then $L_{\mathrm{sk}}(f, 2k-2) = L(f, k)$. Note that under the Shimura correspondence our half-integral weight form f of weight $k = \frac{\kappa}{2}$ with $\kappa > 5$ an odd integer becomes an integer weight form $\tilde{f}$ of weight $\ell = \kappa - 1$. Taking $m = \ell - 2$ as the special value, then $m+1 = \ell - 1 = 2k-2$ corresponds to our value $L(\kappa - 1, \tilde{f})$ which by the Shimura correspondence is equal to $L(k, f)$. We can always find a character ψ such that if we twist the L-functions by ψ then the L-functions are non-vanishing at k, see the main theorem of [12, p. 382]. For such a twist we can then compare the two different periods. The ψ determines which ν we shall need to take, but then we see that

$$i^{2k-3}\pi\Omega_{\tilde{f}}^{(-1)^{2k-3}} \in \langle f, f\rangle_N \mathbb{Q}(f_1, \psi).$$

7.1. Integrality

A final point of interest from this construction is in the determination of the integrality of the measure produced. The measure is integral when it takes values in $\mathbb{Z}_p$, or equivalently when it corresponds to an element of $\mathbb{Z}_p[[T]]$.

As a result of the Shimura correspondence any differences in determining integrality of the measure in our setting then offers up some alternative insights into determining the integrality of the original p-adic measure for the L-function of integer weight modular forms. The periods appearing in the denominator of the measure are naturally pivotal to the integrality of the measures and, as we have seen above, our construction here differs significantly with that found in [7], which uses the Eichler-Shimura isomorphism and modular symbols. In that construction, integrality is determined via congruences between cusp forms and Eisenstein series.

Our construction is much closer in line with the p-adic measure for the adjoint square L-function of modular forms of integer weight, as seen in [3,8], in which $\langle f, f\rangle$ plays the role of the period. In the construction of the p-adic adjoint square, questions of integrality are settled through the congruence module, as seen in [8, p. 296]. The potential upshot of this is that integrality for the p-adic measure constructed in this paper is likely to be through the congruence module which would involve congruences between cusp forms of *half-integer* weight.

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